3-1-2004

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Suggested Citation

Humi, Mayer (2004). Estimation of the Atmospheric Structure Constants from Airplane Data. *Journal of Atmospheric and Oceanic Technology, 21*(3), 495-500.
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Estimation of the Atmospheric Structure Constants from Airplane Data

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(Manuscript received 19 May 2003, in final form 20 August 2003)

ABSTRACT

Meteorological measurements in the upper troposphere and lower stratosphere were carried out in the past by several high-flying airplanes. It is a common practice to use the temperature spectrum of the time series obtained from these measurements to compute an averaged value (over the flight path) of the temperature structure constant C_T^2. In this paper time–frequency techniques are used to obtain an estimate for the local spectra of the temperature, which is utilized to obtain local estimates for C_T^2.

1. Introduction

The structure function of a geophysical variable, for example, temperature, is defined as

$$D(\mathbf{r}) = \langle T'([\mathbf{r}_1 + \mathbf{r}), T'(\mathbf{r}_1)]^2\rangle, \tag{1.1}$$

where T' are the turbulent fluctuations in the temperature.

Kolmogorov showed that for isotropic turbulence in the inertial range this function depends only on $r = |\mathbf{r}|$ and scales as

$$D(r) = C_T^2 r^{2/3}, \tag{1.2}$$

where C_T^2, the proportionality constant in this equation, is referred to as the ''temperature structure constant.''

The determination of the atmospheric structure constants (Tatarskii 1967; Jumper and Beland 2000; Cote et al. 2003) and, in particular, the temperature structure constant C_T^2 is important in many applications, for example, the propagation of electromagnetic signals (Fanta 1985; Dewan 1980). In particular, local peaks in the values of these constants, which are indicative of strong turbulence and reflect on the structure of the atmospheric flow, can have a negative effect on the operation of various optical instruments.

In estimating these structure constants in the upper troposphere or the stratosphere it is a common practice to send high-flying airplanes to collect data about the basic meteorological variables (such as wind, temperature, and pressure) along a flight path that may extend up to 200 km. To estimate the averaged value of the structure constants along this path one must decompose first the meteorological data into mean flow, waves, and turbulent residuals. From the spectrum of the turbulent residuals one can estimate an averaged value of the structure constants using Kolmogorov inertial range scaling and Taylor's frozen turbulence hypothesis (Panofsky and Dutton 1989). For C_T^2 in particular one obtains

$$C_T^2(k) = 4F(k)k^{5/3}, \tag{1.3}$$

where $F(k)$ is the temperature spectral density in the inertial range and k is the wavenumber. An averaged value for C_T^2 (over all wavenumbers in the inertial range) is obtained by averaging these values over k.

However, spectral estimation might be biased by discontinuities in the data, padding, and the choice of the spectral window (Kay 1988). Furthermore, the measurements are usually carried at constant frequency, and in order to convert the spectrum from frequency to wavenumber dependence one applies the prescription given in Panofsky and Dutton (1989): $k = \omega/\overline{V}$, where ω is the frequency and $\overline{V}$ is the averaged value of the true air speed, which (obviously) can vary considerably over 200 km. Furthermore, the averaged value of the structure constant might mask strong local peaks.

An obvious partial solution to these problems is to divide the original time series of measurements into shorter ones (i.e., use windowed spectral analysis) and thus obtain several (''local'') estimates of the structure constant. However, in this procedure one estimates the spectral density using ''short'' time series, which increases the bias of this estimation.

From a statistical point of view, if the time series data were stationary, then these paradigms might still be appropriate. However, it stands to reason that the meteorological data are actually nonstationary, or nonhomogeneous in some sense. Under these circumstances, it is imperative to look at the data either on the time–

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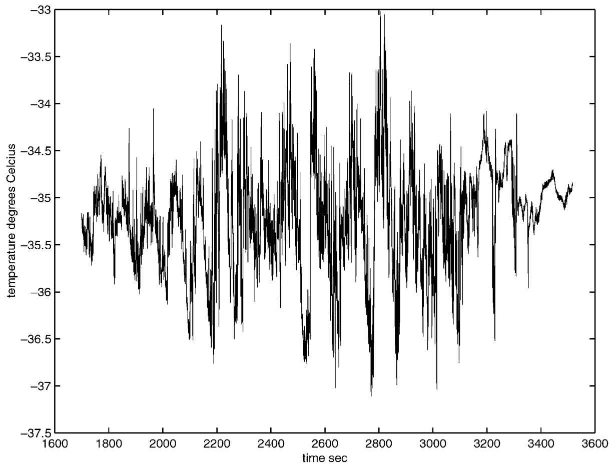

FIG. 1. The time series for the temperature at 9650 m.

