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Plane simple shear of smooth inelastic circular disks: the anisotropy of the second moment in the dilute and dense limits

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We consider a plane, steady, homogeneous flow of circular disks. The disks are identical, smooth, and inelastic. We adopt the assumption of molecular chaos and introduce an anisotropic Maxwellian velocity distribution function based on the full second moment of the velocity fluctuations. In the limits of dilute and dense flows, we determine approximate analytic solutions of the balance law for the second moment that result in stresses whose qualitative behaviour and magnitudes are in good agreement with numerical simulations.

1. Introduction

Existing theories for rapidly flowing granular materials, as reviewed, for example, by Richman (1986) and Jenkins (1987), all exploit the similarities between the colliding grains in such a flow and the agitated molecules of a dense, disequilibrated gas, while incorporating the important difference that collisions between the grains inevitably dissipate energy. When the particles are smooth and round and the amount of energy dissipated in a collision is small, standard arguments of the kinetic theory, slightly modified, may be employed to derive balance laws for the means of the mass density, velocity, and energy of the velocity fluctuations; to determine the velocity distribution function; and to calculate the stress, the flux of fluctuation energy, and its collisional rate of dissipation. This has been done for spheres by Lun *et al.* (1984) and Jenkins & Richman (1985*a*) and for plane flows of circular disks by Jenkins & Richman (1985*b*). To the order of the approximation used in determining the velocity distribution function, the stress and the energy flux are identical to those for elastic particles. However, the presence of the rate of dissipation in the energy balance permits it to have steady solutions in situations where none are possible in the classical theory.

When collisions between smooth spheres or disks involve more significant dissipation, numerical simulations of the detailed particle dynamics in a steady, homogeneous shear flow (Walton & Braun 1986, 1987) indicate that the appropriate theory has a more complicated structure. The simulations show that, at least in relatively dilute systems, the deviatoric part of the second moment of the velocity fluctuations is of the same order as its isotropic part, contrary to an assumption made in deriving the more elementary theory. In order to account for this, the theory may be extended by treating the full second moment as a field variable and adding the balance law for its deviatoric part to those for the density, velocity, and its isotropic part. The velocity distribution function is, then, expected to depend upon the full second moment in a significant way.

Such extensions of the theory for dilute and dense systems of spheres have already been made by, respectively, Goldreich & Tremaine (1978) and Araki & Tremaine (1986). For smooth spheres, they employ the assumption of molecular chaos, or Enskog's extension of it to dense systems, to relate the probable frequency of collisions to the velocity distribution function. They then suppose that the velocity distribution function is an anisotropic Maxwellian based on the full second moment of the velocity fluctuations. For both dilute and dense systems they obtain local numerical solutions of the balance of second moment for the steady plane, inhomogeneous, shear flow in a planetary ring.

Here we consider steady, plane, homogeneous, shear flows of identical, smooth, inelastic, circular disks. Then, proceeding in a fashion similar to Goldreich & Tremaine (1978) and Araki & Tremaine (1986), we adopt the appropriate form of the anisotropic Maxwellian and use it to calculate the relevant means and mean rates. In the dilute and dense limits, we obtain approximate analytic solutions of the balance of second moment that provide the dependence of the second moment on the shear rate, area fraction, and coefficient of restitution. The approximate analytical solutions differ only slightly from the corresponding numerical solutions. With the solutions for the second moment, the components of the stress may be computed numerically or approximated analytically. The stress components exhibit the same qualitative behaviour in the two limits as observed in the numerical simulations and, in the two cases that quantitative comparisons can be made, the differences are within the error of the simulation in the dilute limit and on the order of 10% in the dense limit.

2. General considerations

We restrict our attention here to plane flows of a granular material consisting of identical, smooth, inelastic circular disks of mass m and diameter σ. We operate within the context of kinetic theories for macroscopic dissipative particles that have been developed in the past five years. Such theories involve balance laws for mean values of particle properties and constitutive relations that prescribe how these mean values are changed in collisions and by the motion of particles between collisions. A typical property ψ depends upon the particle through the velocity $\boldsymbol{c}$ of its centre, and its mean value $\langle\psi\rangle$ is calculated using the single particle velocity distribution $f^{(1)}(\boldsymbol{c},\boldsymbol{r},t)$. This function of the particle's velocity, position $\boldsymbol{r}$, and the time t, is defined so that the number n of particles per unit area at $\boldsymbol{r}$ and t is given by

$$n(\boldsymbol{r},t) = \int f^{(1)}(\boldsymbol{c},\boldsymbol{r},t)\,\mathrm{d}\boldsymbol{c}, \tag{1}$$

where the integration is over the entire plane in velocity space. The mean of ψ is then

$$\langle\psi\rangle \equiv \frac{1}{n}\int \psi(\boldsymbol{c}) f^{(1)}(\boldsymbol{c})\,\mathrm{d}\boldsymbol{c}, \tag{2}$$

where the dependence upon $\boldsymbol{r}$ and t is understood and the integration is as before. Of particular importance are the mean mass density $\rho \equiv mn$ and the mean velocity $\boldsymbol{u} \equiv \langle\boldsymbol{c}\rangle$. The fluctuation velocity $\boldsymbol{C}$ is the velocity of a particle relative to the mean, $\boldsymbol{C} \equiv \boldsymbol{c}-\boldsymbol{u}$; through $\boldsymbol{u}$, it is a function of $\boldsymbol{r}$ and t. The kinetic energy of the velocity fluctuations is proportional to the granular temperature $T \equiv \frac{1}{2}\langle \boldsymbol{C}\cdot\boldsymbol{C}\rangle$.

When $\psi = \psi(\boldsymbol{c})$, the rate of change of its mean value in a fixed area element is due to the acceleration of a disk by an external force $\boldsymbol{F}$, the net influx of disks bearing ψ, and changes in ψ in collisions:

$$\frac{\partial}{\partial t}\langle n\psi\rangle = \left\langle n\frac{\boldsymbol{F}}{m}\cdot\frac{\partial\psi}{\partial\boldsymbol{c}}\right\rangle - \nabla\cdot\langle n\boldsymbol{c}\psi\rangle + \mathbb{C}[\psi]. \tag{3}$$

Here $\mathbb{C}[\psi]$ is the collisional rate of production of ψ in a unit area of the plane. It depends upon the change in the property of a particle in a typical binary collision and the probable frequency of such collisions.

In a collision between a pair of smooth circular disks, the velocities $\boldsymbol{c}'_1$ and $\boldsymbol{c}'_2$ of the disks after a collision are related to their velocities $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ before the collision by the unit vector $\boldsymbol{k}$, directed at impact from the centre of the first disk to the centre of the second, and the coefficient of restitution e:

$$2(\boldsymbol{c}'_A - \boldsymbol{c}_A) = 2(\boldsymbol{C}'_A - \boldsymbol{C}_A) = (-1)^A(1+e)(\boldsymbol{k}\cdot\boldsymbol{g})\boldsymbol{k}, \tag{4}$$

where $A = 1,2$ and $\boldsymbol{g} \equiv \boldsymbol{c}_1 - \boldsymbol{c}_2$. The total change $\Delta\psi$ of ψ in a collision is defined as $\Delta\psi = \psi'_1 + \psi'_2 - \psi_1 - \psi_2$, where, for example, $\psi'_1 \equiv \psi(\boldsymbol{c}'_1)$. Then, by (4), $\Delta\boldsymbol{c} = 0$; but, in dyadic notation,

$$2\Delta(\boldsymbol{cc}) = 2\Delta(\boldsymbol{CC}) = (1+e)(\boldsymbol{k}\cdot\boldsymbol{g})[(1+e)(\boldsymbol{k}\cdot\boldsymbol{g})\boldsymbol{kk} - \boldsymbol{gk} - \boldsymbol{kg}]. \tag{5}$$

