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Suggested Citation
Humi, Mayer (2005). Darboux Transformations for Schrodinger Equations in Two Variables. *Journal of Mathematical Physics, 46*(8).
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JOURNAL OF MATHEMATICAL PHYSICS **46**, 083515 (2005)

Darboux transformations for Schrödinger equations in two variables

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(Received 7 June 2005; accepted 17 June 2005; published online 11 August 2005)

Darboux transformations in one variable form the basis for the factorization methods and have numerous applications to geometry, nonlinear equations and SUSY quantum mechanics. In spite of this wide range of applications the theory of Darboux transformations in two variables and its elegant relationship to analytic complex functions has not been recognized in the literature. To close this gap we develop in this paper the theory of Darboux transformation in the context of Schrödinger equations in two variables. This yields a constructive algorithm to determine the relationship between potential functions which are related by Darboux transformations. © *2005 American Institute of Physics.*
[DOI: 10.1063/1.2000727]

I. INTRODUCTION

For over half a century Darboux transformations in one independent variable have found numerous application in various field of mathematics and physics.[1–4] (References 1 and 2 contain an extensive list of references.) In particular the factorization method[5,6] and its generalizations[7–11] which have been instrumental in many physical applications [including SUSY quantum mechanics (QM)[12]] has its roots based on these transformations. Recently however these transformations were generalized and applied to systems of nonlinear equations such as the KdV hierarchy and others.[13] In addition various applications of this method in geometry were worked out and form an important ongoing research area.[2] Extensions of the method to multidimensional oriented Riemann manifolds,[14] time dependent potentials[15] and shape invariant potentials[16] have appeared in the literature.

It is surprising that in spite of this extensive research effort the theory of these transformations in two variables and its elegant relationship to complex analytic function theory has not been worked out (as far as we could ascertain). An exception is the recent paper by Demircioglu *et al.*[17] which considered these transformations under some additional constraints using real variables and polar coordinates. However under these additional constraints only partial results were obtained and the relationship between these transformations and analytic complex functions was lost.

We now give a short overview of Darboux transformations for Schrödinger equation in one variable.

We say that the solutions of two Schrödinger equations with different potentials $u(x)$, $v(x)$, i.e.,

$$\phi'' = (u(x) + \lambda)\phi, \tag{1.1}$$

$$\psi'' = (v(x) + \lambda)\psi, \tag{1.2}$$

are related by a Darboux transformation if there exist $A(x)$, $B(x)$ so that

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$$\psi = \left[A(x) + B(x)\frac{\partial}{\partial x} \right] \phi(x). \tag{1.3}$$

Letting $B(x)=1$ one can easily show that in order for Eqs (1.1) and (1.2) to be related by the transformation (1.3) $A(x)$, $u(x)$, $v(x)$ must satisfy

$$A'' + u' + A(u - v) = 0, \tag{1.4}$$

$$2A' + u - v = 0. \tag{1.5}$$

Eliminating $(u-v)$ between these equations and integration yields

$$A' - A^2 + u = -\nu, \tag{1.6}$$

where ν is an integration constant. Equation (1.6) is a Riccati equation which can be linearized by the transformation $A=-\zeta'/\zeta$ which leads to

$$\zeta'' = (u(x) + \nu)\zeta. \tag{1.7}$$

Thus ζ is an eigenfunction of the original Eq. (1.2) with $\lambda=\nu$. From (1.5) we now infer that

$$v = u - 2(\ln \zeta)'', \tag{1.8}$$

i.e., a Darboux transformation changes the potential function $u(x)$ by $\Delta u=-2(\ln \zeta)''$ where ζ is an arbitrary eigenfunction of (1.1).

Our objective in this paper is to generalize these transformations to Schrödinger equations in two independent variables and determine the relationship between their potentials. From a broader point of view the goal of this project is to derive for two dimensional nonseparable potentials results similar to those that followed from the application of Darboux transformations and the factorization method in one dimension. That is an enumeration of physically important potentials which can be treated and solved by these transformations. This program includes the exploration of the Lie algebraic structure underlying these potentials and may lead to the definition of generic families of "special functions" in two variables. (That is, functions which cannot be expressed as a finite sum of products of functions in one variable.) The present paper represents the first step towards achieving these objectives.

