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The extension of optimisation problems containing controls in the coefficients

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Synopsis

The paper suggests a procedure for direct construction of minimal extensions of constrained optimisation problems, particularly those containing controls in coefficients of elliptic equations. The preliminary version of the procedure has been described in [**1**].

Introduction

We consider non-self-adjoint optimisation problems containing controls in coefficients of elliptic equations which play the role of constraints. These problems are usually ill-posed and known to need relaxation, i.e. construction of a suitable minimal extension of the initial set U of admissible controls (material constants). Such an extension has hitherto been built [**2**] on the basis of a precise knowledge of the G-closure of U, i.e. the set GU of all composites assembled from the elements of U. Such detailed information often turns out to be unnecessary for many applications. Besides that, a complete description of the GU-set itself represents a difficult problem for which, up to now, solutions have been found for only a small number of examples [**3**].

For this reason, in what follows we suggest a method of direct construction of the required minimal extension, adapted to a specific non-self-adjoint problem of optimal control. The approach is not associated with any preliminary information about GU. It provides the required solution in a straightforward way. Formally, we suggest a special transformation of the integrand of the equivalent max-min problem, generalising the polyconvexification applicable for relaxation of non-convex variational problems.

1. Statement and solution of a typical problem

Consider the following problem of optimal control. We are given two isotropic materials whose specific heat conductances are equal to u_-, u_+ ($u_- < u_+$); the materials fill in the rectangle S ($-1 \leqq x \leqq 1$, $0 \leqq y \leqq 1$). Across its upper side $y = 1$, there enters the heat flux $i_y \cdot q = 1$, its lateral sides $x = -1$, $x = 1$ are insulated, and the lower side $y = 0$ is maintained at zero temperature (Fig. 1).

The temperature distribution is thus determined by the boundary-value

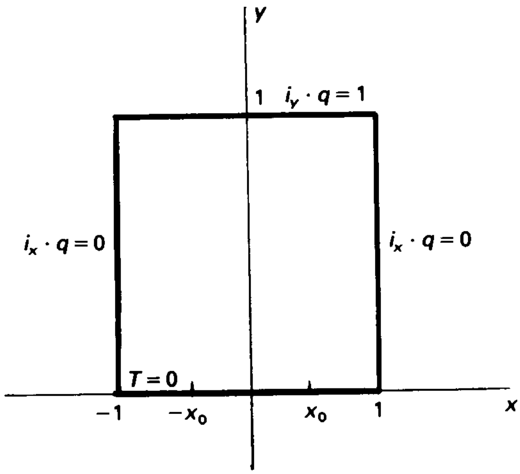

Figure 1

problem

$$q = u\nabla T, \quad \nabla . q = 0, \tag{1.1}$$

$$u = u_-\chi_1(x, y) + u_+\chi_2(x, y), \tag{1.2}$$

$$i_y . q|_{y=1} = 1, \quad i_x . q|_{x=-1} = i_x . q|_{x=1} = 0, \tag{1.3}$$

$$T|_{y=0} = 0. \tag{1.4}$$

Here $\chi_1(x, y)$ and $\chi_2(x, y)$ denote the characteristic functions of regions occupied by materials with heat conductances u_- and u_+, and i_x, i_y denote the unit vectors in Cartesian coordinates.

It is required to distribute the given materials over the rectangle, so as to maximise the functional $I = \int_{-1}^{1} \rho(x) i_y . q|_{y=0}\, dx$, where $\rho(x) \in L_\infty(0, 1)$ denotes some fixed weight function. If this function is set equal to

$$\rho(x) = \begin{cases} 1, & -x_0 \leqq x \leqq x_0, \\ 0, & 1 > |x| > x_0, \end{cases} \tag{1.5}$$

then the problem is reduced to that of maximisation of the heat flux across the "window" of length $2x_0$ on the lower side of the rectangle.

A similar problem for a cylindrical region has been solved in **[3, 4]**. This solution was based on the description of the GU-set, $U = \{u_+, u_-\}$. We give below an immediate solution with no reference to the GU-set.

Introduce the Lagrange multiplier λ corresponding to the heat equation

$$\nabla . u\nabla T = 0 \tag{1.6}$$

and consider the augmented functional

$$J = J(\lambda, T) = I - \int_S \lambda\nabla . u\nabla T\, dx\, dy. \tag{1.7}$$

Equating to zero its first variation with respect to T, we arrive at the conjugate equation

$$\nabla \cdot u\nabla\lambda = 0, \tag{1.8}$$

and the boundary conditions

$$u\nabla\lambda \cdot i_x|_{x=\pm 1} = 0, \quad u\nabla\lambda \cdot i_y|_{y=1} = 0, \tag{1.9}$$

$$\lambda|_{y=0} = -\rho(x) \tag{1.10}$$

After integration by parts with the boundary conditions (1.3), (1.10), the functional (1.7) takes the form

$$J = -\int_{-1}^{1} \lambda(x, 1)\, dx + \int_s \nabla\lambda \cdot u\nabla T\, dx\, dy. \tag{1.11}$$

The latter form of the functional is convenient for further analysis. Variation of the right-hand side with respect to T leads to (1.8) and (1.9) if we take account of the main boundary condition (1.4); if, however, we perform variation with regard to λ and use (1.10), then we arrive at (1.6) and (1.3).

We now show that the problem $\sup_u I$ under the additional constraints (1.1)–(1.4) is equivalent to $\sup_{u,T} \inf_\lambda J$ under the constraints (1.4), (1.10). In fact, since (n denotes outer normal to the boundary ∂S)

$$J = -\int_{-1}^{1} \lambda(x, 1)\, dx + \oint_{\partial s} \lambda q \cdot n\, dt - \int_s \lambda\nabla \cdot u\nabla T\, dx\, dy,$$

the operation $\inf_\lambda J$ under the restriction (1.10) yields $\inf_\lambda J = I$; the conditions (1.3) and (1.6) turn out to be natural in that case.

