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Michael A. Demetriou

*Worcester Polytechnic Institute*, [mdemetri@wpi.edu](mailto:mdemetri@wpi.edu)

Islam I. Hussein

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## ESTIMATION OF SPATIALLY DISTRIBUTED PROCESSES USING MOBILE SPATIALLY DISTRIBUTED SENSOR NETWORK\*

MICHAEL A. DEMETRIOU<sup>†</sup> AND ISLAM I. HUSSEIN<sup>†</sup>

**Abstract.** The problem of estimating a spatially distributed process described by a partial differential equation (PDE), whose observations are contaminated by a zero mean Gaussian noise, is considered in this work. The basic premise of this work is that a set of mobile sensors achieve better estimation performance than a set of immobile sensors. To enhance the performance of the state estimator, a network of sensors that are capable of moving within the spatial domain is utilized. Specifically, such an estimation process is achieved by using a set of spatially distributed mobile sensors. The objective is to provide mobile sensor control policies that aim to improve the state estimate. The metric for such an estimate improvement is taken to be the expected state estimation error. Using different spatial norms, two guidance policies are proposed. The current approach capitalizes on the efficient filter gain design in order to avoid intense computational requirements resulting from the solution to filter Riccati equations. Simulation studies implementing and comparing the two proposed control policies are provided.

**Key words.** spatially distributed systems, sensor control, mobile sensor network, process estimation, diffusion equation

**AMS subject classifications.** 93E10, 93C20, 70B05, 93D05

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**1. Introduction and examples.** Many applications have emerged in recent years that rely on the use of networks of dynamic multiagent, limited-range sensors to collect and process data. Applications include emergency response, aerial mapping, and multiple satellite imaging systems for high-resolution imaging [71]. These and other applications often involve tasks in adversarial, highly dynamic environments that are hazardous to human operators. Hence, there is a pressing need to develop autonomous multiagent sensor network systems that seek to collect and process distributed information under constrained resources.

Sensors of interest, such as infrared and vision-based cameras, and sonar, are used to measure a certain field over some domain  $\mathcal{D}$ . In many applications this is known as the coverage problem. In these problems, the field to be measured satisfies a partial differential equation (PDE). In addition to the task of collecting field measurements, the sensors may also be asked to relay information to the base station or to process the information for ensuing decision making. Data processing may be carried out either in a centralized or decentralized fashion in order to possibly (i) estimate the state, (ii) identify the process parameters, (iii) detect sources, and, in the event of an actuation capability, (iv) provide action in order to alter the process response. Examples include the estimation of the temperature distribution in a wildfire, where the PDE is given by a reaction-convection-diffusion equation [3, 39, 63]. The present work attempts to address the basic goal of using a network of dynamic, limited-range sensors to improve the estimate of the field of interest.

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<sup>†</sup>Department of Mechanical Engineering, Worcester Polytechnic Institute, Worcester, MA 01609 (mdemetri@wpi.edu, ihussein@wpi.edu). The first author gratefully acknowledges the support from the National Science Foundation, grant CMS-0408974, while the second author gratefully acknowledges financial support from WPI Faculty Development Grant.

The theory and application of network control have recently received much attention, as is evident from the plethora of published works in the last few years. In [59], the author provides a lucid overview of the theory, operation, and application of wireless sensor networks. We also refer the reader to [5], where the authors provide a thorough account of the state of the art and of the current challenges in networked real-time systems. Within the same special issue of *Proceedings of the IEEE*, several papers address various issues such as the current state of the technology of networked control systems, the foundations of networked real-time systems, and wireless networks.

In general, there are two classes of sensor coverage control problems. The first class involves spatially *fixed* sensors. The goal, which has been extensively studied in the past, is to optimize sensor locations and sensor domains in fixed-sensor networks, and the problems in this class are considered to be in locational optimization [37, 73]. In such problems, the solution is a Voronoi partition [38], where the optimal sensor domain is a Voronoi cell in the partition and the optimal sensor location is a centroid of a Voronoi cell in the partition.

The second class of problems involves a set of *mobile* sensors. In [14], the authors present a survey of recent activities in the control and design of both static and dynamic sensor networks. In their design criteria, they consider issues such as maximum coverage, detection of events, and minimum communication energy expenditure. In the paper [65], the authors propose a formal model for a network of robotic agents and define notions of network, control, and communication law; coordination; and time and communication complexity. In a subsequent publication, the authors provide upper and lower bounds on the time complexity of basic coordination algorithms running on synchronous robotic networks for rendezvous and deployment over a region of interest [66].

The authors in [41] discuss challenges in the modeling of robotic networks, motion coordination algorithms, sensing and estimation tasks, and complexity of distributed algorithms. In [67], the authors present recent theoretical tools for modeling, analysis, and design of coordination algorithms for networks of mobile autonomous agents for problems with distributed information. The authors discuss motion coordination and the motivation that some recent techniques has received from biological systems. These problems include deployment over a given region, pattern formation, rendezvous, or synchronous rigid-body motions.

In [60], the authors consider a probabilistic network model and a density function to represent the frequency of random events taking place over a mission space. The authors develop an optimization problem that aims to maximize coverage using sensors with limited ranges, while minimizing communication cost. Starting with initial sensor positions, the authors develop a gradient algorithm to converge to a (local) solution to the optimization problem. The sequence of sensor distributions along the solution is seen as a discrete time trajectory of the mobile sensor network until it converges to the local minimum. In [19], the authors address the same question, but instead of converging to a local solution of some optimization problem, the trajectory converges to the centroid of a cell in a Voronoi partition of the search domain. The authors propose stable algorithms in both continuous and discrete time. These algorithms are the dynamic version of the Lloyd algorithm [62], which iteratively achieves the optimal configuration. Voronoi-based approaches, however, require exhaustive computational effort to compute the Voronoi cells continuously during a real-time implementation of the controllers.

In [18], the authors present coordination algorithms for groups of mobile agents performing deployment and coverage tasks under the constraint that each mobile agent

has a limited sensing or communication radius. Based on the geometry of Voronoi partitions and proximity graphs, they propose coverage algorithms in continuous and discrete time that are convergence-guaranteed and are spatially distributed with respect to appropriate proximity graphs. In [40], the authors propose a nonsmooth gradient algorithm for the problem of maximizing the area of the region visible to an observer in a simple nonconvex polygon.

In [64], the authors study optimal sensor placement and motion coordination strategies for mobile sensor networks for target tracking using range sensors. They propose motion coordination algorithms that achieve optimal deployment. The authors in [84] propose an algorithm for monitoring an environmental boundary with mobile agents that use only locally sensed information. Their objective is to approximate the boundary with a polygon. The algorithm proves to be convergent for static boundaries and is shown to perform well for slowly moving boundaries.

The paper [42] uses a novel discrete-event controller for the coordination of cooperating heterogeneous wireless sensor networks containing both unattended ground sensors and mobile sensor robots. Given an environment perception, the discrete-event controller sequences the most suitable tasks for each agent and assigns sensor resources. The authors introduce several new tools for discrete-event controller design and operation. The resulting controller represents a complete dynamical description of the wireless network system and is experimentally demonstrated on a wireless sensor network prototyping system.

In [44], the authors use a stochastic approach to find the sensor schedule that results in the minimum error covariance of a state to be measured. They develop a stochastic sensor selection strategy that is computationally tractable. Applications of this work include the sensor selection problem, where multiple sensors cannot operate simultaneously, as in single frequency band sonar in which sensor trajectory optimization is needed to optimize their trajectories. The algorithm is applied to these problems and illustrated through simple examples.

In the above works, the authors address the *redployment* problem to improve network performance. More recent research results, such as [50, 51], consider the following problem. Given a sensor network and a mission domain (the domain to be sampled)  $\mathcal{D}$ , develop closed-loop control strategies such that each point in  $\mathcal{D}$  is sampled by some agents in the network by an amount of effective coverage equal to  $C^*$ . In the discrete setting, the goal may be understood as the collection of at least  $C^*$  measurements of a physical quantity at each point in  $\mathcal{D}$  using a group of limited-range sensors. The goal is to dynamically survey the mission domain while the agents are moving in the mission space. This problem is known in the robotics literature as the coverage path planning problem, where a single limited-range sensor agent needs to visit all points in the environment (see, for example [1, 16], and references therein).

For the coverage path planning problem for networks of multiple sensor-equipped robots, in [50] a deterministic approach is pursued and a convergent cooperative feedback control law is proposed that achieves a satisfactory coverage of  $\mathcal{D}$ , while avoiding converging to local minima of a defined coverage error. These results were motivated by approaches studied in [15, 47] for (optimal and suboptimal) motion planning of multiple spacecraft interferometric imaging systems (MSIIS). In [51], the authors also guarantee collision avoidance, and in [52] they further modify the control law to guarantee collision-free coverage with a flocking behavior.

Regarding collision avoidance, the authors in [51, 52] mainly rely on the use of barrier-type functions originally developed in [57, 58] for collision avoidance in two-agent systems. Later these results were generalized for multiagent (more than two)

systems in noncooperative [83] and cooperative settings [82]. A decentralized scheme for collision avoidance of multiple independent nonpoint agents was developed in [36] using a methodology based on navigation functions. In this paper, we do not include collision avoidance control. However, once cooperative coverage control strategies have been developed (as in this paper), one can easily append collision avoidance control components to the coverage control law to achieve safe coverage of the domain.

In the stochastic setting, the authors in [17] develop a Kalman filter-based algorithm that aims to use a mobile sensor network for estimating the state of a *single* target (that is, this is not a coverage control problem). The author in [48] uses the Kalman filter for estimating a spatially decoupled (i.e., it does not satisfy a PDE) field and uses the prediction step of the filter for guiding the vehicles to move in directions that improve the field estimate. Moreover, the control algorithm is modified to guarantee satisfactory global coverage of the domain. A similar coverage problem formulation is addressed in [43], but from an information theoretic perspective that does not discuss dwelling into local minima of the metric of choice. Motivated by the employment of sensing devices for the estimation of spatially distributed processes (unsteady diffusion-type parabolic PDEs) in [26], the authors in [49] extend these results to the case where the field of interest satisfies a PDE. The present paper is an extension of [49], where we presently formalize the mathematical approach and address some of the mathematical intricacies in the formulation. We also present the estimation problem in a more general abstract formulation that is important to understand. Such an abstract framework is conducive to optimization that emanates from the subsequent inclusion of communication and decision/actuation considerations.