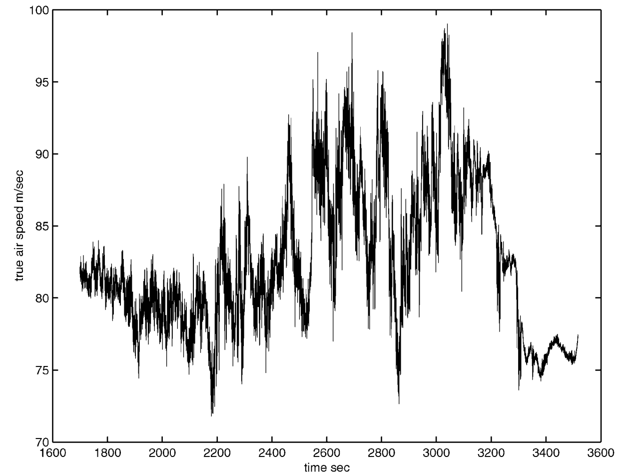

FIG. 2. The time series for the true air speed at 9650 m.

frequency plane or on multiple scales over time. Time–frequency and time-scale (wavelets) methods enable us to estimate the variations in the value of C_T^2 along the flight path and as a function of different scales.

To obtain these local estimates for the structure constants this paper applies a technique borrowed from time–frequency analysis (Cohen 1995). This technique [which is commonly used in signal analysis (Yen 1987; Imberger and Boashash 1986)] yields a local estimate for the spectra as a function of the frequency. Since the spectral estimate is local, one can convert the spectrum accurately from frequency to wavenumber dependence without bias (using the true air speed) and hence obtain a local value for the structure constants. As a result the value of these structure constants as a function along the flight path is obtained.

While the ''robustness'' of this approach is debatable, it is possible to argue that it offers the best methodology available to us today to obtain insight about the fluctuations of the structure constants and hence can be important for applications where these constants are used.

In this paper the meteorological data that were obtained on 6 August 1999 over Australia at a height of 9650 m (Cote et al. 2003) will be analyzed. For this dataset the averaged value, the windowed estimates, the time-scaled estimates, and the time–frequency analysis for C_T^2 are presented. (The potential temperature is not used since the plane was flying at constant height and the pressure was almost constant.)

The plan of the paper is as follows. In section 2 a presentation of the data and their statistical attributes is made. Section 3 presents a short overview of time–frequency and time-scale analysis. In section 4 estimates are obtained for the value of C_T^2 by averaging over the whole time series, by windowed spectral analysis, and by time-scale and time–frequency analysis. A summary and some conclusions are presented in section 5.

2. The data and its statistical attributes

In this paper the temperature data that were collected by Cote et al. (2003) on 6 August 1999 over Australia at approximate heights of 9100, 9300, and 9650 m with a sampling frequency of 55.1 Hz will be analyzed. These measurements were taken along a flight path of 180 km (approximately), and the plane was flying in a straight line at an approximate speed of 103 m s^{-1}. Figures 1 and 2 are plots of the temperature and true air speed along the flight path at 9650 m.

To use these data to estimate C_T^2 one has to split the original measurements into a sum of mean flow, waves, and turbulent residuals. To accomplish this task the Karhunen–Loeve (K–L) decomposition algorithm, which has been used by many researchers (e.g., Penland et al. 1991; Humi 2003), will be used. Humi et al. (2003) applies the K–L decomposition to the same data considered here, and the details of the decomposition that are given there will not be repeated here.

Based on instrument specifications the data noise should be at a relative error level of 10^{-3}. This is confirmed by the eigenvalues obtained in the K–L decomposition where the last few eigenvalues (which reflect the noise level in the data) are of order 10^{-3} of the leading eigenvalue.