Now we estimate the functional $\sup_{u,T} \inf_\lambda J$ from above:

$$\begin{aligned} \sup_{u,T} \inf_\lambda J &= \sup_T \sup_u \inf_\lambda J \leqq \sup_T \inf_\lambda \sup_u J \\ &= \sup_T \inf_\lambda \left[-\int_{-1}^{1} \lambda(x, 1)\, dx + \int_s G(\nabla T, \nabla\lambda)\, dx\, dy \right] \end{aligned} \tag{1.12}$$

where (we accept the notation $\xi = \nabla T$, $\eta = \nabla\lambda$)

$$G(\xi, \eta) = \begin{cases} u_+\xi \cdot \eta & \text{if} \quad \xi \cdot \eta \geqq 0, \\ u_-\xi \cdot \eta & \text{if} \quad \xi \cdot \eta \leqq 0. \end{cases} \tag{1.13}$$

The function $G(\xi, \eta)$ is convex with respect to any of its arguments. The problem $\sup \inf J$, $T \in (1.4)$, $\lambda \in (1.10)$ is ill-posed. It would be well-posed if the integrand $G(\xi, \eta)$ were a saddle function, i.e. concave in ξ for fixed η and convex in η for fixed ξ. The solution would exist and the operations sup and inf would commute. For our specific problem, however, the function $G(\xi, \eta)$ is not saddle. At the same time, the requirement that $G(\xi, \eta)$ be saddle is too strong, since it guarantees existence without any reference to the fact that the variables ξ and η are gradients. With such a reference, it is not necessary for $G(\xi, \eta)$ to be saddle to ensure existence. The class of functions which are in this respect good is wider: it includes functions not saddle in the classical sense. For such functions, the solution exists, i.e. the supremum and infimum are attained: $\sup_T \inf_\lambda = \max_T \min_\lambda$. On the other hand, the operations $\max_T$ and $\min_\lambda$ are then not necessarily commutable. The functions $G(\xi, \eta)$ possessing this property will be called quasisaddle.

To make the problem solvable we will build up a quasisaddle function $G_{qs}(\xi, \eta)$ and use this one to replace $G(\xi, \eta)$ in the integrand of (1.12). The new functional $J_{qs} = -\int_{-1}^{1} \lambda(x, 1)\, dx + \int_s G_{qs}(\xi, \eta)\, dx\, dy$ will be such that $\max_T \min_\lambda J_{qs} = \sup_{u,T} \inf_\lambda J$, and the max min in the left-hand side is attained.

To construct $G_{qs}(\xi, \eta)$, we first build the upper bound for $\sup_{u,T} \inf_\lambda J$. This bound will be a formal one, produced by some new function $G^{**}(\xi, \eta)$ possessing the property $G^{**}(\xi, \eta) \geq G(\xi, \eta)$ and used instead of $G(\xi, \eta)$. With this new integrand we will associate the functional $J^{**} = -\int_{-1}^{1} \lambda(x, 1)\, dx + \int_s G^{**}(\xi, \eta)\, dx\, dy$; obviously, $\sup_{u,T} \inf_\lambda J \leqq \sup_{u,T} \inf_\lambda J^{**}$.

On the other hand, the functional $\sup_T \inf_\lambda J^{**}$ allows a lower estimate, this one provided by some selected microstructure. This will be a laminate; it corresponds to some specific integrand $G_1(\xi, \eta)$ generated by a set of properly oriented lamina. Generally, $G_1(\xi, \eta) \leqq G^{**}(\xi, \eta)$; if, however, the two functions coincide, they define the quasisaddle function $G_{qs}(\xi, \eta)$.

The function $G^{**}(\xi, \eta)$ will be constructed with the aid of a specific pointwise transformation applied to $G(\xi, \eta)$. It will provide an analogue of the polyconvexification [**5, 6**] applicable to multidimensional minimisation problems for non-convex functionals. In the latter case, the transformation included only the sup operations [**5, 6**]; this time, because of the max min character of the original problem, some of the sup operations will give place to inf. Specifically, the requires transformation will be as follows:

$$G^{**}(\xi, \eta) = \sup_A \sup_b \inf_a \{a \cdot \xi + b \cdot \eta + A(\xi_1\eta_2 - \xi_2\eta_1) - \inf_\xi \sup_\eta (a \cdot \xi + b \cdot \eta + A(\xi_1\eta_2 - \xi_2\eta_1) - G(\xi, \eta))]\}. \qquad (1.14)$$

First calculate $\sup_\eta [b \cdot \eta - H(\xi, \eta)]$ with $H(\xi, \eta) = -A(\xi_1\eta_2 - \xi_2\eta_1) + G(\xi, \eta)$:

$$b \cdot \eta - H(\xi, \eta) = \begin{cases} c^+ \cdot \eta & \text{if} \quad \xi \cdot \eta \geqq 0, \\ c^- \cdot \eta & \text{if} \quad \xi \cdot \eta \leqq 0. \end{cases}$$

The vectors $c^\pm$ have the following components: $(c_1^\pm, c_2^\pm) = (b_1 - A\xi_2 - u_\pm \xi_1, b_2 + A\xi_1 - u_\pm \xi_2)$. Let $c^+ \neq k\xi$, $k = \text{const}$; then the halfspace $\xi \cdot \eta \geqq 0$ on the (η, η_2)-plane will contain a sector for which $c^+ \cdot \eta \to +\infty$. If $c^+ = k\xi$, then

$$\sup_{\eta \in \xi \cdot \eta \geq 0} c^+ \cdot \eta = \sup_{\eta \in (\xi \cdot \eta) \geqq 0} k(\xi \cdot \eta) = \begin{cases} +\infty & \text{if} \quad k > 0 \\ 0 & \text{if} \quad k \leqq 0 \end{cases}$$

We see that for vectors η belonging to the halfspace $\xi . \eta \geqq 0$,

$$\sup_{\eta} [b . \eta - H(\xi, \eta)] = \begin{cases} 0 \text{ if } b_1 - A\xi_2 - u\xi_1 = 0, \quad b_2 + A\xi_1 - u\xi_2 = 0, \quad u \leqq u_+, \\ +\infty \text{ otherwise.} \end{cases}$$

Similarly, for vectors η belonging to the halfspace $\xi . \eta \leqq 0.$,

$$\sup_{\eta} [b . \eta - H(\xi, \eta)] = \begin{cases} 0 \text{ if } b_1 - A\xi_2 - u\xi_1 = 0, \quad b_2 + A\xi_1 - u\xi_2 = 0, \quad u \geqq u_- \\ +\infty \text{ otherwise.} \end{cases}$$

Combining both possibilities, we obtain

$$\bar{h}(\xi, b) = \sup_{\eta} [b . \eta - H(\xi, \eta)]$$
$$= \begin{cases} 0 \text{ if } b_1 - A\xi_2 - u\xi_1 = 0, \quad b_2 + A\xi_1 - u\xi_2 = 0, \quad u_- \leqq u \leqq u_+, \\ +\infty \text{ otherwise.} \end{cases} \tag{1.15}$$

Consider now the operation

$$\inf_a \left\{ a . \xi - \inf_{\xi} [a . \xi - (-\bar{h}(\xi, b))] \right\}. \tag{1.16}$$

