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Monochromatic Hamiltonian $t$-tight Berge-cycles in hypergraphs

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Abstract

In any $r$-uniform hypergraph $H$ for $2 \leq t \leq r$ we define an $r$-uniform $t$-tight Berge-cycle of length $\ell$, denoted by $C^{(r)}_{\ell}$, as a sequence of distinct vertices $v_1, v_2, \ldots, v_\ell$, such that for each set $(v_i, v_{i+1}, \ldots, v_{i+t-1})$ of $t$ consecutive vertices on the cycle, there is an edge $E_i$ of $H$ that contains these $t$ vertices and the edges $E_i$ are all distinct for $1 \leq i \leq \ell$ where $\ell + j \equiv j$. For $t = 2$ we get the classical Berge-cycle and for $t = r$ we get the so-called tight cycle. In this note we formulate the following conjecture. For any fixed $2 \leq c, t \leq r$ satisfying $c + t \leq r + 1$ and sufficiently large $n$, if we color the edges of $K^{(r)}_n$, the complete $r$-uniform hypergraph on $n$ vertices, with $c$ colors, then there is a monochromatic Hamiltonian $t$-tight Berge-cycle. We prove some partial results about this conjecture and we show that if true the conjecture is best possible.

1 Introduction

The investigations of Turán type problems for paths and cycles of graphs were started by Erdős and Gallai in [3]. The corresponding Ramsey problems have

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been looked at some years later first in [6] and then later in [4], [5], [8], [12] and [14].

There are several possibilities to define paths and cycles in hypergraphs. In this paper we address the case of the Berge-cycle; probably it is the earliest definition of a cycle in hypergraphs in the book of Berge [1]. Turán type problems for Berge-paths and Berge-cycles of hypergraphs appeared perhaps first in [2]. Other types of hypergraph cycles, loose and tight, have been studied in [11], [13] and [15]. The investigations of the corresponding Ramsey problems started quite recently with [9] and [10] where Ramsey numbers of loose and tight cycles have been determined asymptotically for two colors and for 3-uniform hypergraphs.

Let $H$ be an $r$-uniform hypergraph (some $r$-element subsets of a set). Let $K_r^{(r)} n$ denote the complete $r$-uniform hypergraph on $n$ vertices. In any $r$-uniform hypergraph $H$ for $2 \leq t \leq r$ we define an $r$-uniform $t$-tight Berge-cycle of length $\ell$, denoted by $C^{(r,t)}_\ell$, as a sequence of distinct vertices $v_1, v_2, \ldots, v_\ell$, such that for each set $(v_i, v_{i+1}, \ldots, v_{i+t-1})$ of $t$ consecutive vertices on the cycle, there is an edge $E_i$ of $H$ that contains these $t$ vertices and the edges $E_i$ are all distinct for $i, 1 \leq i \leq \ell$ where $\ell + j \equiv j$. We will denote by $E(C^{(r,t)}_\ell)$ the set of these edges $E_i$ used on the cycle. For $t = 2$ we get Berge-cycles and for $t = r$ we get the tight cycle. When the uniformity is clearly understood we may simply write $C^{(t)}_\ell$ for $C^{(r,t)}_\ell$ or just $C_\ell$. $R_c(C^{(r,t)}_\ell)$ will denote the Ramsey number of the $r$-uniform $t$-tight $\ell$ cycle using $c$ colors. A Berge-cycle of length $n$ in a hypergraph of $n$ vertices is called a Hamiltonian Berge-cycle. It is important to keep in mind that, in contrast to the case $r = t = 2$, for $r > t \geq 2$ a Berge-cycle $C^{(r,t)}_\ell$, is not determined uniquely, it is considered as an arbitrary choice from many possible cycles with the same triple of parameters.

In this note, continuing the investigations from [7], we study Hamiltonian Berge-cycles in hypergraphs. Thinking in terms of graphs, such an attempt seems strange, since in many 2-coloring of $K_n$ there are no monochromatic Hamiltonian cycles. For example, if each edge incident to a fixed vertex is red and the other edges are blue, there is no monochromatic Hamiltonian cycle. However, from the nature of Berge-cycles, this example does not carry over to hypergraphs, if $K^{(3)} n$ is colored this way, there is a red Hamiltonian Berge-cycle (for $n \geq 5$).

