Multirate Digital Signal-Processing Algorithm to Calculate Complex Acoustic Pressure Fields

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Multirate digital signal processing algorithm to calculate complex acoustic pressure fields

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An efficient algorithm to compute complex (magnitude and phase) acoustic pressure field data that uses a multirate digital processing architecture is presented. The algorithm is based on the discretization of the velocity potential function, sampled at a rate that varies as a function of field point location and transducer geometry. The algorithm can be used to determine accurate magnitude and phase information at any field point location, and for any transducer geometry with a closed-form velocity potential function, including planar piston, spherically focused piston, and planar annular array transducers (e.g., the nondiffracting or Jo-Bessel transducer). Numerical simulations based on this algorithm are presented together with exact field calculations wherever possible in order to make absolute error comparisons. Additionally, results based on a standard Gaussian quadrature integration scheme are presented in order to compare computational speed and accuracy in the near field. Results indicate improvements in numerical efficiency of 15 to 30 times over standard numerical integration techniques.

PACS numbers: 43.20.Rz, 43.40.Le, 43.35.Ze

LIST OF SYMBOLS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>transducer radius, or projected radius for spherically focused piston</td>
</tr>
<tr>
<td>(a_i)</td>
<td>outer radius of (i)th ring of annular transducer</td>
</tr>
<tr>
<td>c</td>
<td>speed of propagation in medium</td>
</tr>
<tr>
<td>(d_0)</td>
<td>depth of spherically focused transducer</td>
</tr>
<tr>
<td>(E(f))</td>
<td>envelope of magnitude spectrum of velocity potential function at axial field points, for a given transducer geometry</td>
</tr>
<tr>
<td>(F(\cdot))</td>
<td>Fourier transform of quantity</td>
</tr>
<tr>
<td>FFT</td>
<td>fast Fourier transform</td>
</tr>
<tr>
<td>(f)</td>
<td>frequency, Hertz</td>
</tr>
<tr>
<td>(f_{bb})</td>
<td>upper limit of desired baseband frequency range</td>
</tr>
<tr>
<td>(f_n)</td>
<td>Nyquist frequency, (=f/2)</td>
</tr>
<tr>
<td>(f_{ni})</td>
<td>Nyquist frequency after (i)th stage of decimation, (i=1,2)</td>
</tr>
<tr>
<td>(f')</td>
<td>Nyquist frequency after decimation, e.g., (f'<em>{n1} = f</em>{n1}) and (f'<em>{n2} = f</em>{n1})</td>
</tr>
<tr>
<td>(f_s)</td>
<td>sampling frequency</td>
</tr>
<tr>
<td>(f_{si})</td>
<td>sampling frequency after (i)th stage of decimation, (i=1,2)</td>
</tr>
<tr>
<td>(f_s')</td>
<td>sampling frequency after decimation, e.g., (f_s'<em>{1} = f</em>{s1}) and (f_s'<em>{2} = f</em>{s1})</td>
</tr>
<tr>
<td>(f_g)</td>
<td>global sampling rate allowed for multirate algorithm</td>
</tr>
<tr>
<td>(f_{sr})</td>
<td>local (maximum) sampling rate for a given velocity potential function</td>
</tr>
<tr>
<td>(f_{sm})</td>
<td>minimum allowable global sampling rate</td>
</tr>
<tr>
<td>(f_{sb})</td>
<td>digital filter passband frequency, (= \Omega_{sb}/f_g)</td>
</tr>
<tr>
<td>(f_{sb}')</td>
<td>digital filter stopband frequency, (= \Omega_{sb}/f_{s1})</td>
</tr>
<tr>
<td>(g[n])</td>
<td>coefficients of linear phase lowpass digital filter</td>
</tr>
<tr>
<td>(h(r,t))</td>
<td>velocity potential function in response to an impulsive velocity on surface of transducer</td>
</tr>
<tr>
<td>(h[n])</td>
<td>discretized impulse velocity potential function</td>
</tr>
<tr>
<td>(h_{as}(r,t))</td>
<td>velocity potential for planar annular array transducer</td>
</tr>
<tr>
<td>(h_{pp}(r,t))</td>
<td>velocity potential for planar piston transducer</td>
</tr>
<tr>
<td>(h_{sp}(r,t))</td>
<td>velocity potential for spherically focused piston transducer</td>
</tr>
<tr>
<td>(h'[n])</td>
<td>decimated impulse velocity potential function sequence</td>
</tr>
<tr>
<td>(H[k])</td>
<td>impulse velocity potential in frequency domain</td>
</tr>
<tr>
<td>(L)</td>
<td>number of samples in linear phase digital filter</td>
</tr>
<tr>
<td>(LPDF)</td>
<td>linear phase digital filter</td>
</tr>
<tr>
<td>(M)</td>
<td>integer decimation ratio</td>
</tr>
<tr>
<td>(M_i)</td>
<td>decimation ratio of the (i)th stage decimator</td>
</tr>
<tr>
<td>(N)</td>
<td>number of samples in sequence (h[n])</td>
</tr>
<tr>
<td>(N_g)</td>
<td>number of samples in decimated sequence (h'[n])</td>
</tr>
<tr>
<td>(N_{a})</td>
<td>number of annuli in planar annular array</td>
</tr>
<tr>
<td>(p(r,t))</td>
<td>acoustic pressure</td>
</tr>
<tr>
<td>(P(r,\omega))</td>
<td>acoustic pressure in the frequency domain</td>
</tr>
<tr>
<td>(Q)</td>
<td>field or observation point</td>
</tr>
<tr>
<td>(Q(\cdot))</td>
<td>ratio of undesired aliasing component to true desired frequency component of a signal, at some frequency (f)</td>
</tr>
<tr>
<td>(R)</td>
<td>focal length of transducer</td>
</tr>
<tr>
<td>(r)</td>
<td>general position vector</td>
</tr>
<tr>
<td>(r_{pb})</td>
<td>digital filter passband ripple specification</td>
</tr>
<tr>
<td>(r_{sb})</td>
<td>digital filter stopband ripple specification</td>
</tr>
<tr>
<td>(\tilde{S}(\cdot))</td>
<td>magnitude spectrum of velocity potential function at axial field points, for a given transducer geometry</td>
</tr>
<tr>
<td>(T_g)</td>
<td>phase delay of linear phase digital filter</td>
</tr>
<tr>
<td>(T'_{s})</td>
<td>sampling period, (= 1/f_s)</td>
</tr>
<tr>
<td>(t)</td>
<td>time, seconds</td>
</tr>
<tr>
<td>(t_d)</td>
<td>time delay to midpoint of (h(z,t)); only defined for axial (rectangular) velocity potential functions</td>
</tr>
</tbody>
</table>
INTRODUCTION

Considerable effort is being expended for the development of quantitative ultrasound techniques, in particular for nondestructive acoustic evaluation. To the extent that precise determination of the relevant acoustic parameters of a test sample can be performed, valuable information may be obtained about its structural integrity, homogeneity, or physical dimensions. The explicit numerical nature of quantitative testing allows an objective analysis of the sample parameter being studied. This is in contrast to the more conventional qualitative ultrasonic tests, wherein an observable parameter in the received signal (whether a discriminating feature found in the received electrical signal, or a visual characteristic of an image) is identified and correlated with a physical parameter of interest. This type of analysis utilizes less of the information available in the received signal and is more prone to error since the interpretation of test results are quite subjective in nature.

In nondestructive evaluation applications, such quantitative tests may be used to determine the attenuation and impedance profiles of a material sample, which, for exact results, is predicated on a priori knowledge of the pressure field over the sample region of interest. Once determined, these quantities can be used to predict weld and bond failures, to detect undesirable material inhomogeneities such as bubbles and inclusions, and to identify stress cracks and flaws developing inside a sample.

Likewise, quantitative ultrasound is obviously applicable in diagnostic medicine, where the relevant acoustic parameters determined for various organs and vessels can be used to locate tissue pathologies such as tumors and lesions, or to find atherosclerotic plaque and abnormal vessel constriction, without having to resort to more invasive and potentially harmful techniques.

Quantitative measurements require an accurate knowledge of the temporal and spatial pressure field distribution generated by the ultrasonic transducer, as well as the interaction and scattering of the field by the volume of interest, and the pressure field intercepted (and electrical signal produced) by the receiving transducer in response to this field. The knowledge about the pressure fields is incorporated into the analysis and interpretation of the received data, so that the desired information concerning the actual structure or state of the material being examined can be determined. Each of these topics are active areas of research in quantitative ultrasound, as witnessed by the vast number of papers on these and related areas.1

This paper addresses one of the key aspects of conducting quantitative ultrasonic analyses: the need to accurately compute or predict the acoustic pressure field generated by a source transducer. Many factors can cause the calculation of the pressure field to be computationally intensive, such as requiring the full complex (magnitude and phase) pressure field result. The magnitude of the pressure field has commonly been a distinguishing feature in both qualitative and quantitative measurements. The phase information of the pressure field is becoming an increasingly important quantity in current analyses, however, as researchers begin to unveil the more fundamental relations between acoustic measurement and physical wave interaction. For example, phase information is critical to the modeling of received signals from transducers. The analysis of the acoustic field in the near-field regions of the source transducer is another computationally demanding task. Many practical ultrasonic examinations are conducted in the near field, where the field is varying quickly with respect to both its magnitude and phase, and therefore requiring the calculations to be performed on a much finer spatial grid than in the far field. Additionally, the determination of the complex pressure at a single spatial coordinate in the near field (for a given transducer geometry) is more computationally intensive than for a point located in the far field. This can be attributed to the increasingly oscillatory nature of the Rayleigh integrand (necessary to describe the acoustic pressure at any field point) as the field point moves from far field to near field.2,3