A common aspect of all the results mentioned above is the assumption that the field to be measured is static, especially in the spatial sense. In this paper we consider the case where the field to be measured satisfies some PDE, i.e. *evolves both in time and space*. Applications for both immobile and mobile sensor networks include the following:

- Aerial wildfire control in inaccessible and rugged country, where the temperature distribution, satisfying a PDE [3, 39, 63], has to be estimated to identify critical points that require immediate deposition of fire suppression material.
- Underwater and atmospheric sampling, where the field to be measured (e.g., salinity or temperature) satisfies a particular PDE [9, 54, 78, 79].
- Health monitoring of civil infrastructures such as bridges and “smart” buildings [13], wherein a network of sensors is used to monitor vital structural changes due to wear, fire, oxidation, cyclic loading, and earthquakes.
- Oil spill and ground water contamination, where mobile agents with sensing and possible actuating capabilities are used to contain/encircle moving boundaries of contaminating fluids. For the former, flotillas equipped with computational, sensing, and limited actuation capabilities attempt to encircle and contain moving boundaries caused by contaminating oil spills, relay information to a base station, and possibly take limited action by minimizing the environmental effect of the contaminating substance via an appropriate dispersion of neutralizing agents.
- Other applications, such as MSIIS, surveillance, and aerial mapping, where the PDE does not have a diffusion term (since “information” does not diffuse spatially) and the PDE is elliptic. Basically, these are Poisson-like PDEs which describe the steady-state solutions of unsteady diffusion-advection processes. This is a special subclass of PDEs that also fall under the more general

class of PDEs considered here. When such elliptic PDEs are considered, one simply embeds them in an unsteady or parabolic process and proceeds with the guidance scheme proposed in this paper. The embedding of such elliptic PDEs is similar in spirit to the work in [45] for the adaptive parameter identification in ground water hydrology.

Communications, in particular wireless communication, in networks is a crucial issue in cooperative networks. Issues include lossy communications, fading channels, dynamic communication structures, and communication-induced time delays. While this paper does not address the issue of communications aspects, it deserves much attention and is the subject of current and future research by the authors. However, this paper lays down the abstract mathematical framework in which one can naturally augment communication and actuation aspects via the penalization of an associated performance measure. For more on communication in networked systems, we refer the reader to the review paper [5].

In the context of sensor networks, the papers [69, 70] address distributed sensing and communication under communication constraints and derive tradeoffs between communication and sensing requirements in a decentralized mobile sensor network, respectively. We refer the reader to these papers and references therein for more on constraints imposed by communications (such as channel fading) on sensor network performance.

In a parallel fashion to the above research efforts on sensor and actuator networks, there was a considerable amount of research done by the infinite dimensional systems community, which considered the placement and scheduling of sensing and actuating devices in systems governed by spatially distributed processes; such examples include thermal manufacturing, chemical transport processes, and mechanical structures. The basic idea was to optimally place and schedule such devices within the spatial domain. The PDE interpretation on “move” or “schedule” or “scan” sensors and actuators to improve performance of the filter or the controller and to enhance the identifiability of the parameter estimation scheme translated into studying the well-posedness of an associated evolution equation. More specifically, the placement and scheduling of sensing and actuating devices was equivalent to choosing the output and input operators that were parameterized by the spatial position of these devices. An added dimension to the positioning of sensing and actuating devices in processes described by PDEs was the issue of locations that resulted in partial or complete loss of observability and controllability. For a one-dimensional diffusion equation this amounts to the avoidance of placing pointwise-in-space sensing and actuating devices at the zeros of the associate spatial operator of the system; this then relates to the definition of approximate observability and controllability [21]. Such works started to appear, at least in the open literature, in the late 1970s and throughout the 1980s both in the West and the former Soviet Union. For the former, works in [7, 20, 55, 68, 56] provided the seed for viewing the spatial location of sensing and actuating devices as another level of control and optimization. For the latter, early works by Butkovskii [10, 11, 12] paved the way for the eventual guidance of sensors and actuators in systems governed by PDEs and whose state is a field over a spatial domain.

The process to be estimated is naturally described by a PDE. A system-theoretic approach to studying PDEs has received much attention over the years, addressing various issues such as control and filter design, optimization, and finite dimensional approximation. Such efforts have been reported, for example, in the texts [21, 61, 74]. Related to the work under consideration is the issue of the placement of sensors at fixed positions for improved state and parameter estimation, fault tolerance, and

observer-based closed-loop performance [20, 29, 86]. Closer to the proposed issue of mobile sensors is the work by Uciński and coworkers [81, 87, 88] and Nehorai and coworkers [72]. The above works dealt primarily with optimal motion planning of mobile sensors (robots) for parameter estimation. By utilizing an information theoretic approach, via the use of the Fisher information matrix, in the optimization of an objective function along with additional robot motion constraints, an optimal path planning policy was derived. While similar to the previous works, the current work considers the motion planning of mobile sensors in distributed systems for improved state estimation.

The proposed mobile sensor motion planning is based on Lyapunov stability techniques. In addition to the method used to derive the motion planning, the proposed work considers spatially distributed sensors as opposed to point sensors. Related to the above is the work in [26], where a network of fixed-position pointwise sensors was utilized for detection of a moving source (intrusion detection) for a diffusion process (i.e., the field to be measured was the solution to a diffusion-advection PDE) in a two-dimensional spatial domain. A sensor management scheme was proposed in order to minimize power consumption by having a subset of the available sensors in transmit mode and having the remaining sensors in the network in sleep mode. The detection scheme would activate, over the duration of a given time interval, the relevant sensors within a radius to the moving source and deactivate the sensors that were outside a ball surrounding the centroid of the moving source. A state estimator was subsequently incorporated into the moving source detection scheme in [28] for the same diffusion process, and eventually a containment policy utilizing actuating devices collocated to the mobile sensors was considered in [27]. Such a containment policy aimed at providing limited local-in-space control action of the sensors that were within proximity of the moving source over the duration of a given time interval.

For the sake of exposition, the process under consideration here is governed by a one-dimensional diffusion-advection process, and the results are extendable to the two-dimensional case with minor adjustments for the sensor motion. Moreover, such a multidimensional extension requires attention to some technical issues pertaining to the existence and uniqueness of solutions to certain evolution equations with non-simply connected spatial domains, having nonsmooth boundaries, both internal and external, and the well-posedness of Lyapunov functions and their time derivatives that are subsequently used for the stability analysis. Additional conditions may also have to be imposed on the initial condition as well. However, the abstract framework is the same, as are the sensor navigation policies.

The paper is organized as follows. The diffusion process with its abstract framework formulation, along with the design of a state estimator, are summarized in section 2. The guidance policies (path planning) for the mobile sensors along with the requisite stability results are presented in section 3. Numerical studies of a one-dimensional diffusion-advection process utilizing both proposed guidance policies are reported in section 4, with conclusions and future research following in section 5.

**2. Mathematical formulation and problem statement.** All notation in this paper is standard. For Banach spaces  $X$  and  $Y$ ,  $\mathcal{L}(X, Y)$  denotes the space of bounded linear operators from  $X$  into  $Y$ . All inner products  $\langle \cdot, \cdot \rangle$  are assumed to be linear in their first argument and to be conjugate linear in the second. Additionally,  $\langle \phi, \psi \rangle \triangleq \langle \phi, \psi \rangle_{X, Y}$  denotes the action of the linear functional  $\psi \in Y$  on the element  $\phi \in X$ , and  $\langle \psi, \phi \rangle \triangleq \langle \psi, \phi \rangle_{Y, X}$  denotes the actions of the conjugate linear functional  $\psi \in Y$  on the element  $\phi \in X$ .

**2.1. One-dimensional diffusion process.** The diffusion process under consideration is modeled by a parabolic PDE on the bounded interval  $\Omega = [0, \ell] \subset \mathbb{R}$ . The state of the system is denoted by  $x(t, \xi)$ , where  $\xi \in \Omega$  denotes the spatial variable and  $t \in [0, \infty[$  is the time variable. The PDE is given by

$$(2.1) \quad \frac{\partial x}{\partial t}(t, \xi) = a_1 \frac{\partial^2 x}{\partial \xi^2}(t, \xi) - a_2 \frac{\partial x}{\partial \xi}(t, \xi) - a_3 x(t, \xi) + b_1(\xi)w(t) + b_2(\xi)u(t),$$

where  $a_1, a_2, a_3 > 0$ , along with Dirichlet boundary conditions  $x(t, 0) = x(t, \ell) = 0$  and initial condition  $x(0, \xi) = x_0(\xi) \in L_2(\Omega)$ . The function  $b_1(\xi) \in L_2(\Omega)$  denotes the spatial distribution of the process noise and  $w(t)$  denotes its temporal component. Similarly, the function  $b_2(\xi) \in L_2(\Omega)$  denotes the spatial distribution of the input function, while  $u(t)$  denotes its temporal component. For the case of spatiotemporally moving inputs, or mobile controls, one may consider  $b_2(\xi; \xi_a(t))$ , where  $\xi_a(t)$  denotes the time-varying location of the mobile actuating device. Such mobile actuators and their associated control policies were considered in [12, 23, 24, 25, 27, 30, 31, 32, 33, 34, 35, 53].