In [7] monochromatic Hamiltonian (2-tight) Berge-cycles were studied and the following conjecture was formulated. Assume that $r > 1$ is fixed and $n$ is sufficiently large. Then every $(r-1)$-coloring of $K^{(r)} n$ contains a monochromatic Hamiltonian (2-tight) Berge-cycle. The conjecture was proved for $r = 3$. For general $r$, the statement was proved for sufficiently large $n$ with $\lfloor \frac{r+1}{2} \rfloor$ colors instead of $r-1$ colors. In this note we look at monochromatic Hamiltonian $t$-tight Berge-cycles and we generalize the above conjecture in the following way.

**Conjecture 1.** For any fixed $2 \leq c, t \leq r$ satisfying $c + t \leq r + 1$ and sufficiently large $n$, if we color the edges of $K^{(r)} n$ with $c$ colors, then there is a monochromatic Hamiltonian $t$-tight Berge-cycle.
We will prove that if the conjecture is true it is best possible, since for any values of \(2 \leq c, t \leq r\) satisfying \(c + t > r + 1\) the statement is not true.

**Theorem 2.** For any fixed \(2 \leq c, t \leq r\) satisfying \(c + t > r + 1\) and sufficiently large \(n\), there is a coloring of the edges of \(K^{(c)}_n\) with \(c\) colors, such that the longest monochromatic \(t\)-tight Berge-cycle has length at most \(\lceil \frac{t(c-1)n}{t(c-1)+1} \rceil\).

We know that Conjecture 1 is true for \(c = t = 2\) and \(r = 3\), see [7]. It has also been proved in [7] that Conjecture 1 is asymptotically true for \(c = 3, t = 2\) and \(r = 4\). For the symmetrical case, \(c = 2, t = 3\), we were able to prove only the following weaker but *sharp* result.

**Theorem 3.** For any \(n \geq 7\), if the edges of \(K^{(5)}_n\) are colored with two colors, then there exists a monochromatic Hamiltonian 3-tight Berge-cycle.

Note that Conjecture 1 would imply the same statement with \(r = 4\) instead of \(r = 5\), however, at this point we were unable to prove the statement for \(r = 4\).

Similarly as in [7], for general \(r\) we were able to obtain only the following weaker result, where essentially we replace the sum \(c + t\) with the product \(ct\).

**Theorem 4.** For any fixed \(2 \leq c, t \leq r\) satisfying \(ct + 1 \leq r\) and \(n \geq 2(t + 1)rc^2\), if we color the edges of \(K^{(r)}_n\) with \(c\) colors, then there is a monochromatic Hamiltonian \(t\)-tight Berge-cycle.

In Section 2 we give the simple construction for Theorem 2. In Sections 3 and 4 we present the proofs of Theorems 3 and 4.

## 2 The construction

*Proof. (of Theorem 2)*

Let \(A_1, \ldots, A_{c-1}\) be disjoint vertex sets of size \(\lfloor \frac{n}{t(c-1)+1} \rfloor\). The \(r\)-edges not containing a vertex from \(A_1\) are colored with color 1. The \(r\)-edges that are not colored yet and do not contain a vertex from \(A_2\) are colored with color 2. We continue in this fashion. Finally the \(r\)-edges that are not colored yet with colors \(1, \ldots, c-2\) and do not contain a vertex from \(A_{c-1}\) are colored with color \(c-1\). The \(r\)-edges that contain a vertex from all \(c-1\) sets \(A_1, \ldots, A_{c-1}\) (if such \(r\)-edges exist) get color \(c\). We claim that in this \(c\)-coloring of the edges of \(K^{(r)}_n\) the longest monochromatic \(t\)-tight Berge-cycle has length \(\leq \lfloor \frac{t(c-1)n}{t(c-1)+1} \rfloor\). This is certainly true for Berge-cycles in color \(i\) for \(1 \leq i \leq c-1\), since the subhypergraph induced by the edges in color \(i\) leaves out \(A_i\) (a set of size \(\lfloor \frac{n}{t(c-1)+1} \rfloor\)) completely. Finally, note that in a \(t\)-tight Berge-cycle in color \(c\) (if such a cycle exists) from \(t \geq r + c + 1\) consecutive vertices on the cycle at least one has to come from \(A_1 \cup \ldots \cup A_{c-1}\) and thus the cycle has length at most

\[
\frac{t(c-1)n}{t(c-1)+1} \leq \frac{t(c-1)n}{t(c-1)+1} \leq \lceil \frac{t(c-1)n}{t(c-1)+1} \rceil.
\]

\(\square\)
3 3-tight 5-uniform Berge-cycles

Lemma 5. If the edges of $K_7^{(5)}$ are colored with 2 colors, there exists a monochromatic Hamiltonian 3-tight Berge-cycle.

