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SURFACE SUBGROUPS OF GRAPH GROUPS

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ABSTRACT. Given a graph Γ, define the group F_Γ to be that generated by the vertices of Γ, with a defining relation $xy = yx$ for each pair x, y of adjacent vertices of Γ. In this article, we examine the groups F_Γ, where the graph Γ is an n-gon, $(n \geq 4)$. We use a covering space argument to prove that in this case, the commutator subgroup F'_Γ contains the fundamental group of the orientable surface of genus $1 + (n-4)2^{n-3}$. We then use this result to classify all finite graphs Γ for which F'_Γ is a free group.

To each graph $\Gamma = (V, E)$, with vertex set V and edge set E, we associate a presentation $P\Gamma$ whose generators are the elements of V, and whose relations are $\{xy = yx | x, y \text{ adjacent vertices of } \Gamma\}$. $P\Gamma$ can be regarded as the presentation of a k-algebra $k\Gamma$, of a monoid M_Γ, or of a group F_Γ, called a *graph group.* These objects have been previously studied by various authors [2–8].

Graph groups constitute a subclass of the *Artin groups.* Recall that an Artin group is defined by a presentation whose relations all take the form $xyx\cdots = yxy\cdots$, where the two sides have the same length $n > 1$, and there is at most one such relation for any pair of generators. To each such presentation we can associate a labeled graph Γ, which has a vertex for each generator, and for each relation $xyx\cdots = yxy\cdots$, an edge joining x and y and labeled "n", where n is the length of each side of the relation. Thus, a graph group is an Artin group whose graph has all edges labeled '2'. In the same context, we mention the conjecture of Tits [1], which states that in the Artin group with labeled graph Γ, the subgroup generated by the squares of the generators is isomorphic to the graph group F_{Γ_2}, where Γ_2 is the subgraph of Γ consisting of all the vertices, and all edges labeled '2'. This conjecture has been proved by S. Pride [7] in the case that the graph Γ contains no triangles.

For many graphs Γ it is true that every subgroup of F_Γ is itself a graph group [4], besides the obvious cases where Γ is either complete (F_Γ free Abelian) or

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completely disconnected (F_Γ free). In this article, we use topological techniques to examine the commutator subgroups of certain graph groups, showing in particular that some of them are not graph groups.

For $n \geq 3$, the n-gon is the graph with n vertices $v_1, v_2, \dots, v_n$, and n edges (v_i, v_{i+1}), indices modulo n. We will show that if Γ has a subgraph isomorphic to an n-gon for some $n \geq 4$, then the commutator subgroup F'_Γ has a subgroup isomorphic to the fundamental group of the orientable surface of genus $1 + (n-4)2^{n-3}$. (In particular, the graph group of the pentagon contains a subgroup isomorphic to the group of the five-holed torus.) We will use this result to show that F'_Γ is a free group if and only if Γ contains no full subgraph isomorphic to any n-gon with $n \geq 4$. We also show that if Γ does not contain any squares (i.e., full subgraphs isomorphic to C_4), then F'_Γ can be a graph group only if it is free.

1. Preliminaries

A graph is a pair (V, E), where V is the set of vertices and E is a set of unordered pairs of elements of V. So a graph is undirected with no loops and no multiple edges. A graph $\Sigma = (W, D)$ is called a *subgraph* of Γ if $W \subset V$ and $D \subset E$, and there is a natural homomorphism $f: F_\Sigma \to F_\Gamma$ defined by setting $f(w) = w$ for each $w \in W$. If D contains all unordered pairs of elements of W which are contained in E, we call Σ a *full* subgraph of Γ. In this case, it is clear that the natural homomorphism is one-to-one, so we shall simply regard F_Σ as a subgroup of F_Γ.

We now summarize a result of [8], which completely describes the centralizer of an element of F_Γ. Given an element $u \in F_\Gamma$, the *support* of u, $\operatorname{supp}(u)$, is the set of vertices $v \in V$ such that either v or v^{-1} occurs in all factorizations of u as a product of vertices and their inverses. $\operatorname{Supp}(u)$ is well defined.

Let $1 \neq x \in F_\Gamma$. Then there are sets $A_0, A_1, \dots, A_n$ of vertices of Γ, and a factorization $p^{-1}(x_1^{r_1} \cdots x_n^{r_n})p$ of x such that (1) for $i = 1, 2, \dots, n$, $A_i = \operatorname{supp}(x_i)$, (2) the sets $A_0, A_1, \dots, A_n$ are pairwise disjoint, (3) if $i \neq j$, then every vertex in A_i is adjacent to each vertex in A_j, and (4) $\operatorname{cent}(x) = p^{-1}(G \times \langle x_1 \rangle \times \cdots \times \langle x_n \rangle)p$, where G is the subgroup of F_Γ generated by the elements of A_0. The elements x_i are called the *pure factors* of x.