Spatially distributed measurements from  $m$  sensors are assumed to be available over the spatial intervals  $[\xi_k^s - \Delta\xi \leq \xi \leq \xi_k^s + \Delta\xi]$ ,  $k = 1, 2, \dots, m$ ,

$$y(t, \xi; \xi^s) = \begin{bmatrix} c(\xi; \xi_1^s)x(t, \xi) + d(\xi; \xi_1^s)v(t) \\ c(\xi; \xi_2^s)x(t, \xi) + d(\xi; \xi_2^s)v(t) \\ \vdots \\ c(\xi; \xi_m^s)x(t, \xi) + d(\xi; \xi_m^s)v(t) \end{bmatrix},$$

where  $\xi_k^s$  denotes the  $k$ th sensor position within the domain  $[0, \ell]$ ,  $\xi^s = [\xi_1^s, \dots, \xi_m^s] \in \mathbb{R}^m$  denotes the vector of sensor locations,  $\Delta\xi$  denotes the one-half spatial support of the sensing device, and  $c(\xi; \xi_k^s) \in L_2(0, \ell)$ ,  $k = 1, 2, \dots, m$ , denotes the output shaping function associated with the  $k$ th sensor. The spatial distribution (shaping function) of the measurement noise is denoted by  $d(\xi; \xi_k^s) \in L_2(0, \ell)$  and is similarly defined on the interval  $[\xi_k^s - \Delta\xi \leq \xi \leq \xi_k^s + \Delta\xi]$ . Its temporal component is denoted by  $v(t)$ . Example distributions for the output measurement and output noise shaping functions are depicted in Figure 2.1. Other distributions may also be considered. Examples include a Gaussian function or any other polynomial or trigonometric function. Figure 2.1(a) depicts both the box function and its smoothed approximation. The smoothed approximation is necessary for both regularity and numerical implementation requirements. Regarding regularity, smoothing guarantees well-posedness and certain system-theoretic properties, such as approximate observability, of the infinite dimensional system described below, to easily follow from already established results. On the numerical implementation side, smoothing aims to avoid Gibb's type phenomena in the numerical approximation of nonsmooth functions such as the box function. In fact, the following polynomial representation is used here for the smoothed approximation of the box function:

$$c(\xi; \xi_k^s) = \begin{cases} 1 & \text{if } \xi \in [\xi_k^s - 0.6\Delta\xi, \xi_k^s + 0.6\Delta\xi], \\ 1 - 3\xi_{l_k}^2 - 2\xi_{l_k}^3 & \text{if } \xi \in [\xi_k^s - \Delta\xi, \xi_k^s - 0.6\Delta\xi], \\ 1 - 3\xi_{r_k}^2 + 2\xi_{r_k}^3 & \text{if } \xi \in [\xi_k^s + 0.6\Delta\xi, \xi_k^s + \Delta\xi], \\ 0 & \text{otherwise,} \end{cases}$$

where  $\xi_{r_k} = \frac{\xi - \xi_k^s - 0.6\Delta\xi}{0.4\Delta\xi}$  and  $\xi_{l_k} = \frac{\xi - \xi_k^s + 0.6\Delta\xi}{0.4\Delta\xi}$ . A similar cubic polynomial is used to smooth the spatial distribution of the measurement noise. As shown in Figure 2.1(b),

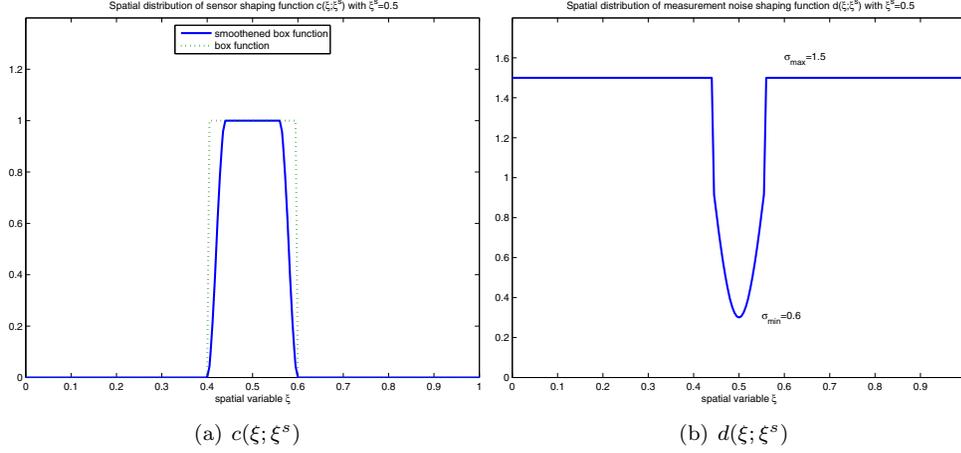


FIG. 2.1. Spatial distributions  $c(\xi; \xi^s)$  and  $d(\xi; \xi^s)$  for  $\xi^s = \ell/2$ .

the value of  $d(\xi; \xi_k^s)$  is equal to  $\sigma_{max}$  outside the sensor range  $[\xi_k^s - \Delta\xi, \xi_k^s + \Delta\xi]$ , but its contribution to the output measurement is removed by multiplying it by the box function, thereby eliminating the introduction of noise outside the sensor range, i.e., excluding noise from the spatial interval  $[0, \ell] \setminus [\xi_k^s - \Delta\xi, \xi_k^s + \Delta\xi]$ . The noise effects are smaller at the center of the sensor range and increase as one moves away from the center. Similarly, the expression for  $d(\xi; \xi^s)$  is

$$d(\xi; \xi_k^s) = \begin{cases} \sigma_{max} & \xi \in [\xi_k^s - \Delta\xi, \xi_k^s - \Delta\xi]^c, \\ \sigma_{max}(1 - 3\xi_{r_k}^2 + 2\xi_{r_k}^3) & \xi \in [\xi_k^s + 0.6\Delta\xi, \xi_k^s + \Delta\xi], \\ \sigma_{max}(1 - 3\xi_{l_k}^2 - 2\xi_{l_k}^3) & \xi \in [\xi_k^s - \Delta\xi, \xi_k^s - 0.6\Delta\xi], \\ \sigma_{min} + 2(\sigma_{max} - \sigma_{min})\Xi^2 - (\sigma_{max} - \sigma_{min})\Xi_k^4 & \text{otherwise,} \end{cases}$$

where  $[\xi_k^s - \Delta\xi, \xi_k^s - \Delta\xi]^c = [0, \ell] \setminus [\xi_k^s - \Delta\xi, \xi_k^s + \Delta\xi]$  and  $\Xi_k = \frac{\xi - \xi_k^s}{\Delta\xi}$ .

When mobile sensors are considered, then one has

$$(2.2) \quad y(t; \xi^s) = \begin{bmatrix} c(\xi; \xi_1^s(t))x(t, \xi) + d(\xi; \xi_1^s(t))v_1(t) \\ c(\xi; \xi_2^s(t))x(t, \xi) + d(\xi; \xi_2^s(t))v_2(t) \\ \vdots \\ c(\xi; \xi_m^s(t))x(t, \xi) + d(\xi; \xi_m^s(t))v_m(t). \end{bmatrix},$$

which explicitly models the sensor position motion via the time-variation of the second argument of the output measurement shaping function  $c(\xi; \xi_i^s(t))$  and noise measurement shaping function  $d(\xi; \xi_i^s(t))$  for the  $i$ th sensing device.

**2.2. Abstract formulation.** For well-posedness and stability of the proposed estimation scheme, the above PDE given in (2.1), (2.2) will be viewed in an abstract framework. Such an abstract framework includes a larger class of PDEs, and hence the results can be easily applied to any member of this class of systems.

We let  $\mathcal{X}$  be a Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and corresponding induced norm  $|\cdot|$ . Let  $\mathcal{V}$  be a reflexive Banach space with norm denoted by  $\|\cdot\|$ , and assume that  $\mathcal{V}$  is embedded densely and continuously in  $\mathcal{X}$  [77, 90]. Let  $\mathcal{V}^*$  denote the conjugate

dual of  $\mathcal{V}$  (in other words, the space of continuous conjugate linear functionals on  $\mathcal{V}$ ) and  $\|\cdot\|_*$  denote the usual uniform operator norm on  $\mathcal{V}^*$ . It follows that

$$(2.3) \quad \mathcal{V} \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{V}^*,$$

with both embeddings dense and continuous [22, 80]. Specifically, we assume that

$$(2.4) \quad |\phi| \leq c\|\phi\|, \quad \phi \in \mathcal{V},$$

for some positive constant  $c$ . The notation  $\langle \cdot, \cdot \rangle$  will also be used to denote the duality pairing between  $\mathcal{V}^*$  and  $\mathcal{V}$  induced by the continuous and dense embeddings given in (2.3); that is, for  $\phi \in \mathcal{V}^*$  and  $\psi \in \mathcal{V}$ ,  $\langle \phi, \psi \rangle$  denotes the action of the bounded linear functional  $\phi$  on the vector  $\psi$ . This quantity simply reduces to  $\langle \phi, \psi \rangle$  if  $\phi \in \mathcal{X}$ ; i.e., the value of  $\phi$  acting on  $\psi$  is equal to the  $\mathcal{X}$  inner product of  $\phi$  and  $\psi$ .

We consider a linear operator  $\mathcal{A} : \mathcal{V} \rightarrow \mathcal{V}^*$  satisfying the following assumptions:

(A1)  $\mathcal{V} \rightarrow \mathcal{V}^*$ -*boundedness*: There exists  $\alpha > 0$  such that

$$|\langle \mathcal{A}\phi, \psi \rangle| \leq \alpha\|\phi\| \|\psi\| \quad \text{for } \phi, \psi \in \mathcal{V}.$$

(A2)  $\mathcal{V}$ -*coercivity*: the operator  $-\mathcal{A}$  is coercive, i.e.,

$$\operatorname{Re} \langle -\mathcal{A}\phi, \phi \rangle \geq \beta\|\phi\|^2 \quad \text{for some positive } \beta \text{ and } \phi \in \mathcal{V}.$$

Additionally, we may impose the following symmetry condition which, while it simplifies the stability analysis, nonetheless restricts the class of systems (i.e., diffusion processes) to which the proposed sensor navigation and state estimation policy is applicable.

(A3) *Symmetry*: the operator  $\mathcal{A}$  is symmetric:

$$\langle \mathcal{A}\phi, \psi \rangle = \overline{\langle \mathcal{A}\phi, \psi \rangle} \quad \text{for all } \phi, \psi \in \mathcal{V}.$$

For ease of exposition, we have assumed that the operator  $\mathcal{A}$  is time invariant. However, it is relatively straightforward to extend all of the results in this paper to the case of a time-dependent operator  $\mathcal{A}(t)$ ,  $t \geq 0$ . One need only make some standard assumptions on the regularity of the map  $t \rightarrow \mathcal{A}(t)$ ,  $t \geq 0$ , for the present results to remain valid [8, 61, 76, 85].

