Monochromatic Hamiltonian $t$-tight Berge-cycles
in hypergraphs

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Abstract

In any $r$-uniform hypergraph $H$ for $2 \leq t \leq r$ we define an $r$-uniform $t$-tight Berge-cycle of length $\ell$, denoted by $C_{\ell}^{(r,t)}$, as a sequence of distinct vertices $v_1, v_2, \ldots, v_{\ell}$, such that for each set $(v_i, v_{i+1}, \ldots, v_{i+t-1})$ of $t$ consecutive vertices on the cycle, there is an edge $E_i$ of $H$ that contains these $t$ vertices and the edges $E_i$ are all distinct for $1 \leq i \leq \ell$ where $\ell + j \equiv j$.

For $t = 2$ we get the classical Berge-cycle and for $t = r$ we get the so-called tight cycle. In this note we formulate the following conjecture. For any fixed $2 \leq c, t \leq r$ satisfying $c + t \leq r + 1$ and sufficiently large $n$, if we color the edges of $K_n^{(r)}$, the complete $r$-uniform hypergraph on $n$ vertices, with $c$ colors, then there is a monochromatic Hamiltonian $t$-tight Berge-cycle. We prove some partial results about this conjecture and we show that if true the conjecture is best possible.

1 Introduction

The investigations of Turán type problems for paths and cycles of graphs were started by Erdős and Gallai in [3]. The corresponding Ramsey problems have

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been looked at some years later first in [6] and then later in [4], [5], [8], [12] and [14].

There are several possibilities to define paths and cycles in hypergraphs. In this paper we address the case of the Berge-cycle; probably it is the earliest definition of a cycle in hypergraphs in the book of Berge [1]. Turán type problems for Berge-paths and Berge-cycles of hypergraphs appeared perhaps first in [2]. Other types of hypergraph cycles, loose and tight, have been studied in [11], [13] and [15]. The investigations of the corresponding Ramsey problems started quite recently with [9] and [10] where Ramsey numbers of loose and tight cycles have been determined asymptotically for two colors and for 3-uniform hypergraphs.

Let $H$ be an $r$-uniform hypergraph (some $r$-element subsets of a set). Let $K(r)_n$ denote the complete $r$-uniform hypergraph on $n$ vertices. In any $r$-uniform hypergraph $H$ for $2 \leq t \leq r$ we define a $t$-tight Berge-cycle of length $\ell$, denoted by $C^{(r,t)}_\ell$, as a sequence of distinct vertices $v_1, v_2, \ldots, v_\ell$, such that for each set $(v_i, v_{i+1}, \ldots, v_{i+t-1})$ of $t$ consecutive vertices on the cycle, there is an edge $E_i$ of $H$ that contains these $t$ vertices and the edges $E_i$ are all distinct for $i$, $1 \leq i \leq \ell$ where $\ell + j \equiv j$. We will denote by $E(C^{(r,t)}_\ell)$ the set of these edges $E_i$ used on the cycle. For $t = 2$ we get Berge-cycles and for $t = r$ we get the tight cycle. When the uniformity is clearly understood we may simply write $C^{(t)}_\ell$ for $C^{(r,t)}_\ell$ or just $C_\ell$. $R_c(C^{(r,t)}_\ell)$ will denote the Ramsey number of the $r$-uniform $t$-tight $\ell$ cycle using $c$ colors. A Berge-cycle of length $n$ in a hypergraph of $n$ vertices is called a Hamiltonian Berge-cycle. It is important to keep in mind that, in contrast to the case $r = t = 2$, for $r > t \geq 2$ a Berge-cycle $C^{(r,t)}_\ell$, is not determined uniquely, it is considered as an arbitrary choice from many possible cycles with the same triple of parameters.

In this note, continuing the investigations from [7], we study Hamiltonian Berge-cycles in hypergraphs. Thinking in terms of graphs, such an attempt seems strange, since in many 2-coloring of $K(r)_n$ there are no monochromatic Hamiltonian cycles. For example, if each edge incident to a fixed vertex is red and the other edges are blue, there is no monochromatic Hamiltonian cycle. However, from the nature of Berge-cycles, this example does not carry over to hypergraphs, if $K(r)_n$ is colored this way, there is a red Hamiltonian Berge-cycle (for $n \geq 5$).

