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Steady States of Self-Gravitating Incompressible Fluid in Two Dimensions

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In this paper we develop a simple model for the steady state of two-dimensional self-gravitating incompressible gas which is based on the hydrodynamic equations for stratified fluid. These equations are then reduced to a system of two equations for the mass density and the gravitational field. Analytical analysis and numerical solutions of these equations under different modeling assumptions (with special attention to the isothermal case) are then used to study the structure of the resulting steady state of the fluid. © *2006 American Institute of Physics.*
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I. INTRODUCTION

The steady states of self-gravitating fluid in three dimensions have been studied by a long list of illustrious mathematical physicists. (For an extensive list of references see Refs. 1–3). The motivation for this research was due to the interest in the shape and stability of celestial bodies. We now know however that many celestial objects such as galaxies exhibit (effectively) "two dimensional structure."[4–7] Similarly, recent discoveries are leading us to believe that systems similar to our solar system are "abundant" in the galaxy and their existence might be due to the collapse of a two-dimensional primordial gas cloud under gravitation.[8–11] This background motivates us to investigate in this paper the steady states of self-gravitating fluid in two dimensions. This problem has been explored by a large number of investigators using analytic methods and computer simulations (see Refs. 8–10 for a complete list of references). What is missing however is a simple analytic model that is able to capture the basic physics of this process and leads to some insights about the "parameters" that govern its outcome.

In this paper we attempt to develop such a model using the basic hydrodynamic equations that govern the steady state flow of an incompressible, stratified (i.e., nonconstant density) fluid in two dimensions under gravity but with no magnetic field.[1–3] Under these assumptions we show that the number of basic equations can be reduced to a system of two coupled equations. One for the mass density and the second for the gravitational field. The only "parameter" in this these equations is a function that encodes the information about the momentum distribution in the interstellar cloud. We then consider radial solutions to these equations with special attention to the steady state of a gas cloud with isothermal equation of state.[10] We find that under proper choices of the parameter function there will exist out-of-core regions where the mass density peaks out locally. These regions might therefore represent the formation of out-of-core structures in the primordial gas cloud. (The region close to the "center" of the cloud where the density and velocities are expected to be large is usually referred to as the core region. Other regions in the cloud are referred to as out-of-core regions.)

We emphasize, however, that the model we develop here is a steady state one. Accordingly, it cannot address questions about the stability of the mass distribution pattern that is predicted by the model equations. It might be argued also that the assumptions of steady state and incompressibility

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are not realistic from an astrophysical point of view. However, our main goal in this paper is to capture analytically (as far as possible) the nonlinear aspects of the processes under consideration. Accordingly, our results might be useful to provide some analytic insights for more realistic work on this topic. Moreover, they provide a natural extension to the results on the equilibrium states of three-dimensional bodies under gravity.

The plan of the paper is as follows: In Sec. II we present the basic hydrodynamic equations and show how one can reduce them to a coupled system of two equations. We also discuss the conditions under which the two-dimensional approximation is justified. In Sec. III we derive the pressure equation for the fluid under consideration with particular emphasis on isothermal conditions. In Sec. IV we discuss radial solutions to these equations and present the results of some simulations under these assumptions. We end up in Sec. V with a summary and conclusions.

II. DERIVATION OF THE MODEL EQUATIONS

Following the standard convention,[1,12–14] we model the steady state flow of an incompressible fluid in two dimensions (x,y) by the hydrodynamic equations of inviscid and incompressible stratified fluid

$$u_x + v_y = 0, \tag{2.1}$$

$$u\rho_x + v\rho_y = 0, \tag{2.2}$$

$$\rho(uu_x + vu_y) = - p_x - \rho\phi_x, \tag{2.3}$$

$$\rho(uv_x + vv_y) = - p_y - \rho\phi_y, \tag{2.4}$$

$$\nabla^2\phi = 4\pi G\rho, \tag{2.5}$$

where subscripts indicate differentiation with respect to the indicated variable, $\mathbf{u}=(u,v)$ is the fluid velocity, ρ is its density, p is the pressure, ϕ is the gravitational field, and G is the gravitational constant.

