The Source of Second Moment in Dilute Granular Flows of Highly Inelastic Spheres

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Synopsis

We extend methods from the kinetic theory of gases to obtain a general constitutive relation for the collisional source of second moment in dilute flows of identical, smooth, highly inelastic spheres. In the derivation, we base all statistical averaging on an anisotropic Maxwellian distribution function, which is sensitive to all components of the full second moment of fluctuation velocity and is not based on the assumption that the fluctuations are nearly isotropic. In the case of homogeneous shear flow, we combine the constitutive relation with the balance equation for full second moment to determine, for prescribed values of shear rate, coefficient of restitution, and solid fraction, both exact numerical and approximate closed-form solutions for the second moment and pressure tensor. Most striking are the resulting normal pressure differences, which are predicted by this theory but not by kinetic theories for nearly elastic particles.

INTRODUCTION

In rapid flows of dry granular materials, quantities such as momentum and energy are transferred by a combination of particle transport and particle collisions. Because of the similarity between the granular motions within these materials and the molecular motions within dense gases, workers have employed the methods of the kinetic theory to obtain statistically averaged constitutive relations that govern the rates of transfer throughout such flows. Typically, the effect of dissipative collisions on the statistical description of the particles' velocities has been treated as a small perturbation from the near-Maxwellian description of molecular velocities in a disequilibrated dense gas. Consequently, the resulting theories apply only to flows in which the collisions between grains are nearly energy conserving. Kinetic theories derived in this manner have been reviewed thoroughly by Richman[1] and Jenkins.[2] They include those obtained for systems of identical, smooth spheres,[3,4] for systems of identical, rough particles,[5–7] and for binary mixtures of smooth spheres.[8,9]

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When the grains are smooth, the mean fields of interest are the solid fraction, the mean velocity, and a measure of the energy associated with the velocity fluctuations known as the granular temperature.

Each of these theories predicts that under all circumstances the normal stresses in homogeneous shear flows are equal. However, the computer simulations of these same flows carried out for two-dimensional systems of frictional, inelastic disks,[10,11] and for three-dimensional systems of smooth, inelastic spheres,[12] exhibit profound normal stress differences when the flows are dilute and highly dissipative. Such stress differences indicate that the mean square of the fluctuation velocity component in the direction of the shearing is several times greater than those components perpendicular to the shearing. This demonstrates that the granular temperature, which is an isotropic measure of the velocity fluctuations, is not sufficient to characterize the energetics within highly dissipative flows. A more complete analysis of these flows should include an exact determination of the full second moment of fluctuation velocity.

In this article, we generalize the work of Jenkins and Richman[13] on two-dimensional flows of circular disks to derive a constitutive relation for the collisional source of second moment in three-dimensional dilute flows of identical, smooth, highly inelastic spheres. We base all statistical averaging on an anisotropic Maxwellian, which is equally sensitive to all components of the full second moment of fluctuation velocity. As an important example, we employ the constitutive relation for the collisional source in the balance equation for the second moment to determine the second moment, induced stresses, and corresponding normal stress differences in homogeneous shear flows.

PRELIMINARIES

Of interest here are dilute granular flows of identical, smooth, highly inelastic spheres of diameter σ and mass m. Following the kinetic theory of gases,[14] we introduce a single particle distribution function f, defined such that $f(\mathbf{c}, \mathbf{x}, t)\, dc\, dx$ gives the number of particles with velocity $\mathbf{c}$ within the range $d\mathbf{c}$ whose centers are located at $\mathbf{x}$ within the range $d\mathbf{x}$ at time t. The particle number density $n(\mathbf{x}, t)$ is found by integrating $f(\mathbf{c}, \mathbf{x}, t)$ over all velocities, the solid fraction $\nu(\mathbf{x}, t)$ is equal to $n\pi\sigma^3/6$, and the mass density $\rho(\mathbf{x}, t)$ of the flow is equal to mn.

Based upon the velocity distribution f, the mean value of any particle property $\psi(\mathbf{c}, \mathbf{x}, t)$ is defined as,

$$\langle \psi \rangle = \frac{1}{n} \int \psi f \, d\mathbf{c}, \tag{1}$$

in which the integration is taken over all velocities. The mean velocity $\mathbf{u}(\mathbf{x}, t)$ of the flow, for example, is equal to $\langle \mathbf{c} \rangle$, and the fluctuation velocity $\mathbf{C}$ is equal to the difference $\mathbf{c} - \mathbf{u}$. A mean field of special significance in highly dissipative systems is the full second moment $\boldsymbol{K}(\mathbf{x}, t)$ of fluctuation velocity, equal to the tensor product $\langle \mathbf{CC} \rangle$. The granular temperature $T(\mathbf{x}, t) \equiv (1/3) tr\boldsymbol{K}$ is equal to the isotropic components of $\boldsymbol{K}$, and the tensor $\boldsymbol{B} \equiv \boldsymbol{K}/T$ is the dimensionless counterpart of $\boldsymbol{K}$.