In [7] monochromatic Hamiltonian (2-tight) Berge-cycles were studied and the following conjecture was formulated. Assume that $r > 1$ is fixed and $n$ is sufficiently large. Then every $(r-1)$-coloring of $K(r)_n$ contains a monochromatic Hamiltonian (2-tight) Berge-cycle. The conjecture was proved for $r = 3$. For general $r$, the statement was proved for sufficiently large $n$ with $\lfloor \frac{r-1}{2} \rfloor$ colors instead of $r - 1$ colors. In this note we look at monochromatic Hamiltonian $t$-tight Berge-cycles and we generalize the above conjecture in the following way.

**Conjecture 1.** For any fixed $2 \leq c, t \leq r$ satisfying $c + t \leq r + 1$ and sufficiently large $n$, if we color the edges of $K(r)_n$ with $c$ colors, then there is a monochromatic Hamiltonian $t$-tight Berge-cycle.
We will prove that if the conjecture is true it is best possible, since for any values of $2 \leq c, t \leq r$ satisfying $c + t > r + 1$ the statement is not true.

**Theorem 2.** For any fixed $2 \leq c, t \leq r$ satisfying $c + t > r + 1$ and sufficiently large $n$, there is a coloring of the edges of $K_n^{(c)}$ with $c$ colors, such that the longest monochromatic $t$-tight Berge-cycle has length at most $\lceil \frac{t(c-1)n}{t(c-1) + 1} \rceil$.

We know that Conjecture 1 is true for $c = t = 2$ and $r = 3$, see [7]. It has also been proved in [7] that Conjecture 1 is asymptotically true for $c = 3$, $t = 2$ and $r = 4$. For the symmetrical case, $c = 2$, $t = 3$, we were able to prove only the following weaker but sharp result.

**Theorem 3.** For any fixed $2 \leq c, t \leq r$ satisfying $ct + 1 \leq r$ and sufficiently large $n$, if the edges of $K_n^{(5)}$ are colored with two colors, then there exists a monochromatic Hamiltonian 3-tight Berge-cycle.

Note that Conjecture 1 would imply the same statement with $r = 4$ instead of $r = 5$, however, at this point we were unable to prove the statement for $r = 4$.

Similarly as in [7], for general $r$ we were able to obtain only the following weaker result, where essentially we replace the sum $c + t$ with the product $ct$.

**Theorem 4.** For any fixed $2 \leq c, t \leq r$ satisfying $ct + 1 \leq r$ and $n \geq 2t + 1 > r$, if we color the edges of $K_n^{(r)}$ with $c$ colors, then there is a monochromatic Hamiltonian $t$-tight Berge-cycle.

In Section 2 we give the simple construction for Theorem 2. In Sections 3 and 4 we present the proofs of Theorems 3 and 4.

## 2 The construction

**Proof.** (of Theorem 2)

Let $A_1, \ldots, A_{c-1}$ be disjoint vertex sets of size $\lfloor \frac{n}{t(c-1)+1} \rfloor$. The $r$-edges not containing a vertex from $A_1$ are colored with color 1. The $r$-edges that are not colored yet and do not contain a vertex from $A_2$ are colored with color 2. We continue in this fashion. Finally the $r$-edges that are not colored yet with colors 1, \ldots, $c-2$ and do not contain a vertex from $A_{c-1}$ are colored with color $c-1$. The $r$-edges that contain a vertex from all $c-1$ sets $A_1, \ldots, A_{c-1}$ (if such $r$-edges exist) get color $c$. We claim that in this $c$-coloring of the edges of $K_n^{(r)}$ the longest monochromatic $t$-tight Berge-cycle has length $\leq \lceil \frac{t(c-1)n}{t(c-1)+1} \rceil$. This is certainly true for Berge-cycles in color $i$ for $1 \leq i \leq c-1$, since the subhypergraph induced by the edges in color $i$ leaves out $A_i$ (a set of size $\lfloor \frac{n}{t(c-1)+1} \rfloor$) completely. Finally, note that in a $t$-tight Berge-cycle in color $c$ (if such a cycle exists) from $t (> r - c + 1)$ consecutive vertices on the cycle at least one has to come from $A_1 \cup \ldots \cup A_{c-1}$ and thus the cycle has length at most

$$t(c-1)\lfloor \frac{n}{t(c-1)+1} \rfloor \leq \frac{t(c-1)n}{t(c-1)+1} \leq \lceil \frac{t(c-1)n}{t(c-1)+1} \rceil.$$ 

□

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3 3-tight 5-uniform Berge-cycles

Lemma 5. If the edges of $K_7^{(5)}$ are colored with 2 colors, there exists a monochromatic Hamiltonian 3-tight Berge-cycle.