We can nondimensionalize these equations by introducing the following scalings:

$$x = L\tilde{x}, \quad y = L\tilde{y}, \quad u = U_0\tilde{u}, \quad v = U_0\tilde{u}, \quad \rho = \rho_0\tilde{\rho}, \quad p = \rho_0 U_0^2\tilde{p}, \quad \phi = U_0^2\tilde{\phi}, \tag{2.6}$$

where L, U_0, ρ_0 are some characteristic length, velocity, and mass density, respectively, that characterize the problem at hand. Substituting these scalings in Eqs. (2.1)–(2.5) and dropping the tildes, these equations remain unchanged (but the quantities that appear in these equations become nondimensional) while G is replaced by $\tilde{G}=G\rho_0 L^2/U_0^2$. (Once again we drop the tilde.)

At this point, we observe that in relativistic physics there is a natural velocity (viz. c-the speed of light) by which one scales the velocities. In the nonrelativistic approach (which we are adopting here) there is no such speed and accordingly U_0 is a "charateristic speed" which one chooses to nondimensionalize the velocities. From a practical (astrophysical) point of view $1 \leqslant U_0 \leqslant 100$ m/s in most typical cases.[3,8] In any case, all quantities (including velocities) in this paper are nondimensional.

Equations (2.1)–(2.5) form a two-dimensional approximation to the three-dimensional analog of these equations (which requires an additional equation for the flow in the z-direction which is similar to Eq. (2.3)). To justify this approximation in our context we assume in the following that we are considering matter distribution whose spatial extension (with characteristic length L) in the x,y directions is much larger than that in z direction (with characteristic "height" H). That is $H/L=O(\epsilon)$ where $0 \leqslant \epsilon \ll 1$. Furthermore we assume that

$$\frac{\partial\rho}{\partial x}=O(1),\quad \frac{\partial\rho}{\partial y}=O(1),\quad \frac{\partial\rho}{\partial z}=O(\epsilon). \tag{2.7}$$

Under these assumptions, the three-dimensional analog of Eq. (2.5) can be approximated by

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=4\pi G\left[\rho(x,y,0)+z\frac{\partial\rho(x,y,0)}{\partial z}\right]. \tag{2.8}$$

Under the scaling introduced by Eq. (2.6) (with $z=L\bar{z}$) we then have

$$\bar{z}\frac{\partial\rho(x,y,0)}{\partial\bar{z}}=O(\epsilon^2) \tag{2.9}$$

while the other terms in Eq. (2.8) are of order $O(1)$. It follows then that we can approximate this equation by its two-dimensional analog

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=4\pi G\rho(x,y,0). \tag{2.10}$$

The justification for the two-dimensional approximation to the other equations follows along similar lines and has been discussed by many authors. (A lucid treatment is given in Ref. 12, pp. 1–12).

In view of Eq. (2.1), we can introduce a stream function ψ so that

$$u=\psi_y,\quad v=-\psi_x. \tag{2.11}$$

Using this stream function we can rewrite Eq. (2.2) as[13,15]

$$J\{\rho,\psi\}=0, \tag{2.12}$$

where for any two (smooth) functions f,g,

$$J\{f,g\}=\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}. \tag{2.13}$$

Equation (2.12) implies that the functions ρ,ψ are dependent on each other and we can express each of them in terms of the other. Thus we can write ψ as $\psi(\rho)$ or ρ as $\rho(\psi)$.

Using ψ the momentum equations (2.3) and (2.4) become

$$\rho(\psi_y\psi_{yx}-\psi_x\psi_{yy})=-p_x-\rho\phi_x, \tag{2.14}$$

$$\rho(-\psi_y\psi_{xx}+\psi_x\psi_{xy})=-p_y-\rho\phi_y. \tag{2.15}$$

To eliminate p from these equations we differentiate Eqs. (2.14) and (2.15) with respect to y,x, respectively, and subtract. This leads to

$$\rho_y(\psi_y\psi_{yx}-\psi_x\psi_{yy})+\rho(\psi_y\psi_{yx}-\psi_x\psi_{yy})_y-\rho_x(-\psi_y\psi_{xx}+\psi_x\psi_{xy})-\rho(-\psi_y\psi_{xx}+\psi_x\psi_{xy})_x=-J\{\phi,\rho\}. \tag{2.16}$$