Proof. We first remark that the hypergraph $K_7^{(5)}$ contains 21 edges, that each pair is contained in exactly 10 edges, and each triple is contained in exactly 6 edges.

Let us consider a coloring of the edges of $K_7^{(5)}$ in two colors, blue and red. We will first consider two favorable cases, when the edges containing a pair or a triple of vertices are mostly colored with the same color.

Case 1: Suppose that there exists a pair of vertices (for instance $\{0, 4\}$) contained in less than 3 edges of a color (for instance blue); that is it is contained in at least 8 red edges. Without loss of generality, we can assume that if there are blue edges containing $\{0, 4\}$, one is $(0, 1, 2, 3, 4)$ and possibly a second one is either $(0, 1, 4, 5, 6)$ or $(0, 1, 2, 4, 5)$.

Let us consider the cycle $(0, 6, 2, 3, 4, 5, 1)$. In Table 1, we give a choice of a red edge for each triple of consecutive vertices of this cycle, all distinct.

<table>
<thead>
<tr>
<th>Table 1: Choice of a red edge for each triple for Lemma 5 Case 1.</th>
</tr>
</thead>
<tbody>
<tr>
<td>${0, 6, 2}$: $(0, 2, 4, 5, 6)$</td>
</tr>
<tr>
<td>${6, 2, 3}$: $(0, 2, 3, 4, 6)$</td>
</tr>
<tr>
<td>${2, 3, 4}$: $(0, 2, 3, 4, 5)$</td>
</tr>
<tr>
<td>${3, 4, 5}$: $(0, 3, 4, 5, 6)$</td>
</tr>
<tr>
<td>${4, 5, 1}$: $(0, 1, 3, 4, 5)$</td>
</tr>
<tr>
<td>${5, 1, 0}$: $(0, 1, 2, 4, 5)$ or $(0, 1, 4, 5, 6)$</td>
</tr>
<tr>
<td>${1, 0, 6}$: $(0, 1, 3, 4, 6)$</td>
</tr>
</tbody>
</table>

Case 2: Suppose now that every pair of vertices is contained in at least 3 edges of each color. Suppose that for some triple of vertices, say $\{0, 1, 2\}$, all the 6 edges containing it are of the same color, for instance red.

Consider the pair $\{3, 6\}$, at least three red edges contains it. One of them is $(0, 1, 2, 3, 6)$, let $(3, 6, \alpha, \beta, \gamma)$ be another one. Necessarily, $\{\alpha, \beta, \gamma\} \cap \{0, 1, 2\} \neq \emptyset$, so we can suppose without loss of generality $\gamma = 2$.

We give in Table 2 a choice of a red edge for each triple of consecutive vertices for the cycle $(0, 3, 6, 2, 4, 1, 5)$. All these edges are obviously distinct, except perhaps for $(2, 3, 6, \alpha, \beta)$. Yet this edge may be equal only to $(0, 1, 2, 3, 6)$, and we chose them to be different. So this cycle with this choice of edges forms a red Hamiltonian 3-tight Berge-cycle in $K_7^{(5)}$.  

Case 3: Finally, we can assume that every pair of vertices is contained in 3 edges of each color and that every triple of vertices is contained in an edge of each color. The hypergraph $K_7^{(5)}$ contains 21 edges, so there must be 11 edges of the same color, suppose red. By the pigeonhole principle, we will prove that there must exist a triple that is contained in at least 4 red edges. Each red edge contains exactly $\binom{5}{3} = 10$ distinct triples, this makes at least 110 pairs $\{e,f\}$ such that $e$ is a red edge and $f$ is a triple with $f \subset e$. There are exactly $\binom{7}{3} = 35$ triples, now $\frac{110}{35} > 3$, so there exists a triple that is contained in at least 4 red edges.

