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Representations and Invariant Equations of E(3)

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Using methods analogous to those introduced by Gel'fand *et al.* [*Representations of the Rotation and Lorentz Groups and Their Applications* (Pergamon, New York, 1963)] for the Lorentz group the matrix elements for the representations of the Lie algebra of the Euclidean group in three dimensions E(3) are explicitly derived. These results are then used to construct invariant equations with respect to this group and to show, in particular, that the nonrelativistic analog to the Dirac equation is not unique.

I. INTRODUCTION

The Euclidean group in three dimensions E(3) plays in classical mechanics the same role that the Lorentz group plays in relativistic mechanics.[1–3] Moreover, from a mathematical point of view, the Lie algebras of these two groups contain six generators J_i, K_i, $1 = 1,2,3$, whose commutation relations (CR) for the Lorentz algebra are

$$[J_i,J_j] = \epsilon_{ijk}J_k, \tag{1.1}$$

$$[J_i,K_j] = \epsilon_{ijk}K_k, \tag{1.2}$$

$$[K_i,K_j] = -\epsilon_{ijk}J_k. \tag{1.3}$$

The CR of the Lie algebra of E(3) differ from those above only in the third set [Eq. (1.3)], which is replaced by

$$[K_i,K_j] = 0. \tag{1.4}$$

[For more details regarding the generators of E(3) and the notations used for the rest of this paper see the Appendix.]

In view of these similarities it is surprising to find that the representations of these algebras and their respective invariant equations were treated rather differently in the literature.

Thus while complete and explicit expressions for the matrix elements of the irreducible representations of the Lorentz algebra and its invariant equations exist in the literature[4] the same is not true for E(3). As for this latter algebra the "general form" of the matrix elements for the finite (indecomposable) irreducible representations were given (somewhat) indirectly in Ref. 5. What has been left open, however, was the computation of the interlocking constants between the various irreducible representations of O(3) that appear in the decomposition of an E(3) representation (especially when can these interlocking constants be nonzero). These constants can be computed by *ad hoc* methods (as suggested in Ref. 5) only in some very simple cases.

In view of these circumstances it is our first objective in this paper (Sec. II) to derive explicitly, the expressions for the matrix elements for some of the irreducible representations of E(3) (both finite and infinite dimensional) by methods that are completely analogous to those used by Gel'fand *et al.* for the Lorentz group.[4]

In Sec. III we present a systematic approach to the construction of first-order invariant equations with respect to E(3) subject to the constraint that each component of the wave function satisfies the Galilean energy momentum relation $2mE = m^2 + p^2$ (Refs. 6–8). As a result we show that the nonrelativistic analog to the Dirac equation is not unique and that there is no nonrelativistic analog to the Majorana equation.[9]

II. REPRESENTATIONS OF E(3)

To construct the representations of E(3) we first observe that Gel'fand *et al.*[4] already found the most general solution of the CR(1.1), (1.2). Adopting the notations used by Gel'fand *et al.* we can write this solution as follows:

$$H_3\xi^{\tau}_{l,m} = m\xi^{\tau}_{l,m}, \tag{2.1}$$

$$H_+\xi^{\tau}_{l,m} = \alpha^l_{m+1}\xi^{\tau}_{l,m+1}, \tag{2.2}$$

$$H_-\xi^{\tau}_{l,m} = \alpha^l_m\xi^{\tau}_{l,m-1}, \tag{2.3}$$

$$F_3\xi^{\tau}_{l,m} = \sum_{\tau'}\{C^{\tau\tau'}_l d(l,m)\xi^{\tau'}_{l-1,m} - A^{\tau\tau'}_l \times m\xi^{\tau'}_{l,m} - C^{\tau\tau'}_{l+1}\, d(l+1,m)\xi^{\tau'}_{l+1,m}\}, \tag{2.4}$$

$$F_+\xi^{\tau}_{l,m} = \sum_{\tau'}\{C^{\tau\tau'}_l g_1(l,m)\xi^{\tau'}_{l-1,m+1} - A^{\tau\tau'}_l g_2(l,m)\xi^{\tau'}_{l,m+1} + C^{\tau\tau'}_{l+1} g_3(l,m)\xi^{\tau'}_{l+1,m+1}\}, \tag{2.5}$$