Using Eq. (2.12), we can rewrite this equation (after some algebra) as

$$\rho J\{\nabla^2\psi,\psi\}+J\left\{\tfrac{1}{2}(\psi_x^2+\psi_y^2),\rho\right\}=-J\{\phi,\rho\}. \tag{2.17}$$

However, in view of Eq. (2.12) $\psi=\psi(\rho)$ and this fact can be used to eliminate ψ from Eq. (2.17). To this end we observe that

$$\psi_x = \psi_\rho \rho_x, \quad \psi_y = \psi_\rho \rho_y, \quad \nabla^2 \psi = \psi_{\rho\rho}[\rho_x^2 + \rho_y^2] + \psi_\rho \nabla^2 \rho \tag{2.18}$$

and note that for any function of $F(\rho)$ we have $J\{F(\rho),\rho\}=0$. This leads after long algebra to the following relation:

$$J\Big\{(\rho\psi_\rho^2)\nabla^2\rho + \tfrac{1}{2}(\rho\psi_\rho^2)_\rho[\rho_x^2 + \rho_y^2] + \phi, \rho\Big\} = 0. \tag{2.19}$$

Hence, we infer that

$$h(\rho)\nabla^2\rho + \frac{1}{2}h'(\rho)[\rho_x^2 + \rho_y^2] + \phi = S(\rho), \quad h' = \frac{\mathrm{d}h(\rho)}{\mathrm{d}\rho}, \tag{2.20}$$

where

$$h(\rho) = \rho\psi_\rho^2 \tag{2.21}$$

and $S(\rho)$ is some function of ρ.

The function $h(\rho)$ can be considered as a parameter function which is determined by the momentum (and angular momentum) distribution in the fluid. From a practical point of view the choice of this function determines the structure of the steady state density distribution. The corresponding flow field can be computed then *a posteriori* (that is after solving for ρ) from the following relations:

$$u = \sqrt{\frac{h(\rho)}{\rho}}\frac{\partial\rho}{\partial y}, \quad v = -\sqrt{\frac{h(\rho)}{\rho}}\frac{\partial\rho}{\partial x}. \tag{2.22}$$

The function $S(\rho)$ that appears in Eq. (2.20) can be determined from the asymptotic values of ρ and ϕ on the boundaries of the domain on which Eqs. (2.5) and (2.20) are solved. When these asymptotic values are imposed or known one can evaluate the left-hand side of Eq. (2.20) on the domain boundaries and re-express it in terms of ρ only to determine $S(\rho)$ (on the boundary of the domain). However, the resulting functional relationship of S on ρ must then hold also within the domain itself since S does not depend on x,y directly. For example assume that on an infinite domain we let $h(\rho)=1$ and

$$\lim_{r\to\infty} \rho(x,y) = e^{-r}, \quad \lim_{r\to\infty} \phi(x,y) = 1/r \tag{2.23}$$

(where $r^2=x^2+y^2$). Under these assumptions, the left-hand side of Eq. (2.20) evaluates asymptotically to $e^{-r}(1-1/r)+1/r$. Rewriting this expression in terms of ρ, we obtain

$$S(\rho) = \rho + \frac{(\rho - 1)}{\ln\rho}. \tag{2.24}$$

When such asymptotic relations are not given, $S(\rho)$ can be viewed as a "gauge." In the following, we let $S(\rho)=0$ under these circumstances.

We observe that Eq. (2.20) can be rewritten in the form

$$h(\rho)^{1/2}\,\nabla\cdot(h(\rho)^{1/2}\,\nabla\rho) + \phi = S(\rho). \tag{2.25}$$

Using this equation, we can eliminate ϕ from Eqs. (2.25) and (2.5) to obtain one fourth-order equation for ρ,

$$\nabla^2(h(\rho)^{1/2}\,\nabla\cdot(h(\rho)^{1/2}\,\nabla\rho)) + 4\pi G\rho = \nabla^2 S(\rho). \tag{2.26}$$

Equation (2.20) can be simplified considerably by introducing a new dependent variable

$$\eta(\rho) = \int^{\rho} h^{1/2}(s)\mathrm{d}s. \tag{2.27}$$

With this new variable, Eq. (2.20) transforms into

$$\nabla^2 \eta + h^{-1/2}(\rho)(\phi - S(\rho)\ = 0, \tag{2.28}$$

where $h(\rho)$ and $S(\rho)$ have to be expressed in terms of η. (We assume implicitly that the transformation given by Eq. (2.27) is invertible).