The likelihood of binary collisions is determined by the complete pair distribution function $f^{(2)}$ defined so that, at time t, $f^{(2)}(\boldsymbol{c}_1, \boldsymbol{r}_1, \boldsymbol{c}_2, \boldsymbol{r}_2, t)\,\mathrm{d}\boldsymbol{c}_1\,\mathrm{d}\boldsymbol{r}_1\,\mathrm{d}\boldsymbol{c}_2\,\mathrm{d}\boldsymbol{r}_2$ is the probable number of pairs of disks with the centre of the first in $\mathrm{d}\boldsymbol{r}_1$ at $\boldsymbol{r}_1$ with velocity in $\mathrm{d}\boldsymbol{c}_1$ about $\boldsymbol{c}_1$ and the centre of the second in $\mathrm{d}\boldsymbol{r}_2$ at $\boldsymbol{r}_2$ with velocity in $\mathrm{d}\boldsymbol{c}_2$ about $\boldsymbol{c}_2$. Then the probable number of collisions per unit time experienced by a disk in $\mathrm{d}\boldsymbol{r}$ at $\boldsymbol{r}$ with velocity in $\mathrm{d}\boldsymbol{c}_2$ at $\boldsymbol{c}_2$ over the element of angle $\mathrm{d}k$ at $\boldsymbol{k}$ from disks with velocities within $\mathrm{d}\boldsymbol{c}_1$ at $\boldsymbol{c}_1$ is $f^{(2)}(\boldsymbol{c}_1, \boldsymbol{r} - \sigma\boldsymbol{k}, \boldsymbol{c}_2, \boldsymbol{r})\,\sigma(\boldsymbol{k}\cdot\boldsymbol{g})\,\mathrm{d}k\,\mathrm{d}\boldsymbol{c}_1\,\mathrm{d}\boldsymbol{c}_2\,\mathrm{d}\boldsymbol{r}$ when $\boldsymbol{k}\cdot\boldsymbol{g} > 0$. Consequently, the collisional production of ψ per unit area at $\boldsymbol{r}$ is

$$\mathbb{C}[\psi] = \iiint (\psi'_2 - \psi_2) f^{(2)}(\boldsymbol{c}_1, \boldsymbol{r} - \sigma\boldsymbol{k}, \boldsymbol{c}_2, \boldsymbol{r})\,\sigma(\boldsymbol{k}\cdot\boldsymbol{g})\,\mathrm{d}k\,\mathrm{d}\boldsymbol{c}_1\,\mathrm{d}\boldsymbol{c}_2, \tag{6}$$

where the integration is to be taken over all values of $\boldsymbol{c}_1$, $\boldsymbol{c}_2$, and $\boldsymbol{k}$ for which a collision is impending, $\boldsymbol{k}\cdot\boldsymbol{g} > 0$. Alternatively, if the first disk is assumed to be located at $\boldsymbol{r}$ and all possible collisions between it and a second are considered, then

$$\mathbb{C}[\psi] = \iiint (\psi'_1 - \psi_1) f^{(2)}(\boldsymbol{c}_1, \boldsymbol{r}, \boldsymbol{c}_2, \boldsymbol{r} + \sigma\boldsymbol{k})\,\sigma(\boldsymbol{k}\cdot\boldsymbol{g})\,\mathrm{d}k\,\mathrm{d}\boldsymbol{c}_1\,\mathrm{d}\boldsymbol{c}_2, \tag{7}$$

integrated over all impending collisions. A more symmetric and suggestive form for the collisional production is obtained by making use of the Taylor series

$$f(\boldsymbol{r} - \sigma\boldsymbol{k}, \boldsymbol{r}) = f(\boldsymbol{r}, \boldsymbol{r} + \sigma\boldsymbol{k}) - (\sigma\boldsymbol{k}\cdot\nabla)\sum_{m=0}^{\infty}\frac{(-\sigma\boldsymbol{k}\cdot\nabla)^m}{(m+1)!} f(\boldsymbol{r}, \boldsymbol{r} + \sigma\boldsymbol{k}), \tag{8}$$

in (7) and taking half of the sum of this and its alternative (6). The result may be written in the form

$$\mathbb{C}[\psi] = \aleph[\psi] - \nabla\cdot\boldsymbol{\Theta}[\psi], \tag{9}$$

where

$$\aleph[\psi] \equiv \frac{1}{2}\iiint \Delta\psi f^{(2)}(\boldsymbol{c}_1, \boldsymbol{r} - \sigma\boldsymbol{k}, \boldsymbol{c}_2, \boldsymbol{r})\,\sigma(\boldsymbol{k}\cdot\boldsymbol{g})\,\mathrm{d}k\,\mathrm{d}\boldsymbol{c}_1\,\mathrm{d}\boldsymbol{c}_2, \tag{10}$$

taken over all $\boldsymbol{k}\cdot\boldsymbol{g} \geqslant 0$, is the collisional source of ψ, and

$$\boldsymbol{\Theta}[\psi] \equiv -\tfrac{1}{2}\sigma \iiint (\psi_1' - \psi_1)\,\boldsymbol{k} \sum_{m=0}^{\infty} \frac{(-\sigma \boldsymbol{k}\cdot\boldsymbol{\nabla})^m}{(m+1)!} f^{(2)}(\boldsymbol{c}_1, \boldsymbol{r}, \boldsymbol{c}_2, \boldsymbol{r}+\sigma\boldsymbol{k})\,\sigma(\boldsymbol{k}\cdot\boldsymbol{g})\,\mathrm{d}k\,\mathrm{d}\boldsymbol{c}_1\,\mathrm{d}\boldsymbol{c}_2, \tag{11}$$

over all $\boldsymbol{k}\cdot\boldsymbol{g} \geqslant 0$, is the collisional flux of ψ. For our purposes it is convenient to have a more compact expression for the collisional flux. To obtain this we integrate the Taylor series

$$f(\boldsymbol{r}-\eta\sigma\boldsymbol{k}, \boldsymbol{r}+\sigma\boldsymbol{k}-\eta\sigma\boldsymbol{k}) = \sum_{m=0}^{\infty} \eta^m \frac{(-\sigma\boldsymbol{k}\cdot\boldsymbol{\nabla})^m}{m!} f(\boldsymbol{r}, \boldsymbol{r}+\sigma\boldsymbol{k}), \tag{12}$$

over η from zero to one and make the correspondence with the integrand of (11). Thus,

$$\boldsymbol{\Theta}[\psi] = -\tfrac{1}{2}\sigma \iiint (\psi_1' - \psi_1)\,\boldsymbol{k} \int_0^1 f^{(2)}(\boldsymbol{c}_1, \boldsymbol{r}-\eta\sigma\boldsymbol{k}, \boldsymbol{c}_2, \boldsymbol{r}+\sigma\boldsymbol{k}-\eta\sigma\boldsymbol{k})\,\mathrm{d}\eta\,\sigma(\boldsymbol{k}\cdot\boldsymbol{g})\,\mathrm{d}k\,\mathrm{d}\boldsymbol{c}_1\,\mathrm{d}\boldsymbol{c}_2. \tag{13}$$

With the decomposition (9), the balance law (3) may be written in the form

$$\frac{\partial}{\partial t}\langle\rho\psi\rangle = \left\langle n\boldsymbol{F}\cdot\frac{\partial\psi}{\partial\boldsymbol{c}}\right\rangle - \boldsymbol{\nabla}\cdot(\langle\rho\boldsymbol{c}\psi\rangle + m\boldsymbol{\Theta}[\psi]) + m\aleph[\psi]. \tag{14}$$

For example, when $\psi = 1$ the balance of mass results:

$$\dot{\rho} + \rho\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \tag{15}$$

where the overdot denotes a time derivative calculated with respect to the mean velocity. With $\psi = \boldsymbol{c}$ in (14) and the use of (15) the familiar form of the balance of linear momentum is obtained:

$$\rho\dot{\boldsymbol{u}} = -\boldsymbol{\nabla}\cdot\boldsymbol{P} + n\boldsymbol{F}, \tag{16}$$

where

$$\boldsymbol{P} \equiv \rho\langle\boldsymbol{C}\boldsymbol{C}\rangle + m\boldsymbol{\Theta}[\boldsymbol{C}], \tag{17}$$

is the pressure tensor.