The plan of the paper is as follows: In Sec. II we derive the basic equations that constrain Darboux transformations in two variables and their solutions. In Secs. III and IV we construct explicitly some Darboux transformations and their related potentials. In Sec. V we consider potential cascades whose functional form is preserved under these transformations. We end up in Sec VI with summary and conclusions.

II. DARBOUX TRANSFORMATIONS IN TWO VARIABLES

We shall say that two Schrödinger equations in two independent variables

$$\nabla^2 \phi = u(x,y,\lambda)\phi, \tag{2.1}$$

$$\nabla^2 \psi = v(x,y,\lambda)\psi, \tag{2.2}$$

are related by a Darboux transformation if there exist (smooth functions) $A(x,y)$, $B(x,y)$, $C(x,y)$ so that their solutions satisfy

$$\psi(x,y) = \left[A(x,y) + B(x,y)\frac{\partial}{\partial x} + C(x,y)\frac{\partial}{\partial y} \right] \phi(x,y). \tag{2.3}$$

For brevity we drop in the following the dependence of the various functions on the independent variables.

Using Eq. (2.3) to substitute for ψ in Eq. (2.2) and eliminating the higher order derivatives of ϕ and $\partial^2\phi/\partial y^2$ using Eq. (2.1) we obtain

$$\left[-2\frac{\partial C}{\partial y}+2\frac{\partial B}{\partial x}\right]\frac{\partial^2\phi}{\partial x^2}+\left[2\frac{\partial B}{\partial y}+2\frac{\partial C}{\partial x}\right]\frac{\partial^2\phi}{\partial x\,\partial y}+\left\{\nabla^2 C+2\frac{\partial A}{\partial y}+C[u-v]\right\}\frac{\partial\phi}{\partial y}$$
$$+\left\{\nabla^2 B+2\frac{\partial A}{\partial x}+B[u-v]\right\}\frac{\partial\phi}{\partial x}+\left\{\nabla^2 A+2\frac{\partial C}{\partial y}u+A[u-v]+B\frac{\partial u}{\partial x}+C\frac{\partial u}{\partial y}\right\}\phi=0. \tag{2.4}$$

To satisfy this equation we treat ϕ and its derivatives as independent variables and let their coefficients be zero. This leads then to the following system of equations:

$$\frac{\partial B}{\partial x}-\frac{\partial C}{\partial y}=0, \tag{2.5}$$

$$\frac{\partial B}{\partial y}+\frac{\partial C}{\partial x}=0, \tag{2.6}$$

$$\nabla^2 B+2\frac{\partial A}{\partial x}+B[u-v]=0, \tag{2.7}$$

$$\nabla^2 C+2\frac{\partial A}{\partial y}+C[u-v]=0, \tag{2.8}$$

$$\nabla^2 A+A[u-v]+2\frac{\partial C}{\partial y}u+B\frac{\partial u}{\partial x}+C\frac{\partial u}{\partial y}=0. \tag{2.9}$$

We observe that Eq. (2.9) can be rewritten in a symmetric form in B, C in view of Eq. (2.5).

Equations (2.5) and (2.6) are Cauchy-Riemann equations for B, C. Hence these functions must be harmonic conjugates and

$$\Phi=B+iC \tag{2.10}$$

is analytic. In view of this fact $\nabla^2 C=\nabla^2 B=0$ and Eqs. (2.7) and (2.8) simplify to

$$2\frac{\partial A}{\partial x}+B[u-v]=0,\quad 2\frac{\partial A}{\partial y}+C[u-v]=0. \tag{2.11}$$

By eliminating $u-v$ we then get the following equation for A:

$$C\frac{\partial A}{\partial x}-B\frac{\partial A}{\partial y}=0. \tag{2.12}$$

This leads us to consider the following equation:

$$B\,\mathrm{d}x+C\,\mathrm{d}y=0. \tag{2.13}$$

Although this equation is not exact an integrating factor is given by $1/(B^2+C^2)$. (This fact follows from Cauchy-Riemann equations for B, C.) The general solution of this equation can be expressed therefore by the standard formula

$$w(x,y) = \int_{x_0}^{x} \frac{B(x,y)}{B(x,y)^2 + C(x,y)^2} \mathrm{d}x + \int_{y_0}^{y} \frac{C(x_0,y)}{B(x_0,y)^2 + C(x_0,y)^2} \mathrm{d}y. \tag{2.14}$$

It follows then that the general solution for A is of the form $A=f(w)$ where f is any smooth function of w.