Thus, we reduced the original nonlinear system of partial differential equations (2.1)–(2.5) to a coupled system of two equations consisting of Eqs. (2.5) and (2.28).

III. EQUATION FOR THE PRESSURE

In order to derive Eq. (2.20) we eliminated the pressure from Eqs. (2.14) and (2.15). However, in some practical astrophysical applications it is important to know the equation of state of the fluid under consideration. For this reason, we derive here an equation analogous to Eq. (2.20) for the pressure. To this end, we divide Eqs. (2.14) and (2.15) by ρ, differentiate the first with respect to y, the second with respect to x, and subtract. This leads to

$$\rho^2 J\{\nabla^2 \psi, \psi\} = J\{p, \rho\}. \tag{3.1}$$

Eliminating ψ from this equation (using Eq. (2.18)) yields

$$J\{\rho^2 \psi_\rho^2 \nabla^2 \rho + \rho^2 \psi_\rho \psi_{\rho\rho}(\rho_x^2 + \rho_y^2) - p, \rho\} = 0. \tag{3.2}$$

Hence,

$$\rho \psi_\rho^2 \nabla^2 \rho + \rho \psi_\rho \psi_{\rho\rho}(\rho_x^2 + \rho_y^2) - \frac{p}{\rho} = R(\rho), \tag{3.3}$$

where $R(\rho)$ is some function of ρ. Re-expressing this equation using $h(\rho)$ we have

$$h(\rho)\nabla^2 \rho + \frac{1}{2}[h'(\rho) - \psi_\rho^2](\rho_x^2 + \rho_y^2) - \frac{p}{\rho} = R(\rho). \tag{3.4}$$

Subtracting this equation from Eq. (2.20) we then have

$$\frac{p}{\rho} = S(\rho) - R(\rho) - \frac{1}{2}\psi_\rho^2(\rho_x^2 + \rho_y^2) - \phi. \tag{3.5}$$

Therefore, the solution of Eqs. (2.20) and (2.5) determines the pressure distribution in the fluid (assuming that the functions R, S have been determined from the boundary conditions).

Conversely, if the pressure distribution is known *a priori*, e.g., if we assume that the fluid is an isothermal gas where $p = c^2 \rho$ (c is the isothermal sound speed which is a constant for this medium) then Eq. (3.5) can be used to eliminate ϕ from Eq. (2.5),

$$\nabla^2(R) = \nabla^2\left[S - \tfrac{1}{2}\psi_\rho^2(\rho_x^2 + \rho_y^2)\right] - 4\pi G \rho. \tag{3.6}$$

It follows then that for an isothermal gas Eqs. (3.3) and (3.6) form a closed system of coupled equations for ρ and R with a parameter function ψ_ρ^2. However, if we eliminate R from these two equations we recover Eq. (2.26).

IV. SOLUTIONS OF Eq. (2.26)

Equation (2.26) is, in general, a nonlinear equation which (to our best knowledge) cannot be solved analytically. The only exception is the case where $h = 1$ under which the resulting equation is linear. (In the following we let $S(\rho) = 0$.) For this choice of h, $\nabla \psi = (1/\sqrt{\rho})\nabla \rho$. That is with the