In medical ultrasound, the frequency range of most practical ultrasonic transducers ranges from less than 1 to 15 MHz or greater. The frequency range of pressure field calculations intended for medical ultrasound analysis should therefore be capable of spanning this bandwidth. For nondestructive materials testing, the range of acoustic frequencies employed is much greater, ranging from below 1 kHz to over 100 MHz. In either case, the frequency resolution of the

\[
\begin{align*}
t_i & \quad \text{time epochs of discontinuities in velocity potential functions (t_0, t_1, etc.)} \\
u(r,t) & \quad \text{velocity excitation of transducer surface} \\
\vec{U}(r,\omega) & \quad \text{normal component of } u(r,t), \text{ in frequency domain} \\
V_i & \quad \text{excitation voltage to } i\text{th ring of annular array transducer} \\
W & \quad \text{surface of transducer (region of integration)} \\
z & \quad \text{unit vector in } z \text{ direction (points axially away from } W \text{ for axisymmetric transducers)} \\
z_Q & \quad \text{z coordinate of field point } Q \\
(\cdot) & \quad \text{denotes a complex quantity} \\
\Delta t & \quad \text{temporal length of nonzero portion of } h(r,t)
\end{align*}
\]

\[
\begin{align*}
\delta(t) & \quad \text{Dirac delta function} \\
\phi(r,t) & \quad \text{velocity potential function} \\
\Phi(r,\omega) & \quad \text{velocity potential in frequency domain} \\
\rho_q & \quad \text{radial coordinate of field point } Q \\
\rho_0 & \quad \text{density of propagation medium} \\
\sigma(t) & \quad \text{Heaviside unit step function} \\
\omega & \quad \text{frequency, radians} \\
\Omega_{ps} & \quad \text{unity-normalized passband cutoff frequency of lowpass digital filter} \\
\Omega_{sb} & \quad \text{unity-normalized stopband cutoff frequency of lowpass digital filter} \\
\Omega(cT) & \quad \text{half-angle of arc used to derive } \phi(r,t) \text{ for piston transducers}
\end{align*}
\]
pressure field over the desired bandwidth is specific to the analysis being conducted, but current quantitative results indicate that increasingly finer frequency resolutions offer distinct advantages that are essential to accurate quantitative reconstructions, particularly for results destined for time-domain analysis and interpretation.

As mentioned above, the number of field point locations at which the pressure field must be evaluated can be quite large, and depends primarily upon the extent of the sample volume of interest and the acoustic wavelength in the medium. As an example, consider the field produced at a frequency of 10 MHz in a medium with a sound speed of 1500 m/s (typical parameters for medical ultrasound); the acoustic wavelength is c/f, or 150 μm. To accurately represent this spatial wavelength, the Nyquist sampling criterion is satisfied when at least two samples per wavelength are determined, or roughly a field point every 75 μm. This indicates that the number of field point locations can be quite large for most samples of practical importance.

I. REVIEW OF CURRENT ALGORITHMS

Many algorithms for calculating acoustic pressure fields emanating from a wide variety of ultrasonic transducers have been reported over the past several decades. Numerical techniques exist that base their results on a direct evaluation of Huygen's integral, a slow but powerful technique that can handle transducers of arbitrary geometries and dimensions. Another algorithm that can generate complex field results for arbitrary transducer geometries is based on the angular spectrum decomposition of the source field. This is a computationally intensive technique, but is a suitable approach for problems involving field diffraction from finite apertures, and field scattering from rough reflectors. Another category of field calculation algorithms are those that determine the pressure field directly in the time domain, including those based on finite-element and boundary-element (Galerkin) formulations.

Algorithms based on formulation of the velocity potential function have been well known for some time, and recent algorithms can accomplish an efficient calculation of the acoustic field using numerical integration of the velocity potential at a given field location and frequency. Some of the velocity potential algorithms gain their efficiency by utilizing simplifying approximations valid only in the far field. Other algorithms directly approximate the velocity potential functions, either through Taylor series expansions or other polynomial parametrizations, and yield results with varying degrees of accuracy. A common characteristic of these algorithms is that they determine the complex pressure field at a single frequency each time the algorithm is executed. This is primarily due to the intrinsic numerical integration of the Fourier transform contained in these algorithms, which must be performed once at each frequency. This is an advantage that can be speeded up if the algorithms are designed to yield broadband frequency results directly, rather than iteratively.

The multirate algorithm presented in this paper is based on the velocity potential formulation, and can be used to increase the efficiency of pressure field calculations in quantitative analyses where the complex field must be determined over a large number of frequencies. The algorithm computes the complex pressure field directly in the frequency domain, using a zoom-FFT and multistage decimator modified to accept variable sampling rates. In the chosen implementation, pressure field components are generated over the entire 0–15.625 MHz range, at a resolution of roughly 30 kHz, for each execution of the algorithm. We will limit our discussions in the remainder of this paper to the medical ultrasound frequency range of 0–15 MHz. Changes in algorithm requirements to handle other frequency ranges will be discussed when appropriate. Section II reviews the essential mathematics of pressure field analysis. Sections III–V detail various aspects of the multirate algorithm. Section VI contains results obtained using the multirate algorithm, and in particular presents pressure field profiles for the more recent "nondiffracting" or J0-Bessel annular array transducer.

Results of accuracy comparisons are given with respect to both exact techniques (whenever available) and a current algorithm utilizing a Gaussian quadrature numerical integration scheme. The numerical efficiency is evaluated by performance comparison to the Gaussian quadrature integration scheme.

II. PRESSURE FIELD ANALYSIS

We will begin by reviewing several basic results of acoustic pressure field analysis. Derivations of the equations are well known and will not be presented here; the interested reader is referred to the cited references where indicated.

The acoustic pressure p in a fluid is related to the velocity potential function φ by

\[ p(r, t) = \rho_0 \frac{\partial \phi(r, t)}{\partial t}, \]  

(1)

where \( \rho_0 \) is the equilibrium density of the propagation medium and \( r \) is the position vector of the field point. The particle velocity \( u \) is proportional to the gradient of the velocity potential and may be expressed as

\[ u(r, t) = -\nabla \phi(r, t). \]  

(2)

Knowledge of \( \phi \) therefore allows a complete description of the acoustic field in a fluid medium; note that \( \phi \) satisfies the linearized scalar wave equation. Equations (1) and (2) assume that a scalar (longitudinal mode) wave theory will accurately describe the propagation phenomenon, which is the case for propagation in a fluid medium (the particle velocity in an inviscid fluid is irrotational). Propagation in a medium that supports additional modes (e.g., longitudinal and shear waves in solids) requires a vector representation of the acoustic field, and is not covered in this paper.

For the case of a radiator surface set in an infinite rigid baffle, the velocity potential \( \phi \) is given by Rayleigh's integral,

\[ \phi(r, t) = \frac{1}{2\pi} \int_\Omega \frac{u(t - \tau/c)}{r} dS, \]  

(3)

where \( W \) is the surface of the radiator, \( r = |r| \), \( c \) is the propagation velocity of the medium and \( u(t) \) is the normal component of the radiator velocity \( u(t) \), assumed to be uniform over the entire surface \( W \). The conditions on \( u \) as may be expressed as


D. P. Orofino and P. C. Pedersen: Complex acoustic fields 565

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where \( \sigma(t) \) is the Heaviside unit-step function. The integration in (3) depends upon the geometry and boundaries of the transducer surface \( W \). We need to concern ourselves only with the case of a time-impulsive velocity \( u(t) = \delta(t) \), since we may obtain the general time-varying velocity solution by convolution. Therefore,

\[
\phi(r,t) = u(t) \otimes h(r,t),
\]

where

\[
h(r,t) = \frac{1}{2\pi} \int \frac{\delta(t - r/c)}{r} dS;
\]

\( h(r,t) \) is thus the velocity potential due to a spatially uniform, time-impulsive velocity on the radiator surface.

The previous equations may conveniently be expressed in the frequency domain. By application of Fourier transform theorems, (1) can be written as

\[
\tilde{P}(\mathbf{r}, \omega) = j\omega \rho \tilde{\Phi}(\mathbf{r}, \omega),
\]

where the tilde superscript denotes a complex quantity. Likewise, (5) and (6) may be expressed in the frequency domain as

\[
\tilde{\Phi}(\mathbf{r}, \omega) = \tilde{U}(\omega) \tilde{H}(\mathbf{r}, \omega)
\]

and

\[
\tilde{H}(\mathbf{r}, \omega) = \frac{1}{2\pi} \int \int_w \frac{\exp(-j\omega r/c)}{r} dS.
\]

Therefore, we may compute the pressure field directly in the frequency domain by finding the Fourier transform of \( h(r,t) \), and multiplying by the various quantities as indicated in (7) and (8). The multirate algorithm for pressure field calculation is essentially a discrete implementation of Eqs. (6)–(8). Note that \( \tilde{H}(\mathbf{r}, \omega) \) can be numerically obtained by Fourier transformation of (6) using an FFT, rather than by analytical Fourier transformation of (6) as expressed in (9).

Since solutions to (6) can be found in closed form, the FFT of (6) constitutes an attractive and efficient alternative to straightforward numerical integration of (9).

In order to find the velocity potential function \( h(r,t) \), Eq. (6) must be evaluated over the surface \( W \) described for a given transducer geometry. Closed-form solutions to (6) are known at an arbitrary field point for several transducer geometries, including the circular planar piston,\(^7\)\(^8\) rectangular planar piston,\(^9\)\(^10\) and spherically focused piston transducer.\(^11\)\(^12\) A small approximation is made in formulating the pressure field due to a spherical transducer in terms of the Rayleigh integral in (3). Each hemispherical wave generated by points on the transducer surface is reflected by other points on the curved surface, causing a secondary diffraction of the field at the given field point; this secondary diffraction is not taken into account in the Rayleigh integral.\(^13\)\(^14\) However, the error produced is generally small, especially for a weakly focused transducer. Derivations of the velocity potential functions are left to the references wherever cited.