2. Topological realization of the commutator subgroup

Let $\Gamma = (V, E)$ be a finite graph, and let X_Γ denote the Cayley complex of the corresponding presentation of F_r; that is, X_Γ has one 0-cell, $*$, an oriented 1-cell for each vertex of Γ, and for each edge $(v, w) \in E$, a 2-cell attached along the loop $vwv^{-1}w^{-1}$. We have $\pi_1(X_\Gamma) = F_\Gamma$. If $\Gamma = (V, E)$ has n vertices, then X_Γ is a subcomplex of the n-fold Cartesian product $(S^1)^n$, where the circle S^1 has one 0-cell and $(S^1)^n$ is the product complex. In particular, if K is the complete graph with vertex set V, then X_K is the entire 2-skeleton of

$(S^1)^n$. Let U_K denote the universal cover of X_K. Since the fundamental group of a complex is carried by its 2-skeleton, it follows that U_K is the 2-skeleton of the cubical complex of R^n, i.e., the complex on R^n whose n-cells are the integer translates of the unit cube I^n.

Now, it is easy to see that the natural homomorphism $\alpha: F_\Gamma \to F_K$ is the Abelianization map, and that the inclusion $i: X_\Gamma \to X_K$ realizes α. Thus, $F'_\Gamma = \ker(\alpha)$ is realized by UG in the pullback diagram

$$\begin{array}{ccc} UG & \longrightarrow & U_K \\ \downarrow & & \downarrow \\ X_\Gamma & \xrightarrow{i} & X_K \end{array}$$

Hence, UG is the subcomplex of U_K obtained by deleting the lifts of all 2-cells of X_K which correspond to nonadjacent vertices of Γ.

Let w be a word on $V^{\pm 1}$ representing an element $[w] \in F_\Gamma$. Then [8] w can be transformed into a word of shortest length representing $[w]$ via a finite sequence of the following moves:

M$_1$. Delete a subword $a^{-1}a$ or aa^{-1}.

M$_2$. Replace a subword $v^{\pm 1}w^{\pm 1}$ with $w^{\pm 1}v^{\pm 1}$ if $(v, w) \in E$.

In particular, if $[w] = 1$, then w can be transformed into the empty word using a finite sequence of these moves.

Let Z be a covering space of X_Γ, and let Z have the induced cell structure. Let $p: I \to Z$ be any (cellular) loop in Z which is path-homotopic to the constant loop. Then, by the above, there is a sequence of path homotopies

$$p = p_1 \to p_2 \to \cdots \to p_k = *$$

connecting p to the constant loop $*$, with each p_i cellular, such that the homotopy $p_i \to p_{i+1}$ is supported either by p_i, in case of a move of type M_1, or, in case of a move of type M_2, by $p_i \cup F$, where F is a face of Z which intersects p_i in at least two incident edges (i.e., two edges having a common endpoint). Let Y be a subcomplex of Z and suppose that every face of Z which intersects Y in at least two incident edges is contained in Y. Then any loop in Y which is path homotopic to the constant loop in Z is path homotopic to the constant loop in Y also. Thus:

Proposition 1. *Let Z be a cover of the Cayley complex of F_Γ, and let Y be a subcomplex of Z with the property that any face of Z which contains at least two incident edges of Y is contained in Y. Then the inclusion $i: Y \to Z$ induces a monomorphism $i_*: \pi_1(Y) \to \pi_1(Z)$.*

3. Surface subgroups of n-gon groups

Let C_n denote the n-gon $(n \geq 3)$, and F_n the corresponding graph group. As in §2, F'_n is realized by a subcomplex U_n of the cubical lattice of R^n.

Consider the subcomplex I^n of R^n. Since I^n is convex, every 2-cell of R^n which intersects I^n in two edges is also a 2-cell of I^n, and so $Y = U_n \cap I^n$ has the same property with respect to U_n. Thus, by Proposition 1, $\pi_1(Y)$ is a subgroup of F'_n.

Now, every edge of I^n corresponds to a vertex v of C_n, and is incident to $n-1$ faces, one for each of the $n-1$ other vertices of C_n. Y contains exactly two of these faces—those corresponding to the two vertices of C_n which are adjacent to v. Thus, Y is a connected 2-complex in which every edge is adjacent to exactly two faces; i.e., Y is a surface. Y is 2-sided since it is a subcomplex of the 2-skeleton of I^n.