Proof. We first remark that the hypergraph $K_7^{(5)}$ contains 21 edges, that each pair is contained in exactly 10 edges, and each triple is contained in exactly 6 edges.

Let us consider a coloring of the edges of $K_7^{(5)}$ in two colors, blue and red. We will first consider two favorable cases, when the edges containing a pair or a triple of vertices are mostly colored with the same color.

Case 1: Suppose that there exists a pair of vertices (for instance $\{0, 4\}$) contained in less than 3 edges of a color (for instance blue); that is it is contained in at least 8 red edges. Without loss of generality, we can assume that if there are blue edges containing $\{0, 4\}$, one is $(0, 1, 2, 3, 4)$ and possibly a second one is either $(0, 1, 4, 5, 6)$ or $(0, 1, 2, 4, 5)$.

Let us consider the cycle $(0, 6, 2, 3, 4, 5, 1)$. In Table 1, we give a choice of a red edge for each triple of consecutive vertices of this cycle, all distinct.

Table 1: Choice of a red edge for each triple for Lemma 5 Case 1.

<table>
<thead>
<tr>
<th>Triple</th>
<th>Red Edge</th>
</tr>
</thead>
<tbody>
<tr>
<td>${0, 6, 2}$</td>
<td>$(0, 2, 4, 5, 6)$</td>
</tr>
<tr>
<td>${6, 2, 3}$</td>
<td>$(0, 2, 3, 4, 6)$</td>
</tr>
<tr>
<td>${2, 3, 4}$</td>
<td>$(0, 2, 3, 4, 5)$</td>
</tr>
<tr>
<td>${3, 4, 5}$</td>
<td>$(0, 3, 4, 5, 6)$</td>
</tr>
<tr>
<td>${4, 5, 1}$</td>
<td>$(0, 1, 3, 4, 5)$</td>
</tr>
<tr>
<td>${5, 1, 0}$</td>
<td>$(0, 1, 2, 4, 5)$ or $(0, 1, 4, 5, 6)$</td>
</tr>
<tr>
<td>${1, 0, 6}$</td>
<td>$(0, 1, 3, 4, 6)$</td>
</tr>
</tbody>
</table>

Case 2: Suppose now that every pair of vertices is contained in at least 3 edges of each color. Suppose that for some triple of vertices, say $\{0, 1, 2\}$, all the 6 edges containing it are of the same color, for instance red.

Consider the pair $\{3, 6\}$, at least three red edges contains it. One of them is $(0, 1, 2, 3, 6)$, let $(3, 6, \alpha, \beta, \gamma)$ be another one. Necessarily, $\{\alpha, \beta, \gamma\} \cap \{0, 1, 2\} \neq \emptyset$, so we can suppose without loss of generality $\gamma = 2$.

Let us consider the cycle $(0, 6, 2, 3, 4, 5, 1)$. In Table 2, we give a choice of a red edge for each triple of consecutive vertices of this cycle, all distinct.

Table 2: Choice of a red edge for each triple for Lemma 5 Case 2.

<table>
<thead>
<tr>
<th>Triple</th>
<th>Red Edge</th>
</tr>
</thead>
<tbody>
<tr>
<td>${0, 6, 2}$</td>
<td>$(0, 2, 3, 4, 5, 6)$</td>
</tr>
<tr>
<td>${6, 2, 3}$</td>
<td>$(0, 2, 3, 4, 6)$</td>
</tr>
<tr>
<td>${2, 3, 4}$</td>
<td>$(0, 2, 3, 4, 5)$</td>
</tr>
<tr>
<td>${3, 4, 5}$</td>
<td>$(0, 3, 4, 5, 6)$</td>
</tr>
<tr>
<td>${4, 5, 1}$</td>
<td>$(0, 1, 3, 4, 5)$</td>
</tr>
<tr>
<td>${5, 1, 0}$</td>
<td>$(0, 1, 2, 4, 5)$ or $(0, 1, 4, 5, 6)$</td>
</tr>
<tr>
<td>${1, 0, 6}$</td>
<td>$(0, 1, 3, 4, 6)$</td>
</tr>
</tbody>
</table>

We give in Table 2 a choice of a red edge for each triple of consecutive vertices for the cycle $(0, 6, 2, 3, 4, 5, 1)$. All these edges are obviously distinct, except perhaps for $(2, 3, 6, \alpha, \beta)$. Yet this edge may be equal only to $(0, 1, 2, 3, 6)$, and we chose them to be different. So this cycle with this choice of edges forms a red Hamiltonian 3-tight Berge-cycle in $K_7^{(5)}$. 