To treat Eq. (2.9) we use Eq. (2.7) to eliminate $u-v$. This leads to

$$B\frac{\partial u}{\partial x} + C\frac{\partial u}{\partial y} + 2\frac{\partial C}{\partial y}u = -\nabla^2 A + \frac{2A\frac{\partial A}{\partial x}}{B}. \tag{2.15}$$

This equation can be used now to determine u for a proper choice of the function $f(w)$. To proceed we now note that [using Eq. (2.14)]

$$-\nabla^2 A + \frac{2A\frac{\partial A}{\partial x}}{B} = \frac{1}{B^2 + C^2}[-f' + f^2]'. \tag{2.16}$$

Hence we set

$$[-f' + f^2]' = g'(w), \tag{2.17}$$

where $g(w)$ is some smooth function. As a result we are led to the following equation for $f(w)$:

$$f' - f^2 + g(w) = 0. \tag{2.18}$$

[We are not adding a constant of integration since $g(w)$ is arbitrary.] This is a Riccati equation which can be linearized by the transformation $f=-q'/q$ and this leads to

$$q'' - g(w)q = 0. \tag{2.19}$$

Since we want to consider only analytic solutions to this equation which can be expressed in terms of known functions the function $g(w)$ must be chosen appropriately. In particular $g(w)$ can be chosen so that Eq. (2.19) is factorizable.[5] Here we consider only three possible choices for $g(w)$,

(1) $g(w)=\text{constant}=c$. If c is negative $c=-\alpha^2$ then $q=D\cos(\alpha w+\eta)$ where D, η are constants. Hence $f=\alpha\tan(\alpha w+\eta)$. If on the other hand $c=0$ then $q=Dw+E$ (E is a constant) and hence $f=-D/(Dw+E)$. Finally if c is positive $c=\alpha^2$ then $q=D\cosh(\alpha w+\eta)$ and $f=-\alpha\tanh(\alpha w+\eta)$.

(2) $g(w)=w^2-(2n+1)$, $n=0, 1,\ldots$. (This is the kernel of the differential equation for the harmonic oscillator.) For $n=0$ this leads to $f=w$ while for $n>0$ we obtain $f=w-[2nH_{n-1(w)}/H_n(w)]$, where $H_n(w)$ are Hermite functions.

(3) $g(w)=n(n-1)/w^2$, $n=2, 3,\ldots$. In this case Eq. (2.19) is a Euler equation and $q=w^n$ or $q=w^{-(n-1)}$. In either case this leads to $f\sim 1/w$.

We observe that different choices of $g(w)$ can lead to the same $f(w)$.

Having made a choice for $g(w)$ Eq. (2.15) becomes

$$B\frac{\partial u}{\partial x} + C\frac{\partial u}{\partial y} + 2\frac{\partial C}{\partial y}u = \frac{g'}{B^2 + C^2}. \tag{2.20}$$

To find the general solution to this equation we must find two independent solutions to the system,

$$\frac{\mathrm{d}x}{B} = \frac{\mathrm{d}y}{C} = \frac{-\,\mathrm{d}u}{2u\dfrac{\partial C}{\partial y} - \dfrac{g'}{B^2 + C^2}}. \tag{2.21}$$

The first equality in this equation leads to

$$C\,\mathrm{d}x - B\,\mathrm{d}y = 0. \tag{2.22}$$

Once again this equation is not exact but an integrating factor is given by $1/(B^2+C^2)$ and the solution of the equation $w_1(x,y)$ can be expressed then by a standard formula similar to Eq. (2.14). Using w_1 we can eliminate x or y from the second equation in (2.21) and find a solution $w_2(u,y)$. The function u is given then implicitly by any smooth function $F(w_1,w_2)=0$. Once u has been determined v can be computed from Eq. (2.7) or (2.8).

To summarize the procedure, one starts by choosing an analytic function $\Phi=B+iC$ then computes w using Eq. (2.14). For a proper choice of $g(w)$ one computes $A=f(w)$ from Eq. (2.19). The determination of u (and hence v) requires then the solution of Eqs. (2.20) and (2.21). We note however that instead of choosing the function g one can choose u so that the left-hand side of Eq. (2.20) multiplied by B^2+C^2 is a function of w only. This will determine g' and hence A from Eq. (2.18).