This is known to put into correspondence with any given function $-\bar{h}(\xi, b)$ its concave (with regard to ξ) envelope, i.e. the least concave function of ξ greater than or equal to $-\bar{h}(\xi, b)$. Particularly, if $-\bar{h}(\xi, b)$ is itself a concave function of ξ, then the operation (1.16) recovers this function. In our specific case, the function $-\bar{h}(\xi, b)$ is not concave with regard to ξ. According to (1.15), $\bar{h} = 0$ along the arc of a circle

$$M(\xi, b) = A(\xi_1^2 + \xi_2^2) + b_2\xi_1 - b_1\xi_2 = 0, \tag{1.17}$$

corresponding to the values of u belonging to the interval $[u_-, u_+]$. This arc rests on the points $\xi^+(\xi_1^+, \xi_2^+)$ and $\xi^-(\xi_1^-, \xi_2^-)$:

$$\xi_1^+ = \frac{u_+ b_1 - Ab_2}{u_+^2 + A^2}, \quad \xi_2^+ = \frac{Ab_1 + u_+ b_2}{u_+^2 + A^2}, \tag{1.18}$$

$$\xi_1^- = \frac{u_- b_1 - Ab_2}{u_-^2 + A^2}, \quad \xi_2^- = \frac{Ab_1 + u_- b_2}{u_-^2 + A^2} \tag{1.19}$$

and, obviously, does not include the origin. For other points of the (ξ_1, ξ_2)-plane, the function $-\bar{h}(\xi, b)$ equals minus infinity.

The concave hull (envelope) of this function is obtained by an obvious geometric construction, which yields minus infinity everywhere except for the points of a segment Ξ bounded by the arc (1.17) and the chord

$$\frac{\xi_1 - \xi_1^-}{\xi_1^+ - \xi_1^-} = \frac{\xi_2 - \xi_2^-}{\xi_2^+ - \xi_2^-} \tag{1.20}$$

passing through the points (1.18) and (1.19) (Fig. 2):

$$\inf_a \left\{ a . \xi - \inf_{\xi} [a . \xi - (-\bar{h}(\xi, b))] \right\} = \begin{cases} 0, \ \xi \in \Xi, \\ -\infty, \ \xi \notin \Xi. \end{cases}$$

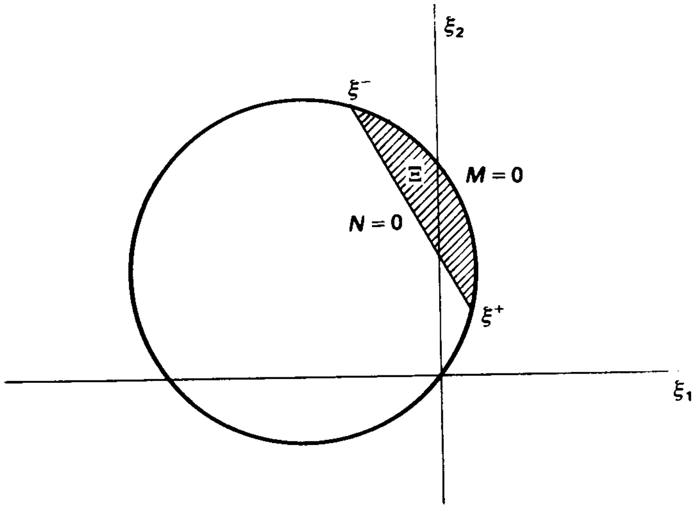

Figure 2

It is quite remarkable that this function depends on the argument b in quite a similar way: the role of the circular segment Ξ on the (ξ_1, ξ_2)-plane is played on the (b_1, b_2)-plane also by a circular segment, this time the segment B (Fig. 3). The latter possesses the vertices $b^+(b_1^+, b_2^+)$ and $b^-(b_1^-, b_2^-)$:

$$b_1^+ = u_+\xi_1 + A\xi_2, \quad b_2^+ = -A\xi_1 + u_+\xi_2, \tag{1.21}$$

$$b_1^- = u_-\xi_1 + A\xi_2, \quad b_2^- = -A\xi_1 + u_-\xi_2 \tag{1.22}$$

(cf. (1.18), (1.19)) determined as points of intersection on the (b_1, b_2)-plane of

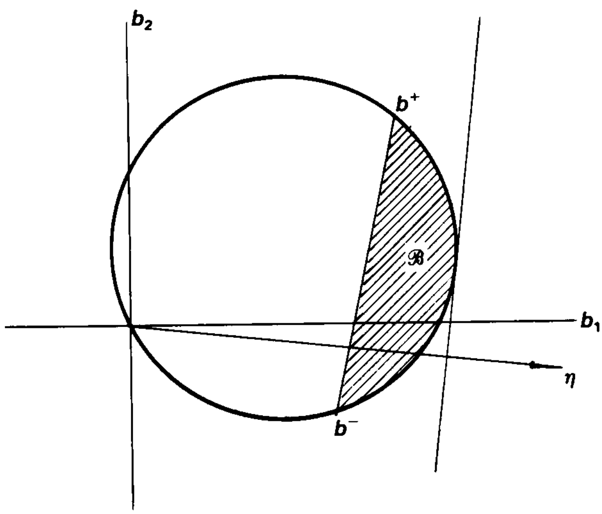

Figure 3

the chord (1.17) and the circular arc (1.20), in its equivalent form

$$N(\xi, b) = A(b_1^2 + b_2^2) + D \cdot b = 0. \tag{1.23}$$

Here D denotes the vector with components

$$D_1 = -r\xi_1 - t\xi_2, \quad D_2 = -r\xi_2 + t\xi_1 \tag{1.24}$$

and symbols r and t are determined as

$$r = A(u_+ + u_-), \quad t = A^2 - u_+u_- \tag{1.25}$$

It is easily seen that the segments Ξ and B on the ξ- and b-planes are determined by the inequalities

$$\begin{aligned} M \leqq 0, \quad N \leqq 0 \quad \text{if } A > 0, \\ M \geqq 0, \quad N \geqq 0 \quad \text{if } A < 0. \end{aligned} \tag{1.26}$$

We thus obtain that

$$\inf_a \left\{ a \cdot \xi - \inf_\xi [a \cdot \xi - (-\bar{h}(\xi, b))] \right\} = \begin{cases} 0, & b \in B, \\ -\infty, & b \notin B. \end{cases}$$

The operation $\sup_b \left(b \cdot \eta + \inf_a \left\{ a \cdot \xi - \inf_\xi [a \cdot \xi - (-\bar{h}(\xi, b))] \right\} \right)$ is now reduced to $\sup_{b \in B} b \cdot \eta$.