When the particle property is a function of the fluctuation velocity $\boldsymbol{C}$, the balance law corresponding to (14) has a slightly more complicated structure. In this case, because ψ depends on $\boldsymbol{r}$ through $\boldsymbol{u}$, there is an additional term in the decomposition (9). This is best expressed in terms of Cartesian components:

$$\mathbb{C}[\psi] = \aleph[\psi] - \frac{\partial}{\partial r_\alpha}\Theta_\alpha[\psi] - \frac{\partial u_\alpha}{\partial r_\beta}\Theta_\beta\left[\frac{\partial\psi}{\partial C_\alpha}\right], \tag{18}$$

where Greek indices take the values 1 and 2. Then, for $\psi = \psi(\boldsymbol{C})$,

$$\frac{\partial}{\partial t}\langle n\psi\rangle = \left\langle n\frac{\mathrm{d}C_\alpha}{\mathrm{d}t}\frac{\partial\psi}{\partial C_\alpha}\right\rangle - \frac{\partial}{\partial r_\alpha}(\langle nc_\alpha\psi\rangle + \Theta_\alpha[\psi]) - \frac{\partial u_\alpha}{\partial r_\beta}\Theta_\beta\left[\frac{\partial\psi}{\partial C_\alpha}\right] + \aleph[\psi], \tag{19}$$

where the time derivative of $\boldsymbol{C}$ is calculated following a particle:

$$\frac{\mathrm{d}\boldsymbol{C}}{\mathrm{d}t} = \frac{\boldsymbol{F}}{m} - \frac{\partial\boldsymbol{u}}{\partial t} - (\boldsymbol{c}\cdot\boldsymbol{\nabla})\,\boldsymbol{u}. \tag{20}$$

With this and (15), the balance law (19) may be written as

$$\rho\langle\dot{\psi}\rangle = -\frac{\partial}{\partial r_\gamma}(\rho\langle C_\gamma \psi\rangle + m\Theta_\gamma[\psi]) - (\rho\dot{u}_\gamma - nF_\gamma)\left\langle\frac{\partial\psi}{\partial C_\gamma}\right\rangle$$
$$-\left(\rho\left\langle C_\mu \frac{\partial\psi}{\partial C_\gamma}\right\rangle + m\Theta_\mu\left[\frac{\partial\psi}{\partial C_\gamma}\right]\right)\frac{\partial u_\gamma}{\partial r_\mu} + m\aleph[\psi]. \quad (21)$$

So, for example, if $\psi = \boldsymbol{CC}$ and $\boldsymbol{K} \equiv \langle \boldsymbol{CC}\rangle$,

$$\rho\dot{K}_{\alpha\beta} = -Q_{\gamma\alpha\beta,\gamma} - P_{\mu\beta}u_{\alpha,\mu} - P_{\mu\alpha}u_{\beta,\mu} + \aleph_{\alpha\beta}, \quad (22)$$

where a comma denotes a partial derivative,

$$Q_{\gamma\alpha\beta} \equiv \rho\langle C_\gamma C_\alpha C_\beta\rangle + m\Theta_\gamma[C_\alpha C_\beta], \quad (23)$$

and

$$\aleph_{\alpha\beta} \equiv m\aleph[C_\alpha C_\beta]. \quad (24)$$

Equation (22) is the balance law for the second moment $\boldsymbol{K}$ of $f^{(1)}$.

In order to relate the complete pair distribution function at collision to the velocity distribution function, we adopt the assumption of molecular chaos and ignore the possible correlations in the velocities of colliding disks and account for the correlations in their position in the simple way proposed by Enskog for dense gases. Then $f^{(2)}$ for a colliding pair may be written as the product of the $f^{(1)}$ of each disk, evaluated at its centre, and the equilibrium radial distribution function g_0 evaluated at the point of contact:

$$f^{(2)}(\boldsymbol{c}_1, \boldsymbol{r}-\sigma\boldsymbol{k}, \boldsymbol{c}_2, \boldsymbol{r}) = g_0 f^{(1)}(\boldsymbol{c}_1, \boldsymbol{r}-\sigma\boldsymbol{k}) f^{(1)}(\boldsymbol{c}_2, \boldsymbol{r}). \quad (25)$$

Verlet & Levesque (1982) have determined an analytic expression for the dependence of g_0 upon the area fraction $\nu = \frac{1}{4}n\pi\sigma^2$ that is in excellent agreement with the numerical simulations of Hoover & Alder (1967) up to a value of $\nu = 0.665$ at which a change of phase to a solid is observed. It is

$$g_0(\nu) = (16-7\nu)/16(1-\nu)^2. \quad (26)$$

In order to carry out an analysis of an unsteady, inhomogeneous flow we would write down an additional balance law for the contracted third moment $\langle C_\alpha C_\beta C_\beta\rangle$ and assume that all other components of the third moment were zero and that all higher moments vanished. We would next introduce an explicit form for the single-particle velocity distribution function that depended upon $\boldsymbol{r}$ and t through the mean fields $n, \boldsymbol{u}, \boldsymbol{K}$, and the contracted third moment. With the assumption of molecular chaos (25), the collisional fluxes and sources could then be calculated as functions of the mean field and their spatial derivatives. Finally, the balance laws, used with appropriate initial and boundary conditions, would determine the $\boldsymbol{r}$ and t dependence of the mean fields.

Here we do this for the simplest case of a steady, homogeneous, shear flow. We anticipate that more complicated unsteady, inhomogeneous flows may be treated as perturbations of this elementary but important uniform steady state.

3. Steady, homogeneous, shearing

In this simple flow n, $\nabla \boldsymbol{u}$, and $\boldsymbol{K}$ are constant and the contracted third moment vanishes. In it, we shall assume that the velocity distribution is an anisotropic Maxwellian,

$$f^{(1)}(\boldsymbol{c}, \boldsymbol{r}, t) = \frac{n}{2\pi \Delta^{\frac{1}{2}}} \exp\left(-\tfrac{1}{2}\boldsymbol{C}\cdot\boldsymbol{K}^{-1}\cdot\boldsymbol{C}\right), \tag{27}$$

where Δ is the determinant of $\boldsymbol{K}$. The dependence of $f^{(1)}$ upon $\boldsymbol{r}$ enters through that of $\boldsymbol{u}$. The appropriate simplification of the balance law (22) for the second moment will be used to determine $\boldsymbol{K}$ in terms of n and $\nabla \boldsymbol{u}$.

We take the x- and y-axes in the plane of shear and, respectively, parallel and perpendicular to the streamlines. The non-vanishing x-component u of the velocity is then

$$u = 2\lambda y, \tag{28}$$

where λ is a constant. The stretching $\boldsymbol{D}$ and the spin $\boldsymbol{W}$ are, respectively, the symmetric and antisymmetric parts of the velocity gradients:

$$D_{\alpha\beta} = \begin{bmatrix} 0 & \lambda \\ \lambda & 0 \end{bmatrix}, \quad W_{\alpha\beta} = \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix}. \tag{29}$$

The eigenvector of $\boldsymbol{D}$ corresponding to the eigenvalue λ is obtained by rotating the unit vector in the x-direction counterclockwise through an angle of $\frac{1}{4}\pi$. The eigenvector of $\boldsymbol{K}$ corresponding to the eigenvalue K_1 is assumed to be related to this vector by a counterclockwise rotation through an angle ϕ. Then the second eigenvector of $\boldsymbol{K}$ makes an angle of $\phi + \frac{1}{4}\pi$ with the y-axis and is associated with the eigenvalue K_2. We introduce the parameters

$$2T \equiv K_{\alpha\alpha} = K_1 + K_2, \quad \alpha \equiv (K_2 - K_1)/2T, \tag{30}$$

and

$$R \equiv \sigma\lambda / 4T^{\frac{1}{2}}. \tag{31}$$

The second moment $\boldsymbol{K}$ is determined when T, α, and ϕ are known. Its components are

$$K_{\alpha\beta} = T \begin{bmatrix} 1 + \alpha \sin 2\phi & -\alpha \cos 2\phi \\ -\alpha \cos 2\phi & 1 - \alpha \sin 2\phi \end{bmatrix}. \tag{32}$$

The parameter R measures the strength of the mean shear relative to the vigour of the velocity fluctuations. We note that the diagonal components of $\boldsymbol{K}$ are, in general, not equal.

