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# Factorization of systems of differential equations

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It is shown that the classical Infeld–Hull factorization method can be extended to coupled systems of second-order equations. A complete solution of the factorization equations in two dimensions is given and a partial enumeration of factorizable systems is made.

## I. INTRODUCTION

As is well known, the classical factorization method of Infeld and Hull<sup>1</sup> plays an outstanding role in the theory<sup>2,3</sup> and applications<sup>4</sup> of the special functions of mathematical physics. The essence of the method is the reduction of

$$y'' + r(x, m)y + \lambda y = 0, \quad (1.1)$$

for some values of the parameter  $m$ , into first-order equations. Solutions for other values of  $m$  can be found then by successive applications of the raising and lowering operators.

In view of the effectiveness of this method, it is not surprising that several authors attempted to generalize and extend the method in various contexts.<sup>5–8</sup> Our objective in this paper is, therefore, to show that this classical method can be extended to coupled systems of second-order equations. Although the application of this extension to specific physical systems will not be considered here, we wish to point out that the coupled systems of Schrödinger equations appear in various physical contexts<sup>9,10</sup> and several authors sought to decouple such systems by means of Darboux transformations.<sup>11,12</sup> The factorization method, however, provides a different approach and new classes of coupled systems that are amenable to analytic treatment.

The plan of the paper is as follows: In Sec. II we discuss the factorization method for coupled systems in  $R^n$  and derive the basic equations. In Sec. III we restrict ourselves to two-dimensional systems and solve in full generality the factorization equations. In Sec. IV we demonstrate that there exist factorizable systems whose factorization kernel has no one-dimensional analog. Finally, in Sec. V we present a partial list of factorizable equations in  $R^2$ .

It should be remarked, however, that some of the “algebra” in this paper was carried out by using MACSYMA.

## II. FACTORIZATION OF SYSTEMS

Following the classical factorization technique, we say that a system of second-order differential equations

$$\underline{y}'' + R(x, m)\underline{y} + \lambda \underline{y} = 0, \quad \underline{y} \in R^n, \quad R(x, m) \in M(n) \quad (2.1)$$

can be factored if it is possible to replace it by both of the following systems

$$H_{m+1}^- H_m^+ \underline{y}(\lambda, m) = [\lambda - L(m+1)] I \underline{y}(\lambda, m), \quad (2.2)$$

$$H_{m-1}^+ H_m^- \underline{y}(\lambda, m) = [\lambda - L(m)] I \underline{y}(\lambda, m), \quad L(m) \in R, \quad (2.3)$$

where

$$H_m^+ = K(x, m+1) - \frac{d}{dx} I, \quad (2.4)$$

$$H_m^- = K(x, m) + \frac{d}{dx} I, \quad (2.5)$$

$K(x, m)$  is a  $n \times n$  matrix, and  $I$  is the  $n \times n$  identity matrix.

To find out for which  $R(x, m)$  the system (2.1) can be factored we carry out explicitly the multiplications of  $H^+$  and  $H^-$  in (2.2) and (2.3) using (2.1). We obtain

$$K^2(x, m+1) + K'(x, m+1) + L(m+1)I = -R(x, m), \quad (2.6)$$

$$K^2(x, m) - K'(x, m) + L(m)I = -R(x, m), \quad (2.7)$$

and hence

$$K^2(x, m+1) - K^2(x, m) + K'(x, m+1) + K'(x, m) = [L(m) - L(m+1)]I. \quad (2.8)$$

To determine those matrices  $K(x, m)$  that satisfy this equation, we examine three possible forms for the dependence of this matrix on  $m$ .

### A. $K(x, m) = K_0(x) + mK_1(x)$

From (2.8) this ansatz leads to

$$(m+1)^2(K_1^2 + K_1') + (m+1)(K_0K_1 + K_1K_0 + 2K_0') - m^2(K_1^2 + K_1') - m(K_0K_1 + K_1K_0 + 2K_0') = (L(m) - L(m+1))I. \quad (2.9)$$

Following the same argument as in Ref. 1, we conclude from (2.9) that

$$K_1' + K_1^2 = -a^2 I, \quad (2.10)$$

$$2K_0' + K_0K_1 + K_1K_0 = \begin{cases} -caI, & a \neq 0, \\ bI, & a = 0, \end{cases} \quad (2.11)$$

and

$$L(m) = \begin{cases} m^2 a^2 + ca, & a \neq 0, \\ -bm, & a = 0. \end{cases} \quad (2.12)$$

In Sec. III we shall consider some explicit solutions of the system (2.10) and (2.11) in the two dimensions.

