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Abstract

Edge colorings of r-uniform hypergraphs naturally define a multicoloring on the 2-shadow, i.e. on the pairs that are covered by hyperedges. We show that in any $(r-1)$-coloring of the edges of an r-uniform hypergraph with n vertices and at least $(1-\epsilon)\binom{n}{r}$ edges, the 2-shadow has a monochromatic matching covering all but at most $o(n)$ vertices. This result implies that for any fixed r and sufficiently large n, there is a monochromatic Berge-cycle of length $(1-o(1))n$ in every $(r-1)$-coloring of the edges of $K_n^{(r)}$, the complete r-uniform hypergraph on n vertices.

1 Introduction

Let $\mathcal{H}$ be an r-uniform hypergraph (a family of some r-element subsets of a set). The *shadow graph* of $\mathcal{H}$ is defined as the graph $\Gamma(\mathcal{H})$ on the same vertex set, where two vertices are adjacent if they are covered by at least one edge of $\mathcal{H}$. A coloring of the edges of an r-uniform hypergraph $\mathcal{H}$, $r \geq 2$, induces a multicoloring on the edges of the shadow graph $\Gamma(\mathcal{H})$ in a natural way; every edge e of $\Gamma(\mathcal{H})$ receives the color of all hyperedges containing e. A subgraph of $\Gamma(\mathcal{H})$ is *monochromatic* if the color sets of its edges have a nonempty intersection.

A set of pairwise disjoint edges of the shadow graph covering $n-o(n)$ vertices is called an *almost perfect matching* of $\Gamma(\mathcal{H})$. Let $K_n^{(r)}$ denote the complete r-uniform hypergraph on n vertices. An r-uniform hypergraph is *almost complete,* if it has at least $(1-o(1))\binom{n}{r}$ edges. We call an r-uniform hypergraph $(1-\epsilon)$-complete if it has at least $(1-\epsilon)n^r/r!$ edges.

In this paper we prove the following conjecture from [3].

Theorem 1.1. *Assume that $r \geq 2$ is fixed, $\mathcal{H}$ is an almost complete r-uniform hypergraph with n vertices, and its edges are colored with $r-1$ colors. Then the induced multicoloring on $\Gamma(\mathcal{H})$ contains a monochromatic almost perfect matching.*

It is worth noting that Theorem 1.1 does not hold if we color with r colors instead of $r-1$. An example in [3] gives an r-coloring of $K_n^{(r)}$ such that the largest number of vertices covered by any monochromatic matching is not larger than $\frac{(2r-2)n}{2r-1}$. In fact that is conjectured to be the best result ([3]) and proved for $r=3$ ([5]).

It was proved in [3] that Theorem 1.1 implies a stronger result, namely that the almost perfect monochromatic matching M guaranteed can be *connected* as well which means that the edges of M are in the same component of the hypergraph defined by the edges of the color of M. Moreover, it was shown in [3] how to combine this strengthening of Theorem 1.1 and a "weak version" of the hypergraph Regularity lemma to get a Ramsey-type result for Berge-cycles. An r-uniform Berge-cycle ([1]) of length ℓ is a sequence of distinct vertices $v_1, v_2, \ldots, v_\ell$ together with a set of distinct edges $e_1, \ldots, e_\ell$ such that e_i contains v_i, v_{i+1} ($v_{\ell+1} \equiv v_1$).

Corollary 1.2. *In every $(r-1)$-coloring of the edges of $K_n^{(r)}$ there is a monochromatic Berge-cycle of length at least $(1-o(1))n$.*

Note that in [3] it was conjectured that for sufficiently large n this statement is true with a monochromatic Berge-cycle of length n, i.e. there is a monochromatic *Hamiltonian* Berge-cycle. However, at the moment we are unable to prove this stronger statement.

The way to obtain the above corollary from Theorem 1.1 illustrates a principle due to Łuczak (suggested in [8]): in many cases the task of finding a monochromatic path or cycle can be reduced to the easier task of finding a monochromatic matching via the Regularity lemma. This principle is applied in many recent Ramsey-type results such as [2], [3], [4], [5], [6], [7].

2 Proof of Theorem 1.1

For $0 < \delta < 1$ fixed, we say that a sequence $L \subset V(\mathcal{H})$ of k distinct vertices was obtained by a *δ-bounded selection* (or sometimes we just say shortly that L *is* a δ-bounded selection) if its elements are chosen in k consecutive steps so that in each step there are at most δn forbidden vertices that cannot be included as the next element. These sets of δn forbidden vertices may depend on the choices of the vertices chosen in the previous steps. Observe that a δ-bounded selection L is also a δ'-bounded selection for any $\delta' > \delta$. The following lemma from [3] proved to be a convenient tool to handle almost complete hypergraphs.