We consider the disturbance operator  $\mathcal{B}_1 : \mathbb{R} \rightarrow \mathcal{V}^*$  and the *input* operator  $\mathcal{B}_2 : \mathbb{R} \rightarrow \mathcal{V}^*$ . When the control and disturbance signals are assumed to be square integrable, i.e., yielding  $\mathcal{B}_1 w + \mathcal{B}_2 u \in L_2(0, t, \mathcal{V}^*)$ , and  $x(0) = x_0 \in \mathcal{X}$ , then the initial value problem (IVP)

$$(2.5) \quad \frac{d}{dt}x(t) = \mathcal{A}x(t) + \mathcal{B}_1 w(t) + \mathcal{B}_2 u(t), \quad x_0 \in \mathcal{X},$$

is well-posed. By a solution to the above (IVP), we mean a weak solution [76]; this means a function  $x \in L_2(0, t; \mathcal{V})$  with  $\frac{d}{dt}x \in L_2(0, t; \mathcal{V}^*)$  for all  $t > 0$  that satisfies (2.5) [76, 90].

Following [21], the PDE in (2.1) may be expressed in the abstract form (2.5). The state space in this case is  $\mathcal{X} = L_2(0, \ell)$ , where  $x(t, \cdot) = \{x(t, \xi), 0 \leq \xi \leq \ell\}$  denotes the state. The space  $\mathcal{V}$  is identified by the Sobolev space  $\mathcal{V} = H_0^1(0, \ell)$ . In the remainder of the paper we will, with a slight abuse of notation, use  $x(t)$  as the solution to the evolution equation (2.5) and use  $x(t, \xi)$  as the solution to the PDE (2.1). Under the

above representation, the system's second order (strongly) elliptic operator  $\mathcal{A}$  and its domains are given by [22]:

$$\begin{aligned}\mathcal{A}\phi &= a_1 \frac{d^2\phi}{d\xi^2} - a_2 \frac{d\phi}{d\xi} - a_3\phi, \quad \phi \in \text{Dom}(\mathcal{A}), \\ \text{Dom}(\mathcal{A}) &= H^2(0, \ell) \cap H_0^1(0, \ell) \\ &= \{\psi \in L_2(0, \ell) \mid \psi, \psi' \text{ abs. continuous and } \psi(0) = 0 = \psi(\ell)\}.\end{aligned}$$

We now verify the boundedness and coercivity assumptions (A1) and (A2) for the above system:

$$\begin{aligned}|\langle \mathcal{A}\phi, \psi \rangle| &= \left| \int_0^\ell \left( a_1 \frac{d^2\phi(\xi)}{d\xi^2} - a_2 \frac{d\phi(\xi)}{d\xi} - a_3\phi(\xi) \right) \psi(\xi) d\xi \right| \\ &\leq a_1 \left| \int_0^\ell \frac{d^2\phi(\xi)}{d\xi^2} \psi(\xi) d\xi \right| + a_2 \left| \int_0^\ell \frac{d\phi(\xi)}{d\xi} \psi(\xi) d\xi \right| + a_3 \left| \int_0^\ell \phi(\xi) \psi(\xi) d\xi \right| \\ &\leq a_1 \sqrt{\int_0^\ell \left( \frac{d\phi(\xi)}{d\xi} \right)^2 d\xi} \sqrt{\int_0^\ell \left( \frac{d\psi(\xi)}{d\xi} \right)^2 d\xi} \\ &\quad + a_2 \sqrt{\int_0^\ell \left( \frac{d\phi(\xi)}{d\xi} \right)^2 d\xi} \sqrt{\int_0^\ell \psi^2(\xi) d\xi} + a_3 \sqrt{\int_0^\ell \phi^2(\xi) d\xi} \sqrt{\int_0^\ell \psi^2(\xi) d\xi} \\ &= a_1 \|\phi\| \|\psi\| + a_2 \|\phi\| \|\psi\| + a_3 \|\phi\| \|\psi\| \\ &\leq a_1 \|\phi\| \|\psi\| + a_2 \|\phi\| c \|\psi\| + a_3 c^2 \|\phi\| \|\psi\| = (a_1 + a_2 c + a_3 c^2) \|\phi\| \|\psi\|,\end{aligned}$$

where we used the triangle inequality in the first step and used the fact that the space  $\mathcal{V}$  is embedded in  $\mathcal{X}$ . This proves boundedness. To show coercivity, note that

$$\begin{aligned}\langle -\mathcal{A}\phi, \phi \rangle &= \int_0^\ell - \left( a_1 \frac{d^2\phi(\xi)}{d\xi^2} - a_2 \frac{d\phi(\xi)}{d\xi} - a_3\phi(\xi) \right) \phi(\xi) d\xi \\ &= -a_1 \int_0^\ell \frac{d^2\phi(\xi)}{d\xi^2} \phi(\xi) d\xi + a_2 \int_0^\ell \frac{d\phi(\xi)}{d\xi} \phi(\xi) d\xi + a_3 \int_0^\ell \phi(\xi) \phi(\xi) d\xi \\ &\geq a_1 \int_0^\ell \left( \frac{d\phi(\xi)}{d\xi} \right)^2 d\xi + a_2 \int_0^\ell \frac{d}{d\xi} \phi^2(\xi) d\xi + a_3 \int_0^\ell \phi^2(\xi) d\xi \\ &= a_1 \|\phi\|^2 + a_2 |\phi'|^2 + a_3 \|\phi\|^2 \geq a_1 \|\phi\|^2.\end{aligned}$$

It should be noted, however, that due to the presence of a nonzero coefficient  $a_2$ , the operator is not symmetric.

The input operator is given by

$$\mathcal{B}_2 u(t) = b_2(\xi) u(t), \quad \mathcal{B}_2 \in \mathcal{L}(\mathbb{R}, \mathcal{X}).$$

The disturbance (process noise) operator  $\mathcal{B}_1$  is given similarly by

$$\mathcal{B}_1 w(t) = b_1(\xi) w(t), \quad \mathcal{B}_1 \in \mathcal{L}(\mathbb{R}, \mathcal{X}).$$

Similarly, the output equation (2.2) may be written as

$$(2.6) \quad y(t; \xi^s) = \mathcal{C}(\xi^s(t)) x(t) + \mathcal{D}(\xi^s(t)) v(t),$$

where the output measurement and noise operators are parameterized by the sensor location vector  $\xi^s$ . These operators are given via  $\mathcal{C}(\cdot) : \mathcal{V} \rightarrow \underbrace{\mathcal{V}^* \times \mathcal{V}^* \times \cdots \times \mathcal{V}^*}_m$  by

$$\langle \mathcal{C}(\xi^s(t))\phi, \psi \rangle = \begin{bmatrix} \int_0^\ell c(\xi; \xi_1^s(t))\phi(\xi)\psi(\xi) d\xi \\ \int_0^\ell c(\xi; \xi_2^s(t))\phi(\xi)\psi(\xi) d\xi \\ \vdots \\ \int_0^\ell c(\xi; \xi_m^s(t))\phi(\xi)\psi(\xi) d\xi \end{bmatrix},$$

and via  $\mathcal{D}(\cdot) : \mathbb{R}^m \rightarrow \underbrace{\mathcal{V}^* \times \mathcal{V}^* \times \cdots \times \mathcal{V}^*}_m$  by

$$\langle \mathcal{D}(\xi^s(t))\nu, \psi \rangle = \begin{bmatrix} \int_0^\ell d(\xi; \xi_1^s(t))\psi(\xi) d\xi \nu_1 \\ \int_0^\ell d(\xi; \xi_2^s(t))\psi(\xi) d\xi \nu_2 \\ \vdots \\ \int_0^\ell d(\xi; \xi_m^s(t))\psi(\xi) d\xi \nu_m \end{bmatrix}.$$

**2.3. Problem statement.** The problem at hand is to propose a state estimator for the evolution system (2.5), with measurements given by (2.2), and to provide a motion planning strategy of the mobile sensors in (2.2) in order to yield a more efficient state estimator.

**2.4. State estimation process with time-varying output operator.** For an arbitrary but fixed sensor location  $\xi^s$ , one may consider the associated state estimator in  $\mathcal{X}$ ,

$$(2.7) \quad \dot{\hat{x}}(t) = \mathcal{A}\hat{x}(t) + \mathcal{B}_2 u(t) + \mathcal{L}(\xi^s)(y(t) - \mathcal{C}(\xi^s)\hat{x}(t)),$$

where  $\hat{x}(0) = \hat{x}_0 \in \mathcal{X}$  with  $\hat{x}(0) \neq x(0)$ , and  $\mathcal{L}(\xi^s) : \underbrace{\mathcal{V}^* \times \mathcal{V}^* \times \cdots \times \mathcal{V}^*}_m \rightarrow \mathcal{V}$  is

the associated  $\xi^s$ -parameterized observer gain derived from either a Kalman or a Luenberger filter design.

The *state estimation error*  $e(t) \triangleq x(t) - \hat{x}(t)$  for (2.5) is governed by the following evolution equation:

$$(2.8) \quad \begin{aligned} \dot{e}(t) &= \mathcal{A}e(t) - \mathcal{L}(\xi^s)(y(t) - \mathcal{C}(\xi^s)\hat{x}(t)) + \mathcal{B}_1 w(t), & e(0) &= x(0) - \hat{x}(0) \in \mathcal{X}. \\ &= (\mathcal{A} - \mathcal{L}(\xi^s)\mathcal{C}(\xi^s))e(t) + \mathcal{B}_1 w(t) - \mathcal{L}(\xi^s)\mathcal{D}(\xi^s)v(t), \end{aligned}$$

The associated distributed *output estimation error*  $\varepsilon(t; \xi^s)$  corresponding to  $m$  sensor locations represented by the vector  $\xi^s$  is given by

$$(2.9) \quad \varepsilon(t; \xi^s) \triangleq y(t; \xi^s) - \mathcal{C}(\xi^s)\hat{x}(t).$$

Next, this output error will be used to generate the navigation policies for the mobile sensors.