### B. $K(x, m) = K_0(x) + mK_1(x) + (1/m)K_2(x)$

Substituting this form of  $K(x, m)$  in (2.8) we obtain

$$[(m+1)^2(K_1^2 + K_1') + (m+1)(2K_0' + K_1K_0 + K_0K_1) + [1/(m+1)](K_2' + K_2K_0 + K_0K_2) + K_2^2/(m+1)^2] - [\text{same terms with } m] = (L(m) - L(m+1))I. \quad (2.13)$$

Hence we infer that

$$K_2^2 = \gamma_1 I, \quad (2.14)$$

$$K_2' + K_2 K_0 + K_0 K_2 = \gamma_2 I, \quad (2.15)$$

$$2K_0' + K_1 K_0 + K_0 K_1 = \gamma_3 I, \quad (2.16)$$

$$K_1' + K_1^2 = -a^2 I. \quad (2.17)$$

Obviously, this system reduces to (2.10) and (2.11) if we set  $K_2 = 0$ , however, we observe that nontrivial solutions with  $K_2 \neq 0$  are also possible if we let  $K_2$  be a matrix with constant entries,  $K_0 = 0$ , while  $K_1$  satisfies Eq. (2.17). Furthermore, since (2.17) and (2.10) are the same it follows, then, that the two-dimensional solutions for  $K_1$ , which will be derived in Sec. III, essentially provide a solution for the form of  $K(x, m)$  under consideration.

### C. $K(x, m) = K_0(x) + mK_1(x) + m^2K_2(x)$

Substitution of this form of  $K$  in (2.8) leads, after some simple algebraic manipulations, to the following system;

$$K_2^2 = \gamma_1 I, \quad (2.18)$$

$$2K_2' + 3\{K_1, K_2\} = \gamma_2 I, \quad (2.19)$$

$$K_1' + \{K_0, K_2\} + K_1^2 = \gamma_3 I, \quad (2.20)$$

$$2K_0' - \frac{1}{2}\{K_1, K_2\} + \{K_0, K_1\} = \gamma_4 I, \quad (2.21)$$

where  $\{A, B\} = AB + BA$  and  $\gamma_i, i = 1, \dots, 4$  are constants. We observe that in contrast to the scalar case<sup>1</sup> one can not deduce from these equations that  $K_2 \neq 0$  leads to a trivial system of coupled equations, viz.  $R(x, m)$  is a matrix with constant entries. In fact, we shall show in Sec. IV that contrary to the negative results in one dimension, the systems (2.14)–(2.17) and (2.18)–(2.21) admit solutions with  $K_0 \neq 0$  and  $K_2 \neq \text{const}$ , respectively, in two dimensions. Consequently, these kernels and their corresponding factorizable systems have no analog in one dimension.

## III. FACTORIZATION OF SYSTEMS IN TWO DIMENSIONS

Although we derived the factorization equations (2.10) and (2.11) in  $R^n$  the number of coupled scalar equations which have to be solved to compute the entries of  $K_0, K_1$  increases rapidly with  $n$ . Due to this, we shall restrict ourselves in this section (and the rest of this paper) to systems in two dimensions. Our objective in this section is, therefore, to show that a complete closed form solution for  $K_0, K_1$  is available when  $n = 2$ . The solutions we find will depend on several arbitrary parameters, which demonstrates that in principle the factorization method is applicable to a large class of systems. From a practical point of view, however, these general expressions are rather cumbersome and one must set several of these parameters to zero in order to bring them to a more manageable (and presentable) form. This task will be carried, however, in Sec. IV where a partial enumeration of factorizable systems in two dimensions will be given.

### A. Calculation of $K_1$

Letting

$$K_1 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad (3.1)$$

and using (2.10) we obtain the following system of equations

$$\alpha' + \alpha^2 + \beta\gamma = -a^2, \quad (3.2)$$

$$\delta' + \delta^2 + \beta\gamma = -a^2, \quad (3.3)$$

$$\beta' + \beta(\alpha + \delta) = 0, \quad (3.4)$$

$$\gamma' + \gamma(\alpha + \delta) = 0. \quad (3.5)$$

Subtracting (3.3) from (3.2) and introducing

$$\mu = \alpha - \delta, \quad \nu = \alpha + \delta, \quad (3.6)$$

we obtain

$$\mu' + \nu\mu = 0. \quad (3.7)$$

From (3.4), (3.5), and (3.7) we see that  $\beta, \gamma$ , and  $\mu$  satisfy the same first-order differential equation and we have