$$F_-\xi^{\tau}_{l,m} = \sum_{\tau'}\{-C^{\tau\tau'}_l g_1(l,-m)\xi^{\tau'}_{l-1,m-1} - A^{\tau\tau'}_l g_2(l,-m)\xi^{\tau'}_{l,m-1} - C^{\tau\tau'}_{l+1} g_3(l,-m)\xi^{\tau'}_{l+1,m-1}\}, \tag{2.6}$$

where

$$\alpha^l_m = [(l+m)(l-m+1)]^{1/2},$$
$$d(l,m) = [l^2 - m^2]^{1/2},$$
$$g_1(l,m) = [(l-m)(l-m-1)]^{1/2},$$
$$g_2(l,m) = [(l-m)(l+m+1)]^{1/2},$$
$$g_3(l,m) = [(l+m+1)(l+m+2)]^{1/2}.$$

It is therefore clear that the only step one must take in order to construct the representations of E(3), rather than O(3,1) is to subject this general solution to the constraints given by Eqs. (1.4) rather than those of Eqs. (1.3). To carry this program out we begin by constructing two special types of representations that are important from a physical point of view. There are the following.

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(1) Representations of E(3) whose decomposition with respect to O(3) contains each irreducible representation of O(3) at most once.

(2) Representations of E(3) in whose decomposition with respect to O(3) only one irreducible representation of this group appears (several times).

Case 1: Since each representation of O(3) appears only once in the decomposition we can drop the degeneracy index τ.

Furthermore, we observe that if the representation is irreducible and l_0, l_1 are, respectively, the lowest and highest l's that appear in the decomposition then each l_0+n, $n=1,2,\ldots,l_1-l_0$ must also appear in the decomposition. In fact, if some representation l_0+n is missing then we can infer [since the generators of the algebra, according to (2.1)–(2.5), can connect states of l_0+n-1 only with those of l_0+n and (l_0+n-2)] that the representations $l_0,\ldots,l_0+n-1$ form an invariant subspace in contradiction to the assumption that the representation is irreducible.

Thus an irreducible representation of E(3) that belongs to this class is a "ladder representation" which is either finite or infinite (presently we show that the ladder must be infinite).

To construct these irreducible representations we apply any of the constraints $[F_+, F_3]=[F_+, F_-]=[F_-, F_3]=0$ to Eqs. (2.4)–(2.6) under the present assumptions and obtain the following equations:

$$[A_l(l+1)-A_{l-1}(l-1)]C_l=0, \tag{2.7}$$

$$[A_{l+1}(l+2)-A_l l]C_{l+1}=0, \tag{2.8}$$

$$[2l-1]C_l^2-[2l+3]C_{l+1}^2-A_l^2=0. \tag{2.9}$$

To solve these equations we first observe [using Eqs. (2.4)–(2.6)] that for an irreducible representation $C_{l+1}=0$, only if $l=l_1$ (if such a finite l_1 exists) and similarly $C_l=0$ only if $l=l_0$. Hence for other l's that appear in the decomposition of the representation with respect to O(3), both C_l, C_{l+1} are nonzero and we infer from Eqs. (2.7) and (2.8) that

$$A_l=A_{l-1}(l-1)/(l+1),$$

which leads to

$$A_l=A_{l_0}[l_0(l_0+1)/l(l+1)], \tag{2.10}$$

where A_{l_0} is an arbitrary constant. Furthermore, to complete the analogy with the Lorentz group we rewrite (2.10) in the form

$$A_l=il_0L/l(l+1). \tag{2.11}$$

To evaluate the C_l's we now multiply Eq. (2.9) by $(2l+1)$ and sum the resulting equations for $l=l_0,\ldots,l$. This yields, after some algebra,

$$C_l^2=L^2[(l^2-l_0^2)/(4l^2-1)l^2]. \tag{2.12}$$

We infer from this relation that contrary to the Lorentz group a ladder representation of E(3) must be an infinite-dimensional representation. In fact (2.12) implies that either $C_l=0$ for all l (the representation is then reducible) or $C_l\neq 0$ for all l_0+n, $n=1,2,\ldots$. We thus proved the following.

Proposition 1: All irreducible ladder representations of E(3) are of infinite dimension (except the trivial one-dimensional representation). The matrix elements of these representations are given by Eqs. (2.1)–(2.6), where A_l, C_l are determined by Eqs. (2.11), (2.12), respectively.

It appears to us that the explicit computation of these matrix elements is new.