3.1. *The collisional source of second moment*

We introduce the vector $\boldsymbol{V} \equiv \sigma(\boldsymbol{k}\cdot\nabla)\,\boldsymbol{u}$ and expand the velocity distribution function $f^{(1)}(\boldsymbol{c}, \boldsymbol{r} - \sigma\boldsymbol{k})$ in a Taylor series about $\boldsymbol{r}$:

$$f^{(1)}(\boldsymbol{c}, \boldsymbol{r} - \sigma\boldsymbol{k}) = \frac{n}{2\pi\Delta^{\frac{1}{2}}} \exp\left[-\tfrac{1}{2}(C_\alpha + V_\alpha)(C_\beta + V_\beta) K^{-1}_{\alpha\beta}\right], \tag{33}$$

where all of the mean fields are evaluated at $\boldsymbol{r}$. For simple shear, this expression is exact. Using it, the complete pair distribution function may be expressed in terms of $\boldsymbol{g}$, $\boldsymbol{V}$ and $\boldsymbol{Q} \equiv \frac{1}{2}(\boldsymbol{C}_1 + \boldsymbol{C}_2)$:

$$f^{(2)}(\boldsymbol{c}_1, \boldsymbol{r} - \sigma\boldsymbol{k}, \boldsymbol{c}_2, \boldsymbol{r}) = \frac{n^2 g_0}{4\pi^2 \Delta} \exp\left\{-\tfrac{1}{4}K^{-1}_{\alpha\beta}\left[(g_\alpha + V_\alpha)(g_\beta + V_\beta) + (2Q_\alpha + V_\alpha)(2Q_\beta + V_\beta)\right]\right\}. \tag{34}$$

Then, upon changing variables from $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ to $\boldsymbol{Q}$ and $\boldsymbol{g}$ in (10) and noting that $\mathrm{d}\boldsymbol{c}_1\,\mathrm{d}\boldsymbol{c}_2 = \mathrm{d}\boldsymbol{g}\,\mathrm{d}\boldsymbol{Q}$, we have an explicit expression for the collisional source $\aleph$ of a property ψ. In the event that $\Delta\psi$ is independent of $\boldsymbol{Q}$, the integration over $\boldsymbol{Q}$ may be carried out immediately. The result is

$$\aleph(\psi) = \frac{\sigma n^2 g_0}{8\pi\Delta^{\frac{1}{2}}} \iint \Delta\psi \exp\left[-\tfrac{1}{4}K^{-1}_{\alpha\beta}(g_\alpha + V_\alpha)(g_\beta + V_\beta)\right](\boldsymbol{k}\cdot\boldsymbol{g})\,\mathrm{d}k\,\mathrm{d}\boldsymbol{g}, \tag{35}$$

integrated over all $\boldsymbol{k}\cdot\boldsymbol{g} > 0$. For example, $\Delta(C_\alpha C_\beta)$ given by (5) has this property.

In order to facilitate the calculation of $\aleph_{\alpha\beta}$, it is convenient to write the total change of $C_\alpha C_\beta$ in terms of the unit vector $j_\alpha = \epsilon_{\alpha\beta} k_\beta$, where $\epsilon_{12} = -\epsilon_{21} = 1$ and $\epsilon_{11} = \epsilon_{22} = 0$:

$$\Delta(C_\alpha C_\beta) = -\tfrac{1}{2}(1+e)(\boldsymbol{k}\cdot\boldsymbol{g})\left[(1-e)(\boldsymbol{k}\cdot\boldsymbol{g})k_\alpha k_\beta + (\boldsymbol{j}\cdot\boldsymbol{g})(j_\alpha k_\beta + j_\beta k_\alpha)\right]. \tag{36}$$

Note that the second term has a zero trace. Consequently we write

$$\aleph_{\alpha\beta} = A_{\alpha\beta} + \hat{B}_{\alpha\beta}, \tag{37}$$

where $A_{\alpha\beta}$ and $\hat{B}_{\alpha\beta}$ are the integrals corresponding to the first and second terms of (36) and the hat denotes the deviatoric part of $B_{\alpha\beta}$ or a quantity with zero trace.

The integrations over $\boldsymbol{g}$ in $A_{\alpha\beta}$ and $\hat{B}_{\alpha\beta}$ are easily carried out. The results are most compactly expressed in terms of two functions of $\chi \equiv \boldsymbol{V}\cdot\boldsymbol{k}/2\tau^{\frac{1}{2}}$, where $\tau \equiv k_\alpha K_{\alpha\beta} k_\beta = T + k_\alpha \hat{K}_{\alpha\beta} k_\beta$,

$$\mathscr{F} \equiv -\pi^{\frac{1}{2}}(\tfrac{3}{2}+\chi^2)\chi\,\mathrm{erfc}\,(\chi) + (1+\chi^2)\exp(-\chi^2), \tag{38}$$

and

$$\mathscr{G} \equiv \pi^{\frac{1}{2}}(\tfrac{1}{2}+\chi^2)\,\mathrm{erfc}\,(\chi) - \chi\exp(-\chi^2). \tag{39}$$

We obtain:

$$A_{\alpha\beta} = \frac{-4\rho\nu g_0(1-e^2)}{\sigma\pi^{\frac{3}{2}}} \int k_\alpha k_\beta \tau^{\frac{3}{2}} \mathscr{F}\,\mathrm{d}k; \tag{40}$$

and

$$\hat{B}_{\alpha\beta} = \hat{E}_{\alpha\beta} + \hat{F}_{\alpha\beta}, \tag{41}$$

with

$$\hat{E}_{\alpha\beta} = -\frac{4\rho\nu g_0(1+e)}{\sigma\pi^{\frac{3}{2}}} \int (j_\alpha k_\beta + j_\beta k_\alpha)(\boldsymbol{j}\cdot\boldsymbol{K}\cdot\boldsymbol{k})\tau^{\frac{1}{2}}\mathscr{F}\,\mathrm{d}k, \tag{42}$$

and

$$\hat{F}_{\alpha\beta} = \frac{2\rho\nu g_0(1+e)}{\sigma\pi^{\frac{3}{2}}} \int (j_\alpha k_\beta + j_\beta k_\alpha)(\boldsymbol{V}\cdot\boldsymbol{K}^{-1}\cdot\boldsymbol{j})\mathscr{G}\varDelta\,\mathrm{d}k. \tag{43}$$

It remains to carry out the k integration.

3.2. *The collisional flux of momentum*

With the assumption of molecular chaos and the adoption of the anisotropic Maxwellian, the complete pair distribution function that appears in (13) for the collisional flux can be written as a function of $\boldsymbol{g}$, $\boldsymbol{V}$, $\boldsymbol{Q}$, and $\xi \equiv \frac{1}{2} - \eta$ as

$$f^{(2)}[\boldsymbol{c}_1, \boldsymbol{r}+\sigma(\xi-\tfrac{1}{2})\boldsymbol{k}, \boldsymbol{c}_2, \boldsymbol{r}+\sigma(\xi+\tfrac{1}{2})\boldsymbol{k}]$$

$$= \frac{n^2 g_0}{4\pi^2\varDelta}\exp\{-\tfrac{1}{4}K^{-1}_{\alpha\beta}[(g_\alpha+V_\alpha)(g_\beta+V_\beta) + 4(Q_\alpha - \xi V_\alpha)(Q_\beta - \xi V_\beta)]\}. \tag{44}$$

The integration over ξ corresponding to that over η is easily carried out:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} f^{(2)}[\boldsymbol{c}_1, \boldsymbol{r}+\sigma(\xi-\tfrac{1}{2})\boldsymbol{k}, \boldsymbol{c}_2, \boldsymbol{r}+\sigma(\xi+\tfrac{1}{2}\boldsymbol{k})]\,\mathrm{d}\xi$$

$$= \frac{n^2 g_0}{8\pi^{\frac{3}{2}}\Delta a^{\frac{1}{2}}}\left[\operatorname{erf}\left(\frac{b}{a^{\frac{1}{2}}}+\frac{a^{\frac{1}{2}}}{2}\right)-\operatorname{erf}\left(\frac{b}{a^{\frac{1}{2}}}-\frac{a^{\frac{1}{2}}}{2}\right)\right]\exp\left\{-\frac{[V^2Q^2-(\boldsymbol{V}\cdot\boldsymbol{Q})^2]}{\Delta a}\right\}$$
$$\times \exp[-\tfrac{1}{4}K^{-1}_{\alpha\beta}(g_\alpha+V_\alpha)(g_\beta+V_\beta)], \quad (45)$$

where $$a \equiv V_\alpha K^{-1}_{\alpha\beta} V_\beta, \quad b \equiv -V_\alpha K^{-1}_{\alpha\beta} Q_\beta, \quad (46)$$

and, for example, $V^2 \equiv \boldsymbol{V}\cdot\boldsymbol{V}$.