$$\mu = c_1 J, \quad \beta = c_2 J, \quad \gamma = c_3 J, \quad (3.8)$$

where

$$J = \exp\left(-\int \nu(x) dx\right).$$

To compute  $\nu$  we add Eqs. (3.2) and (3.3) and use (3.8) to obtain

$$\nu' + \frac{\nu^2}{2} + \rho \exp\left(-2\int \nu(x) dx\right) = -2a^2, \quad (3.9)$$

where  $\rho = c_1^2/2 + 2c_2c_3$ . We now show that Eq. (3.9) can be solved in full generality. To this end, we first substitute (assuming  $\rho \neq 0$ )

$$w = \exp\left(\frac{1}{2}\int \nu(x) dx\right) \quad (3.10)$$

to obtain

$$2w'' = -\frac{d}{dw}\left(a^2w^2 - \frac{\rho}{2}w^{-2}\right). \quad (3.11)$$

Multiplying (3.11) by  $w'$  and using the chain rule, we obtain after integration

$$w' = [(\rho/2)w^{-2} - a^2w^2 + c_4]^{1/2}, \quad (3.12)$$

where  $c_4$  is an integration constant. Finally, we introduce  $z = w^2$ , which yields

$$x = c_5 + \frac{1}{2}\int \frac{dz}{(\rho/2 - a^2z^2 + c_4z)^{1/2}}, \quad (3.13)$$

where the last integral can be evaluated explicitly in terms of elementary functions.<sup>13</sup> After some algebra this yields

$$\nu = \begin{cases} \frac{2ah \cos 2a(x - c_5)}{h \sin 2a(x - c_5) + c_4}, & h \neq 0, \quad \rho \neq 0, \\ \frac{4a^4}{c_4 \exp[2a^2(x - c_5)] + 2a^2}, & h = 0, \quad \rho \neq 0, \end{cases} \quad (3.14)$$

where  $h = \sqrt{2a^2\rho + c_4^2}$ . When  $\rho = 0$ , Eq. (3.9) reduces to a Riccati equation (equivalent to the equation obtained for  $K_1$  in Ref. 1 for one differential equation) whose solutions are

$$\nu_1(x) = 2a \cot(ax + \phi), \quad \nu_2(x) = 2ia, \quad a \neq 0, \quad (3.16)$$

$$\nu_3(x) = 0, \quad \nu_4(x) = 2/(x + c_7), \quad a = 0. \quad (3.17)$$

The expressions for  $J(x)$  which correspond to these  $\nu(x)$  are not hard to obtain and will be omitted for brevity. (Some of these will be computed, whenever necessary, in Sec. IV.)

## B. Calculation of $K_0$

Letting

$$K_0 = \begin{pmatrix} r & s \\ p & q \end{pmatrix}, \quad (3.18)$$

and using (2.11), (3.1), and (3.8) we obtain the following system of equations:

$$2r' + 2ar + J(c_2p + c_3s) = -ca, \quad (3.19)$$

$$2s' + (\alpha + \delta)s + c_2J(r + q) = 0, \quad (3.20)$$

$$2p' + (\alpha + \delta)p + c_3J(r + q) = 0, \quad (3.21)$$

$$2q' + 2\delta q + J(c_2p + c_3s) = -ca. \quad (3.22)$$

(When  $a = 0$ ,  $-ca$  is replaced in these equations by  $b$ .)

Although Eqs. (3.19)–(3.22) represent a coupled system of equations we now show that they always can be solved analytically and the solution can be expressed explicitly in terms of  $J$  and its integrals. To verify this assertion we have to consider, however, several cases.

### 1. $c_1 \neq 0$ , $c_2c_3 \neq 0$ , and $\rho \neq 0$ (the "general case")

To begin with we multiply (3.20) and (3.21) by  $c_3$  and  $c_2$ , respectively, and add and subtract the resulting equations. This yields

$$2N' + (\alpha + \delta)N + 2c_2c_3JS = 0, \quad (3.23)$$

$$2M' + (\alpha + \delta)M = 0, \quad (3.24)$$

where

$$M = c_3s - c_2p, \quad N = c_3s + c_2p, \quad S = r + q. \quad (3.25)$$

Adding and subtracting (3.19) and (3.22) we obtain

$$2A' + (\alpha + \delta)A + c_1JS = 0, \quad (3.26)$$

$$2S' + (\alpha + \delta)S + J(c_1A + 2N) = -2ca. \quad (3.27)$$

Multiplying (3.23) by  $c_1$  and (3.26) by  $2c_2c_3$  and subtracting, leads to