Let the triple $\{0,1,2\}$ be contained in at least 4 red edges. It is also contained in a blue edge, suppose $\{0,1,2,4,5\}$. If there is a second blue edge containing $\{0,1,2\}$, we assume without loss of generality that it is either $\{0,1,2,3,6\}$ or $\{0,1,2,4,6\}$. Consider the pair $\{4,5\}$; it is contained in at least 3 red edges: $e_1$, $e_2$ and $e_3$. Since none are equal to $\{0,1,2,4,5\}$, they all contain the vertex 3 or 6, maybe both. Moreover, since both triples $\{3,4,5\}$ and $\{4,5,6\}$ are contained in a red edge, then at least one contains 3 and one contains 6. Suppose $e_1$ contains 3 and $e_3$ contains 6, $e_2$ contains either 3 or 6. We consider 3 subcases:

1. If $\{0,1,2,4,6\}$ is red: In this case, since the edge $\{0,1,2,3,4\}$ is also red, we may assume without loss of generality that $e_2$ contains 6. The edge $e_3$ contains either 0, 1, or 2; by symmetry, suppose it is 0. We form the cycle $(0,1,2,3,4,5,6)$ with the choice of edges given in table 3, first column.

2. If $\{0,1,2,4,6\}$ is blue and $e_2$ contains 6: The edge $e_3$ necessarily contains a vertex among 0, 1 and 2, suppose it is 0. Then, we form the cycle $(0,1,2,3,4,5,6)$ with the choice of edges given in table 3, second column.

3. If $\{0,1,2,4,6\}$ is blue and $e_2$ contains 3: The edge $e_1$ necessarily contains a vertex among 0, 1 and 2, suppose it is 2. Then, we form the cycle $(0,1,2,3,4,5,6)$ with the choice of edges given in table 3, third column.

Thus in every case, we managed to build a monochromatic Hamiltonian 3-tight Berge-cycle in $K_7^{(5)}$. □
Subcase 1

\( (0,1,2,3,4) \)
\( (0,1,2,3,4) \)
\( (0,1,2,3,5) \)
\( (0,1,2,5,6) \)

Subcase 2

(of Theorem 3)

\( (0,1,2,5,6) \)
\( (0,1,2,3,6) \)
\( (0,1,2,3,6) \)
\( (0,1,2,4,6) \)

Subcase 3

\( (0,1,2,3,5) \)
\( (0,1,2,3,5) \)
\( (0,1,2,5,6) \)

\( \text{two carmine edges containing each of the sets} \)

\( \text{least 2 edges containing azure, except perhaps one. Otherwise, we will say a pair is} \)

\( \text{be found, corresponding to the sets} \)

\( \text{be red.} \)

\( \{a, x, y, z, u\}, \{a, x, y, z, v\}, \text{and} \{a, x, y, z, w\}. \) Two of them have the same color, say carmine, then \( \{x, z\} \) cannot be blue, and if the color is azure, \( \{x, y\} \) cannot be red.

Suppose first that there exists a \( 1 \leq i \leq n-1 \) such that the pairs \( \{i, i+1\}, \{i+1, i+2\}, \text{and} \{i+2, i+3\} \) (with \( n-1+j \equiv j \)) are green or red. For notation convenience, suppose \( i = 1 \). We claim that there is a choice of edges such that \( \{1, 2, a, 3, 4, \ldots, n-1\} \) is a 3-tight monochromatic carmine Hamiltonian cycle. Let us define such a choice of edges. For any \( 3 \leq j \leq n-1 \), choose for the set \( \{j, j+1, j+2\} \) the corresponding edge in \( C \). Three edges still have to be found, corresponding to the sets \( \{1,2,a\}, \{2,a,3\} \) and \( \{a,3,4\} \). For these three sets, we will choose edges containing \( a \), that are therefore different from the edges we took before.

Since the pairs \( \{1,2\}, \{2,3\} \) and \( \{3,4\} \) are green or red, there are at least two carmine edges containing each of the sets \( \{a,1,2\}, \{a,2,3\} \) and \( \{a,3,4\} \).