The latter operation is elementary. Due to the convexity of the set B, the points b^* which realise the supremum are placed at the boundary ∂B, namely, these points either belong to the circular arc (1.23) or coincide with the vertices (1.21), (1.22) of B.

Analytically, $\sup_{b \in B} b \cdot \eta$ is calculated differently in these two cases. If g^* belongs to the arc (1.23), then it is a stationary point of the function $g(b) = b \cdot \eta + \mu(Ab^2 + D \cdot b)$, where μ is the Lagrange multiplier, and $b^2 = b_1^2 + b_2^2$.

In this case

$$\begin{aligned} b_i^* &= -\frac{1}{2\mu A}(\mu D_i + \eta_i), \quad i = 1, 2, \\ g(b^*) &= -\frac{1}{4\mu A}(\mu D + \eta)^2. \end{aligned} \tag{1.27}$$

Equation (1.23) determines the multiplier μ:

$$\mu^2 = \frac{\eta^2}{D^2}, \quad \mu = \pm \frac{|\eta|}{|D|} \tag{1.28}$$

The point b^* realises the maximum of $b \cdot \eta$ if

$$\mu A < 0, \tag{1.29}$$

which means that $g(b^*) > 0$. Eliminating μ from the expression for $g(b^*)$ with the aid of (1.28), we obtain

$$g(b^*) = \mp \frac{|D|\,|\eta|}{2A} - \frac{1}{2A} D \cdot \eta$$

Using the (easily checked) formulae

$$D^2 = \xi^2 s_+ s_-, \quad s_+ = A^2 + u_+^2, \quad s_- = A^2 + u_-^2, \tag{1.30}$$

$$D \,.\, \eta = -rp + tq = -A(u_+ + u_-)p + (A^2 - u_+u_-)q, \tag{1.31}$$

where

$$p = \xi_1\eta_1 + \xi_2\eta_2, \quad q = \xi_1\eta_2 - \xi_2\eta_1, \tag{1.32}$$

we reduce the expression for $g(b^*)$ to the form

$$g(b^*) = \mp \frac{1}{2A}\sqrt{s_+s_-}\,|\xi|\,|\eta| + \tfrac{1}{2}(u_+ + u_-)p - \frac{1}{2A}(A^2 - u_+u_-)q, \tag{1.33}$$

or, in view of (1.28), (1.29), to

$$g(b^*) = \frac{1}{2\,|A|}\sqrt{s_+s_-}\,|\xi|\,|\eta| + \tfrac{1}{2}(u_+ + u_-)p - \frac{1}{2A}(A^2 - u_+u_-)q. \tag{1.34}$$

If the point b^* coincides with b^+(see (1.21)), then

$$g(b^*) = u_+p - Aq; \tag{1.35}$$

if this point coincides with b^- (see (1.22)) then

$$g(b^*) = u_-p - Aq. \tag{1.36}$$

Summing up, we obtain

$$g(b^*) + Aq = g^*(A, \xi, \eta)$$

$$= \begin{cases} u_+p & \text{in case (1.35)}, \\ \dfrac{1}{2\,|A|}\sqrt{s_+s_-}\,|\xi|\,|\eta| + \tfrac{1}{2}(u_+ + u_-)p + \dfrac{1}{2A}(A^2 + u_+u_-)q & \text{in case (1.34)}, \\ u_-p & \text{in case (1.36)}. \end{cases} \tag{1.37}$$

The geometric construction (see Fig. 3) shows that the regimes (1.34) and (1.35) as well as (1.34) and (1.36) are exchanged, leaving the function $g^*(A, \xi, \eta)$ continuous. We will now prove that the equality

$$\frac{1}{2\,|A|}\sqrt{s_+s_-}\,|\xi|\,|\eta| + \tfrac{1}{2}(u_+ + u_-)p + \frac{1}{2A}(A^2 + u_+u_-)q = u_+p \tag{1.38}$$

is possible only provided that $Aq < 0$.

In fact, the first term in the left-hand side of the equality

$$\frac{1}{2}|A|\sqrt{s_+s_-}\,|\xi|\,|\eta| - \tfrac{1}{2}(u_+ - u_-)p + \frac{1}{2A}(A^2 + u_+u_-)q = 0 \tag{1.39}$$

achieves its minimum with regard to A for $A = \pm\sqrt{u_+u_-}$, this minimum being equal to $\frac{1}{2}(u_+ + u_-)\,|\xi|\,|\eta|$. It is now obvious that the difference of the first two terms is always positive, which is the required result. Similarly, the condition $Aq < 0$ is necessary for validity of the equation which differs from (1.38) in its right-hand side now equal to u_-p.

Assuming that $q<0$, $A=-|A|<0$, transform (1.39). Setting $p=|\xi|\,|\eta|\cos\theta$, $q=|\xi|\,|\eta|\sin\theta$ we obtain

$$\frac{1}{|A|}\sqrt{s_+s_-}-(u_+-u_-)\cos\theta-\frac{1}{|A|}(A^2+u_+u_-)\sin\theta=0,$$

or

$$\left[\frac{1}{|A|}\sqrt{s_+s_-}+u_+-u_-\right]\tan^2\frac{\theta}{2}-\frac{2}{|A|}(A^2+u_+u_-)\tan^2\frac{\theta}{2}$$
$$+\frac{1}{|A|}\sqrt{s_+s_-}-(u_+-u_-)=0,$$

or, finally,

$$\tan\frac{\theta}{2}=\frac{A^2+u_+u_-}{\sqrt{s_+s_-}+|A|(u_+-u_-)}. \tag{1.40}$$

For the point dividing the regimes (1.34) and (1.36), we obtain similarly

$$\tan\frac{\theta}{2}=\frac{A^2+u_+u_-}{\sqrt{s_+s_-}-|A|\,(u_+-u_-)} \tag{1.41}$$

The final conclusion about criteria of realisation of regimes (1.34)–(1.36) is given by the operation $\sup_A g^*(A,\xi,\eta)$. Setting $A<0$, $q>0$, we calculate the derivative with respect to A of the function

$$\varphi(A)=-\frac{1}{2A}\sqrt{s_+s_-}\,|\xi|\,|\eta|+\tfrac{1}{2}(u_++u_-)p+\frac{1}{2A}(A^2+u_+u_-)q.$$

The derivative is easily shown to be

$$\frac{A^2-u_+u_-}{2A^2}\left[q-\frac{|\xi|\,|\eta|(A^2+u_+u_-)}{\sqrt{(A^2+u_+^2)(A^2+u_-^2)}}\right]; \tag{1.42}$$

at the stationary point $A=-\sqrt{u_+u_-}$ the second derivative equals

$$-\frac{1}{\sqrt{u_+u_-}}\left[q-\frac{|\xi|\,|\eta|\,2\sqrt{u_+u_-}}{u_++u_-}\right].$$