The six independent components of $\boldsymbol{K}$ determine T and the five independent components of the deviatoric part $\hat{\boldsymbol{B}}$ of $\boldsymbol{B}$. If $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$ are the normalized eigenvectors of $\hat{\boldsymbol{B}}$, then the identity tensor $\boldsymbol{I}$ may be written as the sum,

$$\boldsymbol{I} = \mathbf{pp} + \mathbf{qq} + \mathbf{rr}. \tag{2}$$

If ξ, η, and ζ are the corresponding eigenvalues of $\hat{\boldsymbol{B}}$, then $\hat{\boldsymbol{B}}$ may be written as,

$$\hat{\boldsymbol{B}} = \xi \mathbf{pp} + \eta \mathbf{qq} + \zeta \mathbf{rr}, \tag{3}$$

and the deviatoric part $\hat{\boldsymbol{b}}$ of $\hat{\boldsymbol{B}}^2$ may be written as,

$$\hat{\boldsymbol{b}} = \frac{1}{3}[(2\xi^2 - \eta^2 - \zeta^2)\mathbf{pp} + (2\eta^2 - \xi^2 - \zeta^2)\mathbf{qq} + (2\zeta^2 - \xi^2 - \eta^2)\mathbf{rr}]. \tag{4}$$

Inverting equations (2), (3), and (4), we find that the tensor products $\mathbf{pp}$, $\mathbf{qq}$, and $\mathbf{rr}$ may be decomposed into sums of their isotropic and deviatoric parts as follows:

$$\mathbf{pp} = \frac{1}{3}\boldsymbol{I} + \frac{\zeta - \eta}{M}(\xi \hat{\boldsymbol{B}} + \hat{\boldsymbol{b}}); \tag{5}$$

$$\mathbf{qq} = \frac{1}{3}\boldsymbol{I} + \frac{\xi - \zeta}{M}(\eta \hat{\boldsymbol{B}} + \hat{\boldsymbol{b}}); \tag{6}$$

and

$$\mathbf{rr} = \frac{1}{3}\boldsymbol{I} + \frac{\eta - \xi}{M}(\zeta \hat{\boldsymbol{B}} + \hat{\boldsymbol{b}}); \tag{7}$$

where M is the determinant of the matrix of coefficients in Eqs. (2), (3), and (4), given in factored form by,

$$M = (\xi - \eta)(\eta - \zeta)(\zeta - \xi). \tag{8}$$

Because the full second moment is an anisotropic measure of the velocity fluctuations, it contains more information about the nature of the flow than does the granular temperature. A velocity distribution function that is sensitive to each component of $\boldsymbol{K}$ and is not based on the assumption that $\boldsymbol{K}$ is nearly isotropic is the anisotropic Maxwellian distribution,

$$\mathrm{f}(\mathbf{c}, \mathbf{x}, \mathrm{t}) = \frac{\mathrm{n}}{(8\pi^3\mathrm{K})^{1/2}} \exp\left(\frac{-1}{2}\mathbf{C} \cdot \boldsymbol{K}^{-1} \cdot \mathbf{C}\right), \tag{9}$$

where K is the determinant of $\boldsymbol{K}$. When the deviatoric part of $\boldsymbol{K}$ is small compared to its isotropic part, expression (9) may be approximated by the modified Maxwellian distribution employed by Jenkins and Richman[4] for homogeneous flows of nearly elastic spheres. When the deviatoric part of $\boldsymbol{K}$ vanishes entirely, expression (9) reduces to a simple Maxwellian. The anisotropic Maxwellian given in Eq. (9) has been employed by Goldreich and Tremaine[15] and Araki and Tremaine[16] to study the dynamics of planetary rings, and its two-dimensional counterpart has been employed by Jenkins and Richman[13] to describe planar flows of inelastic disks.

Of special interest are steady, homogeneous, rectilinear granular shear flows, in which the shear rate, solid fraction, and components of $\boldsymbol{K}$ are constants that are related by the balance equation for full second moment of fluctuation velocity,

$$\rho[(\boldsymbol{K} \cdot \nabla)\mathbf{u}] + \rho[(\boldsymbol{K} \cdot \nabla)\mathbf{u}]^{\mathrm{T}} = \boldsymbol{\Gamma}. \tag{10}$$

Here $\rho\boldsymbol{K}$ is the pressure tensor, due entirely to particle transport, and $\boldsymbol{\Gamma}$ is the source of second moment, due entirely to particle collisions. The full second moment equation (10) is the tensor generalization of the balance of energy and the constitutive quantity $-\boldsymbol{\Gamma}$ is the corresponding generalization of the collisional rate of energy dissipation.