---

**Table 3:** Choice of a red edge for each triple for Lemma 5 Case 3.

<table>
<thead>
<tr>
<th>triple :</th>
<th>Subcase 1</th>
<th>Subcase 2</th>
<th>Subcase 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( {0,1,2} ):</td>
<td>( {0,1,2,5,6} )</td>
<td>( {0,1,2,5,6} )</td>
<td>( {0,1,2,3,5} )</td>
</tr>
<tr>
<td>( {1,2,3} ):</td>
<td>( {0,1,2,3,5} )</td>
<td>( {0,1,2,3,4} )</td>
<td>( {0,1,2,3,4} )</td>
</tr>
<tr>
<td>( {2,3,4} ):</td>
<td>( {0,1,2,3,4} )</td>
<td>( {0,1,2,3,4} )</td>
<td>( e_1 )</td>
</tr>
<tr>
<td>( {3,4,5} ):</td>
<td>( e_1 )</td>
<td>( e_1 )</td>
<td>( e_2 )</td>
</tr>
<tr>
<td>( {4,5,6} ):</td>
<td>( e_2 )</td>
<td>( e_2 )</td>
<td>( e_3 )</td>
</tr>
<tr>
<td>( {5,6,0} ):</td>
<td>( e_3 )</td>
<td>( e_3 )</td>
<td>( {0,1,2,5,6} )</td>
</tr>
<tr>
<td>( {6,0,1} ):</td>
<td>( {0,1,2,4,6} )</td>
<td>( {0,1,2,3,6} )</td>
<td>( {0,1,2,3,6} )</td>
</tr>
</tbody>
</table>

**Proof.** (of Theorem 3)

Consider the complete hypergraph \( H = K_n^{(5)} \) whose edges are 2-colored. We will proceed by induction on \( n \), its number of vertices. Lemma 5 establishes the base case for \( n = 7 \). Let \( n \geq 8 \). Suppose the result is true for \( n-1 \).

Let \( a \) be a vertex of \( H \). By the induction hypothesis, the induced subgraph of \( H \) on all its vertices except \( a \) has a monochromatic Hamiltonian 5-uniform 3-tight Berge-cycle. Say its color is **carmine**, the other color being **azure**. Let us name its vertices \( \{1,2,\ldots,n-1\} \) in the order they appear in the cycle.

In the following, we will give a color to any pair \( \{x, y\} \) of vertices of \( V \setminus \{a\} \), depending on the color of the edges containing \( x, y \) and \( a \). We will say a pair \( \{x, y\} \) is **red** if all the edges containing \( x, y \) and \( a \) are carmine, except perhaps one. We will say a pair \( \{x, y\} \) is **blue** if all the edges containing \( x, y \) and \( a \) are azure, except perhaps one. Otherwise, we will say a pair is **green**, meaning at least 2 edges containing \( x, y \) and \( a \) are carmine and at least 2 are azure.

Remark that if a pair containing \( x \) is red, then no pairs containing \( x \) can be blue, and vice versa. To prove it, suppose a pair \( \{x, y\} \) is red while a pair \( \{x, z\} \) is blue. Take three vertices \( u, v, w \notin \{a, x, y, z\} \). Consider the three edges \( \{a, x, y, z, u\}, \{a, x, y, z, v\}, \text{and} \( \{a, x, y, z, w\} \). Two of them have the same color, say carmine, then \( \{x, z\} \) cannot be blue, and if the color is azure, \( \{x, y\} \) cannot be red.

Suppose first that there exists a \( 1 \leq i \leq n-1 \) such that the pairs \( \{i, i+1\}, \{i+1, i+2\}, \text{and} \{i+2, i+3\} \) (with \( n-1+j \equiv j \)) are green or red. For notation convenience, suppose \( i = 1 \). We claim that there is a choice of edges such that \( \{1, 2, a, 3, 4, \ldots, n-1\} \) is a 3-tight monochromatic carmine Hamiltonian cycle. Let us define such a choice of edges. For any \( 3 \leq j \leq n-1 \), choose for the set \( \{j, j+1, j+2\} \) the corresponding edge in \( C \). Three edges still have to be found, corresponding to the sets \( \{1,2,a\}, \{2,a,3\} \) and \( \{a,3,4\} \). For these three sets, we will choose edges containing \( a \), that are therefore different from the edges we took before.

Since the pairs \( \{1,2\}, \{2,3\} \) and \( \{3,4\} \) are green or red, there are at least two carmine edges containing each of the sets \( \{a,1,2\}, \{a,2,3\} \) and \( \{a,3,4\} \).
If the edge \( (1, 2, 3, 4, a) \) is carmine, take it for the set \( \{2, a, 3\} \). Now choose any other carmine edge for \( \{1, 2, a\} \) and \( \{a, 3, 4\} \). There exist such edges since \( \{1, 2\} \) and \( \{3, 4\} \) are green or red, and they are distinct since different from \( \{1, 2, 3, 4, a\} \). Otherwise, take any suiing carmine edge for \( \{2, a, 3\} \) and different carmine edges for \( \{1, 2, a\} \) and \( \{a, 3, 4\} \). All these edges exist since \( \{1, 2\}, \{2, 3\} \) and \( \{3, 4\} \) are green or red, and the edge for \( \{1, 2, a\} \) and \( \{a, 3, 4\} \) are different or it would be \( \{1, 2, 3, 4, a\} \), which is azure.