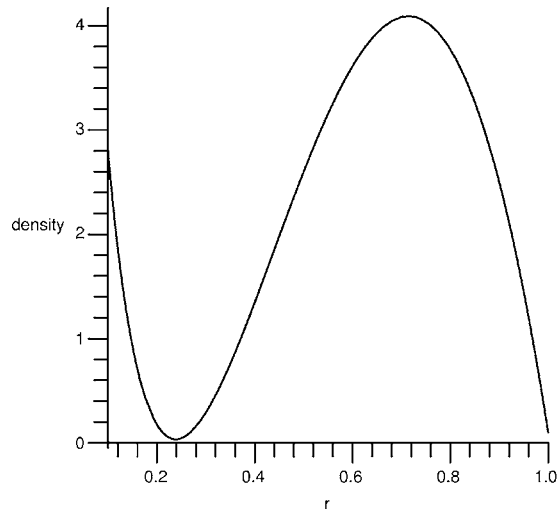

FIG. 1. Radial (analytic) solution of Eq. (2.26) with $h(\rho)=1$ subject to the conditions given by Eq. (4.3).

same gradient of ρ the gradient of ψ will increase as ρ decreases. We conclude then that, in general, matter in regions with low density might have higher momentum than in regions of higher density.

Under the constraint $h=1$ the general solution of Eq. (2.26) can be written in the form

$$\rho = f_0(r) + \sum_{m=1}^{\infty} f_m(r)(E_m \cos m\theta + F_m \sin m\theta), \tag{4.1}$$

where the real form of the solution for $f_m(r)$ is given by

$$f_m(r) = A_m[I_m(\beta R) + I_m(\bar{\beta}R)] + iB_m[I_m(\beta R) - I_m(\bar{\beta}R)] + C_m[K_m(\beta R) + K_m(\bar{\beta}R)] + iD_m[K_m(\beta R) - K_m(\bar{\beta}R)],$$

$$m = 0,1,2, \ldots \; . \tag{4.2}$$

Here $\beta=(\sqrt{2}/2)(-1+i)$, I_m, K_m are the modified Bessel functions of the first and second kinds, overbars denote complex conjugation and $R=(4\pi G)^{1/4}r$. (A_m, B_m, C_m, D_m are arbitrary real constants). However, since K_m have a singularity at the origin we must let $C_m=D_m=0$ if the origin is included in the domain. The remaining arbitary constants that appear in the solution must be adjusted to the boundary conditions one wishes to impose on ρ, ϕ (and subject to the physical requirement that $\rho \geqslant 0$ in the domain under consideration). Obviously various patterns can be obtained for ρ by a proper combination of these functions.

Figure 1 represents the solution given by Eq. (4.2) for $\rho=\rho(r)$, with $h(\rho)=1$ for $r\epsilon[0.1,1]$ subject to the conditions

$$\rho(0.1) = 2.8, \quad \rho(0.5) = 2.6, \quad \rho(0.9) = 2.5, \quad \rho(1) = 0.1. \tag{4.3}$$

This solution exhibits clearly out-of-core region where the density is larger than its surroundings. It demonstrates also the existence of a region in which matter density is almost zero. This is in line with recent findings from simulations about planet formation in rings.[11] We observe that the parameters for this solution were chosen to accentuate the existence of a region in which the density is almost zero thus creating a “gap” between the core and the out-of-core region.

In polar coordinates the flow field is related to the stream function by the relations

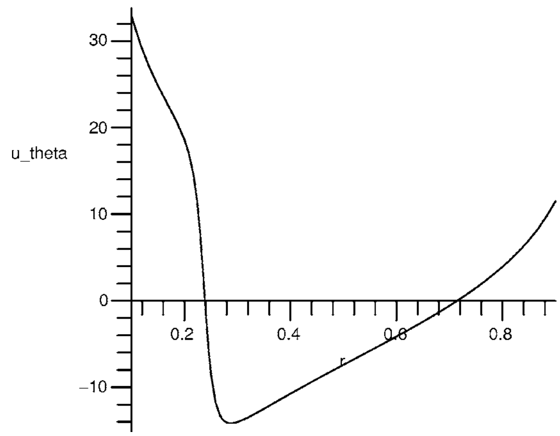

FIG. 2. u_θ as a function of r for solution presented in Fig. 1.

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{\partial \psi}{\partial r}. \tag{4.4}$$

Since $\rho = \rho(r)$ it follows that $u_r = 0$ and

$$u_\theta = -\sqrt{\frac{h(\rho)}{\rho}}\frac{\partial \rho}{\partial r}. \tag{4.5}$$

Observe that u_θ is a nondimensional number. To obtain the actual "dimensional" velocities one has to multiply u_θ by U_0.