In what follows, we first derive a general constitutive relation for $\boldsymbol{\Gamma}$ in terms of ν, T, $\hat{\boldsymbol{B}}$, and the eigenvalues of $\hat{\boldsymbol{B}}$, and then employ this constitutive relation in equation (10) to determine the second moment $\boldsymbol{K}$ and the pressure tensor $\rho\boldsymbol{K}$ in homogeneous shear flows.

SOURCE OF SECOND MOMENT

We consider an inelastic collision between two identical, smooth, spheres, in which $\mathbf{c}_1$ and $\mathbf{c}_2$ are the velocities of the spheres just prior to impact, $\mathbf{k}$ is the unit vector directed from the center of the first sphere to the center of the second at impact, and e is the coefficient of restitution that accounts for the energy dissipated during impact. If $\mathbf{c}_1'$ and $\mathbf{c}_2'$ are the velocities of the spheres just after impact, then the total change $\mathbf{\Delta} \equiv \mathbf{c}_1'\mathbf{c}_1' + \mathbf{c}_2'\mathbf{c}_2' - \mathbf{c}_1\mathbf{c}_1 - \mathbf{c}_2\mathbf{c}_2$ is fixed by the relative incoming velocity $\mathbf{g} \equiv \mathbf{c}_1 - \mathbf{c}_2$, the unit vector $\mathbf{k}$, and the coefficient e according to,

$$\mathbf{\Delta} = \frac{-1}{2}(1 + e)(\mathbf{g} \cdot \mathbf{k})[(1 - e)(\mathbf{g} \cdot \mathbf{k})\mathbf{kk} + (\mathbf{g} \cdot \mathbf{j})(\mathbf{kj} + \mathbf{jk})], \tag{11}$$

in which $\mathbf{j}$ is the unit vector normal to $\mathbf{k}$ that lies in the plane formed by $\mathbf{g}$ and $\mathbf{k}$.

The source of second moment $\mathbf{\Gamma}$ is calculated as a statistical average over all binary collisions of the total change $\mathbf{\Delta}$ per collision weighted by the frequency of each collision. In the integral expression derived by Jenkins and Savage[17] for the collisional source of any particle property, we employ Eq. (11) for the change in second moment per collision and replace the pair distribution function by the product of corresponding single particle distributions to obtain,

$$\mathbf{\Gamma}(\mathbf{x}, t) = \frac{m}{2}\int \mathbf{\Delta} f(\mathbf{c}_1, \mathbf{x}, t) f(\mathbf{c}_2, \mathbf{x}, t)\sigma^2(\mathbf{g} \cdot \mathbf{k})\, d\mathbf{k}\, d\mathbf{c}_1\, d\mathbf{c}_2. \tag{12}$$

In this expression, $d\mathbf{k}$ is an element of solid angle centered about $\mathbf{k}$ at impact, and the integration is over all velocities $\mathbf{c}_1$ and $\mathbf{c}_2$, and points of contact $\mathbf{k}$ for which a collision is impending.

In order to carry out the integration for $\mathbf{\Gamma}$, we introduce the velocity $\mathbf{Q} \equiv [(\mathbf{c}_1 - \mathbf{u}) + (\mathbf{c}_2 - \mathbf{u})]/2$, note that the Jacobian of the transformation from $\mathbf{c}_1$ and $\mathbf{c}_2$ to $\mathbf{g}$ and $\mathbf{Q}$ is equal to one, and replace $d\mathbf{c}_1\, d\mathbf{c}_2$ by $d\mathbf{g}\, d\mathbf{Q}$ in integral (12). With $f(\mathbf{c}, \mathbf{x}, t)$ given by the anisotropic Maxwellian (9), the product of distriubution functions appearing in Eq. (12) is

$$f(\mathbf{c}_1, \mathbf{x}, t) f(\mathbf{c}_2, \mathbf{x}, t) = \frac{n^2}{8\pi^3 K} \exp\left[-\frac{1}{4}(\mathbf{g} \cdot \mathbf{K}^{-1} \cdot \mathbf{g} + 4\mathbf{Q} \cdot \mathbf{K}^{-1} \cdot \mathbf{Q})\right], \tag{13}$$

and may be used with expression (11) for $\mathbf{\Delta}$ to cast the integrand in Eq. (12) entirely in terms of $\mathbf{g}$ and $\mathbf{Q}$. The results of the velocity integrations are expressed compactly in terms of two unit vectors: the first is $\mathbf{k}$, which when written in terms of the eigenvectors $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$ is

$$\mathbf{k} = \sin\theta(\cos\mu\mathbf{p} + \sin\mu\mathbf{q}) + \cos\theta\mathbf{r}, \tag{14}$$

where θ is the angle between $\mathbf{r}$ and $\mathbf{k}$, and μ is the angle between $\mathbf{p}$ and $\mathbf{k} - (\mathbf{k}\cdot\mathbf{r})\mathbf{r}$; and the second is $\mathbf{i}$, which is normal to $\mathbf{k}$ and given by

$$\mathbf{i} = \frac{\sqrt{2}}{2}[(\cos\theta\cos\mu - \sin\mu)\mathbf{p} + (\cos\theta\sin\mu + \cos\mu)\mathbf{q} - \sin\theta\mathbf{r}]. \tag{15}$$