To compute the genus of Y, we observe that I^n has 2^n vertices, $n2^{n-1}$ edges and $\frac{1}{2}n(n-1)2^{n-2}$ faces, with 2^{n-2} faces for each of the $\frac{1}{2}n(n-1)$ pairs of vertices in C_n. Since only n of these pairs are adjacent in C_n, Y has only $n2^{n-2}$ faces, so the Euler characteristic of Y is

$$\chi(Y) = 2^n - n2^{n-1} + n2^{n-2} = (4-n)2^{n-2}$$

and the genus of Y is $1 - \frac{1}{2}\chi(Y) = 1 + (n-4)2^{n-3}$.

Thus, we have shown

Theorem 1. *Let F_n be the graph group of the n-gon graph. Then F'_n has a subgroup isomorphic to the fundamental group of the orientable surface of genus $1+(n-4)2^{n-3}$.*

4. Commutator subgroups of graph groups

Let $\Gamma = (V, E)$ be a graph. Because the exponent sum of each letter of V in each relator of F_Γ is 0, a word w on $V^{\pm 1}$ represents an element of F'_Γ if and only if the exponent sum on each letter of V in w is 0. It follows that if Σ is a full subgraph of Γ, then $F'_\Sigma = F_\Sigma \cap F'_\Gamma$, and that if $x \in F'_\Gamma$, then each pure factor of x lies in F'_Γ.

A graph Γ is called *triangulated* if it contains no full subgraph isomorphic to an n-gon for any $n \geq 4$. (In particular note that trees are triangulated.)

Theorem 2. *If Γ is finite, then F'_Γ is free if and only if Γ is triangulated.*

Proof. If Γ contains a full n-gon with $n \geq 4$, then F'_Γ contains the group of some surface of positive genus, and so cannot be free. Conversely, suppose Γ is triangulated. If Γ is complete, F'_Γ is trivial. Otherwise [4], Γ can be written as the union of two subgraphs X and Y whose intersection, K, is a complete graph (empty in case Γ is disconnected). Comparing presentations, we find that $F_\Gamma = F_X *_{F_K} F_Y$. Now, $F'_\Gamma \cap F_X = F'_X$, $F'_\Gamma \cap F_Y = F'_Y$, and $F'_\Gamma \cap F_K = F'_K = \{1\}$ since F_K is Abelian. But X and Y are full subgraphs of Γ, so they are both triangulated, and so by induction, F'_X and F'_Y are free. Therefore, F'_Γ is free, by the Kurosh subgroup theorem.

Lemma 1. *Let Γ be a graph which contains no full squares (i.e., no full subgraph isomorphic to the square), and let $x \in F'_\Gamma$, $x \neq 1$. Then* $\mathrm{cent}(x)$, *the centralizer of x in F_Γ, is free Abelian, and* $\mathrm{cent}(F'_\Gamma; x)$, *the centralizer of x in F'_Γ, is cyclic.*

Proof. Write $x = p^{-1}(x_1^{r_1} \cdots x_n^{r_n})p$, and let $A_0, A_1, \ldots, A_n$ and G be as in paragraph 1. Since $x \neq 1$, one of the x_i, say x_1, must be nontrivial, and since $x_1 \in F_\Gamma$, A_1 must contain two nonadjacent vertices. But then each of the sets $A_0, A_1, \ldots, A_n$ must induce a complete subgraph of Γ, since Γ contains no squares (for recall that each vertex in any of these sets is adjacent to every vertex of A_1). In particular, G is free Abelian. Further, the support of a nontrivial element of F'_r cannot induce a complete subgraph of Γ, so $x_2 = x_3 = \cdots = x_n = 1$. Thus, $\mathrm{cent}(x) = p^{-1}(G \times \langle x_1 \rangle)p$, which is clearly free Abelian. Finally, $\mathrm{cent}(F'_\Gamma; x) = F'_\Gamma \cap \mathrm{cent}(x) = p^{-1}(\langle x_1 \rangle \times G')p = p^{-1}\langle x_1 \rangle p$, since G is Abelian, so $\mathrm{cent}(F'_\Gamma; x)$ is cyclic.

Theorem 3. *If the finite graph Γ contains no full squares, then F'_Γ is a graph group if and only if it is free.*

Proof. If the graph Σ has an edge joining v and w, then $\mathrm{cent}(v)$ is not cyclic. Thus, if the centralizer of every element of F_Σ is cyclic, Σ must be discrete, which is to say F_Σ must be free.

Theorems 2 and 3 immediately imply

Corollary 1. *If Γ contains no full squares, but it does contain a full n-gon for some $n > 4$, then F'_Γ is not a graph group.*

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