The latter expression is non-positive if

$$\sin\theta\geqq\frac{2\sqrt{u_+u_-}}{u_++u_-}. \tag{1.43}$$

Setting $|A|=\sqrt{u_+u_-}$ in equations (1.40) and (1.41) and comparing with (1.43), we obtain conditions of realisation of the regime (1.34):

$$\sqrt{\frac{u_-}{u_+}}\leqq\tan\frac{\theta}{2}\leqq\sqrt{\frac{u_+}{u_-}}.$$

The corresponding inequalities for regimes (1.35) and (1.36) are, respectively,

$$0\leqq\tan\frac{\theta}{2}\leqq\sqrt{\frac{u_1}{u_+}}\quad\text{and}\quad\sqrt{\frac{u_+}{u_-}}\leqq\tan\frac{\theta}{2}\leqq\infty.$$

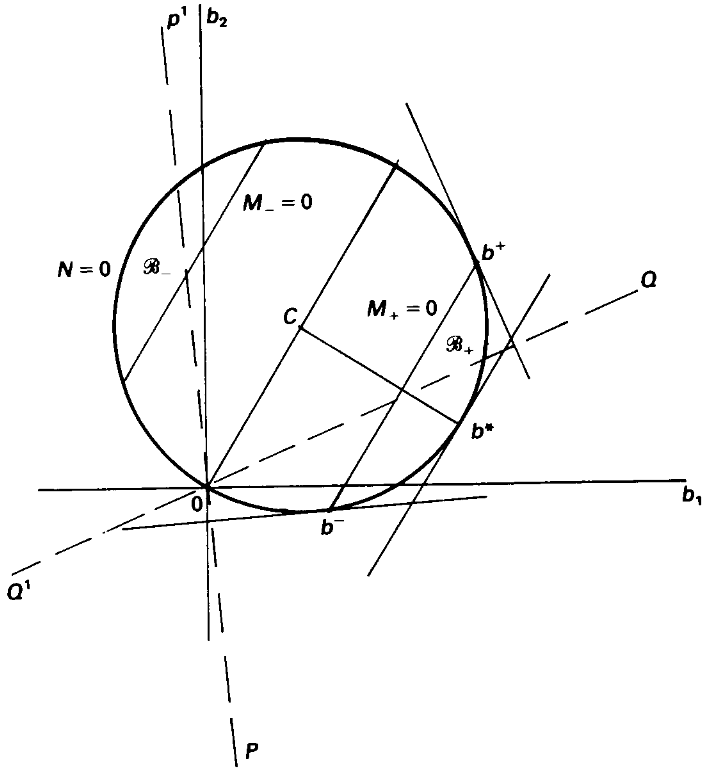

Figure 4

It is clear that the alternative possibility $A>0$, $q<0$ does not influence the final result.

Figure 4 illustrates the construction. For $A=\pm\sqrt{u_+u_-}$, equation (1.23) determines on the (b_1, b_2)-plane one and the same circle $N=0$ with the centre C $(\frac{1}{2}(u_++u_-)\xi_1, \frac{1}{2}(u_++u_-)\xi_2)$ and radius $\frac{1}{2}(u_++u_-)|\xi|$, and equation (1.17) determines two straight lines $M_+=0$ $(A=\sqrt{u_+u_-})$ and $M_-=0$ $(A=-\sqrt{u_+u_-})$ on the same plane. There are also shown two admissible segments $B_+(A=\sqrt{u_+u_-}, M_+\leqq 0, N_+\leqq 0)$ and $B_-(A=-\sqrt{u_+u_-}, M_-\geqq 0, N_-\geqq 0)$ as well as the sectors POQ and $P'OQ'$ which contain the vectors η realising the regime (1.34).

The point b^* on the curvilinear arc restricting the segment $B_+(A=\sqrt{u_+u_-})$ is characterised by the radius-vector (b_1^*, b_2^*) with the components

$$b_1^*=\frac{|D|}{2|\eta|\sqrt{u_+u_-}}\left[\frac{|\eta|}{|D|}\sqrt{u_+u_-}(u_++u_-)\xi_1+\eta_1\right]=\frac{u_++u_-}{2}\left(\xi_1+\frac{|\xi|}{|\eta|}\eta_1\right),$$

$$b_2^*=\frac{|D|}{2|\eta|\sqrt{u_+u_-}}\left[\frac{|\eta|}{|D|}\sqrt{u_+u_-}(u_++u_-)\xi_2+\eta_2\right]=\frac{u_++u_-}{2}\left(\xi_2+\frac{|\xi|}{|\eta|}\eta_2\right),$$

These relationships are illustrated by Figure 4. It is seen that if the vector η changes sign, then the corresponding point b^* will be placed on the curvilinear arc restricting the segment B_-. Since the vector OC is oriented along the vector ξ, Figure 4 shows that $q<0$ for vectors η belonging to the sector POQ, and $q>0$ for those lying within the sector $P'OQ'$.

We have finally

$$G^{**}(\xi,\eta)=\begin{cases} u_+p, & 0\leqq\dfrac{\sqrt{p^2+q^2}-p}{|q|}\leqq\sqrt{\dfrac{u_-}{u_+}},\\ \tfrac{1}{2}(u_++u_-)(p+\sqrt{p^2+q^2})-\sqrt{u_+u_-}\,|q|, & \\ \qquad \sqrt{\dfrac{u_-}{u_+}}\leqq\dfrac{\sqrt{p^2+q^2}-p}{|q|}\leqq\sqrt{\dfrac{u_+}{u_-}} & \\ u_-p, & \sqrt{\dfrac{u_+}{u_-}}\leqq\dfrac{\sqrt{p^2+q^2}-p}{|q|}\leqq\infty, \end{cases} \tag{1.44}$$

or, equivalently ($\chi=\theta/2$),

$$G^{**}(\xi,\eta)=|\xi|\,|\eta|\begin{cases} u_+\cos 2\chi, & 0\leqq\tan\chi\leqq\sqrt{\dfrac{u_-}{u_+}},\\ (u_++u_-)\cos^2\chi-\sqrt{u_+u_-}\sin 2\chi, & \\ \qquad \sqrt{\dfrac{u_-}{u_+}}\leqq\tan\chi\leqq\sqrt{\dfrac{u_+}{u_-}}, & \\ u_-\cos 2\chi, & \sqrt{\dfrac{u_+}{u_-}}\leqq\tan\chi\leqq\infty. \end{cases} \tag{1.45}$$

The expression on the right-hand side of (1.44) actually represents the maximum of the function (1.37) with regard to its argument A. In fact, other stationary points of $\varphi(A)$ (see 1.42)) determined from the equation

$$q-\frac{|\xi|\,|\eta|\,(A^2+u_+u_-)}{\sqrt{(A^2+u_+^2)(A^2+u_-^2)}}=0$$

correspond, as is easily seen, to minima of that function.