The velocity integrations of Eq. (12), which are over all values of $\mathbf{Q}$ and those values of $\mathbf{g}$ for which $\mathbf{g}\cdot\mathbf{k} > 0$, yield

$$\mathbf{\Gamma} = \frac{-6(1 + \mathrm{e})\rho\nu\mathrm{T}^{3/2}}{\sigma\pi^{3/2}}\boldsymbol{\chi}, \tag{16}$$

where $\boldsymbol{\chi}$ is the dimensionless sum $(1 - \mathrm{e})\boldsymbol{A} + 2\hat{\boldsymbol{E}}$ in which $\boldsymbol{A}$ and $\hat{\boldsymbol{E}}$ are the tensor-valued integrals,

$$\boldsymbol{A} = \int \mathbf{kk}(\mathbf{k}\cdot\boldsymbol{B}\cdot\mathbf{k})^{3/2}\,\mathbf{dk}, \tag{17}$$

and

$$\hat{\boldsymbol{E}} = \int (\mathbf{ki} + \mathbf{ik})\,(\mathbf{k}\cdot\boldsymbol{B}\cdot\mathbf{k})^{1/2}(\mathbf{k}\cdot\boldsymbol{B}\cdot\mathbf{i})\,\mathbf{dk}. \tag{18}$$

The two terms in $\boldsymbol{\chi}$ are each symmetric and result from the corresponding two symmetric terms on the right-hand side of Eq. (11).

When the variables of integration in Eqs. (17) and (18) are θ and μ, the element of solid angle $\mathbf{dk}$ is equal to $\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\mu$, and the integrations are on θ from 0 to π and on μ from 0 to 2π. In their expanded forms, the scalars $\mathbf{k}\cdot\boldsymbol{B}\cdot\mathbf{k}$ and $\mathbf{k}\cdot\boldsymbol{B}\cdot\mathbf{i}$ are given in terms of θ, μ, and the dimensionless parameters,

$$\alpha \equiv (\eta - \xi)/2 \qquad \text{and} \qquad \beta^2 \equiv (\eta + \xi)/2, \tag{19}$$

by

$$\mathbf{k}\cdot\boldsymbol{B}\cdot\mathbf{k} = 1 - \alpha\sin^2\theta\cos 2\mu + \beta^2(3\sin^2\theta - 2), \tag{20}$$

and

$$\mathbf{k}\cdot\boldsymbol{B}\cdot\mathbf{i} = \frac{\sqrt{2}}{2}[\alpha\sin\theta(\sin 2\mu - \cos\theta\cos 2\mu) + 3\beta^2\cos\theta\sin\theta]. \tag{21}$$

With the **p**, **q**, and **r** components of **k** and **i** given in Eqs. (14) and (15), integrals (17) and (18) determine the components of $\boldsymbol{A}$ and $\hat{\boldsymbol{E}}$ in the coordinate system whose base vectors are the eigenvectors of $\boldsymbol{B}$. Because the tensors $\boldsymbol{A}$ and $\hat{\boldsymbol{E}}$ are related to $\boldsymbol{B}$ only, they are diagonal in this coordinate system.

A general tensor expression for the dimensionless sum $\boldsymbol{\chi}$ appearing in $\boldsymbol{\Gamma}$ in terms of $\boldsymbol{I}$, $\hat{\boldsymbol{B}}$, and $\hat{\boldsymbol{b}}$ is obtained by employing decompositions (14) and (15) to eliminate **k** and **i** from integrals (17) and (18), and by employing Eqs. (5), (6), and (7) to eliminate **pp**, **qq**, and **rr** from the intermediate results. In this manner, we find that $\boldsymbol{\chi}$ has the general form,

$$\boldsymbol{\chi} = \frac{(1-\mathrm{e})}{3}(tr\boldsymbol{A})\boldsymbol{I} + \frac{1}{\mathrm{M}}\{[(3\beta^4 - \alpha^2)(\chi_{22} - \chi_{11}) - 6\alpha\beta^2\hat{\chi}_{33}]\hat{\boldsymbol{B}} + 3[\beta^2(\chi_{22} - \chi_{11}) + \alpha\hat{\chi}_{33}]\hat{\boldsymbol{b}}\}, \tag{22}$$

where subscripts 1, 2, and 3 refer, respectively, to components in the **p**, **q**, and **r** directions, and overhats denote deviatoric tensors. Integrals (17) and (18) determine the variations of χ_{11}, χ_{22}, and $\hat{\chi}_{33}$ with α and β^2. In terms of α and β, the determinant M is equal to $2\alpha(9\beta^4 - \alpha^2)$.