Now we can suppose that for any \( 1 \leq i \leq n - 1 \), \( \{i, i + 1\}, \{i + 1, i + 2\} \), or \( \{i + 2, i + 3\} \) is blue. Since most edges are now blue, we are tempted to try to form a cycle of color azure. We will still form a carmine cycle in the following case.

Suppose there exists a vertex \( 1 \leq i \leq n - 1 \), such that the edges \( (a, i, i + 1, i + 2, i + 3) \), \( (a, i, i + 1, i + 2, i + 4) \) and \( (a, i, i + 1, i + 2, i + 5) \) are carmine. Then to form a carmine cycle, we insert a between \( i + 1 \) and \( i + 2 \). We get the cycle \( (1, 2, \ldots, i, i + 1, a, i + 2, i + 3, \ldots, n - 1) \). For \( \{i, i + 1, a\} \), we use the edge \( (a, i, i + 1, i + 2, i + 5) \), for \( \{i + 1, a, i + 2\} \), the edge \( (a, i, i + 1, i + 2, i + 4) \), for \( \{a, i + 2, i + 3\} \), the edge \( (a, i + 1, i + 2, i + 3) \), and for all the other triples, we use the corresponding edge of \( C \).

We finally can suppose otherwise that for any \( 1 \leq i \leq n - 1 \), one of the edges \( (a, i, i + 1, i + 2, i + 3) \), \( (a, i, i + 1, i + 2, i + 4) \) and \( (a, i, i + 1, i + 2, i + 5) \) is azure. Then using this edge for the set \( \{i, i + 1, i + 2\} \), we form an azure cycle \( C' \) \( \{1, 2, \ldots n\} \) not containing \( a \). All the edges we used are distinct since \( n - 1 > 6 \). Let us choose a blue pair of consecutive vertices in the cycle. Without loss of generality, suppose the pair is \( \{2, 3\} \). We will insert the vertex \( a \) between 2 and 3 in the cycle \( C' \). Most edges may remain unchanged. For the set \( \{1, 2, a\} \), we can use the edge of \( C' \) formerly used for \( \{1, 2, 3\} \) which contains \( a \) by construction of \( C' \). Likewise, we can use for \( \{a, 3, 4\} \) the edge of \( C' \) formerly used for \( \{2, 3, 4\} \).

We only have to find an edge for \( \{2, a, 3\} \). Since \( \{2, 3\} \) is blue, either \( \{2, a, 3, 5, 6\} \) or \( \{2, a, 3, 5, 7\} \) is azure, and they both are distinct from any edge of \( C' \). So we can find among these two an edge for \( \{2, a, 3\} \), and we get a monochromatic Hamiltonian 3-tight Berge-cycle. □

## 4 Proof of Theorem 4

**Proof.** (of Theorem 4)

We follow the method of [7]. For the sake of completeness we give the details. We first prove the following lemma.

**Lemma 6.** Let \( k \) and \( t \geq 2 \) be fixed positive integers and let \( n > 2(t + 1)tk \). Then a \( (t + 1) \)-uniform hypergraph \( H \) of order \( n \) with at least \( \binom{n}{t+1} - kn \) edges has a Hamiltonian \( t \)-tight Berge-cycle.