Since $(u_++u_-)\cos^2\chi-\sqrt{u_+u_-}\sin 2\chi-u_+\cos 2\chi=(\sqrt{u_-}\cos\chi-\sqrt{u_+}\sin\chi)^2$, the graph of $G^{**}(\xi,\eta)$ lies above that of $G(\xi,\eta)$ for those points where $\sqrt{u_-/u_+}<\tan\chi<\sqrt{u_+/u_-}$. For other points the two graphs coincide. The function of χ determined by the right-hand side of (1.45) is continuous together with its derivative. Note also that for $A=\pm\sqrt{u_+u_-}$ the right-hand side of (1.40) achieves its minimum equal to $\sqrt{u_-/u_+}$ as well as the right-hand side of (1.41) – its maximum equal to $\sqrt{u_+/u_-}$. These values of A thus provide the widest admissible interval of θ for which the regime (1.34) is realised.

Returning to (1.12), we observe that

$$\sup_{u,T}\inf_{\lambda}\leqq\sup_{T}\inf_{\lambda}\left[-\int_{-1}^{1}\lambda(x,1)\,dx+\int_{s}G^{**}(\nabla T,\nabla\lambda)\,dx\,dy\right]. \tag{1.46}$$

On the other hand, the value of $\sup_{u,T}\inf_{\lambda}J$ under the constraints (1.4), (1.10) is equal to that of $\sup_{u}I$ calculated under the side conditions (1.1)–(1.4); the latter functional is estimated from below by its value calculated for laminates [**4**]. At the same time, as shown in [4] (see also [3]), the necessary conditions of optimality derived for the relaxed version of the considered optimisation problem imply that

$$\left.\begin{aligned} d_1 = d_2 = u_+ &\quad \text{if } \tan\chi \leqq \sqrt{\frac{u_-}{u_+}},\\ d_1 = d_2 = u_- &\quad \text{if } \tan\chi \geqq \sqrt{\frac{u_+}{u_-}},\\ \frac{u_+u_-}{d_1^2} = \tan^2\chi &\quad \text{if } \sqrt{\frac{u_-}{u_+}} \leqq \tan\chi \leqq \sqrt{\frac{u_+}{u_-}}. \end{aligned}\right\} \tag{1.47}$$

Here and below d_1 and d_2 denote the eigenvalues of the tensor $D_0 = d_1e_1e_1 + d_2e_2e_2$ of effective heat conductances of a layered composite; these eigenvalues are equal to

$$\left.\begin{aligned} d_1 &= [mu_+^{-1} + (1-m)u_-^{-1}]^{-1},\\ d_2 &= mu_+ + (1-m)u_-, \end{aligned}\right\} \tag{1.48}$$

where m denotes concentration in the composite of layers filled in by the material u_+.

If the layers (i.e. the e_2-axis) bisect the angle $\theta = 2\chi$ between ∇T and $\nabla\lambda$, then the integrand $\nabla T \,.\, D_0 \,.\, \nabla\lambda$ of the relaxed problem (which now stands for the original integrand $u\,\nabla T \,.\, \nabla\lambda$) obtains the form $\nabla T \,.\, D_0 \,.\, \nabla\lambda = |\nabla T|\,|\nabla\lambda|\,(d_2\cos^2\chi - d_1\sin^2\chi) = |\nabla T|\,|\nabla\lambda|\cos^2\chi(d_2 - d_1\tan^2\chi)$. Now using the last equation of (1.47) and the relationship $d_2 = u_+ + u_- - ((u_+u_-)/d_1)$, following from (1.48), we obtain $\nabla T \,.\, D_0 \,.\, \nabla\lambda = |\nabla T|\,|\nabla\lambda|\,[(u_+ + u_-)\cos^2\chi - \sqrt{u_+u_-}\sin 2\chi]$, which is the same as the expression in the second line of (1.45). We thus conclude that the variational problem

$$\sup_T \inf_\lambda \left[-\int_{-1}^{1} \lambda(x, 1)\, dx + \int_S G^{**}(\nabla T, \nabla\lambda)\, dx\, dy \right] \tag{1.49}$$

under the side conditions (1.4), (1.10) is well posed since the corresponding upper bound (1.46) is attained for laminates. This observation enables us, recalling the terminology of quasiconvex analysis [7, 8], to call the function $G^{**}(\nabla T, \nabla\lambda)$ quasiconcave-convex. (Note that the quasiconcave-convex function is convex with regard to the second argument but is not concave with regard to the first (in our case, both functions – $G^{**}(\xi, \eta)$ and $G(\xi, \eta)$ (see (1.44) and (1.13)) – are convex with regard to each of their arguments).

2. Some properties of the transformation (1.14)

We shall now comment on the transformation (1.14) applied to the function $G(\xi, \eta)$ convex with regard to η and arbitrary as a function of ξ. Proceeding as in Section 1, we will show that in this case

$$G^{**}(\xi, \eta) \geqq G(\xi, \eta). \tag{2.1}$$

In fact, the function

$$\bar{h}(\xi, b) = \sup_\eta\,[b \,.\, \eta - H(\xi, \eta)], \quad H(\xi, \eta) = -A(\xi_1\eta_2 - \xi_2\eta_1) + G(\xi, \eta) \tag{2.2}$$

is generally not convex with regard to ξ, and, consequently, the function $-\bar{h}(\xi, b)$ is not concave. But then, by the known property of the concave envelope,

$$\inf_a \left\{a \,.\, \xi - \inf_\xi (a \,.\, \xi - (-\bar{h}(\xi, b))]\right\} \geqq -\bar{h}(\xi, b) \tag{2.3}$$

for all values of b, and, consequently,

$$\sup_b \left(b \,.\, \eta + \inf_a \left\{a \,.\, \xi - \inf_\xi [a \,.\, \xi - (-\bar{h}(\xi, b))]\right\}\right)$$

$$\geqq \sup_b [b \,.\, \eta - \bar{h}(\xi, b)] = \sup_b \left\{b \,.\, \eta - \sup_\eta [b \,.\, \eta - H(\xi, \eta)]\right\} = H(\xi, \eta), \tag{2.4}$$

the latter by the convexity of $H(\xi, \eta)$ with regard to η. We arrive at (cf. (1.14))

$$A(\xi_1\eta_2 - \xi_2\eta_1) + \sup_b \inf_a \left\{a \,.\, \xi + b \,.\, \eta - \inf_\xi \sup_\eta [a \,.\, \xi + b \,.\, \eta + A(\xi_1\eta_2 - \xi_2\eta_1) - G(\xi, \eta)]\right\} \geqq G(\xi, \eta),$$

which holds for all values of A. The supremum over A now leads to (2.1).