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Table 2: Choice of a red edge for each triple for Lemma 5 Case 2.

\[
\begin{align*}
\{0, 3, 6\} &\colon (0, 1, 2, 3, 6) \\
\{3, 6, 2\} &\colon (2, 3, 6, \alpha, \beta) \\
\{6, 2, 4\} &\colon (0, 1, 2, 4, 6) \\
\{2, 4, 1\} &\colon (0, 1, 2, 3, 4) \\
\{4, 1, 5\} &\colon (0, 1, 2, 4, 5) \\
\{1, 5, 0\} &\colon (0, 1, 2, 5, 6) \\
\{5, 0, 3\} &\colon (0, 1, 2, 3, 5)
\end{align*}
\]

Case 3: Finally, we can assume that every pair of vertices is contained in 3 edges of each color and that every triple of vertices is contained in an edge of each color.

The hypergraph $K_5^{(5)}$ contains 21 edges, so there must be 11 edges of the same color, suppose red. By the pigeonhole principle, we will prove that there must exist a triple that is contained in at least 4 red edges. Each red edge contains exactly $\binom{5}{3} = 10$ distinct triples, this makes at least 110 pairs $\{e, f\}$ such that $e$ is a red edge and $f$ is a triple with $f \subset e$. There are exactly $\binom{7}{3} = 35$ triples, now $\frac{110}{35} > 3$, so there exists a triple that is contained in at least 4 red edges.

Let the triple $\{0, 1, 2\}$ be contained in at least 4 red edges. It is also contained in a blue edge, suppose $\{0, 1, 2, 4, 5\}$. If there is a second blue edge containing $\{0, 1, 2\}$, we assume without loss of generality that it is either $\{0, 1, 2, 3, 6\}$ or $\{0, 1, 2, 4, 6\}$. Consider the pair $\{4, 5\}$; it is contained in at least 3 red edges: $e_1$, $e_2$ and $e_3$. Since none are equal to $\{0, 1, 2, 4, 5\}$, they all contain the vertex 3 or 6, maybe both. Moreover, since both triples $\{3, 4, 5\}$ and $\{4, 5, 6\}$ are contained in a red edge, then at least one contains 3 and one contains 6. Suppose $e_1$ contains 3 and $e_3$ contains 6, $e_2$ contains either 3 or 6. We consider 3 subcases:

1. If $\{0, 1, 2, 4, 6\}$ is red:
   In this case, since the edge $\{0, 1, 2, 3, 4\}$ is also red, we may assume without loss of generality that $e_2$ contains 6. The edge $e_3$ contains either 0, 1, or 2; by symmetry, suppose it is 0. We form the cycle $\{0, 1, 2, 3, 4, 5, 6\}$ with the choice of edges given in table 3, first column.

2. If $\{0, 1, 2, 4, 6\}$ is blue and $e_2$ contains 6:
   The edge $e_3$ necessarily contains a vertex among 0, 1 and 2, suppose it is 0. Then, we form the cycle $\{0, 1, 2, 3, 4, 5, 6\}$ with the choice of edges given in table 3, second column.

3. If $\{0, 1, 2, 4, 6\}$ is blue and $e_2$ contains 3: The edge $e_1$ necessarily contains a vertex among 0, 1 and 2, suppose it is 2. Then, we form the cycle $\{0, 1, 2, 3, 4, 5, 6\}$ with the choice of edges given in table 3, third column.

Thus in every case, we managed to build a monochromatic Hamiltonian 3-tight Berge-cycle in $K_7^{(5)}$. □
Table 3: Choice of a red edge for each triple for Lemma 5 Case 3.