To determine if the original time series or the one derived for the turbulent residuals are statistically stationary the autocorrelation function (ACF) test (Brockwell and Davis 2001) is used. According to this test, a time series X_t, $t = 0, 1. . .$ is (weakly) stationary if the first and second moments of X_t are the same as those of X_{t+h}. That is,

$$\mu(t) = E(X_t) \tag{2.1}$$

is independent of t and

$$\mathrm{ACF}(t, t + h) = \mathrm{cov}(X_t, X_{t+h}) \tag{2.2}$$

is independent of t for each lag h,

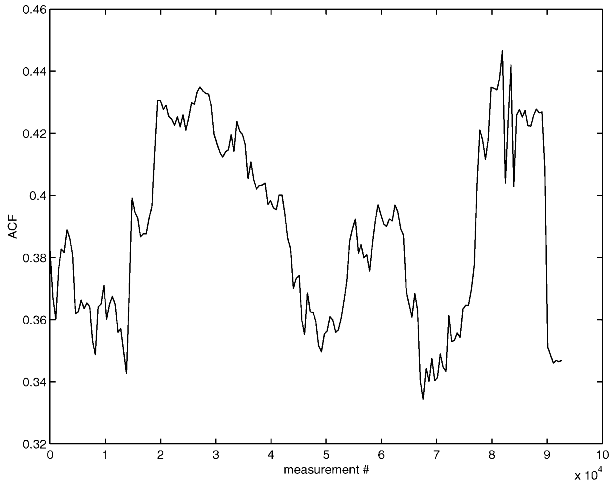

FIG. 3. The results of the ACF test for the temperature at 9650 m.

where $E(X)$ is the expectation value of X and $\mathrm{cov}(X_r, X_s)$ is the covariance of these two random variables. Figure 3 is a plot of the ACF for the temperature data at 9650 m for $h = 8$. [The data were detrended using moving averages to ensure that they satisfy Eq. (2.1)]. It demonstrates clearly that this test fails; that is, the data cannot be considered stationary as there is more than 20% variation in the values of the ACF. As a result, attempts to use statistical methods (e.g., spectral estimates) that apply only under this assumption will, naturally, lead to biased estimators.

3. Time–frequency analysis

Given a signal $s(t)$ its spectrum represents (the time averaged) energy density of the signal. Wigner (1971), and many others since then (Cohen 1995), investigated the possibility of describing the energy density of the signal in time and frequency simultaneously. This is the essence of time–frequency analysis.

To achieve this goal Wigner introduced (what is known today as) the Wigner distribution, which is defined as

$$W(t, \omega) = \frac{1}{2\pi}\int s^*\left(t - \frac{1}{2}\tau\right)s\left(t + \frac{1}{2}\tau\right)e^{-i\tau\omega}\, d\tau. \quad (3.1)$$

It is easy to show that this distribution satisfies the ''marginal constraints''

$$\int_{-\infty}^{\infty} W(t, \omega)\, d\omega = |s(t)|^2, \quad (3.2)$$

$$\int_{-\infty}^{\infty} W(t, \omega)\, dt = |\hat{s}(\omega)|^2, \quad (3.3)$$

that is, the average of W over frequency or time yields the square of the original signal or the spectrum density, respectively. [In addition, the Wigner distribution can provide another test for the nonstationary character of a signal (Flandrin 1999, p. 334).]

It is straightforward to see that for a single wave or impulse the Wigner distribution yields the ''correct'' answers. Thus, for

$$s(t) = e^{i\omega_0 t}, \qquad W(t, \omega) = \delta(\omega - \omega_0) \quad (3.4)$$

and for

$$s(t) = \sqrt{2\pi}\delta(t - t_0), \qquad W(t, \omega) = \delta(t - t_0). \quad (3.5)$$

However, for a superposition of waves interference terms appear in $W(t, \omega)$. Many attempts were made in the literature to ''mitigate'' the effect of these ''ghost'' terms on $W(t, \omega)$, and recent research in this area yielded an optimal time–frequency representation for such signals (Jones and Baraniuk 1995):

$$c(t, \omega) = \frac{1}{4\pi^2}\iint_{R^3}\int s^*\left(u - \frac{1}{2}\tau\right)s\left(u + \frac{1}{2}\tau\right) \times \phi(\theta, \tau)e^{-i(\theta t + \tau\omega - \theta u)}\, du\, d\tau\, d\theta. \quad (3.6)$$

In this equation the function ϕ is referred to as the kernel, and the coordinates θ, τ can be considered as representing inner degrees of freedom of the dynamical system that are being ''hidden'' by the measurement process. To compute the Wigner (and related) distributions in this paper the AOK4 code written by Jones and Baraniuk (1995) is used.