However, in situations in which $\psi_1'-\psi_1$ is independent of $\boldsymbol{Q}$, it is best to make the change of variables to $\boldsymbol{g}$ and $\boldsymbol{Q}$ and carry out the $\boldsymbol{Q}$ integration before the ξ integration. In this case the $\boldsymbol{Q}$ integration yields an expression that is independent of ξ, so the ξ integration poses no problem. For such a property, the result of these integrations is

$$\Theta_\alpha(\psi) = -\frac{\sigma^2 n^2 g_0}{8\pi\Delta^{\frac{1}{2}}}\iint(\psi_1'-\psi_1)\,k_\alpha\exp[-\tfrac{1}{4}K^{-1}_{\alpha\beta}(g_\alpha+V_\alpha)(g_\beta+V_\beta)](\boldsymbol{k}\cdot\boldsymbol{g})\,\mathrm{d}k\,\mathrm{d}\boldsymbol{g}, \quad (47)$$

integrated over all $\boldsymbol{k}\cdot\boldsymbol{g} \geqslant 0$. An expression identical to this would have been obtained if only the first term of the series in (11) had been retained.

The collisional flux that enters into the second moment equation in simple shear is that of linear momentum. Here $\psi_1'-\psi_1$ is given by (4) and is independent of $\boldsymbol{Q}$. So, upon using (4) in (47), multiplying by the mass m, and integrating over $\boldsymbol{g}$, we obtain

$$\Theta_{\alpha\beta} \equiv m\Theta_\alpha[C_\beta] = \frac{2\rho\nu g_0(1+e)}{\pi^{\frac{3}{2}}}\int \tau k_\alpha k_\beta \mathscr{G}\,\mathrm{d}k. \quad (48)$$

It remains to carry out the k integration.

3.3. *The balance of second moment*

In simple shear, the balance law (22) for $\boldsymbol{K}$ reduces to

$$0 = -P_{\mu\beta}u_{\alpha,\mu} - P_{\mu\alpha}u_{\beta,\mu} + \aleph_{\alpha\beta}, \quad (49)$$

where $$P_{\alpha\beta} = \rho K_{\alpha\beta} + \Theta_{\alpha\beta} = P_{\beta\alpha}, \quad (50)$$

with $\aleph_{\alpha\beta}$ and $\Theta_{\alpha\beta}$ given by (37)–(43) and (48) respectively.

At this point it is useful to carry the decomposition (37)–(43) of the collisional source one step further in order to highlight how the symmetric and antisymmetric parts of the velocity gradient enter into (49). We note that $A_{\alpha\beta}$ and $\hat{E}_{\alpha\beta}$ depend on the velocity gradients only through $\boldsymbol{V}\cdot\boldsymbol{k} = \sigma[(\boldsymbol{k}\cdot\nabla)\boldsymbol{u}]\cdot\boldsymbol{k} = \sigma\boldsymbol{k}\cdot\boldsymbol{D}\cdot\boldsymbol{k}$. Consequently, they are functions of $\boldsymbol{D}$ not $\boldsymbol{W}$. However, $\hat{F}_{\alpha\beta}$ depends on $\boldsymbol{W}$ through $\boldsymbol{V}\cdot\boldsymbol{K}^{-1}\cdot\boldsymbol{j}$. We make this dependence explicit by using the identity

$$\Delta K^{-1}_{\alpha\beta} = K_{\gamma\gamma}\delta_{\alpha\beta} - K_{\alpha\beta} \quad (51)$$

to write

$$(j_\alpha k_\beta + k_\alpha j_\beta)(\boldsymbol{V}\cdot\boldsymbol{K}^{-1}\cdot\boldsymbol{j})\Delta = \sigma\tau(k_\alpha k_\gamma W_{\beta\gamma} + k_\beta k_\gamma W_{\alpha\gamma}) + \sigma(j_\alpha k_\beta + j_\beta k_\alpha)k_\mu j_\gamma(TD_{\gamma\mu} - D_{\delta\mu}\hat{K}_{\delta\gamma}). \quad (52)$$

If we use this in the definition (43) of $\hat{F}_{\alpha\beta}$ and employ the definition (48) of $\Theta_{\alpha\beta}$, we have

$$\hat{F}_{\alpha\beta} = \Theta_{\alpha\gamma} W_{\beta\gamma} + \Theta_{\beta\gamma} W_{\alpha\gamma} + \hat{G}_{\alpha\beta}, \tag{53}$$

where $$\hat{G}_{\alpha\beta} \equiv \frac{2\rho\nu g_0(1+e)}{\pi^{\frac{3}{2}}} \int (j_\alpha k_\beta + j_\beta k_\alpha) k_\mu j_\gamma (T D_{\gamma\mu} - D_{\delta\mu} \hat{K}_{\delta\gamma}) \mathscr{G} \, \mathrm{d}k. \tag{54}$$

Note that $\hat{G}_{\alpha\beta}$ depends upon $\boldsymbol{D}$ but not upon $\boldsymbol{W}$. With this, the balance of second moment may be written as

$$\rho K_{\mu\beta}(D_{\alpha\mu} + W_{\alpha\mu}) + \Theta_{\mu\beta} D_{\alpha\mu} + \rho K_{\mu\alpha}(D_{\beta\mu} + W_{\beta\mu}) + \Theta_{\mu\alpha} D_{\beta\mu} = A_{\alpha\beta} + \hat{E}_{\alpha\beta} + \hat{G}_{\alpha\beta}. \tag{55}$$

We solve equation (55) in the two extremes of a dilute system and a dense system. Solutions for values of the area fraction near these two extremes may be obtained as perturbations of these two limits.

3.4. *The dilute limit*

Here we suppose that $R \equiv \sigma\lambda/4T^{\frac{1}{2}}$ and $G \equiv \nu g_0$ are small but that R/G, α, and ϕ are of order one. Then, at lowest order in (55),

$$\Theta_{\alpha\beta} = 0; \tag{56}$$

$$A_{\alpha\beta} = -\frac{4\rho G(1-e^2)}{\sigma\pi^{\frac{3}{2}}} \int k_\alpha k_\beta \tau^{\frac{3}{2}} \, \mathrm{d}k; \tag{57}$$

$$\hat{E}_{\alpha\beta} = -\frac{4\rho G(1+e)}{\sigma\pi^{\frac{3}{2}}} \int (j_\alpha k_\beta + j_\beta k_\alpha)(\boldsymbol{j}\cdot\boldsymbol{K}\cdot\boldsymbol{k}) \tau^{\frac{1}{2}} \, \mathrm{d}k; \tag{58}$$

and $$\hat{G}_{\alpha\beta} = 0. \tag{59}$$

The error we make in (55) by adopting (56) to (59) is of the order of R and G.

The integrations over $\boldsymbol{k}$ in (57) and (58) are effected by introducing the angle θ between $\boldsymbol{k}$ and the first eigenvector of $\boldsymbol{K}$; then $\tau = T(1-\alpha\cos 2\theta)$ and $\mathrm{d}k = \mathrm{d}\theta$. We obtain

$$A_{\alpha\beta} = -\frac{2\rho G(1-e^2) T^{\frac{3}{2}}}{\sigma\pi^{\frac{3}{2}}} \left[\tfrac{3}{2} I(\alpha) \frac{\hat{K}_{\alpha\beta}}{T} + \gamma(\alpha)\,\delta_{\alpha\beta}\right], \tag{60}$$

and $$\hat{E}_{\alpha\beta} = -\frac{4\rho G(1+e) T^{\frac{3}{2}}}{\sigma\pi^{\frac{3}{2}}} I(\alpha) \frac{\hat{K}_{\alpha\beta}}{T}, \tag{61}$$

where $$I(\alpha) \equiv \int_0^{2\pi} \sin^2 2\theta (1-\alpha\cos 2\theta)^{\frac{1}{2}} \, \mathrm{d}\theta, \tag{62}$$

and $$\gamma(\alpha) \equiv \int_0^{2\pi} (1-\alpha\cos 2\theta)^{\frac{3}{2}} \, \mathrm{d}\theta. \tag{63}$$

Using the lowest-order dilute expressions for the pressure tensor and the collisional source of second moment, we may write the trace of the balance of second moment (55) as

$$4\pi^{\frac{3}{2}}\alpha(R/G)\cos 2\phi = (1-e^2)\gamma(\alpha), \tag{64}$$

where ϕ is the counterclockwise angle through which the eigenvectors of $\boldsymbol{D}$ must be rotated to give the eigenvectors of $\boldsymbol{K}$. Equation (64) is the lowest order balance of fluctuation energy for the simple shear of a dilute system.