<table>
<thead>
<tr>
<th>triple</th>
<th>Subcase 1</th>
<th>Subcase 2</th>
<th>Subcase 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>{0, 1, 2}</td>
<td>(0, 1, 2, 5, 6)</td>
<td>(0, 1, 2, 5, 6)</td>
<td>(0, 1, 2, 3, 5)</td>
</tr>
<tr>
<td>{1, 2, 3}</td>
<td>(0, 1, 2, 3, 5)</td>
<td>(0, 1, 2, 3, 5)</td>
<td>(0, 1, 2, 3, 4)</td>
</tr>
<tr>
<td>{2, 3, 4}</td>
<td>(0, 1, 2, 3, 4)</td>
<td>e1</td>
<td>e1</td>
</tr>
<tr>
<td>{3, 4, 5}</td>
<td>e1</td>
<td>e1</td>
<td>e2</td>
</tr>
<tr>
<td>{4, 5, 6}</td>
<td>e2</td>
<td>e2</td>
<td>e3</td>
</tr>
<tr>
<td>{5, 6, 0}</td>
<td>e3</td>
<td>e3</td>
<td>(0, 1, 2, 5, 6)</td>
</tr>
<tr>
<td>{6, 0, 1}</td>
<td>(0, 1, 2, 4, 6)</td>
<td>(0, 1, 2, 3, 6)</td>
<td>(0, 1, 2, 3, 6)</td>
</tr>
</tbody>
</table>

Proof. (of Theorem 3)

Consider the complete hypergraph $H = K_n^{(5)}$ whose edges are 2-colored. We will proceed by induction on $n$, its number of vertices. Lemma 5 establishes the base case for $n = 7$. Let $n \geq 8$. Suppose the result is true for $n - 1$.

Let $a$ be a vertex of $H$. By the induction hypothesis, the induced subgraph of $H$ on all its vertices except $a$ has a monochromatic Hamiltonian 5-uniform 3-tight Berge-cycle $C$. Say its color is carmine, the other color being azure. Let us name its vertices $\{1, 2, \ldots, n - 1\}$ in the order they appear in the cycle.

In the following, we will give a color to any pair $\{x, y\}$ of vertices of $V \setminus \{a\}$, depending on the color of the edges containing $x$, $y$ and $a$. We will say a pair $\{x, y\}$ is red if all the edges containing $x$, $y$ and $a$ are carmine, except perhaps one. We will say a pair $\{x, y\}$ is blue if all the edges containing $x$, $y$ and $a$ are azure, except perhaps one. Otherwise, we will say a pair is green, meaning at least 2 edges containing $x$, $y$ and $a$ are carmine and at least 2 are azure.

Remark that if a pair containing $x$ is red, then no pairs containing $x$ can be blue, and vice versa. To prove it, suppose a pair $\{x, y\}$ is red while a pair $\{x, z\}$ is blue. Take three vertices $u, v, w \notin \{a, x, y, z\}$. Consider the three edges $(a, x, y, z, u)$, $(a, x, y, z, v)$, and $(a, x, y, z, w)$. Two of these have the same color, say carmine, then $\{x, z\}$ cannot be blue, and if the color is azure, $\{x, y\}$ cannot be red.

Suppose first that there exists a $1 \leq i \leq n - 1$ such that the pairs $\{i, i + 1\}$, $\{i + 1, i + 2\}$, and $\{i + 2, i + 3\}$ (with $n - 1 + j \equiv j$) are green or red. For notation convenience, suppose $i = 1$. We claim that there is a choice of edges such that $(1, 2, a, 3, 4, \ldots, n - 1)$ is a 3-tight monochromatic carmine Hamiltonian cycle. Let us define such a choice of edges. For any $3 \leq j \leq n - 1$, choose for the set $\{j, j + 1, j + 2\}$ the corresponding edge in $C$. Three edges still have to be found, corresponding to the sets $\{1, 2, a\}$, $\{2, a, 3\}$ and $\{a, 3, 4\}$. For these three sets, we will choose edges containing $a$, that are therefore different from the edges we took before.

Since the pairs $\{1, 2\}$, $\{2, 3\}$ and $\{3, 4\}$ are green or red, there are at least two carmine edges containing each of the sets $\{a, 1, 2\}$, $\{a, 2, 3\}$ and $\{a, 3, 4\}$. 

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If the edge \((1, 2, 3, 4, a)\) is carmine, take it for the set \(\{2, a, 3\}\). Now choose any other carmine edge for \(\{1, 2, a\}\) and \(\{a, 3, 4\}\). There exist such edges since \(\{1, 2\}\) and \(\{3, 4\}\) are green or red, and they are distinct since different from \((1, 2, 3, 4, a)\). Otherwise, take any suiting carmine edge for \(\{2, a, 3\}\), and different carmine edges for \(\{1, 2, a\}\) and \(\{a, 3, 4\}\). All these edges exist since \(\{1, 2\}\), \(\{2, 3\}\) and \(\{3, 4\}\) are green or red, and the edge for \(\{1, 2, a\}\) and \(\{a, 3, 4\}\) are different or it would be \((1, 2, 3, 4, a)\), which is azure.

