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A NON-ABELIAN 2-GROUP WHOSE ENDMORPHISMS
GENERATE A RING, AND OTHER EXAMPLES OF E-GROUPS

by J. J. MALONE

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1. Introduction

Groups for which the distributively generated near-ring generated by the
dendomorphisms is in fact a ring are known as E-groups and are discussed in (3). R.
Faudree in (1) has given the only published examples of non-abelian E-groups by
presenting defining relations for a family of p-groups. However, as shown in (3),
Faudree's group does not have the desired property when $p = 2$.

In this note, it is shown that most of the groups discussed by D. Jonah and M.
Konvisser in (2) are actually E-groups. These groups, described below in Section 2
are proved by Jonah and Konvisser to be such that all their automorphisms are
central. Here, it is shown that most of these groups are E-groups by proving that each
strict endomorphism (i.e. an endomorphism that is not an automorphism) has its image
in the centre of the group. Since one of the groups treated is a 2-group, this paper
provides the only published example of a non-abelian 2-group which is an E-group.

E-groups are also discussed in (4) and (5). However, no examples are given in
those papers.

2. The Groups of Jonah and Konvisser

The groups treated in (2) are described as follows. Let $\lambda = (\lambda_1, \lambda_2)$ be a vector with
integer entries at least one of which is relatively prime to $p$. Then let $G_\lambda =
(a_1, a_2, b_1, b_2)$ be the $p$-group of class 2 with the additional relations:

$$a_1^p = [a_1, b_1], \quad a_2^p = [a_1, b_2],$$
$$b_1^p = [a_2, b_1b_2], \quad b_2^p = [a_2, b_2], \quad \text{and} \quad [a_1, a_2] = [b_1, b_2] = 1.$$

It is noted in (2) that $G = G_\lambda$ has order $p^8$ and exponent $p^2$, that $Z(G) = G' =
([a_1, b_1], [a_1, b_2], [a_2, b_1], [a_2, b_2])$ is elementary abelian of order $p^4$, and that $G_\mu$ and $G_\lambda$
are isomorphic if and only if $\mu = k\lambda$ for some $k$ relatively prime to $p$. Thus, for each
prime $p$, there are $p + 1$ non-isomorphic groups given as $\lambda$ varies over the set $
((0, 1), (1, 0), (1, 1), \ldots, (1, p - 1))$. However, there is some difficulty with the defining rela-
tions when $\lambda = (1, 0)$ since $a_1^p = a_2^p$ so that $G_{(1, 0)}$ does not have order $p^8$. Therefore, that
case will not be considered in this paper.

If $p = 2$ and $\lambda = (1, 1)$, then application of the defining relations shows that
$(a_1a_2b_2)^2 = 1$. Thus $G^2 \neq G'$. But Theorem 2 of (3) says that, in our context, $G^2 = G'$
is a necessary condition for $G$ to be an E-group. Therefore, when $p = 2$, the case of
$\lambda = (1, 1)$ will also not be considered in this paper.
In (2) it is noted that the normal subgroups $A = \langle a_1, a_2, Z(G) \rangle$ and $B = \langle b_1, b_2, Z(G) \rangle$ are the only abelian subgroups of order $p^2$ over the centre. Furthermore, $A^p \leq [x, G]$ for some $x$ in $A$ while there is no $y$ in $B$ such that $B^p \leq [y, G]$. For $F$ a $p$-group as given in (1) by Faudree, Lemma 6 of (1) implies that the centralisers $C_F(a_2)$ and $C_F(b_3)$ are the only abelian subgroups of order $p^2$ over the centre. Thus, if $F$ and $G$ are to be isomorphic, the two centralisers in $F$ must correspond, in some order, to $A$ and $B$ in $G$. From Lemma 6 we also have that any element of order $p^2$ in $C_F(a_2)$ has the form $a_2^r a_2^s$ and an element of order $p^2$ in $C_F(b_3)$ has the form $b_3^r b_3^s$ with at least one of $r$ and $s$ relatively prime to $p$. But, the defining relations in $F$ indicate there is no $y$ in $C_F(a_2)$ such that $\langle a_2^r, a_2^s \rangle = (C_F(a_2))^p \leq [y, F]$ and no $x$ in $C_F(b_3)$ such that $\langle b_3^r, b_3^s \rangle = (C_F(b_3))^p \leq [x, F]$. Hence $F$ and $G$ are not isomorphic since neither centraliser of order $p^6$ in $F$ can correspond to $A$ in $G$.

3. The Strict Endomorphisms

The groups described in Section 2 for which $p$ is odd and $\lambda$ is in $\{(0,1), (1,1), (1,2), \ldots, (1,p-1)\}$ or $p = 2$ and $\lambda = (0,1)$ will be referred to as $JK$-groups.

Lemma 1. For any $JK$-group, $Z(G) = G' = G^p = U_p(G)$, where $U_p(G)$ is the set of elements whose order divides $p$.