Case 2: When the decomposition of the representation under consideration with respect to O(3) contains only one irreducible representation l of this group with multiplicity n, then obviously the matrix elements of J_i are given by the Kronecker product

$$J_i=I\times J_i(l), \tag{2.13}$$

where $J_i(l)$ are the matrix elements of the irreducible representation l of O(3) and I is the unit matrix of dimension n. The matrix elements of the other generators of the algebra are given by the following proposition.

Proposition 2: Under the present assumption the matrix elements of K_i are given by

$$K_i=N\times J_i(l), \tag{2.14}$$

where N is an $n\times n$ matrix so that $N^2=0$. Moreover, if N admits an invariant (proper) subspace then the representation is reducible otherwise the representation is indecomposable.[10]

Proof: It is easy to show that if J_i, K_i are given by Eqs. (2.13) and (2.14) then all the CR of the algebra are satisfied. In fact,

$$[J_i,K_j]=N\times[J_i(l),J_j(l)]=N\times\epsilon_{ijk}J_k(l)=\epsilon_{ijk}K_k$$

and

$$[K_i,K_j]=N^2\times\epsilon_{ijk}J_k=0.$$

To show that this is the only possible solution of the CR under present assumptions it is enough to observe that the matrix elements of K_3 are [using (2.4)]

$$(K_3)^{\tau\tau'}_{l,l,m,m'}=A^{\tau\tau'}m\delta_{m,m'}, \tag{2.15}$$

i.e., $K_3=N\times J_3(l)$.

An appropriate form of the matrix N which is important in the construction of E(3) invariant equations is given by

$$\mathrm{N}=\begin{bmatrix} 0 & & & & \\ \alpha_2 & & & 0 & \\ \vdots & 0 & \ddots & & \\ \alpha_{n-1} & & & & \\ \epsilon,\beta_2 & & \cdots & \beta_{n-1} & 0 \end{bmatrix}, \tag{2.16}$$

where $\alpha_2\beta_2+\cdots+\alpha_{n-1}\beta_{n-1}=0$ and ϵ is an arbitrary parameter.

We now proceed to discuss the irreducible representations of E(3) in the general case, i.e., when the decomposition of the representation with respect to O(3) contains the representations $l_0 l_0+1\ldots,l_1$, each with multiplicity n_i. (Obviously such a representation is irreducible only if all the representations l_0+n, $n=0,\ldots,l_1-l_0$ appear at least once in the decomposition.) To construct the matrix elements of F_+, F_-, and F_3 in this case we rewrite Eqs. (2.4)–(2.6) in block matrix form:

$$F_3 = \begin{bmatrix} N_{l_0,l_0}\times H_3(l_0), & N_{l_0,l_0+1}\times D(l_0), & 0, & \cdots \\ N_{l_0+1,l}\times D(l_0), & N_{l_0+1,l_0+1}\times H_3(l_0+1), & N_{l_0+1,l_0+2}\times D(l_0+1), & 0\cdots \\ 0, & N_{l_0+2,l_0+1}\times D(l_0+1), & N_{l_0+2,l_0+2}\times H_3(l_0+2), & 0\cdots \\ & \cdots & & \end{bmatrix}, \tag{2.17}$$

where the N_{l_i,l_j} are matrices of dimension $n_i\times n_j$ and

$$[D(l)]_{mm'} = \delta_{mm'}\sqrt{(l+1)^2-m^2}. \tag{2.18}$$

Similar expressions can be written for F_+, F_-. The CR (1.4) now reduce to constraints on the matrices N_{l_i,l_j} in the form

$$N_{l_0,l_0} = N_{l_0,l_0+1}N_{l_0+1,l_0} = N_{l_0,l_0}N_{l_0,l_0+1} = \cdots = 0, \tag{2.19}$$

which is, in general, a redundant system of equation for the matrix elements of these matrices.

A particularly interesting case is obtained when the multiplicity of each l in the representation is the same, i.e., $n_i = n$, $i = 0,...,l_1 - l_0$. In this "degenerate" case all the equations of (2.19) can be satisfied if we choose

$$N_{l,l} = N_{l,l\pm 1} = M,$$

where $M^2 = 0$.

III. INVARIANT EQUATIONS

The study of E(3) invariant equations was initiated by Levi-Leblond[6] who constructed by *ad hoc* methods a nonrelativistic analog of the Dirac equation and studied its properties. In this section, however, we study the construction of such equations from a covariant point of view[7] and especially consider the uniqueness of the nonrelativistic analog of the Dirac equation.