### A. Planar piston

For the circular planar piston, the velocity potential function is found by consideration of the problem geometry. Referring to Fig. 1, \( \rho_q \) denotes the radial distance from the piston center (acoustic axis) to the projection \( \left( Q' \right) \) of field point \( Q \) onto the source, \( z_q \) denotes the axial distance to the field point, and \( a \) is the radius of the circular planar piston transducer. Then, adopting the axisymmetric cylindrical coordinates \( r = (p, z) \),

\[
r_0 = ct_0 = z_q,
\]

\[
r_1 = ct_1 = \left[ (a - \rho_q)^2 + z_q^2 \right]^{1/2},
\]

\[
r_2 = ct_2 = \left[ (a + \rho_q)^2 + z_q^2 \right]^{1/2}.
\]

In the above expression, \( r_1 \) is the distance to the nearest edge and \( r_2 \) the distance to the farthest edge of the piston from the field point \( Q \). Hence, (10) defines the time epochs \( t_0, t_1, \) and \( t_2 \) for a given field point; these time epochs indicate points in the velocity potential where the function is discontinuous.

[The time epochs are explicitly illustrated later in Fig. 5(a).] Based on these notations, the velocity potential function \( h_{pp} \) for the planar piston is found to be (Ref. 7, Table I)

\[
h_{pp}(r,t) = \begin{cases} 
0, & \text{if } \rho_q < a \\
\frac{c}{\pi} \Omega(r,ct), & \text{if } \rho_q > a
\end{cases}
\]

where

\[
h_{pp}(r,t) = \begin{cases} 
0, & \text{if } t < t_0 \\
\frac{c}{\pi} \Omega(r,ct), & \text{if } t_0 < t < t_1 \\
0, & \text{if } t_1 < t < t_2 \\
0, & \text{if } t_2 < t
\end{cases}
\]

and

\[
\Omega(r,ct) = \int_0^{\pi} \int_0^{2\pi} \frac{\exp(-j\omega r/c)}{r} dS.
\]

![FIG. 1. Geometry of planar piston radiator, illustrating angles \( \Omega(r,ct) \) for the cases of \( \rho_q < a \) (with projection \( Q' \)) and \( \rho_q > a \) (with projection \( Q'' \)).](image-url)
It has been shown in the same reference (Ref. 7, Table I) that
\[ \Omega(r,ct) = \cos^{-1}\left(\frac{c^2t^2 - z_q^2 + \rho_q^2 - a^2}{2\rho_q(c^2t^2 - z_q^2)^{1/2}}\right). \]  
\[ (13) \]

**B. Planar annular array**

The velocity potential for a planar, annular concentric array can also be derived in closed form, based on the superposition of solutions to the circular planar piston. This result will be needed when calculating the pressure field due to the annular array implementation of the nondiffracting transducer,\(^{15,16}\) given later in this paper. Typically, an annular array has a uniform excitation voltage assigned to each of \(N_a\) concentric annular rings in the array, where the radial widths of each annulus may vary. The outer and inner radii of the \(i\)th annulus are denoted by \(a_i\) and \(a_{i-1}\), respectively, where \(a_{i-1} < a_i, i = 1,2,...,N_a\), and \(a_0 = 0\). The geometry of the annular array is illustrated in Fig. 2. From the linearity of the wave equation (and hence of the velocity potential function itself), the impulse velocity potential \(h_{i,ar}\) and \(h_{i,ar}^{-1}\) for the first and \(i\)th annular rings, respectively, are

\[ h_{i,ar} = V_i h_{i,pp}, \]

\[ h_{i,ar}^{-1} = V_i h_{i,pp}^{-1} - V_i h_{i,pp}^{-1}, \quad i = 2,3,...,N_a, \]

where \(h_{i,pp}\) denotes the velocity potential function \(h_{i,pp}(r,t)\) for a planar piston transducer of radius \(a_i\). The velocity potential function for the entire annular array, denoted by \(h_{aa}\), is the summation of \(h_{i,ar}\) over all annuli in the array,

\[ h_{aa} = \sum_{i=1}^{N_a} h_{i,ar}, \]

\[ = V_i h_{i,pp} + (V_2 h_{i,pp} - V_2 h_{i,pp}) \]

\[ + \cdots + (V_{N_a-1} h_{i,pp}^{-1} - V_{N_a-1} h_{i,pp}^{-1}) \]

\[ + (V_{N_a} h_{i,pp} - V_{N_a} h_{i,pp}^{-1}). \]

\[ (15) \]

Note that (15) requires the evaluation and summation of \(2N_a - 1\) velocity potential functions. If we regroup terms, then \(h_{aa}\) may be expressed using only \(N_a\) velocity potential functions, as follows:

\[ h_{aa} = V_{N_a} h_{i,pp}^{-1} + \sum_{i=1}^{N_a-1} h_{i,pp} (V_i - V_{i+1}). \]

\[ (16) \]

Equation (16) efficiently computes the closed-form velocity potential function \(h_{aa}\) for an \(N_a\)-ring annular array using \(N_a\) planar piston velocity potential functions.

**C. Spherically focused piston**

The spherically focused transducer has a velocity potential function that may also be expressed in closed form. The geometry of interest is shown in Fig. 3, which describes a transducer of projected radius \(a\) and radius of curvature (focal depth) \(R\). The depth \(d_a\) is a measure of the transducer thickness variation due to the spherical curvature, and is found to be

\[ d_a = R - \sqrt{R^2 - a^2}. \]

\[ (17) \]

The spherically focused piston differs from the planar piston only in the description of the transducer surface geometry; the same conditions for \(u(t)\), as given in (4), describing the velocity on the surface of the transducer still hold. Referring to Eq. (6), the velocity potential function is now obtained by integration over the spherical surface \(W\) of the transducer.

In order to write the impulse response for a given field point \(Q\), we follow the derivation given in Ref. 12. The regions inside and outside the cone defined by the circular boundary of the radiator and the focal point must be considered separately. The field point \(Q\) lies inside this cone if

\[ \frac{\rho_q}{|R - z_q|} < \frac{a}{\sqrt{R^2 - a^2}} \]

where we have adopted an axisymmetric cylindrical coordinate system, where \(r_q = (\rho_q, z_q)\). Four time instants can be

FIG. 2. Geometry of an \(N_a\)-ring planar concentric annular array, indicating annulus outer radii \(a_i\) and voltage excitations \(V_i\). The total array radius is \(a\).

FIG. 3. Geometry of spherically focused transducer, indicating transducer radius \(a\), depth \(d_o\), and focal length \(R\).
defined by the shortest and longest travel times of a wave from different points on the surface of the transducer to the field point. These times are as follows:

\[ t_1 = r_1/c = (1/c) \left[ R - \sqrt{\rho_0^2 + (R - z_q)^2} \right], \]
\[ t_2 = r_2/c = (1/c) \left[ (a - \rho_q)^2 + (z_q - d_0)^2 \right], \]
\[ t_3 = r_3/c = (1/c) \left[ (a + \rho_q)^2 + (z_q - d_0)^2 \right], \]
\[ t_4 = 2R/c - t_1 = (1/c) \left[ R + \sqrt{\rho_0^2 + (R - z_q)^2} \right]. \]

(19)

In the above equations, \( r_1 \) corresponds to the shortest distance from field point to source (transducer surface) when \( z_q < R \) and the field point lies inside the cone, \( r_2 \) is the distance to the closest edge of the source, \( r_3 \) is the distance to the farthest edge of the source, and \( r_4 \) is the farthest distance from field point to source when \( z_q > R \) and the field point lies outside the cone.

For points \( Q \) inside the cone, the impulse velocity potential function \( h_{sp} \) for the spherically focused piston is

\[ h_{sp}(\rho_q, z_q, t) \]

\[ \begin{array}{c|c|c}
\text{If the field point } Q \text{ is outside this cone, then the velocity potential will be} \\
\hline
z_q < R & t < t_1 & t_1 < t < t_2 \\
R/c/\sqrt{\rho_0^2 + (R - z_q)^2} & 1 < t < t_1 & t_1 < t < t_4 \\
(\sqrt{\rho_0^2 + (R - z_q)^2}) \Omega(ct) & t_2 < t < t_3 & t_3 < t < t_4 \\
0 & t_3 < t & t_4 < t . \\
\end{array} \]

(20)

The function \( \Omega(ct) \) is defined to be the half-angle subtended by arc \( L(r) \) from point \( E \). This relation is complicated to derive, and is found to be [Ref. 12, Eq. (8)]

\[ \Omega(ct) = \cos^{-1} \left( \frac{\sqrt{\rho_0^2 + (R - z_q)^2}}{r_0 \rho_q} \right) \times \left[ \sqrt{\rho_0^2(R - z_q)^2 - \rho_0^2(a^2 - r_0^2 - \rho_q^2)} - \rho_0(R - z_q) \right], \]

(22)

where

\[ \alpha = \tan^{-1} \left[ \rho_q / (R - z_q) \right], \]

(23a)

\[ r = [(a - \rho_q)^2 + z_q^2], \]

(23b)

\[ \theta = \cos^{-1} \left( \frac{2Rz_q - z_q^2 - \rho_q^2 + r^2}{2r[\rho_q^2 + (R - z_q)^2]^{1/2}} \right), \]

(23c)

\[ r_0 = r \sin \theta, \]

(23d)

and

\[ \rho_0 = \rho_q + r \cos \theta \sin \alpha. \]

(23e)

Since \( L(r) \) is the length of the intersection of the surface \( W \) and a circular cone with half an apex angle \( \theta \), then, noting that \( r \) lies completely on the surface of this cone, all points on \( L(r) \) must be equidistant from \( Q \). The equation for \( r \) as given in (23b) may then be stated directly. Refer to Fig. 3 for the definitions of angles \( \alpha \) and \( \theta \).