$$2T' + (\alpha + \delta)T = 0, \quad (3.28)$$

where  $T = 2c_2c_3A - c_1N$ . Multiplying (3.26) by  $c_1$  and (3.23) by 2 and adding yields

$$2H' + (\alpha + \delta)H + 2\rho JS = 0, \quad (3.29)$$

where  $H = c_1A + 2N$ . Finally, introducing  $W = (2\rho)^{1/2}S = kS$  in Eqs. (3.27) and (3.29) and adding and subtracting the resulting equations we obtain

$$2X' + (\alpha + \delta + kJ)X = -2cak, \quad (3.30)$$

$$2Y' + (\alpha + \delta - kJ)Y = -2cak, \quad (3.31)$$

where  $X = H + W$ ,  $Y = W - H$ . Equations (3.24), (3.28), (3.30), and (3.31) form a decoupled system of equations which is equivalent to the original system. The explicit solution of the new system is given by

$$M = c_8 J^{1/2}(x), \quad T = c_9 J^{1/2}(x), \quad (3.32a)$$

$$X = J^{1/2}(x)L^{-1}(x) \left[ c_{10} - 2cak \int J^{-1/2}(x)L(x) dx \right], \quad (3.32b)$$

$$Y = J^{1/2}(x)L(x) \left[ c_{11} - 2cak \int J^{-1/2}(x)L^{-1}(x) dx \right], \quad (3.32c)$$

where  $L(x) = \exp((k/2)\int J(x) dx)$ .

The backward transformation from these functions to the original  $s$ ,  $p$ ,  $r$ , and  $q$  is given by

$$r = [(k - c_1)Y + (k + c_1)X + 4T]/4k^2,$$

$$q = [(k + c_1)Y + (k - c_1)X - 4T]/4k^2,$$

$$s = -[c_2c_3(Y - X) + c_1T - k^2M]/2c_3k^2,$$

$$p = -[c_2c_3(Y - X) + c_1T + k^2M]/2c_2k^2.$$

When any of the constants  $c_1$ ,  $c_2c_3$ , or  $\rho$  is equal to zero the solution of the system (3.19)–(3.22) is easier to obtain than in the general case. We discuss now briefly each of these situations.

### 2. $c_2c_3 = 0$ ( $\rho, c_1$ arbitrary)

If (let us say)  $c_2 = 0$ , Eq. (3.20) can be solved for  $s$  and (since  $c_2 = 0$ ) Eqs. (3.19) and (3.22) can then be solved for  $r$  and  $q$ . Equation (3.21) can be solved, then, for  $p$  since  $r$  and  $q$  are known

### 3. $c_1 = 0$ , $c_2c_3 \neq 0$ ( $\rho \neq 0$ )

We proceed as in case 1 up to Eqs. (3.26) and (3.27). Since  $c_1 = 0$ , Eq. (3.26) can be solved for  $A$ . The remaining two equations, (3.23) and (3.27), can then be solved by introducing  $W = kS$  and adding and subtracting these equations.

### 4. $\rho = 0$ , $c_1 \neq 0$ , $c_2c_3 \neq 0$

In this instance we can proceed as in case 1 up to Eq. (3.28). Since  $T$  is proportional to  $H$ , Eq. (3.27) can then be solved for  $S$ , and using this result we can solve Eq. (3.23).

## IV. SOME SPECIAL FACTORIZATIONS IN TWO DIMENSIONS

In this section we discuss solutions to the systems (2.14)–(2.17) and (2.18)–(2.21) in two dimensions. Our primary objective is to show that there exist solutions to these systems with  $K_0 \neq 0$  and  $K_2 \neq 0$  (respectively) which lead to new classes of factorizable systems of equations in contrast to the one-dimensional case. However, for the sake of brevity we shall omit the proofs of our statements, but will provide them in a separate publication elsewhere.

We divide our discussion into two cases. In the first part we consider the aforementioned systems with  $\gamma_1 \neq 0$ , while in the second we let  $\gamma_1 = 0$ .

### A. $\gamma_1 \neq 0$

**Theorem 1:** (1) In two dimensions, if  $\gamma_1 \neq 0$ , then the only solution to the system (2.18)–(2.21) is the trivial solution where  $K_0$ ,  $K_1$ , and  $K_2$  are matrices with constant entries.

(2) Similarly, if  $\gamma_1 \neq 0$  and  $\gamma_2 \neq 0$ , then the only solution to the system (2.14)–(2.17) is the trivial solution.