Suppose now that the function $G(\xi, \eta)$ is concave with regard to ξ and arbitrary in η. Then $\bar{h}(\xi, b)$ (see (2.2)) is convex with regard to ξ, and the transformation (1.16) results in $-\bar{h}(\xi, b)$ which is concave over ξ. The transformation $\sup_b [b \,.\, \eta - \bar{h}(\xi, b)] = \sup_b \left\{b \,.\, \eta - \sup_\eta [b \,.\, \eta - H(\xi, \eta)]\right\}$ determines the convex (over η) envelope $\operatorname{conv}_\eta H(\xi, \eta)$ of $H(\xi, \eta)$. The inequality $\operatorname{conv}_\eta H(\xi, \eta) \leqq H(\xi, \eta)$, or $A(\xi_1\eta_2 - \xi_2\eta_1) + \operatorname{conv}_\eta H(\xi, \eta) \leqq G(\xi, \eta)$, is then fulfilled for all values of A and, consequently,

$$G^{**}(\xi, \eta) \leqq G(\xi, \eta). \tag{2.5}$$

Now let $G(\xi, \eta)$ be concave as the function of ξ and convex with regard to η. Then both inequalities (2.1) and (2.5)) hold, which means that

$$G^{**}(\xi, \eta) = G(\xi, \eta). \tag{2.6}$$

In other words, the concave-convex function is at the same time the quasiconcave-convex.

For the concave-convex function $G(\xi, \eta)$ the extremal problem $\sup_\xi \inf_\eta G(\xi, \eta)$ possesses the solution, and the equality

$$\sup_\xi \inf_\eta G(\xi, \eta) = \inf_\eta \sup_\xi G(\xi, \eta) \tag{2.7}$$

holds, which expresses the saddle-point theorem. The problem does not need relaxation in this case, and (2.6) holds notwithstanding that the vector variables ξ and η are gradients.

For the extremal problem

$$\inf_\xi \sup_\eta G(\xi, \eta), \quad \xi = \nabla T, \quad \eta = \nabla \lambda, \tag{2.8}$$

the role of (1.14) is played by the related transformation

$$G_{**}(\xi,\eta)=\inf_A\inf_b\sup_a\Big\{a\,.\,\xi+b\,.\,\eta+A(\xi_1\eta_2-\xi_2\eta_1)$$
$$-\sup_\xi\inf_\eta[a\,.\,\xi+b\,.\,\eta+A(\xi_1\eta_2-\xi_2\eta_1)-G(\xi,\eta)]\Big\}. \tag{2.9}$$

Suppose that $G(\xi,\eta)$ is concave in η, and its dependence on ξ is arbitrary. In that case, the following inequality holds:

$$G_{**}(\xi,\eta)\leqq G(\xi,\eta). \tag{2.10}$$

To prove it, we notice that the function

$$\underline{h}(\xi,b)=\inf_\eta[b\,.\,\eta-H(\xi,\eta)],\quad H(\xi,\eta)=-A(\xi_1\eta_2-\xi_2\eta_1)+G(\xi,\eta) \tag{2.11}$$

is generally not concave in ξ, and, consequently, $-\underline{h}(\xi,b)$ is not convex. Then, by the property of a convex envelope,

$$\sup_a\Big\{a\,.\,\xi-\sup_\xi[a\,.\,\xi-(-\underline{h}(\xi,b))]\Big\}\leqq-\underline{h}(\xi,b) \tag{2.12}$$

for all values of b and, consequently,

$$\inf_b\Big(b\,.\,\eta+\sup_a\Big\{a\,.\,\xi-\sup_\xi[a\,.\,\xi-(-\underline{h}(\xi,b))]\Big\}\Big)$$
$$\leqq\inf_b[b\,.\,\eta-\underline{h}(\xi,b)]=\inf_b\Big\{b\,.\,\eta-\inf_\eta[b\,.\,\eta-H(\xi,\eta)]\Big\}=H(\xi,\eta).$$

The latter is due to the concavity of $H(\xi,\eta)$ over η. We have thus arrived at the inequality (see (2.9))

$$A(\xi_1\eta_2-\xi_2\eta_1)+\inf_b\sup_a\Big\{a\,.\,\xi+b\,.\,\eta-\sup_\xi\inf_\eta[a\,.\,\xi+b\,.\,\eta$$
$$+A(\xi_1\eta_2-\xi_2\eta_1)-G(\xi,\eta)]\Big\}\leqq G(\xi,\eta),$$

valid for all values of A. The infimum over A yields the required result.

If, further, $G(\xi,\eta)$ is also convex over ξ, then $\underline{h}(\xi,b)$ turns out to be concave with regard to ξ. The convex envelope $\sup_a\Big\{A\,.\,\xi-\sup_\xi[a\,.\,\xi-(-\underline{h}(\xi,b))]\Big\}$ now recovers $-\underline{h}(\xi,b)$, and we obtain the equality $G_{**}(\xi,\eta)=G(\xi,\eta)$. In other words, the convex-concave function is at the same time quasiconvex-concave. The extremal problem (2.8) needs no relaxation in that case, and the saddle-point theorem $\inf_\xi\sup_\eta G(\xi,\eta)=\sup_\eta\inf_\xi G(\xi,\eta)$ holds, notwithstanding that the variables ξ and η are gradients.