Lemma 2.1. *Assume that $\mathcal{H}$ is a $(1-\epsilon)$-complete r-uniform hypergraph and set $\delta = \epsilon^{2^{-r}}$. There are forbidden sets such that for every δ-bounded selection $L \subset V(\mathcal{H})$ of length at most r, at least $(1-\delta)\frac{n^{r-|L|}}{(r-|L|)!}$ edges of $\mathcal{H}$ contain L.*

Now we are ready to prove Theorem 1.1 by induction on r. Let $\epsilon > 0$ be arbitrary, $\mathcal{H}$ is an $(1-\epsilon)$-complete r-uniform hypergraph with n vertices whose edges are colored with $r-1$ colors. We shall prove that there is a monochromatic matching M in $\Gamma(\mathcal{H})$ covering all but at most αn vertices, where α tends to 0, if ϵ tends to 0.

Set $p = \sqrt{\epsilon} n + 1$. For $r = 2$ we have an $(1-\epsilon)$-complete graph (colored with one color). Select a maximum matching M in the graph. Observe that the set of vertices uncovered by M form an independent set. Since $\binom{p}{2} > \frac{\epsilon n^2}{2}$, less than $p = o(n)$ vertices are uncovered by M.

Assume that Theorem 1.1 is true for every $q < r$. Consider an $(r-1)$-coloring of an $(1-\epsilon)$-complete r-uniform hypergraph $\mathcal{H}$ with $r \geq 3$. Set $\delta = \epsilon^{2^{-r}}$ and let G be the defined by those edges of the shadow graph $\Gamma(\mathcal{H})$ that can be obtained as the first two vertices of the δ-bounded selection guaranteed by Lemma 2.1. Note that this implies that G has minimum degree $(1-\delta)n$.

For any $v \in V(G)$ and $1 \leq i \leq r-1$, let A_i be the set of vertices w such that color i is not on the edge $vw \in E(G)$. Assume that $y_i \in A_i$ for $1 \leq i \leq r-1$. Then $e = \{v, y_1, \dots, y_{r-1}\} \notin E(\mathcal{H})$ because no color can be assigned to any r-tuple containing e. Assuming w.l.o.g that $|A_1| \leq |A_2| \leq \dots \leq |A_{r-1}|$, it follows that

$$|A_1|^{r-1} \leq \Pi_{i=1}^{r-1}|A_i| \leq \frac{\delta n^{r-1}}{(r-1)!}$$

implying that v is adjacent in color 1 to at least $(1-\delta-\rho)n$ vertices of G (using the minimum degree condition in G), where $\rho = \left(\frac{\delta}{(r-1)!}\right)^{\frac{1}{r-1}}$. This argument shows that $V(G) = \cup_{i=1}^{r-1} X_i$ where $v \in X_i$ has the property that at least $(1-\delta-\rho)n$ edges of G of color i are incident to v.

Let M_i be a maximum matching in color i in the subgraph of G induced by $V(G) \setminus X_i$ and set $Y_i = V(G) \setminus (V(M_i) \cup X_i)$. Observe that - from the choice of M_i - no edge of G within Y_i is colored with color i.

If

$$p(i-1) + |X_i| \geq |Y_i|$$

holds for some i, $1 \leq i \leq r-1$, then we have the required large matching in color i. Indeed, almost every edge of G incident to X_i has color i thus M_i can be extended to a matching that misses at most $p(i-1)+(\delta+\rho)n = o(n)$ vertices of G.

Assume that $p(i-1) + |X_i| < |Y_i|$ for every i, $1 \leq i \leq r-1$. This implies that

$$\sum_{i=1}^{r-1}|Y_i| > p\sum_{i=1}^{r-1}(i-1) + \sum_{i=1}^{r-1}|X_i| \geq p\binom{r-1}{2} + n.$$

We claim that this inequality implies $|Y_i \cap Y_j| \geq p$ for some $1 \leq i < j \leq r-1$. Indeed, otherwise

$$n \geq | \cup_{i=1}^{r-1} Y_i| > \sum_{i=1}^{r-1}|Y_i| - \binom{r-1}{2}p$$

contradicting to the inequality above. This proves the claim.

Select Y_i, Y_j from the claim, w.l.o.g. $|Y_{r-2} \cap Y_{r-1}| \geq p$. From the definition of p, $\binom{p}{2} > \frac{\epsilon n^2}{2}$ follows, implying that there is $x, y \in Y_{r-2} \cap Y_{r-1}$ such that $xy \in E(G)$. Notice that for $r = 3$ we have a contradiction on this branch of the proof: by Lemma 2.1, there are at least $(1-\delta)n > 0$ edges (triples) containing $\{x, y\}$ and such an edge cannot have a color.

For $r \geq 4$ consider the $(r-2)$-uniform hypergraph $\mathcal{H}^*$ with edge set $\{e \setminus \{x, y\} : e \in \mathcal{H}\}$. Note that x, y are the first two vertices in a δ-bounded selection of Lemma 2.1 thus $\mathcal{H}^*$ is an $(1-\delta)$-complete hypergraph. Moreover, since $x, y \in Y_{r-2} \cap Y_{r-1}$, $\mathcal{H}^*$ is colored with $r-3$ colors. Thus - since $r-2 \geq 2$ - induction applies, $\mathcal{H}^*$ has an almost perfect monochromatic matching M in its shadow graph. Observing that M is a monochromatic matching in the shadow graph of $\mathcal{H}$ as well, the proof is finished. □

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