Definition 1: Let an operator realization of p_λ be given (see the Appendix). We say that the equation

$$(L_\lambda p^\lambda + \tilde{m})\psi = 0, \quad \psi\in\mathbb{R}^n \tag{3.1}$$

is invariant with respect to E(3) if $L_\lambda p^\lambda$ is a scalar of the E(3) algebra, viz.,

$$[L_\lambda p^\lambda, J_{\mu\nu}] = 0. \tag{3.2}$$

We observe that this definition is equivalent to the one introduced in Ref. 4 from a group theoretical point of view for the Lorentz group.

At this point on objection might be raised to the effect that m appears in (3.1) twice, once as a scalar and once as an operator (albeit as a scalar multiplication operator). The answer to this is that this dichotomy is inherent to the definition of mass in nonrelativistic mechanics. In fact m appears as a scalar of the (pure) Galileo group and then is added to the group as an operator through a central extension.[6] In any event since m is constant both as a scalar and as an operator the equations under consideration describe entities with constant mass. Furthermore, by imposing proper constraints on L^μ one can insure (see Proposition 3) that each component of ψ satisfies the Galilean energy-momentum relation. Thus it is appropriate to refer to equations of the form (3.1) as nonrelativistic invariant wave equations.

Corollary 7: Equation (3.1) is invariant with respect to E(3) if and only if

$$[L_\lambda, J_{\mu\nu}] = i[g_{\lambda\mu}L_\lambda - g_{\lambda\nu}L_\mu]. \tag{3.3}$$

Proof: From (3.2) we infer

$$\begin{aligned} 0 &= [L^\lambda p_\lambda, J_{\mu\nu}] = L^\lambda[p_\lambda, J_{\mu\nu}] + [L_\lambda, J_{\mu\nu}]p^\lambda \\ &= iL^\lambda[g_{\lambda\mu}p_\nu - g_{\lambda\nu}p_\mu] + [L_\lambda, J_{\lambda\nu}]p^\lambda \\ &= i[g_{\nu\lambda}L_\mu - g_{\mu\lambda}L_\nu]p^\lambda + [L_\lambda, J_{\mu\nu}]p^\lambda, \end{aligned}$$

from which Eq. (3.3) follows.

Furthermore, to insure the physical meaning of such an invariant equation we require that each component of ψ satisfies the Galilean energy-momentum relation

$$E = m + p^2/2m, \tag{3.4}$$

which leads to the following result.

Proposition 3: Each component of ψ satisfies Eq. (3.4) if

$$\{L_\mu, L_\nu\} = L_\mu L_\nu + L_\nu L_\mu = 2g_{\mu\nu}I. \tag{3.5}$$

Proof: This result is obvious if we multiply (3.1) by $(L^\mu p_\mu - \tilde{m})$ and require that (3.4) is satisfied. Observe, however, that since $g_{\mu\nu}$ is not diagonal Eq. (3.5) implies

$$L_4^2 = 0, \quad \{L_4, L_5\} = 2I. \tag{3.6}$$

We now turn our attention to finite-dimensional equations which describe nonrelativistic particles with definite spin, viz., equations based on finite-dimensional indecomposable representations of the form given by Eqs. (2.13), (2.14), and (2.16).

To begin with we infer from (3.3) that

$$[[L_3, H_+], H_-] = 2L_3. \tag{3.7}$$

Hence using the results in Ref. 4 regarding O(3) invariant equations we obtain

$$L_3 = i\cdot D\times\tilde{J}_3(l) \tag{3.8}$$

and consequently

$$L_1 = i\cdot D\times\tilde{J}_1(l), \quad L_2 = i\cdot D\times\tilde{J}_2(l). \tag{3.9}$$

Furthermore, if we write L_4, L_5 in block form then the CR's

$$[L_4, \tilde{J}_i] = [L_5, \tilde{J}_i] = 0 \tag{3.10}$$

imply, using Schur's lemma, that

$$L_4 = \lambda_4\times I, \quad L_5 = \lambda_5\times I, \tag{3.11}$$

where D, λ_4, λ_5 are $n\times n$ matrices. From Eq. (3.5) it then follows that

$$\begin{gathered} \{\lambda_4, \lambda_5\} = I, \quad \lambda_5^2 = -I, \quad D^2\times\tilde{J}_i^2(l) = I\times I, \\ \lambda_4^2 = \{\lambda_4, D\} = \{\lambda_5, D\} = 0. \end{gathered} \tag{3.12}$$