**3. Navigation of spatially mobile sensors.** The sensor locations  $\xi^s$  considered above are now allowed to vary with time, and thus the above observation and measurement noise operators are time dependent. The associated state estimator is now given by (cf. (2.7))

$$(3.1) \quad \dot{\hat{x}}(t) = \left( \mathcal{A} - \mathcal{L}(\xi^s(t))\mathcal{C}(\xi^s(t)) \right) \hat{x}(t) + \mathcal{B}_2 u(t) + \mathcal{L}(\xi^s(t))y(t),$$

which results in the estimation error equation (cf. (2.8))

$$(3.2) \quad \begin{aligned} \dot{e}(t) &= \left( \mathcal{A} - \mathcal{L}(\xi^s(t))\mathcal{C}(\xi^s(t)) \right) e(t) + \mathcal{B}_1 w(t) - \mathcal{L}(\xi^s(t))\mathcal{D}(\xi^s(t))v(t), \\ e(0) &= x(0) - \hat{x}(0) \in \mathcal{X}. \end{aligned}$$

Similarly, the output estimation error (cf. (2.9)) is given by

$$(3.3) \quad \varepsilon(t; \xi^s(t)) = \mathcal{C}(\xi^s(t))e(t) + \mathcal{D}(\xi^s(t))v(t).$$

One may consider an optimal sensor scheduling, as was developed in [7], to derive the position of the mobile sensors. While the resulting sensor guidance will be optimal, it would nonetheless result in computationally intensive implementation requiring the solution to differential Riccati operator equations. Motivated by computational considerations, we consider sensor guidance schemes that may forsake optimality for ease of implementation and reduction of the computational load. Additionally, we do not necessarily consider a finite horizon problem, and thus one may have to address the issue of observability. Thus, we assume that the sensor guidance scheme navigates the mobile sensors only in the spatial locations that render the system approximately observable [21]. To avoid such locations, we define the set of admissible locations as

$$(3.4) \quad \Theta_{adm} = \left\{ \xi_i^s \in \Omega : (\mathcal{C}(\xi^s), \mathcal{A}) \text{ is approximately observable} \right\}.$$

Any sensor scheduling will then be constrained to the set  $\Theta_{adm}$ . While we will not explicitly impose this condition, one may incorporate such an admissibility condition into a collision avoidance navigation scheme, whereby both undesirable locations and locations that render the system unobservable will be avoided.

The above error provides distributed information of the estimation error throughout the support of a given sensing device, i.e., over  $[\xi_i^s - \Delta\xi, \xi_i^s + \Delta\xi]$ . Using only this spatially distributed error, we propose two guidance policies. The first guidance policy moves the center of the  $i$ th sensing device so as to minimize its spatial  $L_\infty$  norm within the spatial interval  $[\xi_i^s - \Delta\xi, \xi_i^s + \Delta\xi]$ . Such a guidance policy renders the resulting infinite dimensional system a switched system whereby the position of the  $m$  sensors changes at discrete time instances. The second guidance policy considers the global distributed error associated with the nominal noise-free process, and by embedding the sensor position in the process dynamics, a guidance law is derived using Lyapunov stability arguments.

**3.1. Case 1: Guidance using localized measurement error.** Assuming that a given sensor can only move a maximum distance of  $\pm\Delta\xi$  from its current position  $\xi_i^s(t_k)$ , i.e., move anywhere within  $[\xi_i^s(t_k) - \Delta\xi, \xi_i^s(t_k) + \Delta\xi]$ , and taking into account velocity constraints which translate into restrictions on the frequency of the switching positions, we consider the time instances  $t_0 + k\Delta t$ ,  $k = 0, 1, 2, \dots$ . The proposed sensor position switching is given by

$$(3.5) \quad \xi_i^s(t_{k+1}) = \arg \max_{\xi_i^s(t_k) - \Delta\xi \leq \xi \leq \xi_i^s(t_k) + \Delta\xi} |\varepsilon(t_k, \xi; \xi_i^s(t_k))|$$

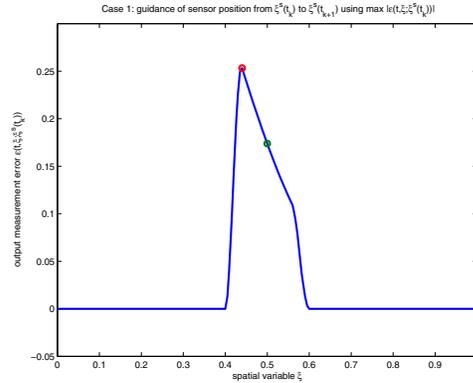


FIG. 3.1. Guidance of moving sensor from position  $\xi^s(t_k)$  (lower dot) to position  $\xi^s(t_{k+1})$  (upper dot) using localized measurement error  $\varepsilon(t, \xi; \xi^s(t_k))$ .

for  $i = 1, 2, \dots, m$ , and which basically finds the maximum of the spatially distributed measurement error over the domain of definition  $[\xi_i^s(t_k) - \Delta\xi, \xi_i^s(t_k) + \Delta\xi]$  of the current sensor position  $\xi_i^s(t_k)$ , and moves the  $i$ th sensor to that maximum. Figure 3.1 depicts a scenario with the current  $\xi^s(t_k)$  and subsequent sensor position  $\xi^s(t_{k+1})$ .

REMARK 3.1. Note that if  $\varepsilon(t, \xi; \xi_i^s(t_k)) = 0$  inside the  $i$ th sensor domain, then sensor  $i$  will not move. If  $\varepsilon(t, \xi; \xi_i^s(t_k)) = 0$  for all sensors  $i$ , then none of the sensors will move, even if the error is nonzero outside the sensor domains. In the spatially decoupled case (i.e., no PDE is satisfied), this is problematic since the error outside all sensor domains is not “transmitted” to the error within the sensor domains. In this case, the sensors are immobile with a nonzero global error. This situation was called Condition **C1** in [51]. To resolve this issue, the authors in [51] propose perturbation control laws that transfer sensors with zero error within the domain to a neighborhood of a point (outside the sensor domain) with nonzero error. Once there, the control law is switched back to (3.5). It was shown in [51] that such a switching policy causes the coverage error to converge to zero and that infinite switching is impossible if the domain  $\mathcal{D}$  is compact.

For the spatially coupled case, under the assumption that the PDE system is approximately observable, which is guaranteed by restricting the sensor motion to the set  $\Theta_{adm}$ , one has that the error inside the sensor domains is zero if and only if the global error is zero. Hence Condition **C1** (that error inside the sensor domain is zero with nonzero global error) described above will never occur and switching is not required.

For the specific case of the observer operator gain  $\mathcal{L}(\xi^s(t)) = \mathcal{C}^*(\xi^s(t))$ , the above state error (2.8) reduces to the switched infinite dimensional system

$$(3.6) \quad \begin{aligned} \dot{e}(t) &= \left( \mathcal{A} - \mathcal{C}^*(\xi^s(t))\mathcal{C}(\xi^s(t)) \right) e(t) + \mathcal{B}_1 w(t) - \mathcal{C}^*(\xi^s(t))\mathcal{D}(\xi^s(t))v(t), \\ e(0) &= x(0) - \hat{x}(0) = e_0 \in \mathcal{X}. \end{aligned}$$

We examine the stability of the above switched system (3.5), (3.6) within the context of switched infinite dimensional systems. We use the notation  $\Xi_k$  to denote the sensor position throughout the time interval  $[t_k, t_k + \Delta t)$ ; i.e., the sensors will maintain the same position  $\xi^s(t)$  for the duration of the time interval  $[t_k, t_k + \Delta t)$ ,

$$\xi^s(t) = \Xi^s, \quad t \in [t_k, t_k + \Delta t).$$

We define the operators  $\mathcal{A}_i = \mathcal{A} - \mathcal{C}^*(\Xi_i)\mathcal{C}(\Xi_i)$  for  $\Xi_i \in \Theta_{adm}$ . In view of the above, we consider the family of infinitesimal generators  $\mathfrak{A} = \{\mathcal{A}_i, i \in \mathcal{I}\}$  on  $\mathcal{X}$  parameterized by some index set  $\mathcal{I}$ . We let  $\sigma : [0, \infty) \rightarrow \mathcal{I}$  be a piecewise constant function of time, termed the *switching signal*. Additionally, we define the operators  $\mathfrak{B}_i : \mathcal{U} = \mathbb{R} \oplus \mathbb{R}^m \rightarrow \mathcal{X}$  and  $\mathfrak{C}_i : \mathcal{X} \oplus \mathbb{R}^m \rightarrow \mathcal{Y}$ ,

$$\begin{aligned} \mathfrak{B}_i v(t) &= \mathfrak{B}_1 w(t) - \mathcal{C}^*(\xi_i^s)\mathcal{D}(\xi_i^s)v(t), \quad v \in \mathcal{U}, \\ \mathfrak{C}_i \chi(t) &= \mathcal{C}(\xi_i^s)e(t) + \mathcal{D}(\xi_i^s)v(t), \quad \chi \in \mathcal{X} \oplus \mathbb{R}^m. \end{aligned}$$

Let  $(S_p)_{p \in \mathcal{I}}$ , for some index set  $\mathcal{I}$ , be a family of linear continuous time systems which, for each fixed  $p \in \mathcal{I}$ , is given by a state linear system  $(\mathcal{A}_p, \mathfrak{B}_p, \mathfrak{C}_p)$ ,

$$(3.7) \quad (S_p) : \begin{cases} \dot{e}(t) = \mathcal{A}_p e(t) + \mathfrak{B}_p v(t), \\ \epsilon(t) = \mathfrak{C}_p \chi(t), \end{cases}$$

where the operator  $\mathcal{A}_p = \mathcal{A} - \mathcal{C}^*(\Xi_p)\mathcal{C}(\Xi_p)$  is the infinitesimal generator of an exponentially stable semigroup  $T_p(t)$  on the Hilbert space  $\mathcal{X}$  for all  $\Xi_p \in \Theta_{adm}$ . For each  $p \in \mathcal{I}$ , the operators  $\mathfrak{B}_p$  and  $\mathfrak{C}_p$  are bounded linear operators from a Hilbert space  $\mathcal{U}$  to  $\mathcal{X}$  and from  $\mathcal{X} \oplus \mathbb{R}^m$  to a Hilbert space  $\mathcal{Y}$ , respectively. To the family  $(S_p)_{p \in \mathcal{I}}$ , we associate the set

$$\Sigma = \{\sigma \mid \sigma : [t_0, \infty) \rightarrow \mathcal{I} \text{ piecewise constant}\}$$

of all possible switches between the given systems. The *family of switched systems*  $((S_p)_{p \in \mathcal{I}}, \Sigma)$  taken under consideration are the hybrid dynamical systems consisting of the family of continuous time systems  $(S_p)_{p \in \mathcal{I}}$  together with all switching rules  $\sigma \in \Sigma$ , all initial states  $e(0) = e_0 \in \mathcal{X}$ , and all inputs  $v \in L_2([t_0, \infty); \mathcal{U})$ . For each given switching function  $\sigma$ , denote the finite set of switching time instants associated to  $\sigma$  by  $t_0 < t_1 < t_2 < \dots$ , where  $k(\sigma) \in \mathbb{N} \setminus \{0\}$ . Here,  $k(\sigma) - 1$  denotes the number of discontinuities for the piecewise continuous function  $\sigma$ . For  $k(\sigma) = 1$ , the no-switching case is obtained. Therefore we impose  $k(\sigma) \geq 2$  as a necessary condition for the nontriviality of the problem; i.e., there exists *at least* one switch.