Finally, the impulse velocity potential function for a field point located at the focal point \( z_q = R \) of the transducer is not given in either (20) or (21); this function is simply [Ref. 12, Eq. (9)]

\[ h_{sp}(0, R, t) = d_0 \delta(t - R/c). \]

(24)

Thus, from (5) and (24), the velocity potential \( \phi(r, t) \) at the focal point of a spherically focused transducer is directly proportional to the velocity excitation \( u(r, t) \) induced on the surface \( W \) of the transducer, as long as the velocity is uniform over \( W \).

Figure 4 reveals the symmetry found in the velocity potential functions for a spherically focused piston transducer with \( R/\alpha = \sqrt{2} \). Figure 4(a)–(c) represent velocity potentials versus radial distance \( \rho \) to the field point for some fixed axial position \( z \). For field points on the source side of the focal point [Fig. 4(a)], the velocity potentials are asymmetric and discontinuous; the functions grow longer in time as the field point moves axially closer and/or radially farther from the transducer center. For field points in the focal plane [Fig. 4(b)], the velocity potentials are symmetric, and tend toward an impulse at the focal point itself. At field points beyond the focus [Fig. 4(c)], there is an apparent time reversal in the potential functions from those given in Fig. 4(a). Axial range relations are shown in Fig. 4(d)–(f), where the radial distance \( \rho \) is fixed to some value, while \( z \) is varied over a given range. The symmetry of the potential functions is clearly revealed in these illustrations.

The velocity potential functions for the planar piston are similar in shape to the velocity potential functions given for the spherically focused piston transducer, if the axial position \( z \) of the field point is confined to the source side of the focal point as illustrated in Fig. 4(a). In fact, the velocity potentials for the spherically focused piston are identical to those of the planar piston, if the focal length is moved out to infinity.

III. Discrete Representation of Velocity Potentials

The goal has been to calculate the complex pressure field via a discrete sampling of the impulse velocity potential function \( h(r, t) \) for the transducer geometry of interest. It has already been shown in (5) that any arbitrary velocity excitation may be incorporated into the velocity potential solution. Once determined, the discretized velocity potential function is Fourier transformed as indicated in (8). The last remaining step is to differentiate and multiply by the density of the propagation medium, as expressed in Eq. (7).

Discretizing \( h(r, t) \) involves sampling it at a rate high enough to render any aliasing effects negligible. The interpretation of this statement is based upon the frequency content of the velocity potential function. In order to estimate
just how high this rate must be, we must examine the worst case situation involving the maximum high-frequency energy content produced by any \( h(r,t) \). This worst case condition occurs on-axis (for planar and focused pistons), when the discontinuities in the derivative of the velocity potential functions are large; the discontinuities are responsible for producing the large bandwidths required to represent the potential function in the frequency domain. For field points on-axis, \( h(r,t) \) assumes the form of a rectangular function,

\[
h(z,t) = \sigma(t - t_0) - \sigma(t - t_1),
\]

where \( \sigma(t) \) once again indicates the unit step function. Referring to Fig. 5(a), we define a rectangular function of amplitude \( A \) and width \( T \) as

\[
A \text{ Rect}(\frac{t}{T}) = A \left[ \sigma(\frac{t - T}{2}) - \sigma(\frac{t + T}{2}) \right].
\]

Since our function is not centered about the time origin but
rather at some delayed time $t_d$, application of the time-delay theorem of Fourier transforms to the transform of a rectangular signal yields

$$F \left[ c \text{Rect} \left( \frac{t - t_d}{\Delta t} \right) \right] = c \frac{\sin(\pi f \Delta t)}{\pi f} \exp(-j \pi f t_d)$$

$$= \tilde{S}(f), \quad (27)$$

where $T$ has been replaced by $\Delta t$, the temporal length of the velocity potential function, and $\tilde{S}(f)$ is the shifted sinc function in the frequency domain. Here, $\Delta t$ can be determined only if a specific transducer geometry and field point location have been defined. Referring to Eq. (10) for a planar piston with an on-axis field point, $\Delta t = t_1 - t_0$ and $t_d = (t_0 + t_1)/2$ as shown in Fig. 5(a) and (b). The frequency domain representation is shown in Fig. 5(c). Note that on-axis, the velocity potentials for the planar piston are of amplitude $c$, the propagation velocity of the medium, while for focused pistons the amplitude is variable [see Eq. (20)]. The envelope of $\tilde{S}(f)$ is denoted $\tilde{E}(f)$, and can be deduced to be

$$\tilde{E}(f) = c/\pi f.$$  \hspace{1cm} (28)

Figure 5(c) reveals the frequency domain interpretation of (27), with some modifications. Here, $\tilde{E}(f)$ has been plotted together with $\tilde{S}(f)$ for convenience, and any value that $\tilde{S}(f)$ may take on will necessarily be less than or equal to $\tilde{E}(f)$.

With the envelope of the frequency content so defined, the effect of sampling rate on signal aliasing may be determined in several ways. We will determine the aliasing in the frequency domain. Note that if $f_s$ is the sampling frequency and $f_n = f_s/2$ is the corresponding Nyquist frequency, then the amplitude of the aliasing signal at a given frequency, $f < f_n$, is approximately equal to

$$\sum_{k=-\infty}^{\infty} \tilde{S}(k f_s - f),$$

(29)

where $(k f_s - f)$ represents the image frequencies $f_{\text{image}}$ such that $f_{\text{image}} = k f_s - f$, $k = \pm 1, \pm 2, \ldots$.

The dominating contribution to the aliasing signal in the positive frequency range $0$ to $f_s$ is from $\tilde{S}(k f_s - f)$ with the image frequency corresponding to $k = 1$. The envelope $\tilde{E}(f_s - f)$ of the magnitude spectrum $\tilde{S}(f_s - f)$ is shown in Fig. 5(c), along with the original magnitude spectrum $\tilde{S}(f)$ and its envelope $\tilde{E}(f)$. This dominant aliasing component can be readily found over the range $0$ to $f_s$ by "wrapping" the original signal spectrum back onto itself, folding the frequency axis at the Nyquist frequency $f_n$. The portion of the spectrum folding back into the lower frequency range approximates very closely the actual aliasing that would occur at various frequencies $f < f_n$. Let $Q(f)$ be defined as the ratio of the envelopes $\tilde{E}(f_s - f)$ and $\tilde{E}(f)$ at some frequency $f < f_n$, such that

$$Q(f) = \frac{\tilde{E}(f_s - f)}{\tilde{E}(f)} = \frac{c}{c/\pi f} = f_s - f.$$ \hspace{1cm} (30)

Solving (30) for $f_s$, we find that to achieve a minimum specified ratio,

$$f_s > f [1 + 1/Q(f)].$$ \hspace{1cm} (31)

For $Q(f) = 0.002$ (i.e., the aliasing signal amplitude is at least 54 dB below the true signal amplitude at $f_s$ or equivalently 0.2% error), and $f = f_{\text{min}} = 15.625$ MHz (the specific maximum extent of the baseband signal for our purposes), we find the following approximate constraint on the sampling frequency:

$$f_s > 8 \text{ GHz}.$$ \hspace{1cm} (32)

Note that the minimum sampling frequency determined in (32) need be maintained only for worst-case velocity potential functions (on-axis for the planar piston). All other velocity potentials exhibit decay over frequency much faster. Additionally, as can be seen in Fig. 5(c), the function $\tilde{S}(f)$ only periodically attains the magnitude of the absolute envelope amplitude $\tilde{E}(f)$ due to its sinc-like behavior. Therefore, a lower $f_s$ than specified by (31) would usually be acceptable to maintain the prescribed limit of aliasing error.

The foregoing analysis of aliasing errors can be applied equally well to both the planar and focused piston trans-

![Diagram](image-url)
decisions, since the worst-case aliasing errors for both occur on-axis, where the velocity potential function discontinuities are at their greatest extent. However, the focused transducer exhibits one singularity in its velocity potential function; at the focal point, the velocity potential impulse response function \( h(r, t) \) is itself an impulse of weight \( d_0 \) as previously found in (24) and shown graphically in Fig. 4(b). Since an impulse cannot be accurately represented with the discrete sampling performed here, the algorithm must detect and bypass this field point. A precise determination of the velocity potential function at the focal point can instead be obtained by direct evaluation of (24). The pressure field at the focal point is obtained by noting that the Fourier transform of (24) is a constant (of amplitude \( d_0 \)) over all frequencies. Differentiation in the frequency domain can then be performed in a straightforward manner.

For the general case (i.e., for field points not at the focal point), the discretization of \( h(r, t) \) is based on the chosen sampling period \( T_s = 1/f_s \), and denoted by

\[
h[n] = h(r, nT_s),
\]

where dependence on \( r \) is assumed. The discrete samples are indicated by \( n = 0, 1, \ldots, N - 1 \), where \( N \) is the total number of samples required to represent the entire temporal function \( h(r, t) \). For the planar piston, the temporal length of \( h(r, t) \) is \( t_2 - t_0 \) [see Eq. (10)], and given the sampling frequency \( f_s \), we find that

\[
N = \lceil(t_2 - t_0)f_s \rceil,
\]

where \( \lceil \cdot \rceil \) is the integer ceiling function, defined to be the smallest integer that is greater than the argument. Similar procedures can be used to find \( N \) for the spherically focused and annular array transducers as well.