In the following section we work out this procedure for the (complex) functions $\Phi=z^n$ and $\Phi=iz^n$.

III. DARBOUX TRANSFORMATIONS WITH $\Phi=z^n$

A. $n\neq 1$

In this case $B=r^n\cos(n\theta)$ and $C=r^n\sin(n\theta)$ hence it is expedient to work in polar coordinates. Equation (2.12) becomes

$$(C\cos\theta - B\sin\theta)\frac{\partial A}{\partial r} - \frac{1}{r}(C\sin\theta + B\cos\theta)\frac{\partial A}{\partial \theta} = 0. \tag{3.1}$$

For the present choice of B, C this yields

$$\sin[(n-1)\theta]\frac{\partial A}{\partial r} - \frac{1}{r}\cos[(n-1)\theta]\frac{\partial A}{\partial \theta} = 0. \tag{3.2}$$

From Eq. (2.14) we then find that

$$w = -\frac{\cos[(n-1)\theta]}{(n-1)r^{(n-1)}}. \tag{3.3}$$

Choosing $g(w)=0$ in Eq. (2.17) we find that $A=f(w)=-D/(Dw+E)$. (In the following we let $D=1$, $E=0$.) Equation (2.20) for u becomes

$$r\frac{\partial u}{\partial r} + \tan[(n-1)\theta]\frac{\partial u}{\partial \theta} + 2nu = 0. \tag{3.4}$$

The general solution of this equation is given implicitly by

$$F\left(r^{2n}u, \frac{r^{(n-1)}}{\sin[(n-1)\theta]}\right) = 0$$

where F is a smooth function. For example, if $F(w_1,w_2)=w_1-w_2$ then

$$u = \frac{1}{r^{(n+1)}\sin[(n-1)\theta]}. \tag{3.5}$$

Using Eq. (2.7) to compute $u-v=-2f'(w)/(B^2+C^2)$ we find for the present choice of f,

$$u - v = -\frac{2(n-1)^2}{r^2\cos^2[(n-1)\theta]}. \tag{3.6}$$

B. $n=1$

In this case $B=x$ and $C=y$. Hence from Eq. (2.14) we deduce that $w=\ln r$ and $A=f(w)$. [Since f is arbitrary we could have written this relation as $A=f(r)$ but this will change the expression for $\nabla^2 A$ in Eq. (2.15)]. Equation (2.15) with the left-hand side reexpressed in polar coordinates becomes

$$r\frac{\partial u}{\partial r} + 2u = \frac{g'(w)}{r^2}. \tag{3.7}$$

We conclude then that in this case u is given by

$$u = \frac{G(\theta)}{r^2} + \frac{1}{r^2}\int \frac{g'(w)}{r}\mathrm{d}r, \tag{3.8}$$

where $G(\theta)$ is a smooth function and $u-v=-2[f'(w)/r^2]$.

IV. DARBOUX TRANSFORMATIONS WITH $\Phi=iz^n$

A. $n\neq 1$

For this choice of Φ we have $B=-r^n\sin(n\theta)$, $C=r^n\cos(n\theta)$ and the roles of B, C have been (essentially) exchanged. In this case w is given by

$$w = \frac{\sin[(n-1)\theta]}{(n-1)r^{(n-1)}} \tag{4.1}$$

and $A=f(w)$. The equation for u with $g'(w)=0$ becomes

$$r\frac{\partial u}{\partial r} - \cot[(n-1)\theta]\frac{\partial u}{\partial \theta} + 2nu = 0 \tag{4.2}$$

whose general solution is of the form

$$F\left(r^{2n}u, \frac{r^{(n-1)}}{\cos[(n-1)\theta]}\right) = 0, \tag{4.3}$$

where $F(w_1,w_2)$ is a smooth function. If we let $f(w)=-1/w$ then

$$u - v = -\frac{2(n-1)^2}{r^2\sin^2[(n-1)\theta]}. \tag{4.4}$$

B. $n=1$

In this case $B=-y$ and $C=x$. Hence $w=\theta$, $A=f(\theta)$ and the equation for u in polar coordinate is

$$\frac{\partial u}{\partial \theta} = \frac{g'(w)}{r^2}. \tag{4.5}$$

Hence

$$u = \frac{g(\theta)}{r^2} + G(r) \tag{4.6}$$

and $u-v=-2f'(\theta)/r^2$.