From our perspective the signal $s(t)$ is represented by a time series of one of the meteorological variables. Wigner distribution (or similar) can be used as a tool to analyze the contributions to the energy density of the signal as a function of time (or spatial extension, in case of simultaneous measurements over a spatial domain). To this end one observes that the time integral in Eq. (2.1) is over a finite time window (due to the finite length of the time series). Under these circumstances it is important to ensure that this time interval is longer than $1/(2\omega_m)$, where ω_m is the maximum measurement frequency. For our present case, $\omega_m = 55.1$ Hz, and this condition is clearly satisfied due to the length of the time series (which contain well over 10^5 points). This ensures that the estimate for $W(t, \omega)$ has the same accuracy for all analyzed ω.

From the Wigner distribution one can estimate the value of C_T^2 as a function of time (or, equivalently, along the flight path) by the following formula:

$$C_T^2(t) = \frac{1}{k_{\max} - k_{\min}}\int_{k_{\min}}^{k_{\max}} k^{5/3}W(t, k)\, dk. \quad (3.7)$$

In this equation the dependence of W on the frequency was converted to wavenumber using $k = \omega/V$, where V is the true air speed at time t. The inertial range boundaries $k_{\min}$ and $k_{\max}$ are somewhat arbitrary, which introduces some error margin in the value of C_T^2 as a function of these boundaries. Furthermore, the value of these two boundaries might be a function of time (and has to be determined dynamically during the computations). It was found, however, that the wavenumber segment [0.05, 0.22] (m^{-1}) was always contained in the inertial range and it was used consistently in Eq. (3.7).

Time-scale analysis provides a paradigm to examine the contents of the signal represented by the time series at different scales. In its ''naive'' form one averages the data over a number of points to obtain a decimated time series in which the time scale (or time resolution) is coarser than in the original time series. For example, if one averages each pair of data points one obtains a time series of half the length, with a time step between measurements that is twice the original. Thus, if one averages over 2, 4, 6. . . points and uses Eq. (1.1), one obtains in each case an estimate for C_T^2 that corresponds to that time (or length) scale.

A more sophisticated approach to the same problem will be to use a continuous wavelet, for example, the Mexican Hat function (the normalized second derivative of a Gaussian) to transform the signal and view it at different scales. Time–frequency analysis, however, has some conceptual advantages over time-scale methods, since the Wigner function (or similar) is intrinsic to the signal whereas the choice of the analyzing wavelet is somewhat arbitrary.

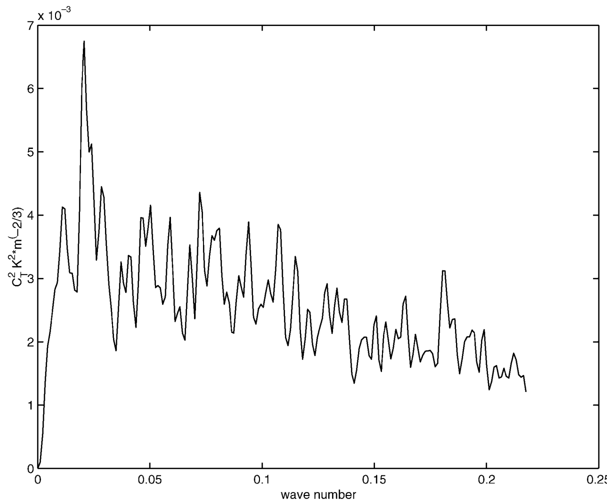

FIG. 4. Spectral estimate of C_T^2. The averaged value of C_T^2 over wavenumbers [0, 05, 0.22] is 2.5×10^{-3}; $\sigma = 4.5 \times 10^{-4}$.

4. Estimators for C_T^2

Four different methods will be used to estimate the value of C_T^2 along the flight path. The first of these is based on the use of Eq. (1.1) under the assumption that the time series under consideration are approximately stationary and that Taylor's frozen turbulence field is valid. With these assumptions this estimator yields an averaged value for C_T^2 over the flight path that is dependent on the wavenumber k (see Fig. 4). By averaging these values over an appropriate ''inertial range'' of wavenumbers over which the spectrum satisfies the Kolmogorov 5/3 law one finally obtains an estimate for the (averaged) value of C_T^2. For the present dataset an appropriate choice of this interval is [0.05, 0.22] (m^{-1}). Observe, however, that C_T^2 varies within the inertial range window by a factor of (almost) 3. Furthermore, the choice of the inertial range (which can be determined only approximately from the spectral plots) can induce a small error in the averaged value of C_T^2. This procedure has been used in the past by Cote et al. (2003) and many others.