The constitutive relation for the collisional source $\boldsymbol{\Gamma}$ in terms of $\boldsymbol{B}$ is given by Eq. (16) with $\boldsymbol{\chi}$ eliminated through Eq. (22). In a homogeneous flow, this constitutive relation may be employed in the second-moment Eq. (10) to determine, for prescribed values of shear rate, solid fraction, and coefficient of restitution, the corresponding values of α, β^2, and T and principal directions of $\boldsymbol{B}$. These, in turn, fix the full second-moment $\boldsymbol{K}$ and the corresponding pressure tensor $\rho\boldsymbol{K}$.

HOMOGENEOUS SHEAR FLOW

We focus attention on steady, homogeneous, rectilinear shear flows that are driven at shear rates $\dot{\gamma}$. In these simple flows, $\dot{\gamma}$, ν, and $\boldsymbol{K}$ are constants, the balance equations of mass and momentum are satisfied identically, and the balance equation of full second moment is given by Eq. (10). In an x-y-z Cartesian coordinate system with the x coordinate in the direction of the mean velocity and the y coordinate in the direction of the velocity gradient, the only nonvanishing components of the stretching tensor or deformation rate $\boldsymbol{D} \equiv (\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}})/2$ and the spin or vorticity tensor $\boldsymbol{W} \equiv (\nabla\mathbf{u} - \nabla\mathbf{u}^{\mathrm{T}})/2$ are $\mathrm{D}_{xy} = \mathrm{D}_{yx} = \dot{\gamma}$ and $\mathrm{W}_{xy} = -\mathrm{W}_{yx} = \dot{\gamma}$. We also introduce a second Cartesian system in which the x_1, x_2, and x_3 coordinate directions coincide respectively with the **p**, **q**, and **r** eigendirections of $\boldsymbol{B}$.

When the constitutive relation obtained from Eqs. (16) and (22) is employed to eliminate Γ, the second-moment Eq. (10) determines the values of α, β^2, and T and the principal directions of $\boldsymbol{B}$ corresponding to prescribed values of $\dot{\gamma}$, e, and ν. The off-diagonal x_1-x_3 and x_2-x_3 components of tensor Eq. (10), for example, are satisfied provided that the x_3-eigendirection of $\boldsymbol{B}$ is normal to the x-y plane of motion. The orientation of the remaining two eigendirections of $\boldsymbol{B}$ is then completely specified by one angle ϕ, defined here such that $\phi + \pi/4$ is the counterclockwise, in-plane angle of rotation from both x to x_1 and y to x_2. With ϕ defined in this manner, the x-y-z components of $\boldsymbol{K}$ are

$$\boldsymbol{K} = \mathrm{T}\begin{pmatrix} 1 + \beta^2 + \alpha \sin 2\phi & -\alpha \cos 2\phi & 0 \\ -\alpha \cos 2\phi & 1 + \beta^2 - \alpha \sin 2\phi & 0 \\ 0 & 0 & 1 - 2\beta^2 \end{pmatrix}. \tag{23}$$

When α, β, and ϕ are small, $\boldsymbol{K}$ is nearly isotropic. The parameter α is then the lowest order measure of the anisotropic part of $\boldsymbol{K}$ relative to its isotropic part. For a given value of α, the angle ϕ is a measure of the difference between the streamwise component K_{xx} and the transverse component K_{yy} of the second moment. When ϕ vanishes, the principal directions of $\boldsymbol{K}$ coincide with those of $\boldsymbol{D}$, and the components K_{xx} and K_{yy} are equal. The parameter β^2 is a measure of the difference between the out-of-plane normal component K_{zz} and the in-plane sum $(K_{xx} + K_{yy})/2$. If β^2 is non-zero, then it is not possible for all three normal components of $\boldsymbol{K}$ to be equal.