Given that \( h[n] \) is a finite-length sequence, then the Fourier transform of \( h[n] \) can be obtained via the discrete Fourier transform (DFT or FFT) as

\[
\tilde{H}[k] = F(h[n]) = \sum_{n=0}^{N-1} h[n] \exp\left[-j \frac{2\pi kn}{N}\right],
\]

where \( k = 0, 1, \ldots, N - 1 \).

We may express the frequencies of evaluation, \( (k/N)f_s \), in normalized form as \( \Omega = f'T_s = k/N \), where \( \Omega \) denotes a sampling rate normalized to unity.

**IV. DECIMATION OF DISCRETIZED VELOCITY POTENTIALS**

Based on the previous discussions, it is clear that once a maximum level of aliasing error \( Q(f) \) has been chosen for the discretization of the velocity potential \( h(r, t) \), a minimum sampling frequency \( f_s \) is dictated as well. Some minimum number of samples \( N \) is required to discretely represent \( h(r, t) \) in its temporal entirety.

Practical limitations prohibit the choice of specifying arbitrarily large sampling frequencies \( f_s \), for the discretization, however. The higher the sampling frequency is chosen to be, the greater the total number of points \( N \) that must be stored in the sampled version of \( h(r, t) \). Not only may there be complications in allocating a sufficient amount of computer memory to process such a long sequence, but there will certainly be difficulty in calculating the Fourier transform \( \tilde{H}[k] \) in a microcomputer environment. For a planar piston transducer with radius \( a = 1 \) cm, \( c = 1500 \) m/s, and field point located at \( z_a = 4 \) cm and \( \rho_a = 1.5 \) cm, the temporal length \( \Delta t \) of the velocity potential function is 5.3 \( \mu \)s; using \( f_s = 8 \) GHz, there are nearly 42 500 samples in its discretization. A frequency resolution problem also becomes evident when the sampling rate is increased to such an extent; too few frequency-domain samples are left in the relatively narrow frequency range of interest for accurate spectral characterization of the pressure at the given field point.

A solution to this dilemma is to reduce the number of samples in the sequence \( h[n] \) by a process known as decimation. Decimation attempts to reduce the discretized sequence length by some integer factor \( M \), without disturbing the frequency content of the original signal. That is, if the number of samples in the original sequence is \( N \), then the decimated sampled sequence will contain \( N' \) samples at a new sampling frequency \( f'_s \), where

\[
N' = N/M \quad (36a)
\]

and

\[
f'_s = f_s/M. \quad (36b)
\]

The Nyquist frequency associated with the new discretized sampling rate is \( f'_s = f_s/2 \). The decimated velocity potential sequence is denoted by \( h'[n] \), where

\[
h'[n] = h[nM], \quad n = 0, 1, \ldots, N' - 1. \quad (37)
\]

The apostrophe will be included wherever the decimated sequence is to be specifically distinguished from the original (undecimated) velocity potential function \( h[n] \). In general, the notations \( f'_s, f'_n \), and \( h'[n] \) are used to denote the next-stage (future) parameters and results of a multistage decimation.

As noted earlier, \( h[n] \) has frequency content such that it requires discretization at the predecimation sampling frequency \( f_s \). Hence, the spectrum of \( \tilde{H}[k] \) is typically nonzero for frequencies above the next-stage decimated Nyquist frequency \( f'_s < f_s \). In order to decimate the sequence while minimizing any additional signal aliasing, \( h[n] \) must be processed with a lowpass filter, \( g[n] \), prior to the decimation operation. This digital filter must have a stopband cutoff frequency \( f_{\text{ab}} \) below the Nyquist frequency \( f'_s \) of the subsequent decimation stage. This digital filter is applied to the original (undecimated) sequence \( h[n] \). Note that the passband cutoff frequency \( f_{\text{ap}} \) (see Fig. 7) of the filter must be greater than the desired baseband, and that the ripple in the filter passband must be kept as small as possible to prevent degradation of the desired signal content.

Decimation is then achieved by keeping every \( M \) th sample from the filtered output sequence. These steps are illustrated in Fig. 6(a). In this figure, \( x[n] \) is the lowpass filtered version of the input signal \( h[n] \). The decimator block (also known as a compressor) is denoted by \( \downarrow M \), and its output \( h'[n] \) is the desired decimated sequence.

To provide a numerical illustration, the velocity potential function at some observation point is discretized with an initial sampling frequency \( f_s \) equal to 4 GHz. The corresponding Nyquist frequency (2 GHz) is much greater than...
our maximum frequency of interest (assumed to be approximately 15 MHz), and we may therefore decimate this sequence by a theoretical maximum value of \( M \approx 133 \). By appropriate filtering and downsampling of the data, we may reduce the required sequence length to \( N/133 \), as long as these operations do not affect the magnitude or phase of the pressure field over the frequencies of interest. This decimation operation may be split into several separate decimation stages, each stage decimating by an integer factor of the total desired decimation ratio \( M \), to yield a more efficient multi-stage decimation algorithm.\(^{19}\)

A brief look at the signal processing implementation of the decimator structure is worthwhile at this point. A straightforward implementation is shown in Fig. 6(a). In this structure, the input is sampled at a frequency \( f_s \), and \( M - 1 \) out of every \( M \) samples of the filter output are discarded by the \( M \)-to-1 sampling rate compressor. Figure 6(b) shows a similar structure but assumes an \( L \)-point FIR filter, \( g[n] \), realized in direct form. This FIR filter is designed to have a lowpass characteristic, with a stopband cutoff frequency \( f_{sb} \) to prevent aliasing of the decimated output. If less than the entire nonaliased baseband \( \Delta f = f_s - f_{sb} \) is required, then the computational load of the digital filtering

operation may be decreased somewhat by allowing the filter cutoff to be greater than \( f_s/2 \). Since the number of coefficients required for a lowpass digital filter becomes smaller as the transition bandwidth (from passband to stopband edges) of the filter increases, then increasing the stopband cutoff frequency (while maintaining a constant passband cutoff frequency) widens the transition bandwidth and yields a shorter length filter. This ultimately leads to a faster filter computation. To determine how wide this transition band can be made, we note that for some desired baseband spectrum whose upper frequency limit is \( f_{bb} \), the digital filter cutoff may be selected such that

\[
f_{sb} < f_s/2 = f_s/M - f_{bb},
\]

where \( f_{bb} \) is the stopband frequency of the lowpass decimator filter. A graphical interpretation of (38) is given in Fig. 7, and indicates that no aliasing will occur in the desired baseband frequency range \( f_c \) to \( f_{bb} \) only if no frequency components exist in the frequency range above \( f_s/2 - f_{bb} \) prior to decimating the sequence.

Note that \( L \) multiplications are required to calculate each output sample in the filter implementation of Fig. 6(b). A better realization of this filter is demonstrated in Fig. 6(c), where the symmetry found in \( g[n] \) for linear phase FIR filter designs is exploited. This structure achieves a reduction in the number of required multiplications by a factor of two over the structure in Fig. 6(b). [In particular, Fig. 6(c) assumes \( L \) to be even.] Even greater efficiency can be achieved by commuting the decimation stage of Fig. 6(c).
into each of the filter branches; the symmetry in \( g[n] \) is still exploited for linear phase designs. This realization is shown in Fig. 6(d). In this final implementation, the filter operates on the decimated data at a rate reduced by an additional factor of \( M \); therefore, this reduces the number of multiplications required to implement the decimator by a factor of \( M \) as well. In total, the decimator given in Fig. 6(d) reduces the number of required multiplications per output sample by a factor of \( 2M \) over the simple approach taken in Fig. 6(b). Note that \( M \) can be unity for any given stage of the decimation under certain circumstances. More on structures for FIR decimators can be found in Crochiere and Rabiner (Ref. 19, Sec. 3.3).

V. MULTIRATE ALGORITHM

The specific multirate algorithm employed to compute the broadband pressure field will now be discussed. No mention has yet been made as to the “multirate” nature of the present algorithm; “multirate” is used here to indicate that the discretization of \( h(r,t) \) occurs at a variable sampling rate, based specifically on spatial coordinate \( r \); this section will explain this procedure in detail. These changes in sampling frequencies will be used to obtain further computational savings.

An outline of the algorithm is described below.

- Discretization of \( h(r,t) \) into \( h[n] \).
- Decimation of \( h[n] \) into \( h'[n] \) to minimize number of samples.
- Frequency transformation of \( h'[n] \) into \( H[k] \).
- Differentiation of \( H[k] \) to yield complex acoustic pressure (times a scale factor).

At this point it may be mentioned that the technique of decimation of data followed by Fourier transformation is sometimes referred to as a zoom FFT.

Throughout the following discussions, a planar piston transducer will be assumed whenever specific velocity potential calculations are referenced. The potential function \( h_{pp}(r,t) \) will thus be indicated in these situations. Note that the subsequent developments are applicable as well to both the annular array and spherically focused piston with only slight changes in notation. Therefore, whenever a statement applies globally to all transducer geometries (with known velocity potential functions), then the more general \( h(r,t) \) without subscript will be used.

**A. Sampling and decimator design**

The first step indicated in the outline is the discretization of the velocity potential. In order to accomplish this, the velocity potential function must be calculated at the sample points. The temporal length of the function at the observation point \( r \) is calculated in order to determine the highest achievable sampling rate, given a maximum sequence length \( N \) that is constrained by the limits of the particular computer used. Note that we still must adhere to the minimum sampling frequency, as it describes the maximum sampling frequency obtainable at the specific observation point. The local sampling frequency is thus

\[
f_{sr} = N / \Delta t.
\]

However, the actual discretization of \( h_{pp}(r,t) \) in \( h[n] \) is performed at a global sampling frequency \( f_{sg} \), which is the largest frequency less than or equal to \( f_{sr} \) conforming to one of several predetermined algorithm sampling rates. Each global sampling rate is an even multiple of the lowest possible \( f_{sm} \), and as such allows a more efficient multistage down-sampling scheme to be employed. Furthermore, limiting the global sampling frequency to factors of two of the lowest possible sampling rate will give us identical frequency domain resolutions when \( H[k] \) is generated, as will be seen later. This is beneficial when using the broadband acoustic pressure field results in a system analysis, where identical analysis frequencies and intervals are often required. The particular translation from \( f_{sr} \) to \( f_{sg} \) used in this algorithm is given in Table I.