By a solution, we mean that given a switching function  $\sigma$ , an initial condition  $e_0$  and an input  $v$ , then, on each interval  $\{[t_i, t_{i+1}]\}_{i=0}^{k(\sigma)-1}$ , the state  $e_\sigma(t)$  of the switched system  $((S_p)_{p \in \mathcal{P}}, \sigma)$  is the mild solution of the Cauchy problem (3.7) [21], i.e. for  $t_i \leq t \leq t_{i+1}$ ,

$$(3.8) \quad \begin{aligned} e_\sigma^i(t) &= T_\sigma(t) e_\sigma(t_i) + \int_{t_i}^t T_{\sigma(t_i)}(t-s) \mathfrak{B}_{\sigma(t_i)} v(s) ds, \\ \epsilon_\sigma^i(t) &= \mathfrak{C}_{\sigma(t_i)} e_\sigma^i(t) + \mathcal{D}_{\sigma(t_i)} \nu(t). \end{aligned}$$

We make the assumption that the resulting hybrid system is not a jump system.

**ASSUMPTION 3.1.** *The initial conditions for the error state at the beginning of each interval  $\{[t_i, t_{i+1}]\}_{i=1}^{k(\sigma)-1}$  are given by  $\{e_\sigma(t_i)\}_{i=1}^{k(\sigma)-1}$ , and they are considered to be the end values of the solution on the preceding time interval, i.e.,*

$$\overbrace{e_\sigma^{i+1}(t_{i+1})}^{\text{initial value at } [t_{i+1}, t_{i+2}]} = \overbrace{e_\sigma^i(t_{i+1})}^{\text{final value at } [t_i, t_{i+1}]}. \quad .$$

Based on the above assumptions, the following then leads to the existence of solutions of the error system (3.6). We state only the result since the proof follows in

a similar fashion to the one presented in [53] for scheduled actuators. As a consequence of the well-posedness, which along with the square integrability of the input signals and the exponential stability of the semigroups associated with each sensor position within  $\Theta_{adm}$ , one also has convergence of the state estimation error  $e$  to zero.

**THEOREM 3.1.** *Assume that the operator  $\mathcal{A}$  satisfies the coercivity and boundedness conditions (A1) and (A2). Furthermore, assume that for each  $\xi^s \in \Theta_{adm}$ , the operator  $\mathcal{A} - \mathcal{C}^*(\xi^s)\mathcal{C}(\xi^s)$  generates an exponentially stable semigroup on  $\mathcal{X}$  and that the control input  $u$ , along with the process and measurement noise, is square integrable, in the sense of  $\mathcal{B}_1 w + \mathcal{B}_2 u \in L_2(0, \infty; \mathcal{X})$  and  $\mathcal{B}_1 w - \mathcal{C}^*(\xi^s)\mathcal{D}(\xi^s)v \in L_2(0, \infty; \mathcal{X})$ . Then the state error system (3.6), along with the switching policy (3.5), is well posed. As a consequence of that, the state estimator (3.1), with the switching policy (3.5), is also well posed.*

**REMARK 3.2.** *It should be noted that the fact that the operator in the evolution equation (3.7) is the infinitesimal generator of an exponentially stable semigroup (uniformly for each  $\Xi_p \in \Theta_{adm}$ ), along with the fact that the forcing term  $\mathfrak{B}_p v$  is square integrable, immediately yields exponential stability of (3.7) for the nonswitched case. For the switched case with the guidance policy given by the switching rule for the sensor location in (3.5) and excluding jump systems using Assumption 3.1, one can similarly show convergence of the state estimation error to zero using similar arguments that were used in the stability of diffusion systems with scheduled actuators in [53].*

**3.2. Case 2: Guidance using global estimation error.** Since the estimation error is available only at the spatial support of the sensors, we consider the idealized process, given by

$$(3.9) \quad \begin{aligned} \dot{\bar{x}}(t) &= \mathcal{A}\bar{x}(t) + \mathcal{B}_2 u(t), \\ \bar{y}(t) &= \mathcal{C}(\xi^s(t))\bar{x}(t), \end{aligned}$$

and define the *nominal estimation error*  $\bar{e}(t) \triangleq \bar{x}(t) - \hat{x}(t)$  governed by

$$(3.10) \quad \begin{aligned} \dot{\bar{e}}(t) &= \left( \mathcal{A} - \mathcal{L}(\xi^s(t))\mathcal{C}(\xi^s(t)) \right) \bar{e}(t) + \mathcal{L}(\xi^s(t)) (\bar{y} - y) \\ &= \mathcal{A}_o(\xi^s(t))\bar{e}(t) + \mathcal{L}(\xi^s(t)) \left( \mathcal{C}(\xi^s(t)) (\bar{x} - x) - \mathcal{D}(\xi^s(t))v(t) \right), \end{aligned}$$

with  $\bar{e} \in \mathcal{X}$ , and where  $\mathcal{A}_o(\xi^s(t)) \triangleq \mathcal{A} - \mathcal{L}(\xi^s(t))\mathcal{C}(\xi^s(t))$ . The above error is generated online and simulates the idealized process (2.5), (2.6) in the absence of process and measurement noise and possibly exogenous/disturbance inputs. In a similar fashion as in the case of the process operator  $\mathcal{A}$ , we make similar boundedness and coercivity assumptions for  $\mathcal{A}_o$  in (3.10) with the constants  $\alpha, \beta$  now replaced by  $\alpha_o, \beta_o$ .

We consider the weighted  $L_2(0, \ell)$  inner product

$$(3.11) \quad \langle \bar{e}, \bar{e} \rangle_g \triangleq \langle \bar{e}(t), g\bar{e}(t) \rangle,$$

where the normalized weighting function  $g(\xi) > 0$  for all  $\xi \in \Omega$ ,  $g \in \mathcal{L}_\infty(\Omega)$ , is also known as the *distribution density function*. This function may be used to emphasize the need to cover some intervals in  $\Omega$  more than others. Similar to the notation for the boundedness and coercivity constants, when the weighted inner product is used, those constants will include  $g$  as a second subscript. For the specific PDE under consideration, the weighted inner product becomes

$$\langle \bar{e}, \bar{e} \rangle_g = \langle \bar{e}(t), \bar{e}(t) \rangle_{L_2, g} = \int_0^\ell \bar{e}^2(t, \xi) g(\xi) d\xi,$$

To obtain the sensor guidance using Lyapunov stability-based arguments, we consider the following Lyapunov-like functional:

$$(3.12) \quad \begin{aligned} V(t; \bar{e}, \xi^s(t)) &= -\frac{1}{2} \left( \langle \bar{e}(t), \mathcal{A}_o(\xi^s) \bar{e}(t) \rangle_g + \langle \mathcal{A}_o(\xi^s) \bar{e}(t), \bar{e}(t) \rangle_g \right) \\ &= -\left\langle \bar{e}(t), \left( \frac{\mathcal{A}_o(\xi^s) + \mathcal{A}_o^*(\xi^s)}{2} \right) \bar{e}(t) \right\rangle_g. \end{aligned}$$

This function is a modified version of that used in [49] and has the following explanation. The function  $V$  represents the negative of the derivative of the weighted error norm (3.11) along the trajectories of the nominal estimation error (3.10). The reason we chose  $-\frac{d}{dt} \|\bar{e}\|^2$  instead of  $\|\bar{e}\|^2$  itself for the Lyapunov-like function is that if the latter is chosen, the expression for  $\dot{V}$  will not involve the control variable  $\dot{\xi}^s$ .

For brevity, we suppress the dependence of the operator  $\mathcal{A}_o(\xi^s)$  on the sensor position and we simply write  $\mathcal{A}_o$ . Since  $\mathcal{A}_o$  is assumed coercive, then we see that the function  $V$  is positive for all nonzero  $\bar{e}$  and is zero if and only if  $\bar{e}$  is zero. The derivative of  $V$  along the trajectories of the nominal estimation error (3.10) is then given by