Thus to complete the construction of the invariant equations under consideration we must solve for D, λ_4, λ_5 using the remaining CR. However, from

$$[L_4, J_{34}] = 0, \quad [L_5, J_{34}] = -iL_3, \tag{3.13}$$

$$[L_3, J_{34}] = -iL_4, \tag{3.14}$$

$$[[L_5, F_+], F_-] = -2L_4, \tag{3.15}$$

it follows that

$$\lambda_4 N - N\lambda_4 = 0, \tag{3.16}$$

$$\lambda_5 N - N\lambda_5 = D, \tag{3.17}$$

$$(ND - DN)\times \tilde{J}_3^2(l) = \lambda_4, \tag{3.18}$$

and

$$N\lambda_5 N \times \{H_+(l), \quad H_-(l)\} = 2\lambda_4 \times I. \tag{3.19}$$

(The remaining CR lead to the same constraints on λ_4, λ_5, D.)

Proposition 4: Invariant equations of the form (3.1) based on the indecomposable representations (2.13) and (2.14) can be constructed only for $l = \frac{1}{2}$.

Proof: From Eqs. (3.18) and (3.19) it follows that to construct E(3) invariant equations we must have

$$\{H_+(l), H_-(l)\} = \alpha I, \quad \alpha,\beta \in R,$$

and

$$\tilde{J}_3^2 = \beta I. \tag{3.20}$$

However, this can be satisified only for $l = \frac{1}{2}$, where

$$\{H_+[\tfrac{1}{2}], H_-[\tfrac{1}{2}]\} = I, \quad \tilde{J}_3^2 = \tfrac{1}{4}I. \tag{3.21}$$

We conclude from this proposition that invariant equations describing particles with spin $\neq \frac{1}{2}$ must be based on "mixed" indecomposable representations ($l \neq$const) and will contain, therefore, some extraneous components that must be eliminated by some subsidiary conditions (this is similar to the situation in relativistic mechanics).

Corollary: For $l = \frac{1}{2}$,

$$N\lambda_5 N = 2\lambda_4, \quad [N, D] = 4\lambda_4, \quad D^2 = 4I. \tag{3.22}$$

This is a direct consequence of (3.12), (3.18), (3.19), (3.21).

The nonrelativistic analog to the Dirac equation derived in Refs. 1 and 6 is based on a representation of E(3) in the form (2.13)–(2.15) with dim $N = 2$. These equations are[6]

$$(\boldsymbol{\sigma}\cdot P)\varphi + 2m\chi = 0, \quad (\boldsymbol{\sigma}\cdot \mathbf{P})\chi + E\varphi = 0, \tag{3.23}$$

where φ, χ are two component functions and $\boldsymbol{\sigma}$ are Pauli matrices. We observe however that in these equations φ, χ satisfy the energy-momentum relation in the form $E = P^2/2m$ rather than Eq. (3.4). Hence E should be replaced by $E - m$ to conform to our notation. Using this observation we can rewrite Eq. (3.23) in matrix form as

$$\left\{(I\times\boldsymbol{\sigma})\cdot\mathbf{P} + \left[\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\times I\right]E + m\begin{bmatrix}0 & 2\\ -1 & 0\end{bmatrix}\times I\right\}\begin{bmatrix}\varphi\\ \chi\end{bmatrix} = 0. \tag{3.24}$$

On the other hand, if we use Eqs. (3.12), (3.16)–(3.19), and (3.22) to determine a solution for D, λ_4, λ_5 based on the same representation of E(3), we find

$$\lambda_4 = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \quad \lambda_5 = \begin{bmatrix}a & 2\\ c & -a\end{bmatrix}, \quad D = \begin{bmatrix}2 & 0\\ \gamma & -2\end{bmatrix} \tag{3.25}$$

subject to the constraints

$$a^2 + 2c = -1, \quad 2a + \gamma = 0. \tag{3.26}$$

In matrix form the corresponding equations can be written as

$$\left\{\left(iD\times\frac{\sigma_i}{2}\right)p^i + [\lambda_4\times I]\tilde{E} + [(\lambda_5 + I)\times I]\tilde{m}\right\}\begin{bmatrix}\varphi\\ \chi\end{bmatrix} = 0, \tag{3.27}$$

and it is easily verified that (3.27) is not equivalent to (3.24).