V. CASCADES

One of the important (and interesting) features of the factorization method in one independent variable is that the application of the ladder operator (or equivalently a Darboux transformation) on a potential $u(x)$ leads to potentials with the same dependence on x but with different parameters. As a result one can apply these operators on "essentially the same potential" a finite or infinite number of times creating a cascade of potentials whose solutions are interrelated by Darboux transformations.

To explore the existence of such cascades in two independent variables we shall assume that $\beta(u-v)=u$. Using this relation to substitute in Eq. (2.7) we obtain

$$u = -\frac{2\beta\dfrac{\partial A}{\partial x}}{B}. \tag{5.1}$$

Substituting this expression for u in Eq. (2.9) and using Eqs. (2.5) and (2.12) to simplify we infer that A must satisfy

$$(2\beta-1)\nabla^2 A + \frac{2A\dfrac{\partial A}{\partial x}}{B} = 0 \tag{5.2}$$

and therefore [since $A=f(w)$]

$$[(2\beta-1)f'(w)+f^2(w)]' = 0. \tag{5.3}$$

Hence

$$(2\beta-1)f'(w)+f^2(w) = c, \tag{5.4}$$

where c is a constant. We deduce then (following the discussion in Sec. II of a similar equation) that $f(w)$ can take any of the following forms:

(1) $c=0$ then $f(w)=(2\beta-1)/(w+c_1)$,
(2) $c=\gamma^2$ then $f(w)=\gamma\tanh[(\gamma w+c_2)/(2\beta-1)]$,
(3) $c=-\gamma^2$ then $f(w)=-\gamma\tan[(\gamma w+c_3)/(2\beta-1)]$.

[We assumed that $2\beta\neq 1$ since otherwise $f(w)=\text{constant}$.] Here c_1, c_2, c_3 are arbitrary constants and the corresponding potential u can be computed from Eq. (5.1).

For $c=c_1=0$ and $\Phi=z^n(n\neq 1)$ this leads to

$$u = \frac{2\beta(2\beta-1)(n-1)^2}{r^2\cos^2[(n-1)\theta]}. \tag{5.5}$$

Similarly for $\Phi=iz^n$ it follows that

$$u = \frac{2\beta(2\beta-1)(n-1)^2}{r^2 \sin^2[(n-1)\theta]}. \tag{5.6}$$

Similar but more complicated expressions can be obtained for the other choices of c.

For $n=1$ we infer from Eqs. (3.8) and (4.6) that cascades exist but the corresponding potentials are essentially in one variable. In particular for $\Phi=iz$ Eq. (4.6) implies that a cascade exists when $G(r)=0$ and

$$g(\theta) = -2\beta f'(\theta). \tag{5.7}$$

Substituting this relation in eq. (2.18) it follows that

$$(1-2\beta)f'(\theta) - f^2(\theta) = 0 \tag{5.8}$$

and therefore $f(\theta)=-[(1-2\beta)/(\theta+c_4)]$ where c_4 is a constant.

VI. SUMMARY AND CONCLUSIONS

In this paper we showed that Darboux transformations in two independent variables have strong affinity to the theory of analytic complex functions. This relationship enabled us to analyze these transformations in full. It allowed us also to give a constructive algorithm for the application of these transformations. This algorithm was used to make a partial classification of Darboux transformations for two classes of analytic functions and their related potentials. Further (exhaustive) enumeration of other potential functions especially those that related to the factorization method through Eq. (2.19) [by the choice of the function $g(w)$] is needed. Moreover it will be important to identify classes of physically interesting nonseparable potentials in two variables and find out if they are amenable to treatment by Darboux transformations through the application of Eq. (2.20). We discussed also the existence of potential cascades whose form is preserved under these transformations. However there are other possible definitions of this property, e.g., $u-v$ =constant. The differential equations that correspond to these cascades are the exact analogs of factorizable equations in one dimension and their algebraic structure from group theoretical point of view remains an important open question.

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