Now we can suppose that for any \(1 \leq i \leq n - 1\), \(\{i, i + 1\}\), \(\{i + 1, i + 2\}\), or \(\{i + 2, i + 3\}\) is blue. Since most edges are now blue, we are tempted to try to form a cycle of color azure. We will still form a carmine cycle in the following case.

Suppose there exists a vertex \(1 \leq i \leq n - 1\), such that the edges \((a, i, i + 1, i + 2, i + 3)\), \((a, i, i + 1, i + 2, i + 4)\) and \((a, i, i + 1, i + 2, i + 5)\) are carmine. Then to form a carmine cycle, we insert \(a\) between \(i + 1\) and \(i + 2\). We get the cycle \((1, 2, \ldots, i, i + 1, a, i + 2, i + 3, \ldots, n - 1)\). For \(\{i, i + 1, a\}\), we use the edge \((a, i, i + 1, i + 2, i + 5)\), for \(\{i + 1, a, i + 2\}\), the edge \((a, i, i + 1, i + 2, i + 4)\), for \(\{a, i + 2, i + 3\}\), the edge \((a, i, i + 1, i + 2, i + 3)\), and for all the other triples, we use the corresponding edge of \(C\).

We finally can assume otherwise that for any \(1 \leq i \leq n - 1\), one of the edges \((a, i, i + 1, i + 2, i + 3)\), \((a, i, i + 1, i + 2, i + 4)\) and \((a, i, i + 1, i + 2, i + 5)\) is azure. Then using this edge for the set \(\{i, i + 1, i + 2\}\), we form an azure cycle \(C'\) \(\{1, 2, \ldots n\}\) not containing \(a\). All the edges we used are distinct since \(n - 1 > 6\). Let us choose a blue pair of consecutive vertices in the cycle. Without loss of generality, suppose the pair is \(\{2, 3\}\). We will insert the vertex \(a\) between \(2\) and \(3\) in the cycle \(C'\). Most edges may remain unchanged. For the set \(\{1, 2, a\}\), we can use the edge of \(C'\) formerly used for \(\{1, 2, 3\}\) which contains \(a\) by construction of \(C'\). Likewise, we can use for \(\{a, 3, 4\}\) the edge of \(C'\) formerly used for \(\{2, 3, 4\}\).

We only have to find an edge for \(\{2, a, 3\}\). Since \(\{2, 3\}\) is blue, either \(\{2, a, 3, 5, 6\}\) or \(\{2, a, 3, 5, 7\}\) is azure, and they both are distinct from any edge of \(C'\). So we can find among these two an edge for \(\{2, a, 3\}\), and we get a monochromatic Hamiltonian 3-tight Berge-cycle. □

4 Proof of Theorem 4

Proof. (of Theorem 4)

We follow the method of [7]. For the sake of completeness we give the details. We first prove the following lemma.

Lemma 6. Let \(k\) and \(t \geq 2\) be fixed positive integers and let \(n > 2(t + 1)tk\). Then a \((t + 1)\)-uniform hypergraph \(\mathcal{H}\) of order \(n\) with at least \(\binom{n}{t+1} - kn\) edges has a Hamiltonian \(t\)-tight Berge-cycle.

Proof. By averaging there exists a vertex \(x \in V(\mathcal{H})\) contained in at least \(\binom{n-1}{t} - (t+1)k\) edges of \(\mathcal{H}\). Thus apart from at most \((t+1)k\) exceptional sets all subsets
Let us denote the union of the vertices in the exceptional subsets by \( U \). Thus \(|U| \leq (t + 1)kt \). Take a cyclic permutation on the remaining vertices where two vertices from \( U \) are never neighbors. Since \( n > 2(t + 1)tk \), this is possible.

But then this cyclic permutation is actually a \( t \)-tight Berge-cycle, i.e. \( C_{n-1}^{(t+1,t)} \).

Indeed, any set of \( t \) consecutive vertices on the cycle contains a non-exceptional vertex and thus it forms an edge with \( x \). Furthermore, since \( n > 2(t+1)tk \), there must be two non-exceptional vertices, denoted by \( x_1 \) and \( y_1 \), that are neighbors on the cycle. Consider the \( 2t \) consecutive vertices along the cycle that include \( x_1 \) and \( y_1 \) in the middle, and denote these vertices by \( x_1, \ldots, x_t, y_1, \ldots, y_t \).