The shortest temporal velocity potential functions for a given transducer (e.g., those functions corresponding to field points close to the acoustic axis and far from the source) will allow the highest sampling frequencies \( f_{sr} \). It should be noted that some minimum global sampling frequency \( f_{sm} \) has been specified for the maximum acceptable aliasing error. This implies that at certain field points and for

<table>
<thead>
<tr>
<th>Possible local sampling rates ( f_{sr} ) (GHz)</th>
<th>Corresponding global sampling rate ( f_{sg} ) (GHz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_{sr} &gt; 8 )</td>
<td>8</td>
</tr>
<tr>
<td>( 8 &gt; f_{sr} &gt; 4 )</td>
<td>4</td>
</tr>
<tr>
<td>( 4 &gt; f_{sr} &gt; 2 )</td>
<td>2</td>
</tr>
<tr>
<td>( 2 &gt; f_{sr} &gt; 1 )</td>
<td>1</td>
</tr>
<tr>
<td>( 1 &gt; f_{sr} &gt; 0.5 )</td>
<td>( 0.5 = f_{sm} )</td>
</tr>
<tr>
<td>( 0.5 &gt; f_{sr} )</td>
<td>Invalid</td>
</tr>
</tbody>
</table>

D. P. Orofino and P. C. Pedersen: Complex acoustic fields
large transducers, there may be \( h_{pp}(r,t) \) whose durations are too long to be completely sampled by this algorithm; regions where this condition exists are termed invalid sampling regions.

To determine exactly when an observation point is in an invalid sampling region, we note that this limit is determined by selection of \( N \) and \( f_{sm} \). Given these parameters, we may derive the following relations for the valid sampling regions for planar piston and annular array transducers. For focused transducers, these relationships hold approximately true:

\[
f_{sm} < \frac{N}{\Delta t} = \frac{N}{t_2 - t_1}. \tag{41}
\]

Note that from (39), \( t_n \) is either \( t_0 \) or \( t_1 \), depending upon whether \( \rho_q < a \) or \( \rho_q > a \), respectively. If \( \rho_q > a \), which can be shown to yield the largest \( \Delta t \) for a given transducer, then for some axial distance \( z_q \),

\[
f_{sm} < \frac{N}{(t_2 - t_1)}, \tag{42}
\]

or

\[
\frac{Nc}{f_{sm}} > \sqrt{(\rho_q + a)^2 + z_q^2} - \sqrt{(\rho_q - a)^2 + z_q^2}. \tag{43}
\]

Combinations of \( \rho \) and \( z \) may be checked against the relation in (43) to determine whether an observation point is in a computable region. Similarly, if \( \rho_q < a \), we see that

\[
\frac{Nc}{f_{sm}} > \sqrt{(\rho_q + a)^2 + z_q^2} - \sqrt{(\rho_q - a)^2 + z_q^2}. \tag{44}
\]

Spatial boundary contours specifying the limits of validity for the sampling technique are given in Fig. 8(a) and (b), where data for planar transducers of radius 1.27 and 2.54 cm, respectively, are given for various values of \( N \). These data were determined by plotting the surface contours of (43) and (44) based on \( f_c = f_{sm} = 0.5 \text{ GHz} \). Note that for a planar piston transducer with a given radius \( a \), the maximum \( \Delta t \) anywhere is

\[
\Delta t_{\text{max}} = 2a/c. \tag{45}
\]

Thus, if

\[
\frac{Nc}{f_{sm}} > 2a \tag{46}
\]

or

\[
a < \frac{Nc}{2f_{sm}}, \tag{47}
\]

then no invalid sampling regions exist. For \( N = 16384 \), \( c = 1500 \text{ m/s} \), and \( f_{sm} = 500 \text{ MHz} \), the invalid sampling regions vanish completely for transducers of radius \( a < 2.46 \text{ cm} \) (i.e., approximately a 2-in.-diam transducer).

**B. Decimation of velocity potentials**

Once \( h(r,t) \) has been discretized, the digital filtering and decimation procedure discussed in Sec. IV may be applied. A two-stage decimator design will be used as a trade-off between overall computational efficiency and total storage requirements.\(^{19}\)

The parameters of the two-stage decimator are shown in Table II. Generally, a decimator of this type would require two lowpass filters for each of the possible sampling rates

\[
\begin{array}{cccccccc}
\text{Global sampling rate} & \text{First stage} & \text{Second stage} \\
& \text{Filter} & \text{Decimator} & \text{Filter} & \text{Decimator} \\
& f_{sa} & f_{se} & f_{sm} & f_{s1} & M_1 & f_{s1} & f_{s2} & f_{s2} \\
8000 & 4000 & 256 & 440 & 16 & 250 & 16 & 27.5 & 8 & 31.25 \\
4000 & 2000 & 128 & 220 & 8 & 250 & 16 & 27.5 & 8 & 31.25 \\
2000 & 1000 & 64 & 110 & 4 & 250 & 16 & 27.5 & 8 & 31.25 \\
1000 & 500 & 32 & 55 & 2 & 250 & 16 & 27.5 & 8 & 31.25 \\
500 & 250 & 16 & 27.5 & 1 & 250 & 16 & 27.5 & 8 & 31.25
\end{array}
\]

\[^{19}\] D. P. Orofino and P. C. Pedersen: Complex acoustic fields
(one filter for each stage). For efficiency of design and computational speed, the final filtering and downsampling rates are to be maintained identical to one another over all possible sampling frequencies. Given the five possible global sampling rates, this would stipulate the design and implementation of six digital filters (since the second stage decimators will all be identical), consuming considerable storage for the filter coefficients. The control logic necessary to implement this number of filters would also be cumbersome in the final design. If we constrain the design to use just a single digital filter, this would reduce the storage requirements of the algorithm and at the same time create a more efficient decimator section. This constraint is possible to achieve along with the previous restriction in global sampling frequencies because a lowpass digital filter having a given set of normalized design parameters will exhibit various passband and stopband cutoff frequencies by adjusting the sampling frequency of its input sequence.

The first and second stage downsample ratios were manipulated until a satisfactory design was achieved. This design allows the same filter to be used for the second stage with sufficient nonaliased bandwidth on the output of the filter for our broadband ultrasonic applications. Table III gives the parameters of a lowpass digital filter capable of performing the necessary filtering required for the two-stage decimator. As discussed in Sec. IV, this filter minimizes the aliasing introduced by the decimation operation. The normalized passband and stopband cutoff frequencies are denoted by \( \Omega_{bp} \) and \( \Omega_{sb} \), respectively.

The values of possible global sampling frequencies \( f_{sg} \) are given in column 1 of Table II, as well as their corresponding Nyquist frequencies \( f_{nyq} \) in column 2. Column 3 gives the first-stage filter passband cutoff frequencies \( f_{pb1} \) obtained by multiplying the normalized filter parameter \( \Omega_{bp} \) by \( f_{nyq} \). Column 4 gives the filter stopband frequencies \( f_{sb1} \) calculated in a similar manner. Column 5 shows the first-stage downsampling ratios (\( M_1 \)), which yield identical first-stage output Nyquist frequencies \( f_{nyq1} \) given in column 6. These are calculated by dividing \( f_{nyq} \) by \( M_1 \). All second stage parameters are derived similarly, but use the first stage decimated sampling frequency \( f_{s1} = 2f_{s0} \) as the input sampling rate.

The determination of the bounds for the normalized filter cutoff frequencies is as follows. Equation (38) constrains the filter stopband cutoff frequency to be less than the maximum sampling frequency into any decimator stage; thus, the most restrictive constraint imposed by (38) may be written as

\[
\Omega_{sb} < \frac{1}{M_1} \frac{f_{sb}}{f_{nyq}}, \tag{48}
\]

<table>
<thead>
<tr>
<th>Cutoff frequency</th>
<th>( \Omega_{sb} = 0.0325 )</th>
<th>( \Omega_{sb} = 0.055 )</th>
<th>( L = 130 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ripple ( r_{sb} )</td>
<td>0.01</td>
<td>0.001</td>
<td></td>
</tr>
</tbody>
</table>

| TABLE III. Parameters of unity-normalized lowpass filter, designed using a linear-phase Parks–McClellan algorithm. |

The limit imposed by (50) is found to be \( \Omega_{sb} > (15.625)/(500) = 0.03125 \). For maximum computational efficiency, we would like to make \( \Omega_{sb} \) as small as possible while keeping it greater than 0.03125. It would be advantageous to select the widest possible transition band (\( \Omega_{sb} - \Omega_{pb} \)) that achieves the previous constraints, in order to generate the shortest possible filter (see Sec. IV). Basing the design on the Parks–McClellan algorithm, a lowpass linear phase digital filter (LPDF) was constructed with a stopband attenuation level and passband ripple as given in Table III. The passband and stopband frequencies meet the constraints stated in (48) and (50). The ripple parameters yield at least 60 dB[ = 20 log(\( r_{sb} \)) ] stopband attenuation and a passband ripple of maximally \( \pm 0.04 \text{dB} = 20 \text{log}(1 - r_{pb}) \).