A second estimator for C_T^2 is obtained by the use of windowed spectral estimates. To this end one considers around each point x_i of the time series a window of length $2m + 1$ centered around this point. In this window one computes the spectrum, identifies dynamically the inertial range, and uses Eq. (1.1) to obtain a ''local'' value for C_T^2 at x_i. Since the window length in this procedure is considerably shorter than that of the whole series it stands to reason that the assumption of statistical stationarity is more justified than that of the previous method. Also, the variations in true air speed in each window are not as large as in the whole series. Nevertheless, the cumulative bias in the value of C_T^2 over the whole time series might be the same as in the previous method. [This can be justified rigorously by computing the ACF in each window using Eqs. (2.1) and (2.2).] However, it is obvious that this technique cannot be applied to the first and last m points of the time series. Furthermore, the fact that ''short'' time series are used to estimate the spectrum introduces additional bias in the derived value of C_T^2 along the flight path. Moreover, implementation of this procedure is computationally intensive, as there is a need to compute the spectrum around each data point. This algorithm has been applied in windows of $4K$, $8K$, $16K$, and $32K$ points ($K = 1024$). As explained above, short windows introduce a bias in the spectral estimate while longer windows require a more intensive computational effort. In fact the use of $4K$, $8K$. . . points is dictated by the need to apply a fast Fourier transform (FFT) on the data under consideration, which can be done only if the data in each window consists of 2^n points. Figure 5 presents the estimates for C_T^2 along the flight path using this methodology in windows of $m = 16K$ points. This window length strikes a balance between the required computational effort and the reliability of the spectral estimates.

The third method used in this paper to estimate C_T^2 is based on the Wigner (1971) distribution function and the Jones and Baraniuk (1995) optimal adaptive kernel. These estimates, combined with Eq. (3.8), yield estimates for C_T^2 along the flight path. The results for the dataset under consideration are presented in Fig. 6. One observes that while the overall averaged estimate for C_T^2 is somewhat low, the local peaks show that C_T^2 can be as high as 7×10^{-3} ($K^2\ m^{-2/3}$). It should be observed

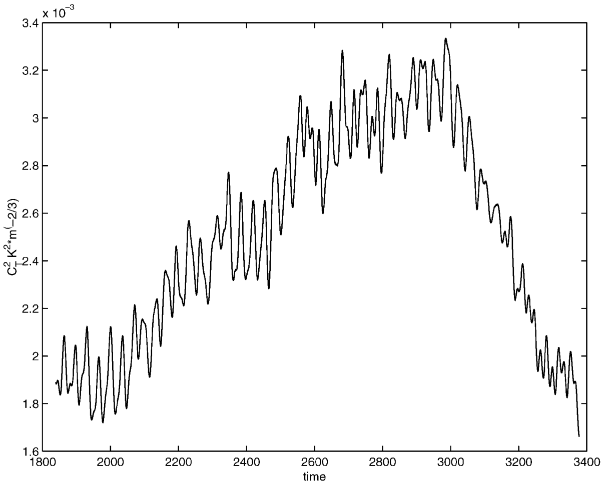

FIG. 5. The temperature structure constant C_T^2 along the flight path using windowed spectral analysis with $m = 16K$. Average $C_T^2 = 2.5 \times 10^{-3}$; $\sigma = 4.5 \times 10^{-4}$.

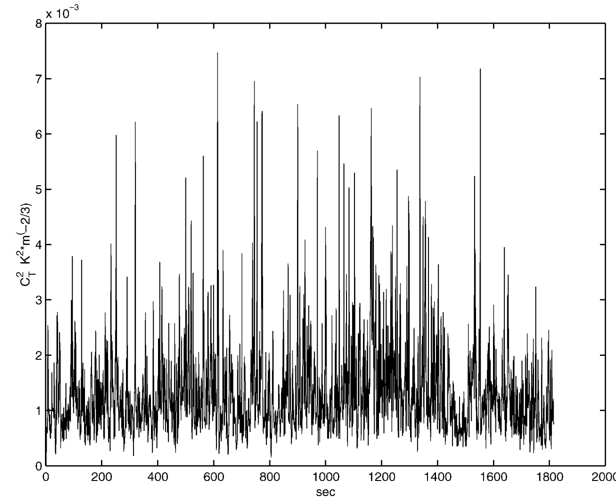

FIG. 6. Time–frequency evaluation of C_T^2 along the flight path. Average $C_T^2 = 1.3 \times 10^{-3}$; $\sigma = 1. \times 10^{-4}$.