Figure 2 presents a plot of u_θ vs r for this case. From this figure we see that matter in the 2-d disk can be divided into three "rings." In the first ring where $0.1 \leqslant r \leqslant 0.24$ matter circulates counterclockwise. This is followed by a very narrow region where the u_θ is changing rapidly. (Observe that the density in this region is almost zero.) In the second ring where $0.24 \leqslant r \leqslant 0.72$ the circulation is clockwise. Finally for $0.72 \leqslant r \leqslant 1$ the circulation is counterclockwise again. (However, we have here a "smooth" transition between the second and third rings.)

For other choices of the function $h(\rho)$ one has to resort to numerical integration of Eq. (2.26). This was carried out in two cases for which $\rho = \rho(r)$. In the first case we let $h(\rho) = \rho^\alpha$ with $\alpha = -1, -0.5, 0, 0.5, 1$. In the second case, we choose $h(\rho) = \rho[1 + A \sin(n\pi\rho)]$ with $n = 2$ and $A = \pm 0.5, 0$.

The solutions to Eq. (2.26) for these two cases on the interval $r \epsilon [0.1, 1]$ with the boundary conditions

$$\rho(0.1) = 1, \quad \rho'(0) = 0, \quad \rho(1) = 0.1(0.2), \quad \rho'(1) = 0 \tag{4.6}$$

are presented in Figs. 3 and 4, respectively. The corresponding flow fields for these density profiles are presented in Figs. 5 and 6. These profiles and flow fields are physically "reasonable" in the sense that they contain no singularities and the circulation is unidirectional.

V. SUMMARY AND CONCLUSIONS

As a first task in this paper, we showed how to reduce the governing equations for the steady state of an incompressible stratified (two-dimensional) fluid under gravity (which comprise a set of five coupled nonlinear partial differential equations) to two equations. The resulting equations contain only one parameter function $h(\rho)$. We also derived a separate equation for the pressure in order to investigate the equation of state of the fluid under consideration. We then explored both analytically and numerically radial solutions to these equations with different choices of $h(\rho)$.

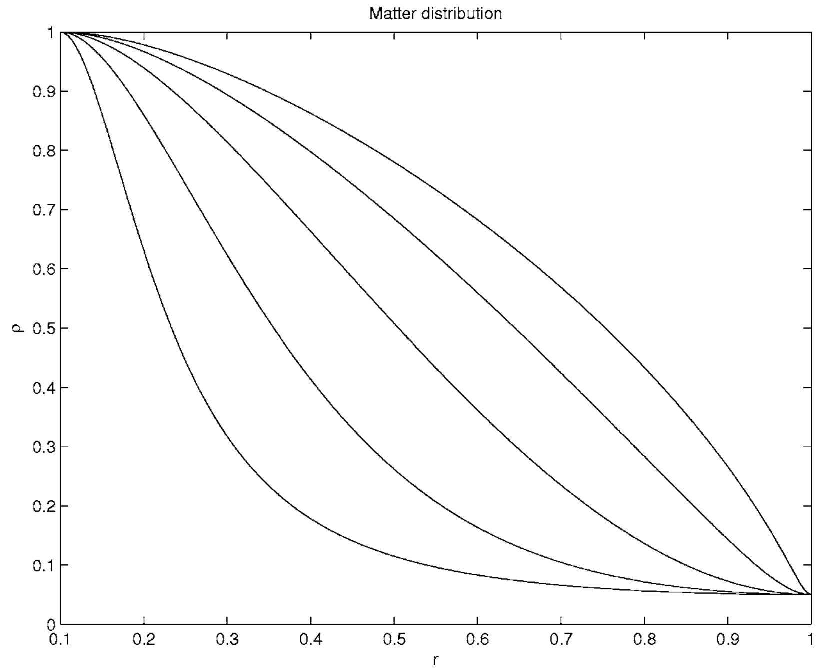

FIG. 3. Numerical solution of Eq. (2.26) with $h(\rho)=\rho^{\alpha}$ and $\alpha=-1,-0.5,0,0.5,1$. The lowest curve corresponds to $\alpha=-1$. The other curves correspond in progression to the other values of α.