**Theorem 2:** In two dimensions, if  $\gamma_1 \neq 0$ ,  $\gamma_2 = 0$ , and  $\gamma_3 = 0$ , then there exist nontrivial solutions to (2.14)–(2.17) with  $K_0 \neq 0$ .

One particular class of nontrivial solutions which satisfies the conditions of Theorem 2 is given by

$$K_0 = J^{1/2} \begin{pmatrix} 0 & d_2 \\ d_3 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} q_1 & 0 \\ 0 & -q_1 \end{pmatrix}, \quad (4.1)$$

and  $K_1$  is the general solution of (2.17) (which was discussed in Sec. III) subject to the condition  $c_2d_3 + c_3d_2 = 0$ .

### B. $\gamma_1 = 0$

The general form of  $K_2$  in this case is

$$K_2 = \begin{pmatrix} q_1(x) & q_2(x) \\ q_3(x) & -q_1(x) \end{pmatrix}, \quad q_1^2 + q_2q_3 = 0. \quad (4.2)$$

However, to simplify the following discussion, we consider only the forms

$$K_2 = b(x) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = b(x)A, \quad (4.3)$$

$$K_2 = b(x) \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = b(x)B.$$

For each of these forms of  $K_2$  there exist nontrivial solutions for the systems (2.14)–(2.17) and (2.18)–(2.21).

**Theorem 3:** The system (2.14)–(2.17) in two dimensions admits nontrivial solutions with  $\gamma_1 = \gamma_2 = 0$ ,  $K_2 = b(x)A$  or  $b(x)B$ , and  $K_0 \neq 0$ .

The general form of the solution for the second case [ $K_2 = b(x)B$ ] is given by

$$K_0 = \begin{pmatrix} r & s \\ s & r \end{pmatrix}, \quad K_1 = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \quad (4.4)$$

where  $\alpha, \beta, r, s$ , and  $b$  must satisfy the equations

$$\begin{aligned} b' + 2rb &= 0, & \beta' + 2\alpha\beta &= 0, & \alpha' + \alpha^2 + \beta^2 &= -a^2, \\ s' + \alpha s + \beta r &= 0, & 2r' + 2\alpha r + 2\beta s &= \gamma_3. \end{aligned} \quad (4.5)$$

**Theorem 4:** The system (2.18)–(2.21) in two dimensions has nontrivial solutions with (1)  $\gamma_1 = \gamma_2 = 0$ ,  $K_2 = b(x)A$ , and (2)  $\gamma_1 = \gamma_2 = \gamma_4 = 0$ ,  $K_2 = b(x)B$ .

The general form of the solution for the second case of this theorem is given by

$$K_1 = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}, \quad K_0 = \begin{pmatrix} r & s \\ -s & -r \end{pmatrix} \quad (4.6)$$

where  $\alpha, \beta, r, s$ , and  $b$  satisfy

$$\begin{aligned} b' + 3ab &= 0, & \beta + 2\alpha\beta &= 0, \\ \alpha' + \alpha^2 + \beta^2 + 2b(r-s) &= \gamma_3, \\ 2r' + 2\alpha r &= ab, & 2s' + 2\alpha s &= \alpha b. \end{aligned} \quad (4.7)$$

## V. PARTIAL ENUMERATION OF FACTORIZABLE SYSTEMS IN TWO DIMENSIONS

In Sec. IV we saw that the analytic expressions for the entries of  $K_0$  and  $K_1$  contain several parameters and thereby lead to a large class of factorizable kernels  $R(x, m)$ . From a practical point of view one must set, therefore, some of these parameters to zero in order to obtain tractable expressions. We shall, therefore, assume in the following that either  $K = K_0 + mK_1$  or  $K = mK_1 + K_2/m$  with  $c_1 = c_2 = c_3 = 0$ , and give a complete enumeration of factorizable systems in two dimensions under these assumptions. (The resulting six classes are the two-dimensional analogs of those in Ref. 1.)

Furthermore, as final examples we shall present two special classes of factorizable systems which are related to those discussed in Sec. IV.

Using Eqs. (3.1)–(3.8) and (3.19)–(3.22) we see immediately that the entries of  $K_0$  and  $K_1$  are given by the following expressions:

$$\alpha = \delta = \nu/2, \quad \beta = \gamma = 0, \quad (5.1)$$

$$r = J^{1/2} \left[ d_1 - \frac{ca}{2} \int J^{-1/2} dx \right], \quad (5.2)$$

$$q = J^{1/2} \left[ d_2 - \frac{ca}{2} \int J^{-1/2} dx \right], \quad (5.3)$$

$$s = d_3 J^{1/2}, \quad p = d_4 J^{1/2}. \quad (5.4)$$

(When  $a = 0$ , one has to replace  $-ca$  by  $b$ .)