Consider also a vertex \( z \) along the cycle that is not among these \( 2t \) vertices. We claim that \( x \) can be inserted between \( x_1 \) and \( y_1 \) on the cycle and thus giving a Hamiltonian \( t \)-tight Berge-cycle in \( \mathcal{H} \). Indeed, for those sets of \( t \) consecutive vertices which do not include \( x \), we can add \( x \) to get the required edge \( E_i \).

If a set of \( t \) consecutive vertices includes \( x \), then it also must include either \( x_1 \) or \( y_1 \) (or maybe both), i.e. a non-exceptional vertex. But then we can add \( z \) to get the required edge. It is easy to check that all the used edges are distinct. \( \square \)

For \( S \subseteq V(K_n^G) \), \(|S| < g \), let \( E_S = \{ e | e \in E(K_n^G) \text{ with } S \subseteq e \} \), the set of edges containing \( S \). Thus \(|E_S| = \binom{n-|S|}{g-|S|}\). It is enough to prove Theorem 4 for \( r = ct + 1 \). Indeed, for \( r > ct + 1 \), one can have a color transfer by any injection of the \((ct + 1)\)-element subsets of the \( n \) vertices into their \( r \)-element supersets \((n \geq 2r \text{ is ensured})\). Then Theorem 4 will easily follow from the following stronger theorem.

**Theorem 7.** Let \( c, t \geq 2 \) and let \( n \geq 2(t + 1)tc^2 \). Furthermore let \( S \subseteq V(K_n^{(ct+1,t)}) \) such that \( S \) is of order divisible by \( t \) (possibly empty) with \(|S| \leq (c - 1)t\). Set \( u = c - \frac{|S|}{t^2} \geq 1 \). Color \( m \geq f(n, u, S) \) edges of \( E_S \) with \( u \) colors.

If \( f(n, u, S) \geq \left(\binom{n-|S|}{g-|S|}\right) - (c-u)(n+t) > 0 \), then \( E_S \) contains a monochromatic Hamiltonian \( t \)-tight Berge-cycle.

*Proof.* Let \( F_S \subseteq E_S \), \(|F_S| = m \), be the set of colored edges in \( E_S \). Fix \( t \geq 2 \). The proof will be by induction on \( u \), \( 1 \leq u \leq c \). If \( u = 1 \), then \(|S| = (c - 1)t\) so that \( \left(\binom{n-|S|}{g-|S|}\right) - (c-1)(n+t) = \binom{n-(c-1)t}{t+1} - (c-1)(n+t) \geq \binom{n-(c-1)t}{t+1} - c(n-(c-1)t) \) when \( u \geq tc^2 \). Define the \((t+1)\)-uniform hypergraph \( \mathcal{H}_S \) with \( V(\mathcal{H}_S) = V(K_n^{(ct+1,t)}) \setminus S \) and \( E(\mathcal{H}_S) = \{ e | e \in F_S \} \). Therefore since \( n - (c - 1)t > 2(t + 1)tc \) by Lemma 6 \( \mathcal{H}_S \) contains a Hamiltonian \( t \)-tight Berge-cycle \( C_{n-(c-1)t}^{(ct+1,t)} \). Then we get the corresponding \( t \)-tight Berge-cycle \( C_{n-(c-1)t}^{(ct+1,t)} \) in \( E_S \). But each edge of \( E_S \) contains \( S \) and only \( n - (c - 1)t \) edges are used on this \( C_{n-(c-1)t}^{(ct+1,t)} \), so that it is easy to insert all of \( S \) in place of any edge of \( C_{n-(c-1)t}^{(ct+1,t)} \) giving the monochromatic \( C_{n}^{(ct+1,t)} \). Indeed, insert all the vertices of \( S \) in arbitrary order between two consecutive vertices on the cycle. Consider a set \( T \) of \( t \) consecutive vertices on the new cycle. If \( T \) does not contain a vertex from \( S \), then we can use the edge \( E_i \) from \( E(C_{n-(c-1)t}^{(ct+1,t)}) \). If \( T \) does have at least one vertex from \( S \), then it has at most \((t-1)\) vertices outside \( S \), and thus at
least \( ct + 1 - |S| - (t - 1) = 2 \) more vertices are “free”, so in \( E_S \) the number of edges containing \( T \) that we can still use (not missing or not used on the cycle yet) is at least

\[
\binom{n - |S \cup T|}{2} - (c + 1)(n - (c - 1)t) \geq \\
\geq \frac{(n - ct)^2}{2} - (c + 1)(n - (c - 1)t).
\]

Thus we can select a distinct edge \( E_i \) for each such \( T \) if

\[
\frac{(n - ct)^2}{2} - (c + 1)(n - (c - 1)t) \geq ct,
\]

which is certainly true for \( n \geq 2(t + 1)ct^2 \).