Such a filter enables the use of a faster decimation algorithm because we may exploit the symmetrical layout of the linear phase filter, mentioned at the end of Sec. IV. Moreover, since the complex pressure function is required, we must maintain either a zero-phase characteristic (e.g., a noncausal filter implementation), or be prepared to compensate for the nonzero phase relationship over all components of the desired baseband frequency spectrum. The linear phase of the filter makes this compensation straightforward.

The second stage of the decimator has specifications similar to the first stage. Recall that the same digital filter as used for the first stage is used again here. To yield a minimum number of samples in our final decimated output, we set the second stage decimation ratio \( M_2 \) (Table II, column 9) to the largest value that will still allow the filter specifications to meet the criteria in (48).

The second stage filter cutoff values are given in columns 7 and 8, based on the same filter used for the first stage. Note that the selection of first stage decimation ratios yields identical first stage downsampling Nyquist rates \( f_{nyq1} \), column 6), and therefore the second stage decimation factor is constant \( \Omega_{sb} \) for all given global sampling rates. Dividing the first stage downsampling Nyquist rates \( f_{nyq1} \) by the
second stage downsample ratio $M_2$ yields the second stage downsample Nyquist rate $f_{s2}$ as given in column 10.

Selection of global sampling rates is primarily based upon the acceptable level of aliasing. When the maximum number of samples capable of being processed and stored on a given computer is specified, the bounds on the sampling regions are thus defined. In general, variations on the sampling scheme may be used either to extend the valid sampling regions, or, along with appropriate downsampling changes, to increase the global sampling rates to achieve greater accuracy in the final results.

It is possible to implement the two-stage decimator as a cascade of two separate decimator sections, instead of reusing the same decimator section for both stages. This implementation will increase the length of the algorithm, but does not require storage of all $N$ samples of the velocity potential function at one time. The samples of the velocity potentials may be moved directly into the cascaded decimator structure, which at a minimum decimates the sequence by an overall factor of 8. Therefore storage of only $N/8$ points can be afforded by this implementation. Note that if the two decimator stages are not cascaded, then this benefit is lost because the first decimator stage may decimate by a unity factor under some circumstances.

C. Transformation, compensation, and differentiation

The decimated velocity potential impulse response function $h'[n]$ must be transformed into the frequency domain, as noted earlier in (7) and (8). The most efficient way of accomplishing the frequency transformation is via the fast Fourier transform, or FFT. The implementation of the FFT used here requires a sequence $h'[n]$ that is some power of 2 in length. The downsampled sequence lengths are not necessarily integer powers of two, and thus some amount of zero padding must be performed.

For the specific decimation algorithm given in Table II, and with $N = 16384$ samples maximum, no decimated sequence exceeds $2^{11}$ or 2048 samples in length. We therefore choose to zero pad all decimated sequences suitably in order to employ a 2048-point FFT. Since the velocity potential is a real valued function, a real-valued FFT is implemented for each frequency $k = 0, 1, ..., N - 1$, by multiplication, as indicated in (7). Given $H_c[k]$, then this is accomplished and multiplied by the propagating medium density $\rho_p$ to determine the amount of compensation necessary at each baseband frequency $f$ of interest. Note that the group delay for a causal LPDF is positive, and when divided by the sampling rate may be added to the previous delay term $t_{k,start}$. The total amount of phase compensation at a given frequency $f$ is thus

$$2\pi f(t_{k,start} + T_c/f_g)$$

and should be added to the phase terms of the frequency domain data once the FFT has been performed. The phase-compensated velocity potential function is denoted by $\tilde{h}_c[k]$ in the frequency domain.

To determine the discrete frequency domain pressure field function $\tilde{P}[k]$ (where dependence on spatial location $r$ is implied), the velocity potential function must be differentiated and multiplied by the propagating medium density $\rho_p$ as indicated in (7). Given $\tilde{h}_c[k]$, then this is accomplished for each frequency $k = 0, 1, ..., N - 1$, by multiplication,

$$\tilde{P}[k] = j\rho_p(2\pi f k / N)\tilde{h}_c[k].$$

VI. PERFORMANCE EVALUATION

The multirate algorithm was implemented on both PC and workstation computing environments. Several representative results from pressure field simulations are presented in this section for the spherically focused and nondiffracting (annular array) transducers. Error analyses for the planar piston and spherically focused piston transducers are presented by means of an exact (closed-form) axial solution to the pressure field, determining absolute error in both magnitude and phase. To determine the computational efficiency of the multirate algorithm, a comparison of execution time to a commonly used Gaussian quadrature numerical integration technique was made. The relative efficiency was computed for several transducer geometries and field point observation regions. The relative pressure field error between the multirate and numerical integration algorithm is presented.
A. Results from the multirate algorithm

1. Spherical focused piston

The multirate algorithm was used to generate the pressure field for a spherically focused transducer with parameters $R/a = \sqrt{2}$, $a/\lambda = 2.5$, and $a/\lambda = 5$; the magnitude pressure fields for each case of $a/\lambda$ are shown in Fig. 9(a) and (b), respectively, with $c = 1500$ m/s. Note that the observation region spans $0 < \rho < 1.5R$ and $0 < \varphi < 0.4R$ in both cases. The focusing effect on the magnitude acoustic pressure field can be clearly seen along the axis in each case. Note that the fields have their peak magnitude pressures on-axis at $z_q < R$. In general, the maximum pressure magnitude occurs not at the geometrical focus, $R$, but at some effective focal point $R_{\text{eff}}$ located to the source side of the geometrical focus (Ref. 21, p. 193). For a fixed ratio $R/a$, the maximum pressure occurs closest to the geometrical focus ($R_{\text{eff}} \approx R$) when the quantity $a/\lambda$ is large (strongly focused piston, large aperture), and occurs much closer to the source transducer ($R_{\text{eff}} < R$) when $a/\lambda$ is small (weakly focused piston, small aperture).

2. Jo-Bessel transducer

Numerical simulations for a nondiffracting transducer were performed using the multirate algorithm, by implementing the transducer using an annular array. The theoretical development and annular array implementation of the nondiffracting transducer have been described in recent literature.\textsuperscript{15,16} The nondiffracting transducer requires a continuous $J_0$-Bessel apodization distribution over a planar piston. In order to implement this transducer as a simple planar annular array with constant voltage excitation per annulus, an approximation to the Bessel apodization is made. The most common approximation made is to choose the annulus radii $a_i$ to be the zeros of the $J_0$ function, such that each annulus spans a single lobe of the $J_0$ function. Furthermore, the excitation voltage $V_i$ is chosen to be the maximum amplitude of the lobe of the $J_0$ function that the $i$th annulus spans. This results in a "squared-off" approximation to the original continuous apodization function, but for a large number of annuli, introduces only a limited amount of noise in the extreme nearfield of the nondiffracting array transducer.

In order to discretize the velocity potential function for the annular array, it has been previously shown in (16) that a total of $N_a$ planar piston velocity potential functions $h_{\text{pp}}^i(t,t)$ are required. These velocity potentials must be individually scaled by a factor $V_i - V_{i+1}$, discretized, and summed together. Each function in the sum has associated with it a radius $a_i$, corresponding to the outer radius of the $i$th annulus in the array. From (10) it can be seen that the time epochs $t_1$ and $t_2$ will be different for each function $h_{\text{pp}}^i(t,t)$. Furthermore, the time epoch $t_0$ (which is constant for all velocity potentials describing the array annuli) may not mark the beginning of the (nonzero) portion of any single velocity potential, since from (11) and (12) it can be deduced that the temporal length $\Delta t$ of any given $h_{\text{pp}}^i(t,t)$ is $\Delta t = t_2 - t_0$ if $\rho < a_i$, and $\Delta t = t_0 - t_1$ if $\rho > a_i$. Therefore, a calculation must be made to determine the starting time delay for the velocity potential discretization array $h[n]$ before sampling of each function $h_{\text{pp}}^i(t,t)$ can be performed.

A simulated transducer array of radius $a = 50.75$ mm was used, and the number of annuli in the array was chosen to be $N_a = 51$ for comparison to published results.\textsuperscript{16} The annulus radii $a_i$ and excitation voltages $V_i$ are found from the zeros and extrema of the $J_0$ Bessel function, $J_0(\alpha p)$, respectively, where $p$ is radial distance measured from the array center, and $\alpha$ is a compression factor that maps the Bessel array-apodization function to the actual transducer annulus radii. The compression $\alpha$ is expressed in units of m$^{-1}$. For a transducer with infinite aperture, i.e., $a \to \infty$, the beam does not undergo any diffraction effects. However, for a practical finite aperture transducer, the beam eventually diffracts at some axial distance $z_{\text{max}}$. Using geometrical op-
tics, it can be shown [Ref. 14, Eq. (5)] that a useful approximation to the axial extent of the diffraction-free region of the transducer is

$$z_{\text{max}} = a\sqrt{(2\pi/\alpha \lambda)^2 - 1}.$$  \hspace{1cm} (53)

The tradeoff in selection of small $\alpha$ values is that small values of $\alpha$ tend to increase the amount of spurious sidelobe generation of the array. Thus, one must trade axial length of the diffraction-free region of the beam for reasonable suppression of sidelobe production.