**Proof.** By averaging there exists a vertex \( x \in V(H) \) contained in at least \( \binom{n-1}{t} - (t+1)k \) edges of \( H \). Thus apart from at most \( (t+1)k \) exceptional sets all subsets
of size $t$ on the remaining $n-1$ vertices form an edge of $\mathcal{H}$ together with $x$. Let us denote the union of the vertices in the exceptional subsets by $U$. Thus $|U| \leq (t+1)kt$. Take a cyclic permutation on the remaining vertices where two vertices from $U$ are never neighbors. Since $n > 2(t+1)tk$, this is possible. But then this cyclic permutation is actually a $t$-tight Berge-cycle, i.e. $C^{(t+1, t)}_{n-1}$. Indeed, any set of $t$ consecutive vertices on the cycle contains a non-exceptional vertex and thus it forms an edge with $x$. Furthermore, since $n > 2(t+1)tk$, there must be two non-exceptional vertices, denoted by $x_1$ and $y_1$, that are neighbors on the cycle. Consider the $2t$ consecutive vertices along the cycle that include $x_1$ and $y_1$ in the middle, and denote these vertices by $x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_t$. Consider also a vertex $z$ along the cycle that is not among these $2t$ vertices. We claim that $x$ can be inserted between $x_1$ and $y_1$ on the cycle and thus giving a Hamiltonian $t$-tight Berge-cycle in $\mathcal{H}$. Indeed, for those sets of $t$ consecutive vertices which do not include $x$, we can add $x$ to get the required edge $E_x$. If a set of $t$ consecutive vertices includes $x$, then it also must include either $x_1$ or $y_1$ (or maybe both), i.e. a non-exceptional vertex. But then we can add $z$ to get the required edge. It is easy to check that all the used edges are distinct.

For $S \subseteq V(K^{(t+1)}_n), |S| < g$, let $E_S = \{e | e \in E(K^{(t+1)}_n) \text{ with } S \subseteq e\}$, the set of edges containing $S$. Thus $|E_S| = \binom{n-|S|}{g-|S|}$. It is enough to prove Theorem 4 for $r = ct + 1$. Indeed, for $r > ct + 1$, one can have a color transfer by any injection of the $(ct+1)$-element subsets of the $n$ vertices into their $r$-element supersets ($n \geq 2r$ is ensured). Then Theorem 4 will easily follow from the following stronger theorem.

**Theorem 7.** Let $c, t \geq 2$ and let $n \geq 2(t+1)tc^2$. Furthermore let $S \subseteq V(K^{(ct+1)}_n)$ such that $S$ is of order divisible by $t$ (possibly empty) with $|S| \leq (c-1)t$. Set $u = c - \frac{|S|}{t} (\geq 1)$. Color $m = f(n, u, S)$ edges of $E_S$ with $u$ colors. If $f(n, u, S) \geq \binom{n-|S|}{ct+1-|S|} - (c-u)(n+t) > 0$, then $E_S$ contains a monochromatic Hamiltonian $t$-tight Berge-cycle.

**Proof.** Let $F_S \subseteq E_S$, $|F_S| = m$, be the set of colored edges in $E_S$. Fix $t \geq 2$. The proof will be by induction on $u$, $1 \leq u \leq c$. If $u = 1$, then $|S| = (c-1)t$ so that $\binom{n-|S|}{ct+1-|S|} - (c-1)(n+t) = \binom{n-(c-1)t}{t+1} - (c-1)(n+t) \geq \binom{n-(c-1)t}{t+1} - c(n-c-1)t$ when $u \geq tc^2$. Define the $(t+1)$-uniform hypergraph $\mathcal{H}_S$ with $V(\mathcal{H}_S) = V(K^{(t+1)}_n) \setminus S$ and $E(\mathcal{H}_S) = \{e \setminus S \mid e \in F_S\}$. Therefore since $n - (c-1)t > 2(t+1)tc$ by Lemma 6 $\mathcal{H}_S$ contains a Hamiltonian $t$-tight Berge-cycle $C^{(ct+1, t)}_{n-(c-1)t}$. Then we get the corresponding $t$-tight Berge-cycle $C^{(ct+1, t)}_{n-(c-1)t}$ in $E_S$. But each edge of $E_S$ contains $S$ and only $n - (c-1)t$ edges are used on this $C^{(ct+1, t)}_{n-(c-1)t}$ so that it is easy to insert all of $S$ in place of any edge of $C^{(ct+1, t)}_{n-(c-1)t}$ giving the monochromatic $C^{(ct+1, t)}_n$. Indeed, insert all the vertices of $S$ in arbitrary order between two consecutive vertices on the cycle. Consider a set $T$ of $t$ consecutive vertices on the new cycle. If $T$ does not contain a vertex from $S$, then we can use the edge $E_T$ from $E(C^{(ct+1, t)}_{n-(c-1)t})$. If $T$ does have at least one vertex from $S$, then it has at most $(t-1)$ vertices outside $S$, and thus at
least \( ct + 1 - |S| - (t - 1) = 2 \) more vertices are “free”, so in \( E_S \) the number of edges containing \( T \) that we can still use (not missing or not used on the cycle yet) is at least
\[
\binom{n - |S \cup T|}{2} - (c + 1)(n - (c - 1)t) \geq \frac{(n - ct)^2}{2} - (c + 1)(n - (c - 1)t).
\]
Thus we can select a distinct edge \( E_i \) for each such \( T \) if
\[
\frac{(n - ct)^2}{2} - (c + 1)(n - (c - 1)t) \geq ct,
\]
which is certainly true for \( n \geq 2(t + 1)t^2 \).