Therefore assume the theorem holds for \( u - 1 \) colors with \( c \geq u \geq 2 \) and color the \( m \) edges of \( E_S \) by \( u \) colors, \( m \geq f(n, u, S) \geq (\frac{n - |S|}{ct + 1 - |S|}) - (c - u)(n + t) > 0, |S| = (c - u)t \). In \( F_S \) select a maximum length monochromatic \( t \)-tight Berge-cycle. Suppose first that this is \( C_t^{(ct+1,t)} = (z_1, z_2, \ldots, z_t) \) in color 1, with \( 2t - 2 \leq \ell < n \). We will handle the case \( \ell < 2t - 2 \) later. Let \( z \in V(K_n^{(ct+1)}) \setminus V(C_t^{(ct+1,t)}) \). Consider the vertices \( \{z_1, z_2, \ldots, z_{2t-2}\} \) (using \( 2t - 2 \leq \ell \)) and the \( t \) subsets \( T_1, \ldots, T_t \) consisting of \( t - 1 \) consecutive vertices in this interval. If for each \( i, 1 \leq i \leq t \) the set \( T_i \cup \{z\} \) is contained in at least \( t \) distinct edges in \( E_S \setminus E(C_t^{(ct+1,t)}) \) in color 1, then clearly we could insert \( z \) into the cycle between \( z_{t-1} \) and \( z_t \), a contradiction. Hence we may assume that for some \( T_i \) (say \( T_1 \)) without loss of generality) apart from at most \( (c - u)(n + t) + t \) exceptional edges all edges in \( E_{S \cup T_1 \cup \{z\}} \setminus E(C_t^{(ct+1,t)}) \) are in color 2, 3, \ldots, \( u \).

Assume now the second case, \( \ell < 2t - 2 \). Consider arbitrary vertices \( \{z_1, z_2, \ldots, z_{2t}\} \in V(K_n^{(ct+1)}) \setminus S \) in a cyclic order and the \( 2t \) subsets \( T_1, \ldots, T_{2t} \) consisting of \( t \) consecutive vertices in this cyclic order. If for each \( i, 1 \leq i \leq t \) the set \( T_i \) is contained in at least \( 2t \) distinct edges in \( E_S \) in color 1, then we would have a \( t \)-tight Berge-cycle of length \( 2t \) in color 1 in \( F_S \), a contradiction. Hence we may assume that for some \( T_i \) (say \( T_1 \)) without loss of generality) apart from at most \( (c - u)(n + t) + 2t \) exceptional edges all edges in \( E_{S \cup T_i} \) are in color 2, 3, \ldots, \( u \).

Let \( S' \) be any set of \( |S| + t = (c - u + 1)t \) vertices containing \( S \cup T_1 \cup \{z\} \) in the first case and \( S \cup T_1 \) in the second case. Thus in both cases at least \( |E_{S'}| - (c - u + 1)(n + t) \) edges of \( E_{S'} \) are colored by at most \( u - 1 \) colors. But if \( f(n, u - 1, S') \geq |E_{S'}| - (c - u + 1)(n + t) = (\frac{n - (|S| + t)}{ct + 1 - (|S| + t)}) - (c - (u - 1))(n + t) > 0, 1 \leq u - 1 = c - \frac{|S'|}{t} \), and \( |S'| = (c - u + 1)t \), so by the induction assumption \( E_{S'} \) contains a monochromatic Hamiltonian \( t \)-tight Berge-cycle, \( C_n^{(ct+1,t)} \), contradicting the assumption that \( E_S \) contains no monochromatic \( C_n^{(ct+1,t)} \). Therefore for any \( u, 1 \leq u \leq c, E_S \) contains a monochromatic \( C_n^{(ct+1,t)} \). \( \square \)

Now the proof of Theorem 4 is concluded by applying Theorem 7 with \( S = \emptyset \). \( \square \)
References