The ratio $\mathrm{R} \equiv \sigma\dot{\gamma}/4\sqrt{\mathrm{T}}$ is a dimensionless measure of the energy associated with the mean velocity relative to that associated with the fluctuation velocity. For prescribed values of σ and $\dot{\gamma}$, the full second-moment $\boldsymbol{K}$ is completely determined by R, α, β^2, and ϕ. For prescribed values of e and ν, these four dimensionless quantities are determined by four remaining independent components of Eq. (10). With Γ given by Eq. (16), the in-plane, off-diagonal x_1-x_2 component of Eq. (10), for example, is simply

$$\sin 2\phi = \frac{\alpha}{1 + \beta^2}. \tag{24}$$

Convenient linear combinations of the three diagonal components Eq. (10) are its isotropic part,

$$8\pi^{3/2}(R/\nu)\alpha \cos 2\phi = 3(1 - e^2)tr\boldsymbol{A}, \tag{25}$$

the x_3-x_3 component of its deviatoric part,

$$-(1 - e)tr\mathbf{A} = 3\hat{\chi}_{33}, \tag{26}$$

and the difference between its x_2-x_2 and x_1-x_1 components,

$$-3(1 + \beta^2)\hat{\chi}_{33} = \alpha(\chi_{22} - \chi_{11}). \tag{27}$$

In Eq. (26) we employed Eq. (25), and in Eq. (27) we employed Eqs. (25) and (26) to eliminate the product $(R/\nu)\cos 2\phi$ from each. Because the components of $\boldsymbol{A}$, $\hat{\boldsymbol{E}}$, and therefore $\boldsymbol{\chi}$ depend on neither R nor ϕ, Eqs. (26) and (27) are uncoupled from Eqs. (24) and (25), and simultaneously determine the variations of α and β^2 with e. Equations (24) and (25) then determine the corresponding variations of ϕ and R/ν.

Because α and β^2 appear in the integrands of expressions (17) and (18) for $\boldsymbol{A}$ and $\hat{\boldsymbol{E}}$, it is not possible to obtain exact closed-form solutions for α, β^2, ϕ, and R. However, accurate approximate solutions may be obtained by retaining in Eqs. (25), (26), and (27), terms up to those quartic in α and β. The corresponding approximations for $tr\boldsymbol{A}$ and the components of $\boldsymbol{\chi}$ are,

$$tr\boldsymbol{A} = \frac{2\pi}{35}\left(70 + 7\alpha^2 + 21\beta^4 - 2\alpha^2\beta^2 + \frac{\alpha^4}{4}\right), \tag{28}$$

$$\hat{\chi}_{33} = \frac{-4\pi}{105}(3 - e)\left(42\beta^2 + 2\alpha^2 - 6\beta^4 - \alpha^2\beta^2 + \frac{\alpha^4}{11}\right), \tag{29}$$

and

$$\chi_{22} - \chi_{11} = \frac{8\pi}{35}(3 - e)\alpha\left(7 + 2\beta^2 - \frac{\alpha^2}{6}\right). \tag{30}$$

In Eqs. (27) and (26), we employ these approximations and expand α^2 in the truncated series,

$$\alpha^2 = \alpha_1\beta^2 + \alpha_2\beta^4. \tag{31}$$

Then, to within an error of order β^6, Eqs. (27) and (26) reduce, respectively, to

$$\left(36 - 3\alpha_1 + \frac{14}{33}\alpha_1^2 - 12\alpha_2\right)\beta^4 + 6(7 - 2\alpha_1)\beta^2 = 0, \tag{32}$$

which is identically satisfied provided that $\alpha_1 = 7/2$ and $\alpha_2 = 1013/396$, and

$$(2960 - 9393\epsilon)\beta^4 - 792[7(8 + 3\epsilon)\beta^2 - 20\epsilon] = 0, \tag{33}$$

in which the parameter ϵ is equal to $1 - e$. With β^2 determined for any ϵ by Eq. (33), α is given by Eq. (31), ϕ is fixed by Eq. (24), and R/ν is found from Eq. (25) in which tr$\boldsymbol{A}$ is approximated by expression (28). Over the range of ϵ between 0 and 1, the approximate values of β^2, α, ϕ, and R/ν calculated in this manner differ from their exact, numerically determined values by at most one percent. When the particles are nearly elastic, ϵ is small and the lowest order approximate solutions are $\beta^2 = 5\epsilon/14$, $\alpha = 2\phi = \sqrt{5\epsilon}/2$, and $R/\nu = 6\sqrt{\epsilon/5\pi}$. Figure 1 shows the exact variations with ϵ of β^2, α, ϕ, and R/ν normalized by their small ϵ behaviors.