that this plot is rather noisy compared to the plots obtained from the previous two methods, and one might argue that it provides an averaged value that is ''inconsistent'' with the previous two methods. Furthermore, the peaks in the value if C_T^2 do not correspond to those of the original series shown in Fig. 1. To answer this criticism one observes (as was discussed above) that the previous two methods provide ''manifestly biased estimators'' for C_T^2. Also, the peaks in the value of this quantity are related to the turbulent content of the time series presented in Fig. 1 (i.e., the turbulent residuals that were derived from the decomposition of the time series). Moreover, from a practical point of view (at least as far as optical turbulence is concerned), one is more interested in the peaks of this quantity (which might cause instrument failure) rather than its mean value. To put it more formally, one is interested in knowing the maximum value of this structure constant rather than its statistical average. Time–frequency analysis provides an unbiased estimate of these ''narrow peaks'' in the value of C_T^2. From a physical point of view, these peaks correspond to strong patches of turbulence in the atmosphere and reflect on the nature and structure of this flow.

The fourth estimator for C_T^2 uses time-scale (or length scale) methodology. The time series ''decimation'' approach has been used to compute C_T^2 as a function of the ''measurement length scales.'' Figure 7 presents a log–log plot of the estimates for C_T^2 with time averaging of the original time series from 1 to 10. The least squares slope of the line in Fig. 6 is -1.72 (which is close to the Kolmogorov $-5/3$ rule). The interpretation of this result is that as one increases the measurement scale the signal is being smoothed out and information about turbulence is lost. This leads to lower values of C_T^2. Our conjecture is, however, that the slope is a natural reflection of the Kolmogorov $-5/3$ rule in the inertial range and hence intrinsic to nature of turbulence in the atmosphere.

Similar results were obtained for other structure constants with slopes that varied around $-5/3$. To avoid confusion it should be observed that this ''measurement scaling'' of C_T^2 is different from the scaling of the corresponding structure function for the potential temperature with the distance r that is given by $r^{2/3}$.

This observation about the functional relation between the value of C_T^2 and the measurement scale might have an important practical application. In fact, many previous datasets of measurements in the stratosphere were made at lower frequencies (1–5 Hz) and spatial separations of 40 m or more between the data points. (This was due to the slow response of the instruments on board the plane.) If our conjecture regarding the universal slope of this curve is correct, then this will enable us to extrapolate the values of C_T^2 that were com-

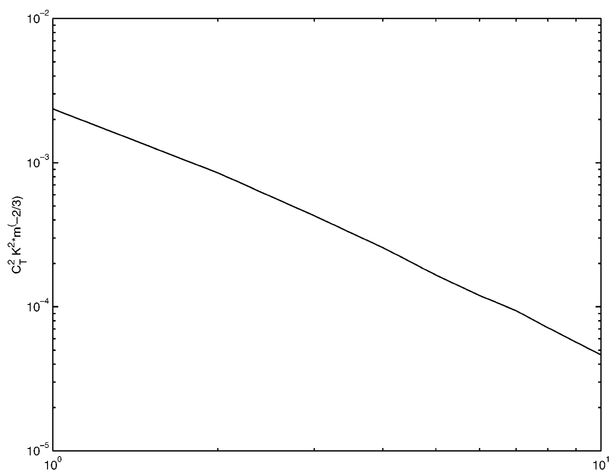

FIG. 7. Time-scale evaluation of C_T^2 (by time series decimation).

puted from these previous measurements to smaller scales as needed for particular applications.

5. Conclusions

Due to the nonstationary nature of the time series encountered in atmospheric research, extra care needs to be exercised in the estimation of the atmospheric structure constants. This is compounded by the fact that in this process one must convert between sampling frequency and wavenumber (which depends on the local true air speed).

This paper explores time–frequency methodology in an attempt to resolve these issues and compares the resulting values of C_T^2 with those obtained from the averaged and windowed spectral estimators. Our analysis also uncovers large variations in the value of this structure constant that are usually ignored in practice. To provide information about these variations a standard deviation analysis must be appended to the estimated value of the structure constant.