Furthermore, one can find solutions to Eqs. (3.12) and (3.16)–(3.19) based on higher-dimensional representations of E(3) as illustrated by the following proposition.

Proposition 5: Let N be a 4×4 matrix in the form (2.16) with $\epsilon = \alpha_2 = \alpha_3 = \beta_2 = -\beta_3 = 2$, then the matrices $\lambda_4 = N$ and

$$\lambda_5 = \begin{bmatrix}0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ \alpha & 1 & 0 & -1\\ -1 & \alpha & 1 & 0\end{bmatrix}, \quad D = 2\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & -2 & 1 & 0\\ 2+2\alpha & 0 & 0 & -1\end{bmatrix}, \tag{3.28}$$

where α is an arbitrary parameter, are solutions of equations (3.12) and (3.16)–(3.19).

Thus we showed the existence of a new nonrelativistic analog to the Dirac equation and demonstrated that the number of components in such an equation is indeterminate from a purely nonrelativistic point of view.

We now turn to an attempt to construct a nonrelativistic analog to the Majorana equation based on the self-coupling of a ladder representation l_0, $l_0 + 1$,... (case 1 of Sec. II).

To begin with we introduce an orthogonal basis $\xi_{l,m}$ on the representation space and set

$$L_4\xi_{l,m} = \sum D_{l,m;l',m'}\xi_{l',m'}. \tag{3.29}$$

However, since $[L_4, J_{\mu\nu}] = 0$ and the representation is irreducible it follows from Schur's lemma that

$$D_{l,m;l',m'} = \rho\delta_{ll'}\delta_{mm'}. \tag{3.30}$$

To calculate the other matrices L_μ we observe that it is sufficient to find L_5. Setting

$$L_5\xi_{l,m} = \sum c_{l,m;l',m'}\xi_{l',m'} \tag{3.31}$$

we infer from $[L_5, H_3] = 0$ that

$$c_{l,m;l'm'} = c_{l,l',m}\delta_{mm'}. \tag{3.32}$$

Furthermore, from $[L_5, H_\pm] = 0$ we deduce that

$$\alpha_{l,m+1}c_{l,l',m+1} - \alpha_{l',m+1}c_{l,l',m} = 0, \tag{3.33}$$

$$\alpha_{l,m}c_{l,l',m-1} - \alpha_{l',m}c_{l,l',m} = 0. \tag{3.34}$$

By simple algebraic manipulations it then follows that

$$c_{l,m;l',m'} = c_l\delta_{ll'}\delta_{mm'}. \tag{3.35}$$

To compute the c_l's and ρ we now invoke the relation (3.15). By applying this relation to $\xi_{l,m}$ we obtain after a long algebra that $\rho=0$ and $c_l=c=\text{const}$, viz., $L_4=0$, $L_5=cI$. Using the CR

$$[L_5,J_{i4}]=-iL_i \tag{3.36}$$

this implies that $L_i=0$. We conclude then that the relativistic Majorana equation has no nonrelativistic analog.

APPENDIX: COMMUTATION RELATIONS FOR E(3)

In this appendix we describe the relationship between the different notations for the generators of the Lie algebra of the Galilean group G and its E(3) subgroup. We also give an explicit differential realization for the commutation relations (CR) of the extended (physical) Galilean group.[6]

1. Real basis[1,6]

The Lie algebra of the (pure) Galilean group has ten generators which we denote by J_i, K_i, P_i, $i=1, 2, 3$, and E. The J_i's are the generators of the Lie algebras of O(3) the K_i's are the boost generators, and P_i, E are the generators for the translations in space and time, respectively [E(3) is generated by J_i, k_i, $i=1, 2, 3$].

The nonzero CR of G are

$$\begin{aligned}&[J_i,J_j]=\epsilon_{ijk}J_k, \quad [J_i,K_j]=\epsilon_{ijk}K_k,\\ &[J_i,P_j]=\epsilon_{ijk}P_k, \quad [K_iE]=P_i.\end{aligned} \tag{A1}$$

The CR for the extended ("physical") Lie algebras of G (which is obtained by a central extension) are the same as in (A1) except that

$$[K_i,P_j]=0$$

is replaced by

$$[K_i,P_j]=m\delta_{ij}. \tag{A2}$$

Thus the extended group has an additional generator m. However, m is a scalar of the algebra as it commutes with all other generators of G.