The x-y-z Cartesian components of the dimensionless pressure tensor $\boldsymbol{P} = 16\nu^2\boldsymbol{K}/\sigma^2\dot{\gamma}^2$ are given by the right-hand side of Eq. (23) with T replaced by $(\nu/R)^2$. When ϵ is small, the lowest order approximations for these components of $\boldsymbol{P}$ are $P_{xx} = P_{yy} = P_{zz} =$

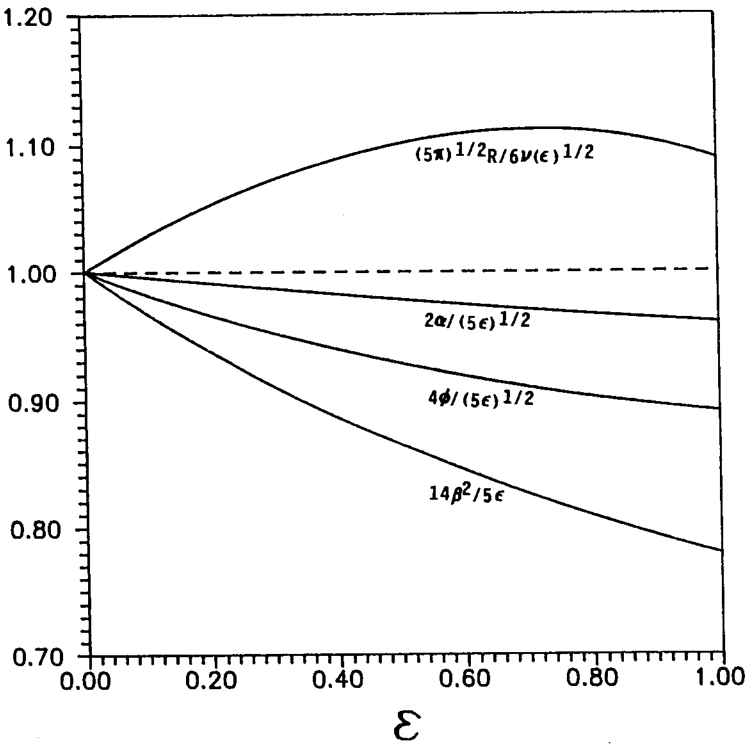

Fig. 1. The variations with dissipation parameter ϵ of the dimensionless quantities α, β^2, R/ν, and ϕ normalized by their small ϵ behaviors.

$5\pi/36\epsilon$ and $P_{xy} = -5^{3/2}\pi/72\epsilon^{1/2}$. The variations with ϵ of the exact values of P_{xx}, P_{yy}, P_{zz}, and $-P_{xy}$ normalized by their small ϵ behaviors are shown in Figure 2. Most striking are the differences between the three normal pressures. In Figure 3, the first and second normal pressure differences, $P_{xx} - P_{yy}$ and $P_{yy} - P_{zz}$, normalized by the average pressure $P \equiv (P_{xx} + P_{yy} + P_{zz})/3$ are plotted as functions of ϵ.

CONCLUSION

We have derived a constitutive relation for the collisional source Γ of second moment in terms of the second-moment $\boldsymbol{K}$ and its eigenvalues that applies to general dilute flows of identical, smooth, highly inelastic spheres. In the case of homogeneous shear flow, we have combined this constitutive relation with the

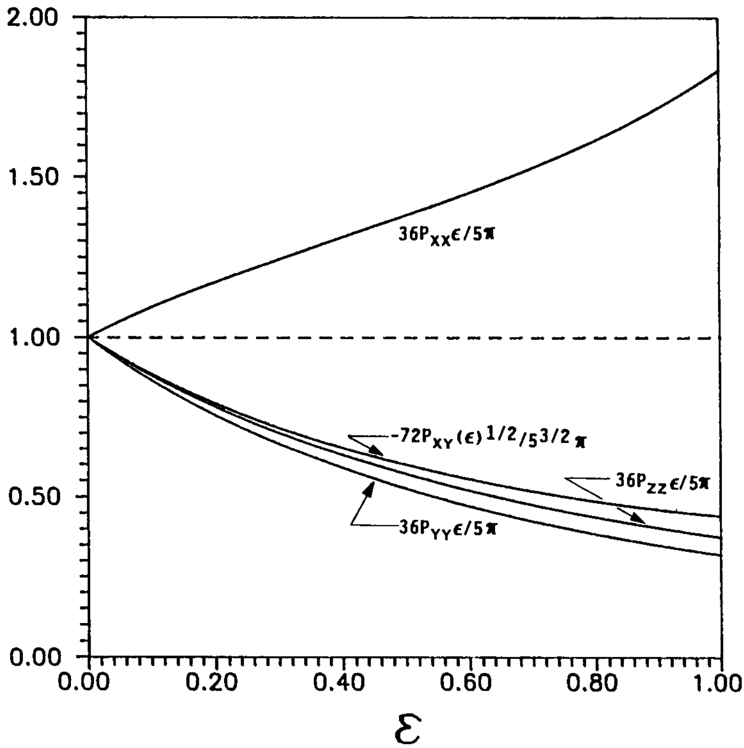

Fig. 2. The variations with dissipation parameter ϵ of the normal pressures P_{xx}, P_{yy}, P_{zz}, and shear stress $-P_{xy}$ normalized by their small ϵ behaviors.