$$\begin{aligned} \frac{d}{dt} V &= -\left\{ \langle \dot{\bar{e}}, \mathcal{A}_o \bar{e} \rangle_{L_2, g} + \langle \bar{e}, \mathcal{A}_o \dot{\bar{e}} \rangle_{L_2, g} + \left\langle \bar{e}, \dot{\xi}^s \frac{\partial \mathcal{A}_o}{\partial \xi^s} \bar{e} \right\rangle_{L_2, g} \right. \\ &\quad \left. + \langle \mathcal{A}_o \dot{\bar{e}}, \bar{e} \rangle_{L_2, g} + \left\langle \dot{\xi}^s \frac{\partial \mathcal{A}_o}{\partial \xi^s} \bar{e}, \bar{e} \right\rangle_{L_2, g} + \langle \mathcal{A}_o \bar{e}, \dot{\bar{e}} \rangle_{L_2, g} \right\} \\ &= -\left\{ \langle \mathcal{A}_o \bar{e}, \mathcal{A}_o \bar{e} \rangle_{L_2, g} + \langle \bar{e}, \mathcal{A}_o(\mathcal{A}_o \bar{e}) \rangle_{L_2, g} \right. \\ &\quad + \left\langle \bar{e}, \dot{\xi}^s \frac{\partial (\mathcal{A} - \mathcal{L}(\xi^s) \mathcal{C}(\xi^s))}{\partial \xi^s} \bar{e} \right\rangle_{L_2, g} + \langle \mathcal{A}_o(\mathcal{A}_o \bar{e}), \bar{e} \rangle_{L_2, g} \\ &\quad + \left\langle \dot{\xi}^s \frac{\partial (\mathcal{A} - \mathcal{L}(\xi^s) \mathcal{C}(\xi^s))}{\partial \xi^s} \bar{e}, \bar{e} \right\rangle_{L_2, g} + \langle \mathcal{A}_o \bar{e}, \mathcal{A}_o \bar{e} \rangle_{L_2, g} \\ &\quad + \langle \mathcal{L}(\xi^s) (\mathcal{C}(\xi^s)(\bar{x} - x) - \mathcal{D}v), \mathcal{A}_o \bar{e} \rangle_{L_2, g} \\ &\quad + \langle \bar{e}, \mathcal{A}_o \mathcal{L}(\xi^s) (\mathcal{C}(\xi^s)(\bar{x} - x) - \mathcal{D}v) \rangle_{L_2, g} \\ &\quad + \langle \mathcal{A}_o \mathcal{L}(\xi^s) (\mathcal{C}(\xi^s)(\bar{x} - x) - \mathcal{D}v), \bar{e} \rangle_{L_2, g} \\ &\quad \left. + \langle \mathcal{A}_o \bar{e}, \mathcal{L}(\xi^s) (\mathcal{C}(\xi^s)(\bar{x} - x) - \mathcal{D}v) \rangle_{L_2, g} \right\} \\ &= -\left\{ \|\mathcal{A}_o \bar{e}\|_{L_2, g}^2 + \langle \bar{e}, \mathcal{A}_o(\mathcal{A}_o \bar{e}) \rangle_{L_2, g} - \left\langle \bar{e}, \dot{\xi}^s \frac{\partial (\mathcal{L}(\xi^s) \mathcal{C}(\xi^s))}{\partial \xi^s} \bar{e} \right\rangle_{L_2, g} \right. \\ &\quad + \langle \mathcal{A}_o(\mathcal{A}_o \bar{e}), \bar{e} \rangle_{L_2, g} - \left\langle \dot{\xi}^s \frac{\partial (\mathcal{L}(\xi^s) \mathcal{C}(\xi^s))}{\partial \xi^s} \bar{e}, \bar{e} \right\rangle_{L_2, g} + \|\mathcal{A}_o \bar{e}\|_{L_2, g}^2 \\ &\quad + \langle \mathcal{L}(\xi^s) (\mathcal{C}(\xi^s)(\bar{x} - x) - \mathcal{D}v), \mathcal{A}_o \bar{e} \rangle_{L_2, g} \\ &\quad + \langle \bar{e}, \mathcal{A}_o \mathcal{L}(\xi^s) (\mathcal{C}(\xi^s)(\bar{x} - x) - \mathcal{D}v) \rangle_{L_2, g} \\ &\quad + \langle \mathcal{A}_o \mathcal{L}(\xi^s) (\mathcal{C}(\xi^s)(\bar{x} - x) - \mathcal{D}v), \bar{e} \rangle_{L_2, g} \\ &\quad \left. + \langle \mathcal{A}_o \bar{e}, \mathcal{L}(\xi^s) (\mathcal{C}(\xi^s)(\bar{x} - x) - \mathcal{D}v) \rangle_{L_2, g} \right\}. \end{aligned}$$

We will examine each of the terms above separately: the second and fourth terms which, due to the symmetry of the weighted inner product, are identical and are

given by

$$\langle \bar{e}, \mathcal{A}_o(\mathcal{A}_o \bar{e}) \rangle_{L_2, g} + \langle \mathcal{A}_o(\mathcal{A}_o \bar{e}), \bar{e} \rangle_{L_2, g} = 2\langle \bar{e}, \mathcal{A}_o^2 \bar{e} \rangle_{L_2, g}.$$

Using the Sobolev embedding theorem [2] (or equivalently the definition of the domain of the operator and integration by parts along with Friedrich's inequality [4]), one may show that

$$2\langle \bar{e}, \mathcal{A}_o^2 \bar{e} \rangle_{L_2, g} \geq 2c_1 \|\bar{e}\|_{L_2, g}^2 \geq 0$$

for some positive  $c_1$  which is related to the embedding constant  $c$  in (2.4). The third and fifth terms are

$$\begin{aligned} & - \left\langle \bar{e}, \dot{\xi}^s \frac{\partial(\mathcal{L}(\xi^s)\mathcal{C}(\xi^s))}{\partial \xi^s} \bar{e} \right\rangle_{L_2, g} - \left\langle \dot{\xi}^s \frac{\partial(\mathcal{L}(\xi^s)\mathcal{C}(\xi^s))}{\partial \xi^s} \bar{e}, \bar{e} \right\rangle_{L_2, g} \\ & = -2 \left\langle \bar{e}, \dot{\xi}^s \frac{\partial(\mathcal{L}(\xi^s)\mathcal{C}(\xi^s))}{\partial \xi^s} \bar{e} \right\rangle_{L_2, g}, \end{aligned}$$

and, for  $\gamma$  any positive gain, can be made positive by the choice

$$(3.13) \quad \dot{\xi}_i^s = -\gamma \left\langle \frac{\partial(\mathcal{L}_i(\xi_i^s)\mathcal{C}_i(\xi_i^s))}{\partial \xi_i^s} \bar{e}, \bar{e} \right\rangle_{L_2, g}, \quad i = 1, 2, \dots, m.$$

Finally, we examine the last four terms in the expression for  $\dot{V}$ . Using similar arguments as made above, we have

$$\begin{aligned} (3.14) \quad & \langle \mathcal{L}(\xi^s)(\mathcal{C}(\xi^s)(\bar{x} - x) - \mathcal{D}v), \mathcal{A}_o \bar{e} \rangle_{L_2, g} + \langle \bar{e}, \mathcal{A}_o \mathcal{L}(\xi^s)(\mathcal{C}(\xi^s)(\bar{x} - x) - \mathcal{D}v) \rangle_{L_2, g} \\ & + \langle \mathcal{A}_o \mathcal{L}(\xi^s)(\mathcal{C}(\xi^s)(\bar{x} - x) - \mathcal{D}v), \bar{e} \rangle_{L_2, g} + \langle \mathcal{A}_o \bar{e}, \mathcal{L}(\xi^s)(\mathcal{C}(\xi^s)(\bar{x} - x) - \mathcal{D}v) \rangle_{L_2, g} \\ & = 2 \left( \langle \mathcal{L}(\xi^s)(\mathcal{C}(\xi^s)(\bar{x} - x) - \mathcal{D}v), \mathcal{A}_o \bar{e} \rangle_{L_2, g} + \langle \bar{e}, \mathcal{A}_o \mathcal{L}(\xi^s)(\mathcal{C}(\xi^s)(\bar{x} - x) - \mathcal{D}v) \rangle_{L_2, g} \right) \\ & = 2 \langle (\mathcal{A}_o + \mathcal{A}_o^*) \bar{e}, \mathcal{L}(\xi^s)(\mathcal{C}(\xi^s)(\bar{x} - x) - \mathcal{D}v) \rangle_{L_2, g} \\ & = 2 \left( \langle (\mathcal{A}_o + \mathcal{A}_o^*) \bar{e}, \mathcal{L}(\xi^s)\mathcal{C}(\xi^s)(\bar{x} - x) \rangle_{L_2, g} - \langle (\mathcal{A}_o + \mathcal{A}_o^*) \bar{e}, \mathcal{L}(\xi^s)\mathcal{D}v \rangle_{L_2, g} \right). \end{aligned}$$

In the noise-free setting ( $v(t) = 0$  and  $w(t) = 0$  for all  $t \geq 0$ ), we would have  $\bar{x} = x$  and, hence,  $\bar{e} = e$ . The last four terms are all zero. In this case, we have

$$\begin{aligned} \dot{V} & \leq -2 \left\{ (1 + c_1) \|\bar{e}\|_{L_2, g}^2 + \gamma \left( \left\langle \frac{\partial(\mathcal{L}(\xi^s)\mathcal{C}(\xi^s))}{\partial \xi^s} \bar{e}, \bar{e} \right\rangle_{L_2, g} \right)^2 \right\} \\ & \leq -2(1 + c_1) \|\bar{e}\|_{L_2, g}^2 \leq -c_2 V, \end{aligned}$$

where  $c_2 > 0$  is a constant. The last inequality follows from the application of the  $\mathcal{V} \rightarrow \mathcal{V}^*$ -boundedness property (A1). Hence, we see that the time derivative of  $V$  is negative definite. Under the control law (3.13), the error  $\bar{e}$  is guaranteed to converge to zero. Within the set of all possible choices of  $\mathcal{L}$  (say through a Luenberger or a Kalman filter design) that render the observer dynamics stable, we can further dictate the motion in an attempt to improve the state estimate. The control law (3.13) is essentially a gradient-type control law that seeks to improve the state estimate beyond the capability of a static set of sensors.

Let us now consider the case where we have nonzero process and measurement noise signals. First, consider the dynamics of the error between the idealized process and the actual process. A simple computation gives

$$(3.15) \quad \frac{d}{dt}(x - \bar{x}) = \mathcal{A}(x - \bar{x}) + \mathcal{B}_1 w.$$

Using the fact that the operator  $\mathcal{A}$  is the infinitesimal generator of an exponentially stable  $C_0$  semigroup and the  $L_2$ -boundedness of  $\mathcal{B}_1 w$  (made in Theorem 3.1), well-posedness of (3.15) immediately follows [75]. In fact, we have asymptotic convergence of  $x - \bar{x}$  with respect to the  $\mathcal{X}$  norm. Given the well-posedness of the above equation, we apply the triangle inequality to obtain  $\dot{V} \leq -c_9 V + c_5 + c_7 \|v\|^2$ . In the above, all constants are positive and are found by successive application of the  $\mathcal{V} \rightarrow \mathcal{V}^*$ -boundedness of the operator  $\mathcal{A}_o$ . This shows that  $V$  converges to the residual set bounded by

$$\frac{c_5 + c_7 \|v\|^2}{c_9}.$$

Since  $v$  is bounded, we readily see that  $V$  is bounded. While this does not mean (norm) convergence of the error  $\bar{e}$  to zero, in the noisy case, the control law (3.13) drives the error to a neighborhood of zero, which furnishes stability in the sense of Lyapunov.

However, we are interested in the true error  $e$ . To show that  $e$  converges to a neighborhood of zero, we note that

$$\|\hat{x}(t) - x(t)\|_{L_{2,g}} = \|\hat{x}(t) - \bar{x}(t) + \bar{x}(t) - x(t)\|_{L_{2,g}} \leq \|\hat{x}(t) - \bar{x}(t)\|_{L_{2,g}} + \|\bar{x}(t) - x(t)\|_{L_{2,g}}.$$

We have already argued that  $\bar{x} - x$  converges asymptotically to zero, and that  $\bar{e} = \hat{x} - x$  converges to a neighborhood of zero. Hence, the state estimate  $\hat{x}$  converges to a neighborhood of the true state  $x$ . This neighborhood is a function of the bounds on the noise. This gives the following theorem.