We next consider the deviatoric part of the lowest-order dilute balance of second

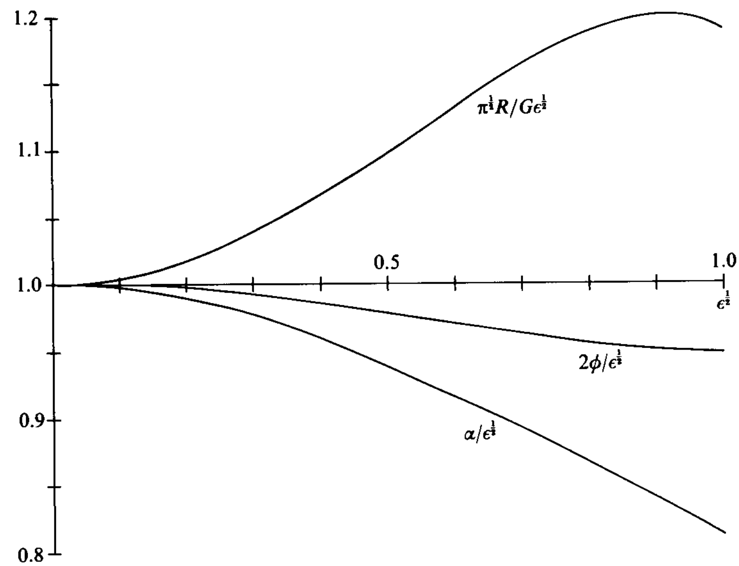

FIGURE 1. The variation of α, R/G, and ϕ, normalized by their behaviour for small ϵ, with $\epsilon^{\frac{1}{2}}$ in the dilute limit.

moment and write its components with respect to the orthonormal basis composed of the eigenvectors of $\boldsymbol{K}$. The diagonal components of the resulting equation require that

$$2\pi^{\frac{3}{2}}(R/G)\cos 2\phi = (1+e)\left[1+\tfrac{3}{4}(1-e)\right]\alpha I(\alpha), \tag{65}$$

while the off-diagonal components impose the condition that

$$\sin 2\phi - \alpha = 0. \tag{66}$$

Equations (64), (65), and (66) determine R/G, ϕ, and α as functions of $\epsilon \equiv (1-e)$. Equivalently, these equations determine T, α, and ϕ (and hence $\boldsymbol{K}$) in terms of λ, G, and ϵ. In these, $(R/G)\cos 2\phi$ may be eliminated between equations (64) and (65) yielding a single equation for the determination of α:

$$\epsilon\gamma(\alpha) = 2(1+\tfrac{3}{4}\epsilon)\,\alpha^2 I(\alpha). \tag{67}$$

This equation may be solved numerically to obtain α as a function of ϵ. The graph of such a solution is shown in figure 1. Also shown are the graphs of ϕ and R/G, determined through (66) and (65), that correspond to this solution.

An alternative to the numerical solution is an approximate series solution. This is facilitated by the expansions of the integrals $I(\alpha)$ and $\gamma(\alpha)$:

$$I(\alpha) = \pi\,\Gamma(\tfrac{3}{2}) \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{2^{2n} n!\,(n+1)!\,\Gamma(\frac{3}{2}-2n)}, \tag{68}$$

and

$$\gamma(\alpha) = 2\pi\,\Gamma(\tfrac{5}{2}) \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{2^{2n}(n!)^2\,\Gamma(\frac{5}{2}-2n)}. \tag{69}$$

The crudest approximate solution of (67) turns out to be remarkably good. When powers of α higher than the third are ignored in (67), its solution is

$$\alpha^2 = 16\epsilon/(16+9\epsilon). \tag{70}$$

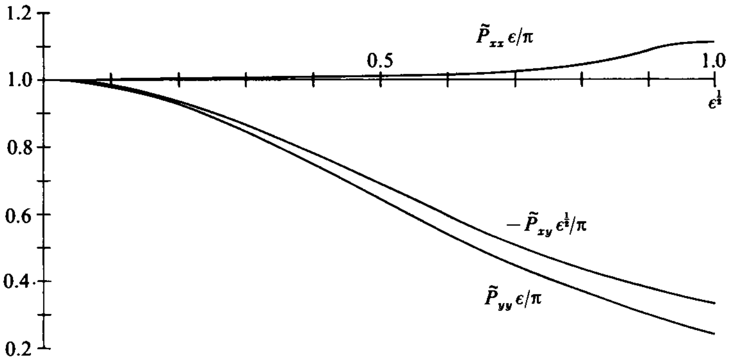

FIGURE 2. The variation of the components of the non-dimensional stress, normalized by their behaviour at small ϵ, with $\epsilon^{\frac{1}{2}}$ in the dilute limit.

Then ϕ is given by (66) and, through (65),

$$R/G = \frac{(2-\epsilon)(4+3\epsilon)}{8\pi^{\frac{1}{2}}}\left(\frac{\alpha^2}{1-\alpha^2}\right)^{\frac{1}{2}}. \tag{71}$$

These differ from the exact solution by, at most, several per cent.

With (50), (32), and (66), the pressure tensor for the simple shear of a dilute system is, at lowest order,

$$\boldsymbol{P} = \rho T\begin{bmatrix} 1+\alpha^2 & -\alpha(1-\alpha^2)^{\frac{1}{2}} \\ -\alpha(1-\alpha^2)^{\frac{1}{2}} & 1-\alpha^2 \end{bmatrix} \equiv \rho T\mathscr{A} \tag{72}$$

We use $\rho T = m\nu\lambda^2/4\pi R^2$ and introduce a non-dimensional pressure tensor

$$\tilde{\boldsymbol{P}} \equiv (4\pi G^2/\nu m\lambda^2)\,\boldsymbol{P} = (G/R)^2\,\mathscr{A}. \tag{73}$$

Then, in figure 2, we plot the non-dimensional shear stress and normal stresses versus ϵ using the numerical determinations of α and R/G. The approximate values, determined by employing (70) and (71) in (73), are within several per cent of these.

3.5. *The dense limit*

Here we suppose that G^{-1} is small but that R and α are of order one. In this case at lowest order the collisional terms dominate the balance of second moment,

$$P_{\alpha\beta} = \Theta_{\alpha\beta}, \tag{74}$$

and $\Theta_{\alpha\beta}$ and $\aleph_{\alpha\beta}$ are given by their complete expressions (48) and (37)–(43).

First we wish to show that, in this limit, $\boldsymbol{D}$ and $\boldsymbol{K}$ have the same eigenvectors. To do this we consider the balance of second moment at lowest order:

$$\Theta_{\alpha\gamma}D_{\gamma\beta} + \Theta_{\beta\gamma}D_{\gamma\alpha} = A_{\alpha\beta} + \hat{E}_{\alpha\beta} + \hat{G}_{\alpha\beta}. \tag{75}$$

This is a tensoral relation between two second-rank tensors, $\boldsymbol{D}$ and $\boldsymbol{K}$. Consequently, when one is diagonal, so is the other. The importance of this result is that at lowest order in the dense limit, ϕ is zero, and only α and R remain to be determined.