2. Complex basis

If we consider G over the complex numbers and define

$$\begin{aligned}&\tilde{J}_j=iJ_j, \quad \tilde{K}_j=iK_j, \quad iP_j, \quad \tilde{E}=iE,\\ &j=1,2,3, \quad \tilde{m}=im,\end{aligned} \tag{A3}$$

then the CR of G (extended) take the form

$$\begin{aligned}&[\tilde{J}_i,\tilde{J}_j]=i\epsilon_{ijk}\tilde{J}_k, \quad [\tilde{J}_i,\tilde{K}_j]=i\epsilon_{ijk}\tilde{K}_k,\\ &[\tilde{J}_i,\tilde{P}_j]=i\epsilon_{ijk}P_k, \quad [\tilde{K}_i,\tilde{E}]=iP_i, \quad [\tilde{K}_i,\tilde{P}_i]=i\tilde{m}\delta_{ij}.\end{aligned} \tag{A4}$$

3. Covariant notation[7-11]

By introducing the nonsingular "metric"

$$g_{\alpha\beta}=\left[\begin{array}{c|cc}-I & 0 & \\ \hline & 0 & 1\\ 0 & & \\ & 1 & -1\end{array}\right], \tag{A5}$$

where I is the unit matrix in three dimensions and

$$\begin{aligned}&J_{ij}=\epsilon_{ijk}\tilde{J}_k, \quad J_{i4}=\tilde{K}_i,\\ &p_i=\tilde{P}_i, \quad p_4=\tilde{m}, \quad p_5=\tilde{E}-\tilde{m},\end{aligned} \tag{A6}$$

we can write the CR of the extended G in covariant notation

$$[J_{\mu\nu},J_{\rho\sigma}]=i[g_{\nu\rho}J_{\mu\sigma}-g_{\mu\rho}J_{\nu\sigma}+g_{\mu\sigma}J_{\nu\rho}-g_{\nu\sigma}J_{\mu\rho}], \tag{A7}$$

$$[p_\mu,J_{\rho\sigma}]=i[g_{\mu\rho}p_\sigma-g_{\mu\sigma}p_\rho] \tag{A8}$$

(note that $g_{44}=0$). We observe that the metric tensor $g_{\alpha\beta}$ can be used only to raise and lower indices of vectors over E(3) (Ref. 11), e.g., if $p_{\alpha'}=[p_1,p_2,p_3,\tilde{m},\tilde{E}-\tilde{m}]$ then

$$p^\alpha=(-p_1,-p_2,-p_3,\tilde{E},\tilde{m}). \tag{A9}$$

Furthermore, $p_\alpha p^\alpha=\tilde{m}^2$ is the Galilean energy-momentum-mass relation.

A differential realization of the generators (A6) and their CR is given by

$$\begin{aligned}&J_{ij}=i\left[x_j\frac{\partial}{\partial x_i}-x_i\frac{\partial}{\partial x_j}\right], \quad J_{i4}=i\left[t\frac{\partial}{\partial x_i}+mx_i\right],\\ &p_i=-i\frac{\partial}{\partial x_i}, \quad p_4=im, \quad p_5=i\left[\frac{\partial}{\partial t}-m\right].\end{aligned} \tag{A10}$$

4. A basis of raising and lowering operators

Since E(3) and the Lorentz group O(3,1) have the same number of generators (and six of the nine CR are the same) it is convenient to introduce a basis of E(3) similar to the one used by Gel'fand *et al.*[4] to construct the representations and invariant equations of O(3,1). Thus in Gel'fand's notation

$$A_{12}=J_1, \quad A_{13}=J_2, \quad A_{23}=J_1, \quad B_i=K_i, \quad i=1,2,3. \tag{A11}$$

Hence

$$\begin{aligned}&H_+=\tilde{J}_1+i\tilde{J}_2, \quad H_-=\tilde{J}_1-i\tilde{J}_2, \quad H_3=\tilde{J}_3,\\ &F_+=\tilde{K}_1+i\tilde{K}_2, \quad F_-=\tilde{K}_1-i\tilde{K}_2, \quad F_3=\tilde{K}_3.\end{aligned} \tag{A12}$$

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