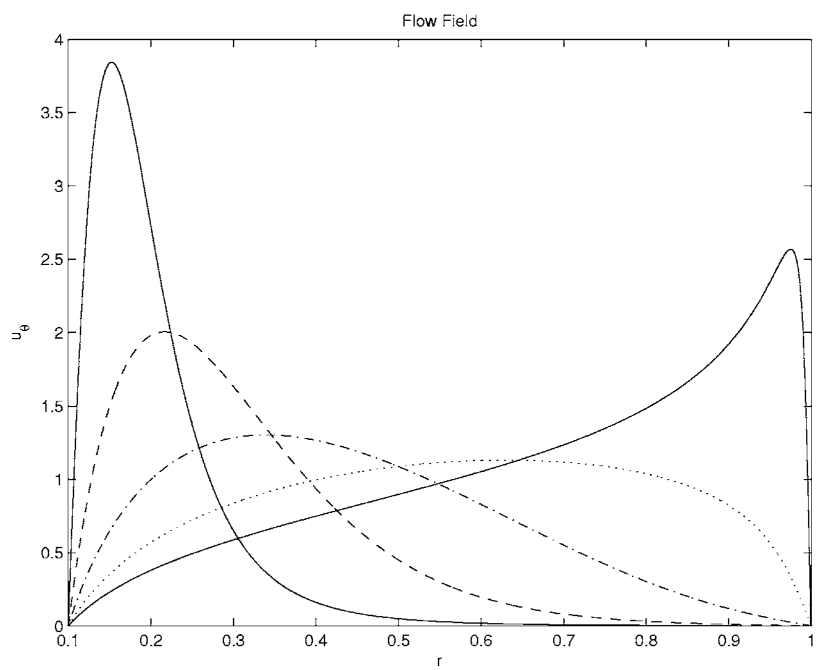

FIG. 4. u_θ as a function of r for the solutions presented in Fig. 3. The dashed, dash-dot and the dot-dot curves represent respectively u_θ for $\alpha=-0.5,0,0.5$. The solid line with a peak near the origin corresponds to $\alpha=-1$ while the other solid line corresponds to $\alpha=1$.

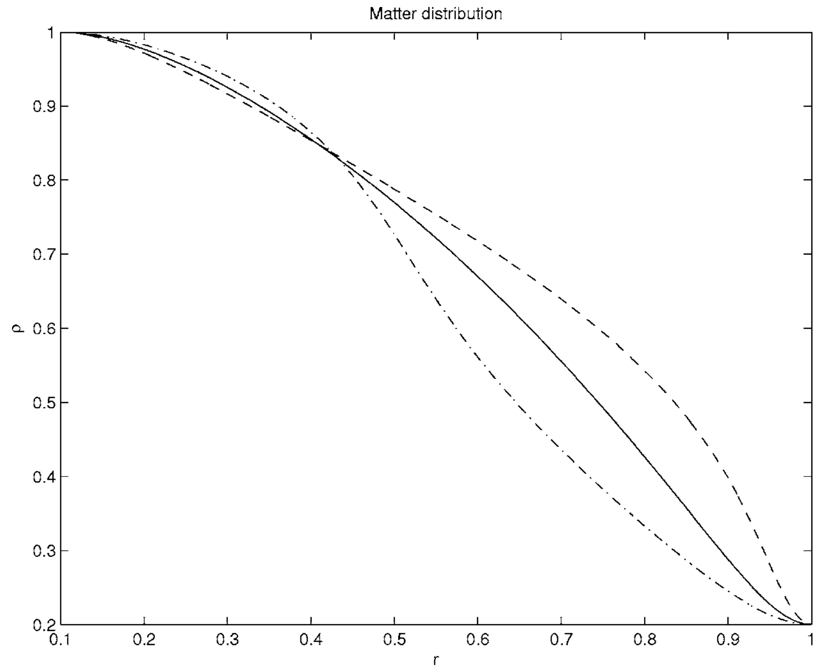

FIG. 5. Numerical solution of Eq. (2.26) with $h(\rho)=\rho(1+A\sin(n\pi\rho))$ with $n=2$. The dashed curve corresponds to $A=0.5$. The dash-dot curve corresponds to $A=-0.5$ and the solid one to $A=0$.