In the Appendix we show that, at lowest order, $A_{\alpha\beta}$, $\hat{E}_{\alpha\beta}$, $\hat{G}_{\alpha\beta}$, and $\Theta_{\alpha\beta}$ may be expressed in terms of six functions, γ, H, I, J, N, and S. The expressions are:

$$A_{\alpha\beta} = -\frac{2\rho G(1-e^2)T^{\frac{3}{2}}}{\sigma\pi^{\frac{3}{2}}}\left[\gamma(\alpha,R)\,\delta_{\alpha\beta}+H(\alpha,R)\frac{\sigma\hat{D}_{\alpha\beta}}{4T^{\frac{1}{2}}}\right], \tag{76}$$

$$\hat{E}_{\alpha\beta} = \frac{4\rho G(1+e)\,T^{\frac{3}{2}}}{\sigma\pi^{\frac{3}{2}}}I(\alpha,R)\frac{\sigma\hat{D}_{\alpha\beta}}{4T^{\frac{1}{2}}}, \tag{77}$$

$$\hat{G}_{\alpha\beta} = \frac{8\rho G(1+e)\,T^{\frac{3}{2}}}{\sigma\pi^{\frac{3}{2}}}J(\alpha,R)\frac{\sigma\hat{D}_{\alpha\beta}}{4T^{\frac{1}{2}}}, \tag{78}$$

and
$$\Theta_{\alpha\beta} = \frac{\rho G(1+e)\,T}{\pi^{\frac{3}{2}}}\left[N(\alpha,R)\,\delta_{\alpha\beta}-S(\alpha,R)\frac{\sigma\hat{D}_{\alpha\beta}}{4T^{\frac{1}{2}}}\right]. \tag{79}$$

With these representations, the isotropic part of the balance of second moment is, at lowest order,

$$4R^2S(\alpha,R) = (1-e)\,\gamma(\alpha,R). \tag{80}$$

The corresponding deviatoric part is

$$2N(\alpha,R) = -\tfrac{1}{2}(1-e)\,H(\alpha,R)+I(\alpha,R)+2J(\alpha,R). \tag{81}$$

Using standard techniques for numerical integration and the determination of roots, these equations may be solved numerically for α and R as functions of ϵ. The graphs of such solutions are shown in figure 3.

Again an alternative to the numerical solution is an approximate series solution. Using the approximations to the functions of α and R that are derived in the Appendix, the energy equation (80) may be written, up to an error involving powers of R greater than the fifth, as

$$8R^2[4+\pi a+R^2(4-\tfrac{3}{2}\pi a^2)] = \epsilon[4+3R^2(8+4\pi a+a^2)+3R^4(4-3\pi a^2+\tfrac{3}{16}\pi^2a^4)], \tag{82}$$

where $a \equiv \alpha/2\pi^{\frac{1}{2}}R$. In the same way, the deviatoric second-moment equation (81) is, up to an error involving powers of R greater than the third,

$$\begin{aligned}4[1+4R^2(1+a)] = 3\epsilon[1+a+R^2(2+3a-\tfrac{1}{8}\pi a^3)]\\ +a[4+R^2(12-\tfrac{1}{2}\pi a^2)]+2[1+2R^2(1-a)].\end{aligned} \tag{83}$$

We write $a = a_0+a_2R^2$ and determine what values of a_0 and a_2 satisfy (83) identically up to terms involving powers of R greater than the third. These are

$$a_0 = (2-3\epsilon)/(4+3\epsilon), \tag{84}$$

and
$$a_2 = \tfrac{1}{8}\pi a_0^3+\frac{(8-9\epsilon)}{(4+3\epsilon)}a_0+\frac{6(2-\epsilon)}{(4+3\epsilon)}. \tag{85}$$

Then, with an error of the same order as that already made in writing down (82), it becomes a quadratic equation for the determination of R^2:

$$\begin{aligned}R^4[2(8-3\epsilon)+2\pi(2-3\epsilon)\,a_2+\tfrac{3}{2}\pi(3\epsilon-4)\,a_0^2-3\pi\epsilon a_0a_2-\tfrac{9}{32}\pi^2\epsilon a_0^4]\\ +R^2[4(4-3\epsilon)+2\pi(2-3\epsilon)\,a_0-\tfrac{3}{2}\pi\epsilon a_0^2]-2\epsilon = 0.\end{aligned} \tag{86}$$

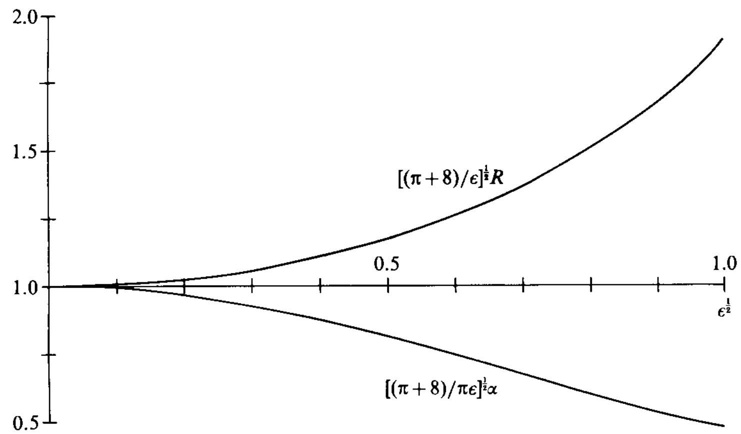

FIGURE 3. The variation of α and R, normalized by their behaviour at small ϵ, with $\epsilon^{\frac{1}{2}}$ in the dense limit.

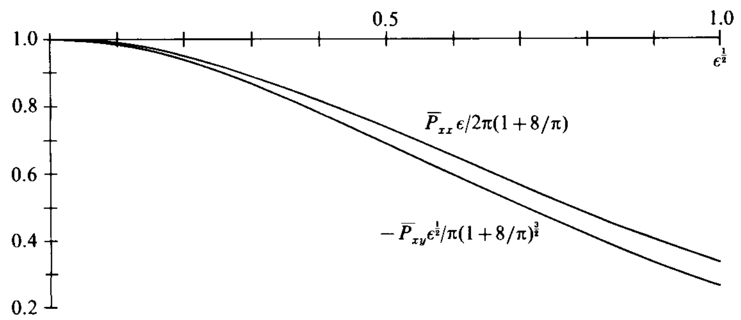

FIGURE 4. The variation of the components of the non-dimensional stress, normalized by their behaviour at small ϵ, with $\epsilon^{\frac{1}{2}}$ in the dense limit.

We introduce a non-dimensional pressure tensor

$$\bar{\mathbf{P}} \equiv (4\pi/\nu G m\lambda^2)\,\boldsymbol{\Theta}, \tag{87}$$

and, in figure 4, graph its components, based on the numerical determinations of α and R, versus ϵ. Alternative approximate expressions for these may be obtained by using the expansions for N and S given in (A 21) and (A 22) of the Appendix. They are

$$\bar{P}_{xx} = \bar{P}_{yy} = \frac{(2-\epsilon)}{\pi^{\frac{3}{2}}}\frac{N}{R^2} \tag{88}$$

$$\doteq \frac{(2-\epsilon)}{R^2}[1+4R^2(1+a_0+a_2R^2)-R^4a_0(4-\tfrac{3}{2}\pi a_0)], \tag{89}$$

up to powers of R greater than the third, and

$$-\bar{P}_{xy} = \frac{(2-\epsilon)}{\pi^{\frac{3}{2}}}\frac{S}{R} \tag{90}$$

$$\doteq \frac{(2-\epsilon)}{\pi^{\frac{1}{2}}R}[4+\pi(a_0+a_2R^2)+R^2(4-\tfrac{3}{2}\pi a_0^2)], \tag{91}$$

up to powers of R greater than the second. In these a_0 and a_2 are given by (84) and (85) and R is determined as a solution of (86). The approximations differ from the non-dimensional components based on the numerical solution by, at most, several per cent.

4. Discussion

The figures provide a direct measure of the difference between the results of the present theory and those for nearly elastic disks. There is one important qualitative similarity in the two theories; both predict that the components of the stress vary with the square of the shear rate. The important qualitative difference is in the behaviour of the deviatoric part of the second moment, non-dimensionalized by T. For small ϵ in both the dilute and dense limits, its off-diagonal terms are proportional to $\epsilon^{\frac{1}{2}}$; in the dense limit its diagonal terms vanish, in the dilute limit they are proportional to $\epsilon^{\frac{1}{2}}$. Consequently, at lowest order, the diagonal terms are zero or negligible, the off-diagonal terms are small, and $\hat{\boldsymbol{K}}/T$ is proportional to $\boldsymbol{D}$. For more dissipative disks, the components of $\hat{\boldsymbol{K}}/T$ are not small and, in the dilute limit, they are not so simply related to those of $\boldsymbol{D}$.