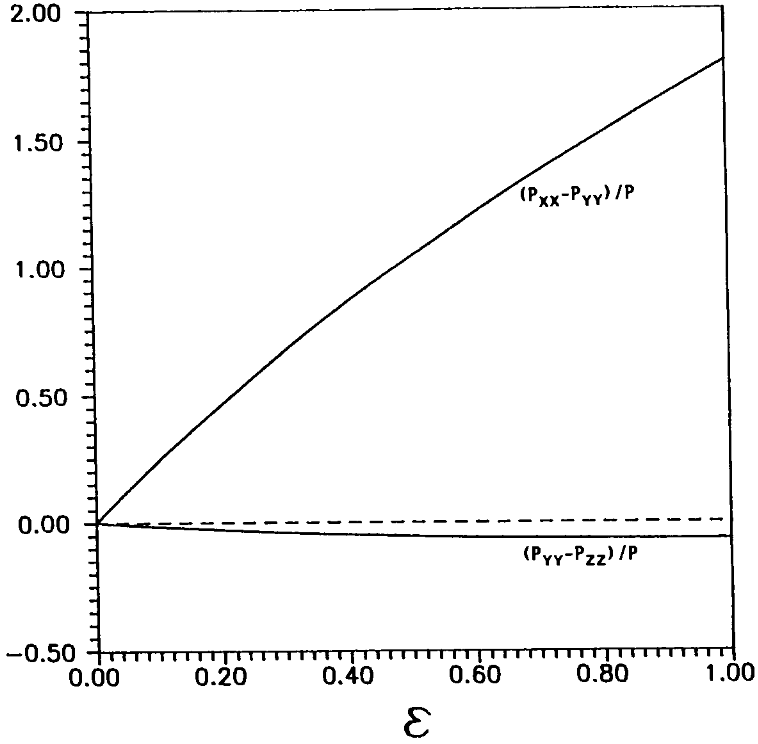

Fig. 3. The variations with dissipation parameter ϵ of the normal pressure differences $P_{xx} - P_{yy}$ and $P_{yy} - P_{zz}$ normalized by the average pressure $P \equiv (P_{xx} + P_{yy} + P_{zz})/3$.

balance equation for second moment to determine, for prescribed values of shear rate, coefficient of restitution, and solid fraction, both exact numerical and approximate closed-form solutions for the second moment and pressure tensor.

Figure 1 indicates that the parameters α, β^2, and R/ν, which determine the eigenvalues of $\boldsymbol{K}$, and the angle ϕ, which determines the in-plane eigendirections of $\boldsymbol{K}$, increase monotonically from zero as ϵ increases from zero to one. The maximum value of β^2, for example, is equal to .278. For this reason, the closed-form solutions, which contain errors of order β^6, are almost indistinguishable from their exact, numerically determined counterparts over the complete range of ϵ. The maximum values of α, ϕ, and R/ν are 1.073, .498, and 1.648, respectively.

Although Figure 1 demonstrates that the small ϵ behaviors for α, β^2, R/ν, and ϕ may be extended with reasonable accuracy to highly inelastic particles, Figure 2 demonstrates that the same is not true of the small ϵ behaviors of P_{xx}, P_{yy}, P_{zz}, and P_{xy}. This is because only when ϵ is small are the normal pressures insensitive to α, ϕ, and β^2 and the shear stress insensitive to ϕ. Consequently, as ϵ, α ϕ, and β^2 increase to their maximum values, the exact pressure tensor diverges from its small ϵ behavior.

The magnitudes of the first and second normal pressure differences,

$$\frac{P_{xx} - P_{yy}}{P} = 2\alpha \sin 2\phi \qquad \text{and} \qquad \frac{P_{yy} - P_{zz}}{P} = 3\beta^2 - \alpha \sin 2\phi \,, \tag{34}$$

shown in Figure 3, both increase monotonically from zero as ϵ increases from zero. At the same time, the pressure ratio P_{xx}/P_{yy} increases dramatically from 1 to 5.77 while the ratio P_{zz}/P_{yy} increases only slightly from 1 to 1.18. By contrast, the small ϵ approximations of the normal pressures are insensitive to α, ϕ, and β^2 and predict that the pressure differences vanish and the pressure ratios equal one for all values of ϵ.

Although the present work is concerned only with dilute flows, we may anticipate the qualitative behavior of the normal pressure differences in denser flows. Jenkins and Richman[13] focused on homogeneous, planar, shear flows of inelastic circular disks and showed that for any value of ϵ, as the solid fraction ν increases to its maximum value, the angle ϕ decreases to zero. Because this result extends to the homogeneous shearing of spheres, we anticipate that in this limit the first normal pressure difference will vanish. However, because the parameter β^2 will not approach zero as ν approaches its maximum value, we anticipate that the second normal pressure difference will not vanish in the dense limit.

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