Evaluating these expressions explicitly for each of the possible forms of  $\nu(x)$  as given by (3.16) and (3.17), and letting

$$R(x, m) = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}, \quad (5.5)$$

we obtain the following formulas. [In (a), (b), (c), and (d) we assume that  $K = K_0 + mK_1$ , while in (e) and (f) we let  $K = mK_1 + (1/m)K_2$ .]

(a)  $\nu = 2a \cot(ax + \phi)$ .

$$L(m) = ma(ma + c),$$

$$K_1 = a \cot(ax + \phi)I,$$

$$K_0 = \frac{1}{2 \sin(ax + \phi)}$$

$$\times \begin{pmatrix} c \cos(ax + \phi) + 2d_1 & 2d_3 \\ 2d_4 & c \cos(ax + \phi) + 2d_2 \end{pmatrix},$$

$$\begin{aligned} R_{11} &= -\frac{1}{4 \sin^2(ax + \phi)} \{ c^2 \cos^2(ax + \phi) \\ &+ 4d_1 [a(2m + 1) + c] \cos(ax + \phi) \\ &+ 4am [a(m + 1) + c] + 2ac + 4(d_1^2 + d_3d_4) \}, \end{aligned}$$

$$\begin{aligned} R_{21} &= -\frac{d_4}{\sin^2(ax + \phi)} \\ &\times \{ [a(2m + 1) + c] \cos(ax + \phi) + d_1 + d_2 \}, \end{aligned}$$

$$\begin{aligned} R_{12} &= -\frac{d_3}{\sin^2(ax + \phi)} \\ &\times \{ [a(2m + 1) + c] \cos(ax + \phi) + d_1 + d_2 \}, \end{aligned}$$

$$\begin{aligned} R_{22} &= -\frac{1}{4 \sin^2(ax + \phi)} \{ c^2 \cos^2(ax + \phi) \\ &+ 4d_2 [a(2m + 1) + c] \cos(ax + \phi) \\ &+ 4am [a(m + 1) + c] + 2ac + 4(d_2^2 + d_3d_4) \}. \end{aligned}$$

(b)  $\nu = 2/(x + c_7)$ .

$$L(m) = -bmI, \quad K_1 = \frac{1}{x + c_7} I,$$

$$\begin{aligned} K_0 &= \frac{1}{x + c_7} \\ &\times \begin{pmatrix} [bx(x + 2c_7) + 4d_1]/4 & d_3 \\ d_4 & [bx(x + 2c_7) + 4d_2]/4 \end{pmatrix}. \end{aligned}$$

Due to the length of the expressions we present  $R$  only when  $c_7 = 0$ :

$$R_{11} = - \left[ \frac{b^2 x^2}{16} + \frac{(m+d_1)(m+d_1+1) + d_3 d_4}{x^2} + \frac{b}{4} (2d_1 - 2m - 1) \right],$$

$$R_{21} = -d_4 \left( \frac{b}{2} + \frac{d_1 + d_2 + 2m + 1}{x^2} \right),$$

$$R_{12} = -d_3 \left( \frac{b}{2} + \frac{d_1 + d_2 + 2m + 1}{x^2} \right),$$

$$R_{22} = - \left[ \frac{b^2 x^2}{16} + \frac{(m+d_2)(m+d_2+1) + d_3 d_4}{x^2} + \frac{b}{4} (2d_2 - 2m - 1) \right].$$

(c)  $\nu = 0$ .

$$L(m) = -bmI,$$

$$K_0 = \begin{pmatrix} bx/2 + d_1 & d_3 \\ d_4 & bx/2 + d_2 \end{pmatrix}, \quad K_1 = 0,$$

$$R_{11} = -(bx + 2d_1)^2/4 + b(m + \frac{1}{2}) - d_3 d_4,$$

$$R_{21} = -d_3(d_1 + d_2 + bx),$$

$$R_{12} = -d_4(d_1 + d_2 + bx),$$

$$R_{22} = -(bx + 2d_2)^2/4 + b(m + \frac{1}{2}) - d_3 d_4.$$

(d)  $\nu = 2ia$ . Following Ref. 1 we replace  $ia$  by  $a$ ,  $c$  by  $-ic$ , and add  $-a^2 c^2 I$  to  $L(m)$ . We obtain