Therefore assume the theorem holds for \( u - 1 \) colors with \( c \geq u \geq 2 \), and color the \( m \) edges of \( E_S \) by \( u \) colors, \( m \geq f(n, u, S) \geq \binom{n - |S|}{ct + 1 - |S|} - (c - u)(n + t) > 0, |S| = (c - u)t \). In \( F_S \) select a maximum length monochromatic \( t \)-tight Berge-cycle. Suppose first that this is \( C_{\ell}^{(ct+1,t)} = (z_1, z_2, \ldots, z_t) \) in color 1, with \( 2t - 2 \leq \ell < n \). We will handle the case \( \ell < 2t - 2 \) later. Let \( z \in V(K_n^{(ct+1)}) \setminus V(C_{\ell}^{(ct+1,t)}) \). Consider the vertices \( \{z_1, z_2, \ldots, z_{2t-2}\} \) (using \( 2t - 2 \leq \ell \)) and the \( t \) subsets \( T_1, \ldots, T_t \) consisting of \( t - 1 \) consecutive vertices in this interval. If for each \( i, 1 \leq i \leq t \) the set \( T_i \cup \{z\} \) is contained in at least \( t \) distinct edges in \( E_S \setminus E(C_{\ell}^{(ct+1,t)}) \) in color 1, then clearly we could insert \( z \) into the cycle between \( z_{t-1} \) and \( z_t \), a contradiction. Hence we may assume that for some \( T_i \) (say \( T_1 \) without loss of generality) apart from at most \( (c - u)(n + t) + t \) exceptional edges all edges in \( E_{S \cup T_1 \cup \{z\}} \) are in color 2, 3, \ldots, \( u \).

Assume now the second case, \( \ell < 2t - 2 \). Consider arbitrary vertices \( \{z_1, z_2, \ldots, z_{2t}\} \in V(K_n^{(ct+1)}) \setminus S \) in a cyclic order and the \( 2t \) subsets \( T_1, \ldots, T_{2t} \) consisting of \( t \) consecutive vertices in this cyclic order. If for each \( i, 1 \leq i \leq 2t \) the set \( T_i \) is contained in at least \( 2t \) distinct edges in \( E_S \) in color 1, then we would have a \( t \)-tight Berge-cycle of length \( 2t \) in color 1 in \( F_S \), a contradiction. Hence we may assume that for some \( T_i \) (say \( T_1 \) without loss of generality) apart from at most \( (c - u)(n + t) + 2t \) exceptional edges all edges in \( E_{S \cup T_1} \) are in color 2, 3, \ldots, \( u \).

Let \( S' \) be any set of \( |S| + t = (c - u + 1)t \) vertices containing \( S \cup T_1 \cup \{z\} \) in the first case and \( S \cup T_2 \) in the second case. Thus in both cases at least \( |E_{S'}| - (c - u + 1)(n + t) \) edges of \( E_{S'} \) are colored by at most \( u - 1 \) colors. But \( f(n, u - 1, S') \geq |E_{S'}| - (c - u + 1)(n + t) = \binom{n - |S| + t}{ct + 1 - (|S| + t)} - (c - (u - 1))(n + t) > 0, 1 \leq u - 1 = c - \frac{|S|}{t} \), and \( |S'| = (c - u + 1)t \), so by the induction assumption \( E_{S'} \) contains a monochromatic Hamiltonian \( t \)-tight Berge-cycle, \( C_n^{(ct+1,t)} \), contradicting the assumption that \( E_S \) contains no monochromatic \( C_n^{(ct+1,t)} \). Therefore for any \( u, 1 \leq u \leq c, E_S \) contains a monochromatic \( C_n^{(ct+1,t)} \).

Now the proof of Theorem 4 is concluded by applying Theorem 7 with \( S = \emptyset \).
References