These solutions show that different choices of the parameter function $h(\rho)$ can lead to density profiles which contain out-of-core bands of matter whose density is higher than that of their surroundings. This is suggestive of the early stages of structure formation in these gas clouds. However this result might be sensitive to the choice of the function $h(\rho)$.

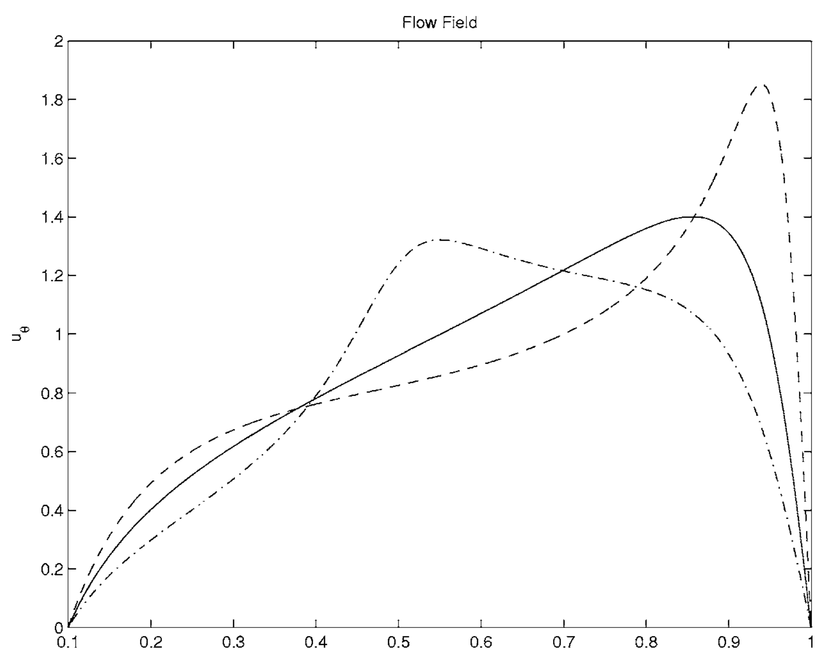

FIG. 6. u_θ as a function of r for the solutions presented in Fig. 5. The lines legend is the same as in Fig. 5.

A problem that our results raise but leave open is the determination of the general conditions on $h(\rho)$ which lead to a solution for ρ which is non-negative and oscillatory.

[1] L. Spitzer, Jr., *Diffuse Matter in Space* (Interscience, New York, 1998).
[2] S. Chandrasekhar, *Ellipsoidal Figures of Equilibrium* (Dover, New York, 1987).
[3] G. P. Kuiper, *Astrophysics: A Topical Symposium*, edited by J. A. Hynek (McGraw-Hill, New York, 1951), pp. 357–424.
[4] H. P. Kunzle and J. M. Nester, J. Math. Phys. **25**, 1009 (1984).
[5] P. S. Letelier and S. R. Oliveira, J. Math. Phys. **28**, 165 (1987).
[6] L. L. Smalley and J. P. Krisch, J. Math. Phys. **36**, 778 (1995).
[7] T. Matsumoto and T. Hanawa, Astrophys. J. **521**, 659 (1999).
[8] J. J. Lissauer, Annu. Rev. Astron. Astrophys. **31**, 120 (1993).
[9] M. Perryman, Rep. Prog. Phys. **63**, 1209 (2000).
[10] J. Silk and Y. Suto, Astrophys. J. **335**, 295 (1988).
[11] R. H. Durisen, K. Cai, A. C. Meja, and M. K. Pickett, Icarus **173**, 417 (2005).
[12] C.-S. Yih, *Stratified Flows* (Academic, New York, 1980).
[13] R. R. Long, Tellus **5**, 42 (1953).
[14] C.-S. Yih, J. Fluid Mech. **29**, 539 (1967).
[15] M. L. Dubreil-Jacotin, J. Math. Pures Appl. **13**, 217 (1934).