$$L(m) = -a^2(m + c)^2 I, \quad K_1 = aI,$$

$$K_0 = e^{-ax} \begin{pmatrix} \frac{2d_1 - ce^{ax}}{2} & d_3 \\ d_4 & \frac{2d_2 - ce^{ax}}{2} \end{pmatrix},$$

$$R_{11} = c(2a + 1)[4ma + (2a - 1)c] + 4d_1 e^{-ax} [c - a(2m + 1)] - 4e^{-2ax}(d_3 d_4 + d_1^2),$$

$$R_{21} = -e^{-2ax} d_4 \{ e^{ax} [a(2m + 1) - c] + d_1 + d_2 \},$$

$$R_{12} = e^{-2ax} d_3 \{ e^{ax} [a(2m + 1) - c] + d_1 + d_2 \},$$

$$R_{22} = c(2a + 1)[4ma + (2a - 1)c] + 4d_2 e^{-ax} [c - a(m + 1)] - 4e^{-2ax}(d_3 d_4 + d_2^2).$$

As to the next two factorizations, where

$$K = mK_1 + m^{-1}K_2,$$

it follows immediately from (2.14) and the assumption that  $K_2$  is a matrix with constant entries that

$$K_2 = \begin{pmatrix} q_1 & q_2 \\ q_3 & -q_1 \end{pmatrix}$$

and  $q_1^2 + q_2 q_3 = \gamma_1$ . Using this relationship and Eqs. (2.11) and (2.13) we obtain the following.

(e)  $\nu = 2a \cot(ax + \phi)$ .

$$L(m) = m^2 a^2 - \frac{\gamma_1}{m^2},$$

$$K_1 = a \cot(ax + \phi) I, \quad K_0 = 0,$$

$$R_{11} = - \left[ \frac{a^2 m(m+1)}{\sin^2(ax + \phi)} + 2aq_1 \cot(ax + \phi) \right],$$

$$R_{21} = -2aq_3 \cot(ax + \phi),$$

$$R_{12} = -2aq_2 \cot(ax + \phi),$$

$$R_{22} = - \left[ \frac{a^2 m(m+1)}{\sin^2(ax + \phi)} - 2aq_1 \cot(ax + \phi) \right].$$

(f)  $\nu = 2/(x + c_7)$ .

$$L(m) = -\gamma_1/m^2,$$

$$k_1 = \frac{1}{x + c_7} I, \quad K_0 = 0,$$

$$R_{11} = - \frac{2q_1(x + c_7) + m(m+1)}{(x + c_7)^2},$$

$$R_{12} = - \frac{2q_2}{x + c_7},$$

$$R_{22} = \frac{2q_1(x + c_7) - m(m+1)}{(x + c_7)^2}, \quad R_{21} = - \frac{2q_3}{x + c_7}.$$

**SF-A:** This class of special factorizable (SF) systems corresponds to those described by Theorem 2. To satisfy the condition  $c_2 d_3 + c_3 d_2 = 0$  we let  $c_2 = c_3 = 0$ . Furthermore we set  $\alpha = \delta$ . The form of the matrices  $K_0$  and  $K_2$  is given by Eq (4.1) and  $K_1 = \alpha I$  where

$$\alpha' + \alpha^2 = a^2. \quad (5.6)$$

[We replace here, and in the following example,  $-a^2$  by  $a^2$  in (2.17).] Using the special solution  $\alpha = a$  for  $\alpha$  we find then that

$$L(m) = -(\gamma_1/m + m^2 a^2)$$

and

$$R_{11} = -(e^{-2ax}/m)[m(2aq_1 e^{2ax} + d_2 d_3) + (d_2 q_3 + d_3 q_2) e^{ax}],$$

$$R_{12} = -ae^{-ax}[2q_2 e^{ax} + d_2(2m + 1)],$$

$$R_{21} = -ae^{-ax}[2q_3 e^{ax} + d_3(2m + 1)],$$

$$R_{22} = (e^{-2ax}/m)[m(2aq_1 e^{2ax} - d_2 d_3) - (d_2 q_3 + d_3 q_2) e^{ax}].$$