In the dilute limit, the principal axes of $\hat{\boldsymbol{K}}$ are driven away from those of $\boldsymbol{D}$ by the spin of the mean motion. This results in a difference in the normal stress on planes parallel to and perpendicular to the direction of flow. In the dense limit, $\hat{\boldsymbol{K}}$ is determined by the condition that its total collisional production vanish. When based on the anisotropic Maxwellian, this collisional production is independent of the spin of the mean motion. Consequently, the principal axes of $\hat{\boldsymbol{K}}$ and $\boldsymbol{D}$ coincide and are oriented at angles of 45° and 135° to the flow. Because of this, even though the principal values of $\hat{\boldsymbol{K}}$ are different, the normal stresses on planes parallel and perpendicular to the flow are the same; however they do differ on planes oriented, for example, at $\pm 45°$ to the flow.

Quantitative comparisons can be made with the results of two numerical simulations, one dilute and one dense. For $e = 0.8$ ($\epsilon = 0.2$) and $\nu = 0.025$, Walton & Braun (1987) find that $P_{xx}/P_{yy} = 1.484$ and $-P_{xy}/P_{yy} = 0.463$. The corresponding values of these ratios based on the analytic approximations are 1.439 and 0.468, respectively. For $e = 0.8$ and $\nu = 0.65$, Walton & Braun find that $-P_{xy}/P_{yy} = 0.357$. In this case the analytic approximation gives a value of 0.397. Recent calculations by Richman (paper in preparation) indicate that the greater error (11.2%) in this dense limit is due to the neglect of higher-order terms involving transport and that it may be reduced substantially by including at least some of them.

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Appendix

In the dense limit, $\phi = 0$,

$$\chi = \frac{2R\cos 2\theta}{(1-\alpha\cos 2\theta)^{\frac{1}{2}}}, \tag{A 1}$$

and, through χ, $\mathscr{F}$ and $\mathscr{G}$ are functions of θ and the parameters α and R. In this case the integrals can be written in the forms (75)–(79), with

$$\gamma(\alpha, R) \equiv \int_0^{2\pi} (1-\alpha\cos 2\theta)^{\frac{3}{2}} \mathscr{F}\, \mathrm{d}\theta, \tag{A 2}$$

$$H(\alpha, R) \equiv \frac{1}{R}\int_0^{2\pi} \cos 2\theta (1-\alpha\cos 2\theta)^{\frac{3}{2}} \mathscr{F}\, \mathrm{d}\theta, \tag{A 3}$$

$$I(\alpha, R) \equiv \frac{\alpha}{R}\int_0^{2\pi} \sin^2 2\theta (1-\alpha\cos 2\theta)^{\frac{1}{2}} \mathscr{F}\, \mathrm{d}\theta, \tag{A 4}$$

$$J(\alpha, R) \equiv \int_0^{2\pi} \sin^2 2\theta \mathscr{G}\, \mathrm{d}\theta, \tag{A 5}$$

$$N(\alpha, R) \equiv \int_0^{2\pi} (1-\alpha\cos 2\theta) \mathscr{G}\, \mathrm{d}\theta, \tag{A 6}$$

and

$$S(\alpha, R) \equiv -\frac{1}{R}\int_0^{2\pi} \cos 2\theta (1-\alpha\cos 2\theta) \mathscr{G}\, \mathrm{d}\theta. \tag{A 7}$$

In these we expand the integrands in powers of α and R and integrate term by term. The resulting series are

$$\gamma = 3\pi^{\frac{3}{2}}\alpha R + 6\pi \sum_{n=0}^{\infty} A_n R^{2n} \sum_{p=0}^{\infty} \frac{2^{n-p}}{(n+p)!} \Lambda(\tfrac{3}{2}-n, 2p)\,\alpha^{2p}, \tag{A 8}$$

where

$$A_n \equiv \frac{(-1)^n}{n!\,(2n-1)\,(2n-3)} \tag{A 9}$$

and

$$\Lambda(m, n) \equiv \frac{\Gamma(m+1)}{n!\,\Gamma(m+1-n)}; \tag{A 10}$$

$$H = -3\pi^{\frac{3}{2}}(1+2R^2) - \frac{3\pi\alpha}{2R} \sum_{n=0}^{\infty} A_n R^{2n} \sum_{p=0}^{\infty} \frac{2^{n+1-p}}{(n+p+1)!} \Lambda(\tfrac{3}{2}-n, 2p+1)\,\alpha^{2p}; \tag{A 11}$$

$$I = 2\pi^{\frac{3}{2}}\alpha^2 R^2 \sum_{p=0}^{\infty} \frac{2^{-p}}{(p+3)!} \Lambda(-1, 2p+1)\,\alpha^{2p}$$
$$+\frac{3\pi\alpha}{R} \sum_{n=0}^{\infty} A_n R^{2n} \sum_{p=0}^{\infty} \frac{2^{n-p}}{(n+p+1)!} \Lambda(\tfrac{1}{2}-n, 2p)\,\alpha^{2p}; \tag{A 12}$$

$$J = \tfrac{1}{2}\pi^{\frac{3}{2}} + 2\pi^{\frac{3}{2}}R^2 \sum_{p=0}^{\infty} \frac{2^{-p}}{(p+2)!} \Lambda(-1, 2p)\,\alpha^{2p}$$
$$-\pi\alpha R \sum_{n=0}^{\infty} B_n R^{2n} \sum_{p=0}^{\infty} \frac{2^{n+1-p}}{(n+p+2)!} \Lambda(-\tfrac{1}{2}-n, 2p+1)\,\alpha^{2p}, \tag{A 13}$$

where

$$B_n \equiv \frac{(-1)^n}{n!\,(2n-1)\,(2n+1)}; \tag{A 14}$$

$$N = \pi^{\frac{3}{2}}(1+4R^2) - 2\pi\alpha R \sum_{n=0}^{\infty} B_n R^{2n} \sum_{p=0}^{\infty} \frac{2^{n+1-p}}{(n+p+1)!} \Lambda(\tfrac{1}{2}-n, 2p+1)\,\alpha^{2p}; \tag{A 15}$$

and
$$S = \frac{\pi^{\frac{3}{2}}\alpha}{2R} - 2\pi \sum_{n=0}^{\infty} B_n R^{2n} \sum_{p=0}^{\infty} \frac{2^{n+1-p}}{(n+p+1)!} A(\tfrac{1}{2}-n, 2p)\, \alpha^{2p}. \tag{A 16}$$

In these we need only the terms that permit us to satisfy the energy equation (80) and the deviatoric second-moment equation (81) at, respectively, fourth order and second order in α and R. The approximations are conveniently expressed in terms of $a \equiv \alpha/2\pi^{\frac{1}{2}}R$ and R:

$$\gamma = \tfrac{1}{2}\pi[4+3R^2(8+4\pi a+\pi a^2)+3R^4(4-3\pi a^2+\tfrac{3}{16}\pi^2 a^4)], \tag{A 17}$$

$$H = -3\pi^{\frac{3}{2}}[1+a+R^2(2+3a-\tfrac{1}{8}\pi a^3)], \tag{A 18}$$

$$I = \tfrac{1}{2}\pi^{\frac{3}{2}}a[4+R^2(12-\tfrac{1}{2}\pi a^2)], \tag{A 19}$$

$$J = \tfrac{1}{2}\pi^{\frac{3}{2}}[1+2R^2(1-a)], \tag{A 20}$$

$$N = \pi^{\frac{3}{2}}[1+4R^2(1+a)-\underline{R^4 a(4-\tfrac{3}{2}\pi a)}], \tag{A 21}$$

and
$$R^2 S = \pi R^2[4+\pi a+R^2(4-\tfrac{3}{2}\pi a^2)]. \tag{A 22}$$

The higher-order terms underlined in (A 21) are not involved in the determination of α and R. They are necessary for the approximation to the non-dimensional normal stress.

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