Proof. From the defining relations it is immediate that $G' = \langle a_1^p, a_2^p, b_1^p, b_2^p \rangle = G^p$, $G' \leq Z(G)$, and $G' \leq U_p(G)$ with $|G| = p^k$ and $|G'| = p^k$. If $|Z(G)| \geq p^3$, then $|G/Z(G)| \leq p^3$. But $G/Z(G)$ is generated by $\langle a_1 Z(G), a_2 Z(G), b_1 Z(G), b_2 Z(G) \rangle$. So two generators of $G$ are congruent mod $Z(G)$ and $G/Z(G)$ is generated by at most three elements. Thus $|G'| \leq p^3$, a contradiction. Hence $Z(G) = G'$. Also, for $x$ and $y$ in $G$, $(xy)^p = x^p y^p$ by Proposition VI.1.k(4) of (6). Since, for odd $p$, $G'$ has exponent $p$, it follows that $(xy)^p = x^p y^p$. Then $(x^r y)^p = a_1^r a_2^t b_1^u b_2^v$. The exponent of $a_1^r a_2^t b_1^u b_2^v$ is in $G^p$. Hence $U_p(G) = G^p$ for odd $p$. For $p = 2$ we show directly from the generating relations that $G^2 = U_2(G)$. Let $g = a_1^r a_2^t b_1^u b_2^v$ be an element of order 2 in $G$. Then $g^2 = a_1^{2r} a_2^{2t} b_1^{2u} b_2^{2v}$ is in $G^p$. For $p = 2$ we show directly from the generating relations that $G^2 = U_2(G)$. Let $g = a_1^r a_2^t b_1^u b_2^v$ be an element of order 2 in $G$. Then $e = g^2 = a_1^{2r} a_2^{2t} b_1^{2u} b_2^{2v}$ has order 2 in $G$. Hence $U_2(G) = G^2$ and the Lemma is proved.

Lemma 2. Let $m_1, m_2, n_1, n_2, \lambda_1, \lambda_2$ be integers modulo $p$ and $\lambda_2^{-1}$ be the inverse of $\lambda_2$ in the field of order $p$. Then the matrix $A$ has rank 2 or greater over the field of order $p$ if at least one of $m_1, m_2, n_1, n_2$ is not congruent to 0 modulo $p$.

$$A = \begin{bmatrix}
m_1 - \lambda_1 \lambda_2^{-1} m_2 & \lambda_2^{-1} m_2 & 0 & 0 \\
0 & 0 & m_1 & -m_1 + m_2 \\
- n_1 & 0 & - n_2 & n_2 \\
\lambda_1 \lambda_2^{-1} n_1 & - \lambda_2^{-1} n_1 & 0 & - n_2
\end{bmatrix}$$
Proof. If either \( m_1 \) or \( m_2 \not\equiv 0(\text{mod. } p) \), the first two rows of \( A \) are linearly independent. If either \( n_1 \) or \( n_2 \not\equiv 0(\text{mod. } p) \), the last two rows of \( A \) are linearly independent.

Theorem. Any JK-group is an E-group.

Proof. In (2) it is shown that all automorphisms of JK-groups are central. The theorem will be established if it is demonstrated that any strict endomorphism of one of these groups has its image in the centre of the group. This will be shown by arguing as was done in Lemma 5 of (1).

Let \( \theta \) be a strict endomorphism of \( G \). Then \( (h)\theta \in G' \) for some \( h \not\in G' \) and \( h = a_1^{\alpha_1}a_2^{\alpha_2}b_1^{\beta_1}b_2^{\beta_2} \) with at least one exponent \( \not\equiv 0(\text{mod. } p) \). Also, \( [(c, h)\theta] = [(c)\theta, (h)\theta] = 1 \) so that

\[
[(c, h) : c \in \{a_1, a_2, b_1, b_2\}] \subseteq \text{Ker} \theta \text{. Note that}
\]

\[
[a_1, h] = a_1^{\alpha_{m_1} - p\alpha_{\ell_1} \ell_{m_2} a_2^{p\alpha_{\ell_1} \ell_{m_2}},
\]

\[
[a_2, h] = b_1^{\alpha_{m_1} b_2^{p(m_2 - m_1)},
\]

\[
[b_1, h] = a_1^{-\beta_{m_1} b_1^{\beta_{m_2}} b_2^{\beta_{m_2}},
\]

\[
[b_2, h] = a_2^{\beta_{m_1} \ell_{m_1}} a_2^{p\beta_{m_1} \ell_{n_1} b_2^{n_2}}.\]

The matrix of the powers of \( a_1^\xi, a_2^\xi, b_1^\xi, b_2^\xi \) in (*) is \( A \). Hence \( |\text{Ker} \theta \cap G'| \geq p^2 \) and there exists \( \{h_i : 1 \leq i \leq 4\} \) such that \( G = \langle h_1, h_2, h_3, h_4 \rangle \) and \( (h_i)\theta = (h_i)\theta = 1 \). Thus for \( i = 1 \) or \( 2 \), \( (h_i)\theta \in G' \) and \( ((G)\theta)' = ((h_1)\theta, (h_2)\theta)' \). Hence \( |((G)\theta)'| = |((G)\theta)'| \leq p \) and \( |\text{Ker} \theta \cap G'| \geq p^3 \). We can then additionally assume that \( (h_1)\theta = 1 \) and \( (h_2)\theta \in G' \). But now, \( ((G)\theta)' = (1) \). Hence \( (G)\theta \) is abelian, \( G' \leq \text{Ker} \theta \) and \( (G)\theta \leq G' = Z(G) \).

REFERENCES


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