**SF-B:** These factorizable systems correspond to those obtained from Theorem 3 with  $K_2 = b(x)A$  and  $s = p = r + q = \gamma_3 = 0$ ,  $\alpha = \delta$ . Under these conditions one infers that the general form for  $L(m)$  and the matrices  $K_0, K_1$ , and  $K_2$  is

$$L(m) = -a^2 m^2,$$

$$K_0 = \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix}, \quad K_1 = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}, \quad K_2 = bA,$$

where  $b$  is a constant. The differential equations for  $\alpha, \beta$ , and  $r$  are

$$\alpha' + \alpha^2 = a^2, \quad \beta' + 2\alpha\beta = 0,$$

$$r' + \alpha r = 0,$$

i.e.,

$$\beta = c_2 J, \quad r = d_1 J^{1/2}.$$

Using  $\alpha = a$  as a solution for  $\alpha$  we obtain for  $R(x, m)$

$$R_{11} = -e^{-2ax} \{d_1 [(2m+1)ae^{ax} + d_1] + bc_2\},$$

$$R_{12} = -2m(m+1)ac_2e^{-2ax},$$

$$R_{21} = -2ab,$$

$$R_{22} = e^{-2ax} \{d_1 [(2m+1)ae^{ax} - d_1] - bc_2\}.$$

To solve these factorizable systems of equations we have to consider them at the top (or bottom) of the ladder in  $m$  where  $\lambda = L(m+1)$  and, therefore,  $\underline{y}(\lambda, m)$  must satisfy

$$H_m^+ \underline{y}(\lambda, m) = \left( K(x, m+1) - \frac{d}{dx} I \right) \underline{y}(\lambda, m) = 0. \quad (5.7)$$

(We note that the proof that  $H^+$  and  $H^-$  are raising and lowering operators follows exactly as in the scalar case.<sup>1</sup>) Equation (5.6) represents a coupled system of first-order equations which can be reduced by elimination to two uncoupled second-order equations for each of the components of  $\underline{y}$  (and then solved by standard techniques). The explicit form of these solutions and the investigation of their properties will be deferred, however, to another publication.

## VI. CONCLUSIONS

In this paper we generalized the classical factorization method to systems of coupled second-order equations and enumerated in Sec. IV some of these systems in two dimensions. In view of the close relationship between the special

functions of mathematical physics and the factorization method it is perhaps appropriate to consider the solutions of factorizable systems in  $n$  dimensions as "generalized special functions." The properties and Lie algebraic contents of such factorizable systems deserve further investigation.

From a physical point of view these factorizable systems might be useful as exactly solvable models for physical systems which are represented by coupled Schrödinger equations. The exact solution of these factorizable equations could then lead to a better insight and understanding of the more realistic models for these physical systems.

<sup>1</sup>L. Infeld and T. E. Hull, *Rev. Mod. Phys.* **23**, 21 (1951).

<sup>2</sup>W. Miller, Jr., *Lie Theory and Special Functions* (Academic, New York, 1968); N. J. Vilenkin, *Special Functions and the Theory of Group Representations* (Am. Math. Soc., Providence, RI, 1968); *Group Theoretical Aspects of Special Functions and Applications*, edited by R. A. Askey (Reidel, Boston, 1984).

<sup>3</sup>B. Kaufman, *J. Math. Phys.* **7**, 447 (1966).

<sup>4</sup>The applications of the factorization method are too numerous. We quote here only two recent articles: N. M. Atakishiyev, in *Group Theoretical Methods in Physics*, edited by M. Serdaroglu and E. İnönü (Springer, Berlin, 1983); B. Mielnik, *J. Math. Phys.* **25**, 3387 (1984).

<sup>5</sup>M. Humi, *J. Math. Phys.* **9**, 1258 (1968).

<sup>6</sup>M. Humi, *Proc. Cambridge Philos. Soc.* **68**, 439 (1970).

<sup>7</sup>W. Miller, Jr., *J. Math. Anal. Appl.* **28**, 383 (1969); C. Boyer and W. Miller, Jr., *J. Math. Phys.* **15**, 1484 (1974).

<sup>8</sup>C. A. Coulson and A. Joseph, *Rev. Mod. Phys.* **39**, 829, 838 (1967); **40**, 845 (1968).

<sup>9</sup>N. F. Mott, and H. S. Massey, *The Theory of Atomic Collisions* (Oxford U. P., London, 1965).

<sup>10</sup>N. F. Lane, *Rev. Mod. Phys.* **52**, 29 (1980).

<sup>11</sup>X. Ch. Cao, *J. Phys. A* **14**, 1069 (1981); *ibid.* **15**, 2727 (1982).

<sup>12</sup>M. Humi, *J. Phys. A* **18**, 1085 (1985).

<sup>13</sup>It is interesting to note the relationship between Eq. (3.11) and those obtained for a motion in central force. See H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, MA, 1981), 2nd ed.