Time–frequency methods take into account the nonstationary nature of the signal and enable unbiased conversion between frequency and wavenumber. Furthermore, the kernel used by the Wigner (1971) functions (or similar) is self-adaptive and depends naturally on the actual data that are being investigated. Thus, these methods provide a better conceptual paradigm for the analysis of atmospheric time series. Furthermore, they yield unbiased estimators for the variation (maxima) of the structure constants along the flight path.

The paper examines also the dependence of the structure constants on the length-scale resolution of the measurements (or, equivalently, their frequency). It is shown that the computed value of the structure constant has a power-law dependence of $-5/3$ on this scale (which is reminiscent of the Kolmogorov law).

Finally, for completeness, it should be pointed out that the availability of the true air speed at each measurement point makes it possible to convert the original time series to a nonuniform space series. The spectrum of such series can be estimated using the Lomb–Scargle algorithm (Scargle 1982), or the series can be converted into equispaced series by the proper interpolation method. However, both of these procedures have their own shortcomings. First, the Lomb–Scargle algorithm has a high error threshold (it is used usually to extract only the most significant modes in the data). Furthermore, it has a tendency to shift the true frequencies of these modes. The interpolation approach, on the other hand, tends to smooth out the turbulence fluctuations and yields lower estimates for the spectral density. In general, these techniques are not highly recommended unless one has no other alternatives.

Acknowledgments. The author is deeply indebted to Dr. R. Beland and to O. Cote, of the U.S. Air Force Research Laboratory, and his team for access to the relevant data and for guidance regarding the issues involved in this research.

REFERENCES

Brockwell, P. J., and R. A. Davis, 2001: *Introduction to Time Series and Forecasting.* Springer-Verlag, 434 pp.

Cohen, L., 1995: *Time Frequency Analysis.* Prentice Hall, 299 pp.

Cote, O., J. Roadcap, D. E. Wroblewski, R. Dobosy, and T. L. Crawford, 2003: Aircraft measurements of turbulence: Length scales, spectra, budgets, and the prediction problem. Preprints, *12th Symp. on Meteorological Observations and Instrumentation,* Long Beach, CA, Amer. Meteor. Soc., CD-ROM, 1.1.

Dewan, E., 1980: Optical turbulence forecasting. Air Force Geophysical Laboratory Rep. AFGL-TR-80-0030, 63 pp.

Fanta, R. L., 1985: Wave propagation in random media. *Prog. Opt.,* **22,** 342–399.

Flandrin, P., 1999: *Time-Frequency/Time Scale Analysis.* Academic Press, 386 pp.

Humi, M., 2003: Approximate simulation of stratospheric flow from aircraft data. *Int. J. Numer. Methods Fluids,* **41,** 209–223.

Imberger, J., and B. Boashash, 1986: Application of the Wigner-Ville distribution to temperature gradient microstructure. *J. Phys. Oceanogr.,* **16,** 1997–2012.

Jones, D. L., and R. G. Baraniuk, 1995: An adaptive optimal kernel time-frequency representation. *IEEE Trans. Signal Process.,* **43,** 2361–2371.

Jumper, G. Y., and R. R. Beland, 2000: Progress in the understandings and modeling of atmospheric optical turbulence. *Proc. 31st AIAA Plasmadynamics and Lasers Conf.,* Denver, CO, AIAA, 1609–1616.

Kay, S. M., 1988: *Modern Spectral Estimation.* Prentice Hall, 543 pp.

Panofsky, H. A., and J. A. Dutton, 1989: *Atmospheric Turbulence.* Wiley, 397 pp.

Penland, C., M. Ghil, and K. M. Weickmann, 1991: Adaptive filtering and maximum entropy spectra with applications to changes in atmospheric angular momentum. *J. Geophys. Res.,* **96,** 22 659–22 671.

Scargle, J. D., 1982: Studies in astronomical time series analysis. II—Statistical aspects of spectral analysis of unevenly spaced data. *Astrophys. J.,* **263,** 835–853.

Tatarskii, V., 1967: *Wave Propagation in a Turbulent Medium.* Dover, 285 pp.

Wigner, E. P., 1971: Quantum mechanical distribution functions revisited. *Perspectives in Quantum Theory,* W. Yourgrau and A. Van der Merwe, Eds., MIT Press, 25–36.

Yen, N., 1987: Time frequency representation of acoustic signals by means of the Wigner distribution. *J. Acoust. Soc. Amer.,* **81,** 1841–1850.