This tradeoff can be seen in the magnitude pressure fields shown in Fig. 10(a) (for $\alpha = 3141.5 \text{ m}^{-1}$) and (b) (where $\alpha = 6283.2 \text{ m}^{-1}$). Adjustment of $\alpha$ is here accomplished simply by changing the radius of the transducer, and allows the number of annuli in the array to remain fixed at $N_a = 51$. For the simulation performed in Fig. 10(a), the calculated length of the diffraction-free region according to (53) is $z_{\text{max}} \approx 33.5 \text{ cm}$; note that the axial magnitude pressure field quickly drops off after this point. This figure agrees with those published in the literature. For the simulation performed in Fig. 10(b), the transducer radius was halved to 25.375 mm, thus increasing $\alpha$ by a factor of two as well. The length of the diffraction-free region is now $z_{\text{max}} \approx 8.1 \text{ cm}$; the sudden drop off in the axial magnitude pressure field verifies this result. Note that the sidelobes generated in the nearfield of the transducer in Fig. 10(b) are much smaller than those seen in Fig. 10(a); this evidence supports the claim previously made regarding the tradeoff of diffraction-free region and nearfield sidelobe production.

**B. Error analysis**

1. **Absolute error on axis**

An absolute error analysis can be made by use of the exact closed form solutions to the pressure field equation when field points are located on the axis of symmetry. Therefore we will explore the errors in both the planar and focused pressure algorithms for the case of the fields along the axis.

For the planar transducer, the exact pressure field on axis ($\rho = 0$) is given in Kinsler and Frey to be

$$P(\rho, z) = \rho_0 c e^{-jkz} - e^{-jk/((\sqrt{\rho^2 + a^2})^2)},$$  \hspace{1cm} (54)

where the transducer surface velocity has been set to unity, and harmonic dependence on $t$ has been dropped for simplicity. The pressure magnitude and phase on axis are therefore

$$|P(\rho, z)| = 2\rho_0 c \sin(kz \sqrt{1 + (a/z)^2 - 1}),$$  \hspace{1cm} (55a)

$$\angle P(\rho, z) = -\tan^{-1}\left(\frac{\sin(kz) - \sin(k \sqrt{\rho^2 + a^2})}{\cos(kz) - \cos(k \sqrt{\rho^2 + a^2})}\right).$$  \hspace{1cm} (55b)

These equations were programmed and compared to the results of axial pressure field calculations from the multirate technique. The exact solutions to the magnitude and phase axial pressure are given in Fig. 11(a) and (b). Figure 11(c) and (d) show the result of comparing the multirate technique to (55a) and (55b), respectively. The magnitude pressure fields were independently normalized such that the maximum pressure magnitude in either result was unity before taking the difference to obtain Fig. 11(c). A total of 500 points along the axis of symmetry were evaluated, for the range $0 < z < 1.5a^2/\lambda$ and $a/\lambda = 2.5$. The small amount of error in these results indicates that the multirate technique yields very accurate results. The phase error can be observed to vary in a steplike fashion with $z$, correspond to two different sampling rates, $f_{sg} = 4 \text{ GHz}$ (for $z < 0.15a^2/\lambda$) and 8 GHz.

To determine the on-axis response for the focused transducer, we follow Luukkala and Penttinen. The integral form of the pressure field, as expressed in (7)-(9), is evaluated using the velocity potential function $h(r,t)$ given in (20), for field points on axis. For these field points, $t_2 = t_3$ and the impulse response is nonzero only in the time interval $t_1$ to $t_2$. From (20), the impulse response is deduced to be a constant, and using (7)-(9) it can be shown that

![FIG. 10. Magnitude pressure field for nondiffracting transducer implemented as a planar annular array. There are $N_a = 51$ annuli in the array, (a) $a = 50.75 \text{ mm}, \alpha = 3141.5 \text{ m}^{-1}$, and (b) $a = 25.375 \text{ mm}, \alpha = 6283.2 \text{ m}^{-1}$.](image)
FIG. 11. Comparison of exact axial pressure solution due to planar piston transducer to corresponding multirate solution. (a) Exact pressure magnitude and (b) exact pressure phase for $0 < z < 1.5 a^2/\lambda$, $a/\lambda = 2.5$, and $c = 1500$ m/s. Absolute error between multirate and exact solution is given in (c) for normalized magnitude and in (d) for phase terms, respectively.

$$
\bar{P}(0,z_0,\omega) = -j \omega \rho_0 \frac{RC}{(R - z_q)} \int_{t_0}^{t_1} e^{-j\omega t} dt
$$

where

$$
s = \sqrt{a^2 + (z_q - d_q)^2} + z_q,
$$

$$
\delta = \sqrt{a^2 + (z_q - d_q)^2} - z_q,
$$

$$
= j \omega \rho_0 e^{-j\omega \delta / \sqrt{2}} e^{-j\omega \delta / \sqrt{2}},
$$

and (b). The closed form solution of the pressure field as given in (56) and (57) is compared to the multirate technique. Figure 12(c) and (d) show the result of comparing the individual magnitude and phase components of (56) to the multirate technique. As before, the magnitude pressure fields were normalized to 1.0 before taking the difference to obtain Fig. 12(c). A total of 500 points along the axis of symmetry were evaluated.

The large spike in the error curve occurs in the vicinity of the focal point, where the velocity potential functions approach an impulse function in the time domain. The velocity potentials introduce a detectable amount of aliasing in the frequency domain, manifesting itself as a pressure field error in the immediate vicinity of the focal point. As with the planar piston transducer, the effect of global sampling frequency can be observed in the phase error curves in Fig. 12(d), where $f_s = 2$ GHz for $z < 0.25R$, 4 GHz for $0.25R < z < 0.5R$, and 8 GHz for the remainder of the axial range.
The result of the multirate pressure field calculator is next compared to another known algorithm to judge its speed and accuracy. A suitably chosen reference is a numerical integration method, implementing the Fourier transform indicated in (7)-(9). The integral in both of these cases is in general
\[ \tilde{P}(r, \omega) = j \omega \rho_0 \tilde{H}(r, \omega) \]
\[ = j \omega \rho_0 \int_{t_s}^{t_b} h(r, t) e^{-j \omega t} dt, \]  
where \( h(r, t) \) is the appropriate velocity potential for a given transducer. Note that the transform needs to be integrated only from \( t_s \) to \( t_b \), which determines the temporal extent of the \( h(r, t) \).

The choice of a specific integration algorithm was motivated by the need for high accuracy with a fast computation time, as well as selecting a reference algorithm that is consistent with the other comparisons made in related literature. The velocity potential functions to be integrated are generally made up of a smooth, continuous interval, along with either a flattop section near axial points of the planar transducer, or an asymptotic region near the focal point of the focused transducer. If the ability to integrate polynomials of highest possible order is used as a criterion of performance, then the Gaussian quadrature numerical integration technique is the most appropriate choice. 23 Since the number of quadrature points is directly related to the order of polynomial integration, the highest order of quadrature readily available was chosen, this being a 96-point Gaussian quadrature integration. 23 This Gaussian quadrature integration al-

2. Relative error over a plane

The result of the multirate pressure field calculator is next compared to another known algorithm to judge its speed and accuracy. A suitably chosen reference is a numerical integration method, implementing the Fourier transform indicated in (7)-(9). The integral in both of these cases is in general

\[ \tilde{P}(r, \omega) = j \omega \rho_0 \tilde{H}(r, \omega) \]
\[ = j \omega \rho_0 \int_{t_s}^{t_b} h(r, t) e^{-j \omega t} dt, \]  
where \( h(r, t) \) is the appropriate velocity potential for a given transducer. Note that the transform needs to be integrated only from \( t_s \) to \( t_b \), which determines the temporal extent of the \( h(r, t) \).
TABLE V. Improvement in calculation time of the multirate technique versus the optimized numerical integration. Time speedup is for difference in exec. time as defined in Table IV.

<table>
<thead>
<tr>
<th>Speed improvement</th>
<th>Planar</th>
<th>Focused</th>
</tr>
</thead>
<tbody>
<tr>
<td>a/A</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>15.8</td>
<td>33.0</td>
</tr>
<tr>
<td>2.5</td>
<td>16.1</td>
<td>24.6</td>
</tr>
<tr>
<td>5.0</td>
<td>17.2</td>
<td>23.9</td>
</tr>
<tr>
<td>10.0</td>
<td>19.8</td>
<td>17.8</td>
</tr>
</tbody>
</table>

velocity potential functions \( h_{np}(r,t) \) approach an impulse function in the time domain, and introduce a measurable level of aliasing in the frequency domain.

C. Calculation speed comparisons

The calculation times of the multirate algorithm are given in Table IV below. The calculation times of the numerical integration technique for the same observation regions are also given in Table IV below for relative comparison. The total run time is the time it takes to calculate the complex acoustic pressure for an entire 50 × 50 grid of field points. For the multirate technique, 512 discrete frequencies are calculated with each run of the algorithm, whereas only a single frequency is calculated by the numerical integration technique; therefore, the total run time results must be scaled accordingly for a comparison to be made. The execution time takes this into account by multiplying the total run time for the numerical integration technique by 512. Execution time also divides both the multirate and numerical integration calculation times by the total number of field points at which the pressure is calculated, namely 50 × 50 or 2500 field points in each case. This averages the computation time over the field so that a reasonable per-field-point calculation time can be found.

Table V shows the improvement in calculation time of the multirate technique over the numerical integration algorithm. The improvement is calculated by dividing the execution time of the multirate technique by the execution time of the appropriate numerical integration technique. The improvement in calculation speed over numerical integration varied from \( \times 15 \) to \( \times 33 \), depending upon transducer geometry and aperture width \( a/\lambda \).

VII. CONCLUSIONS

In this paper, an efficient technique for generating broadband complex acoustic pressure field data has been presented. This technique, based on a multirate, multistage digital decimator structure, is shown to be capable of calculating the pressure fields due to planar, annular, and spherically focused piston transducers. The accuracy of pressure field data is comparable to that of other standard numerical techniques, such as Gaussian quadrature numerical integration, and affords a computation efficiency of up to \( \times 30 \) greater than these more traditional processing techniques.


4. Reference 2, Sec. 8.2; see also Ref. 3, Sec. 4.3.