Lastly, if $G(\xi,\eta)$ is convex with regard to ξ and its dependence on η is arbitrary, then the inequality

$$G_{**}(\xi,\eta)\geqq G(\xi,\eta) \tag{2.13}$$

TABLE 1

$G(\xi, \eta)$	Inequality
Convex over η	$G^{**}(\xi, \eta) \geqq G(\xi, \eta)$
Concave over ξ	$G^{**}(\xi, \eta) \leqq G(\xi, \eta)$
Concave over η	$G_{**}(\xi, \eta) \leqq G(\xi, \eta)$
Convex over ξ	$G_{**}(\xi, \eta) \geqq G(\xi, \eta)$

holds. In fact, under these conditions the function $\underline{h}(\xi, b)$ is concave, and $-\underline{h}(\xi, b)$ convex with regard to ξ. Inequality (2.12) now becomes the equality, which means that

$$\inf_b \left(b \,.\, \eta + \sup_a \left\{a \,.\, \xi - \sup_\xi [a \,.\, \xi - (-\underline{h}(\xi, b))]\right\}\right) = \inf_b [b \,.\, \eta - \underline{h}(\xi, b)]$$

$$= \inf_b \left\{b \,.\, \eta - \inf_\eta [b \,.\, \eta - H(\xi, \eta)]\right\} \geqq H(\xi, \eta),$$

the latter by the well known property of a concave envelope. The final stage of the proof is similar to the preceding. The results of this section are be summed up in Table 1.

$$G^{**}(\xi, \eta) = \sup_A \sup_b \inf_a \left\{a \,.\, \xi + b \,.\, \eta + A(\xi_1\eta_2 - \xi_2\eta_1) - \inf_\xi \sup_\eta [a \,.\, \xi + b \,.\, \eta + A(\xi_1\eta_2 - \xi_2\eta_1) - G(\xi, \eta)]\right\}, \quad (1.14)$$

$$G_{**}(\xi, \eta) = \inf_A \inf_b \sup_a \left\{a \,.\, \xi + b \,.\, \eta + A(\xi_1\eta_2 - \xi_2\eta_1) - \sup_\xi \inf_\eta [a \,.\, \xi + b \,.\, \eta + A(\xi_1\eta_2 - \xi_2\eta_1) - G(\xi, \eta)]\right\}. \quad (2.9)$$

Returning to the problem of Section 1, we might attempt to build an analogue of the estimate (1.46) with the aid of the inequality $G_{**}(\xi, \eta) \geqq G(\xi, \eta)$ (see the last line of Table 1). Changing ξ for η and a for b, and conversely, we would arrive at the transformation

$$G_{**}(\xi, \eta) = \inf_A \inf_a \sup_b \left\{a \,.\, \xi + b \,.\, \eta - A(\xi_1\eta_2 - \xi_2\eta_1) - \sup_\eta \inf_\xi [a \,.\, \xi + b \,.\, \eta - A(\xi_1\eta_2 - \xi_2\eta_1) - G(\xi, \eta)]\right\} \quad (2.14)$$

which differs from (1.14) in that the operation sup is exchanged with inf.

Applied to the function (1.13), this transformation yields, however, a trivial result. In fact, the operation $\inf_\xi [a \,.\, \xi - A(\xi_1\eta_2 - \xi_2\eta_1) - G(\xi, \eta)]$, performed for vectors ξ belonging to the half-space $\xi \,.\, \eta \geqq 0$, results in the expression

$$0 \text{ if } a_1 - A\eta_2 - u\eta_1 = 0, \quad a_2 + A\eta_1 - u\eta_2 = 0, \quad u \geqq u_+$$
$$-\infty \quad \text{otherwise.}$$

If, however, the vectors ξ are such that $\xi \cdot \eta \leqq 0$, then this operation yields

$$0 \text{ if } a_1 - A\eta_2 - u\eta_1 = 0, \quad a_2 + A\eta_1 - u\eta_2 = 0, \quad u \leqq u_-$$
$$-\infty \quad \text{otherwise.}$$

We see that there are no values of u which might result in a zero value of the operation inf performed with regard to any vector ξ. Consequently, this infimum always equals minus infinity. The transformation (2.14) thus generates a rough estimate of (1.13), and the required result is given by (1.14).

By a similar reasoning we prove that the transformation (1.14) appearing in the second line of Table 1 (more exactly, its version associated with the interchange of ξ and η as well as of a and b) is rough if applied to the functions $G(\xi, \eta)$ of the type (1.13), this time, however, concave with regard to each argument. For such functions, the extremal problem (2.8) is relaxed with the aid of the transformation (2.9) corresponding to the third row of Table 1.

3. Discussion

The transformation (1.14) and the related transformations presented in Table 1 generalise to max-min (min-max) problems the known transformations [**5**, **6**] generating the polyconcave (polyconvex) envelopes of integrands in maximisation (minimisation) problems for non-concave (non-convex) functionals. For self-adjoint problems (when $\xi = \eta$), they are trivially reduced to the latter transformations. In the same way as the polyconcavity (polyconvexity) transformation leaves concave (convex) functions immutable, the transformation (1.14) recovers concave-convex, and the transformation (2.9) convex-concave functions.

The transformations of the type (1.14) and (2.9) can, of course, be applied to general integrands depending on two vector variables and convex (concave) with regard to one of them. Imitating the terminology of [**5**, **6**], one may call the results of these transformations polyconcave-convex (polyconvex-concave) functions. These functions furnish upper (lower) estimates of the initial integrand; the problem of attainability of these bounds at some specific microstructures is subject to special analysis in each individual case.

Returning to problems of the type presented in Section 1, note that it will be easy to account for restrictions fixing the overall amount of one of the initial materials. For a self-adjoint optimisation problem for a combined bar in torsion, a similar approach has been illustrated in [**9**].

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References

1 K. A. Lurie. *On the non-self-adjoint problems of optimal control in coefficients of elliptical equations* (Preprint-1175, A. F. Ioffe Physical-Technical Institute, Academy of Sciences of the U.S.S.R., Leningrad, 1987) (in Russian).

2 K. A. Lurie, A. V. Fedorov and A. V. Cherkaev. Regularization of optimal design problems for bars and plates, Parts 1 and 2. *J. Optim. Theory Appl.* **37** (1982), 499–521; 523–543.

3 K. A. Lurie and A. V. Cherkaev. Effective characteristics of composite materials and optimum design of constructive elements. *Uspekhi Mekhaniki* **9** (1986), 3–81 (in Russian).
4 A. V. Cherkaev, L. V. Gibianskii and K. A. Lurie. Optimum focusing of heat flux by means of a non-homogeneous heat-conducting medium (DCAMM Report No. 305, The Technical University of Denmark, July 1985).
5 J. M. Ball. Convexity conditions and existence theorems in non-linear elasticity. *Arch. Rational Mech. Anal.* **63** (1977), 337–407.
6 G. Strang. *The polyconvexification of* $F(\nabla u)$ (Research Report CMA-R09-83, The Australian National University, 1983).
7 R. T. Rockafellar. *Convex Analysis* (Princeton: Princeton University Press, 1970).
8 B. Dacorogna. *Weak Continuity and Weak Lower Semi-continuity for Non-linear Functionals*, Lecture Notes in Mathematics 922 (Berlin: Springer, 1982).
9 J. Goodman, R. V. Kohn and L. Reyna. Numerical study of a relaxed variational problem from optimal design. *Comput. Methods Appl. Engrg.* **57** (1986), 107–127.

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