**THEOREM 3.2.** *The control law (3.13) with  $\mathcal{L}(\xi^s) = \mathcal{C}^*(\xi^s)$  drives the state estimation error  $e$  governed by (3.6) to a neighborhood of zero as time goes to infinity. In the noise-free case, the state estimation error converges to zero asymptotically.*

**REMARK 3.3.** *The benefits of the choice  $\mathcal{L}(\xi^s) = \mathcal{C}^*(\xi^s)$  are twofold: (i) It simplifies the observer gain design by avoiding the solution to either Lyapunov or Riccati operator equations, and (ii) it minimizes the computational complexity due to the gradient of both  $\mathcal{L}(\xi^s)$  and  $\mathcal{C}(\xi^s)$  in (3.13). With the above choice only the gradient of  $\mathcal{C}(\xi^s)$  with respect to  $\xi^s$  is required. In this case, the control law is simply written as*

$$(3.16) \quad \dot{\xi}_i^s = -2\gamma \left\langle \frac{\partial \mathcal{C}_i(\xi_i^s)}{\partial \xi_i^s} \bar{e}, \mathcal{C}_i(\xi_i^s) \bar{e} \right\rangle_{L_{2,g}}, \quad i = 1, 2, \dots, m.$$

**4. Numerical results.** We simulated the PDE in (2.1) with Dirichlet boundary conditions, having initial conditions  $x(0, \xi) = \sin(\frac{\pi \xi}{\ell}) e^{-7\xi^2}$  and  $\hat{x}(0, \xi) = 0$  as depicted in Figure 4.1. It should be noted that for this initial condition, the “bulk” of the initial state error is in the interval  $[0, 0.6\ell]$  and one expects that the sensors be moving in this region as they will be collecting more useful information.

For the specific PDE, the embedding constant is chosen as  $c = \pi^{-1}$  [4]. The parameters in the elliptic operator are taken to be  $a_1 = 5 \times 10^{-3}$ ,  $a_2 = 1.5 \times 10^{-1}$ ,

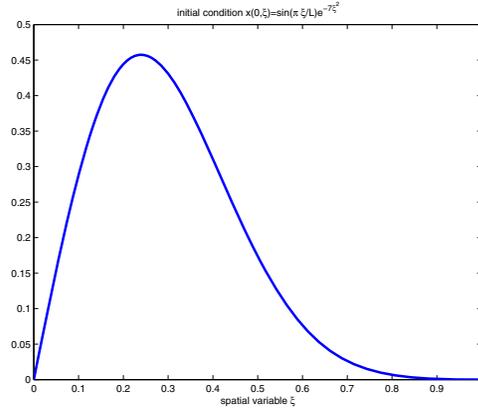


FIG. 4.1. *Distribution of the initial condition  $x(0, \xi)$  considered in the simulation studies.*

$a_3 = 3 \times 10^{-3}$ , and the length of the spatial domain was taken to be  $\ell = 1$ . The spatial support of the sensing devices was chosen as  $\Delta\xi = \ell/10$ .

We approximate (2.1) using linear B-splines [46]. For  $n = 1, 2, \dots$ , let  $\{\varphi_i^n\}_{i=0}^n$  be the standard B-splines on the interval  $[0, \ell]$ , defined with respect to the uniform mesh  $\{0, \frac{\ell}{n}, \dots, \ell\}$ ,

$$\varphi_i^n(\xi) = \begin{cases} 1 - \left| \frac{n\xi}{\ell} - i \right|, & \xi \in \left[ \frac{(i-1)\ell}{n}, \frac{(i+1)\ell}{n} \right], \\ 0, & \xi \in [0, \ell] \setminus \left[ \frac{(i-1)\ell}{n}, \frac{(i+1)\ell}{n} \right]. \end{cases}$$

We consider a sequence of finite dimensional spaces  $\mathcal{X}^n = \text{span}\{\varphi_i^n\}_{i=1}^{n-1}$  and, for each  $n = 1, 2, \dots$ , let  $P_n$  be the orthogonal projection of  $\mathcal{V} = H_0^1(0, \ell)$  into  $\mathcal{X}^n$ . We let  $X^n(t) \in \mathbb{R}^{n-1}$  be the coordinate vector for  $x^n(t)$  with respect to the basis  $\{\varphi_i^n\}_{i=1}^{n-1}$ ,

$$x^n(t) = P_n x(t) = \sum_{i=1}^{n-1} X_i^n(t) \varphi_i^n(\xi).$$

Let  $\hat{x}^n(t) \in \mathbb{R}^{n-1}$  be the coordinate vector for the finite dimensional approximation of  $\hat{x}(t)$ , with  $\hat{x}^n(t) = \sum_{j=1}^{n-1} \hat{X}_j^n(t) \varphi_j^n(\xi)$ . We denote by  $M^n$  the Gram matrix corresponding to  $\{\varphi_i^n\}_{i=1}^{n-1}$ , and thus we obtain

$$M^n = [M_{ij}^n] = \left[ \int_0^\ell \varphi_i^n(\xi) \varphi_j^n(\xi) d\xi \right].$$

Additionally, we let  $K^n, L^n$  be the  $(n-1) \times (n-1)$  matrices

$$K^n = [K_{ij}^n] = \left[ \int_0^\ell d\varphi_i^n(\xi) d\varphi_j^n(\xi) d\xi \right], \quad L^n = [L_{ij}^n] = \left[ \int_0^\ell d\varphi_i^n(\xi) \varphi_j^n(\xi) d\xi \right].$$

The matrix representations of (2.1) and (3.1) then become

$$M^n \dot{X}^n(t) = (-a_1 K^n - a_2 L^n - a_3 M^n) X^n(t) + B_1^n(t) w(t) + B_2^n u(t),$$

$$\begin{aligned} M^n \dot{\hat{X}}^n(t) &= (-a_1 K^n - a_2 L^n - a_3 M^n) \hat{X}^n(t) \\ &\quad + B_2^n u(t) + L^n(\xi^s) \left( y^n(t; \xi^s) - C^n(\xi^s) \hat{X}^n(t) \right). \end{aligned}$$

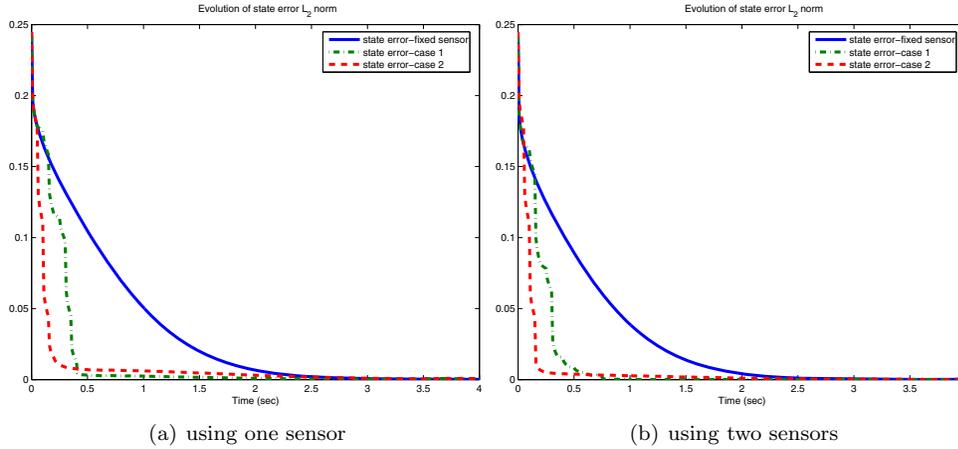
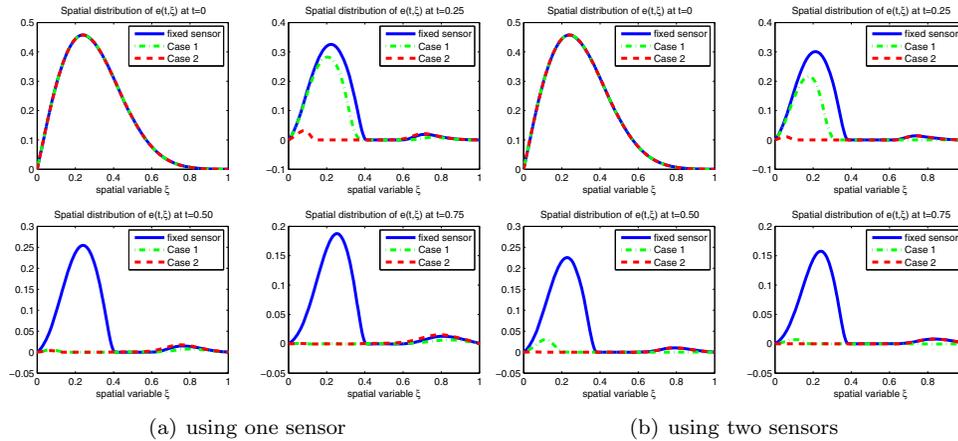


FIG. 4.2. Evolution of state error norms.

FIG. 4.3. Spatial distributions of the error  $e(t, \xi)$  at different time instances.

In both cases, the filter gain  $\mathcal{L}(\xi^s)$  was taken to be equal to  $\mathcal{C}^*(\xi^s)$ . We simulated the  $N$ -dimensional system with 80 basis elements [46] that preserve exponential detectability [6]. The computations were carried out via codes written in MATLAB run on a dual processor DELL workstation (Xeon 2.8GHz,  $2 \times 2$ GB). The resulting finite dimensional system of ordinary differential equations (ODEs) was integrated using the stiff ODE solver from the MATLAB ODE library, routine `ode23s` based on a 4th Runge–Kutta scheme. All spatial integrals were computed numerically via a composite two point Gauss–Legendre quadrature rule [4].

The evolution of the state error norm for the mobile and fixed-sensor cases is presented in Figure 4.2. It is observed that when the sensor is allowed to move within the spatial domain, the estimation error converges to zero faster. This is true for both proposed guidance policies: *localized measurement error* and *global estimation error*. The spatial distribution of the state error at different time instances is depicted in Figure 4.3, where one can once again observe the ability of a mobile sensor to estimate the state faster. In both the case of one and two mobile sensors, Case 2 (based on the

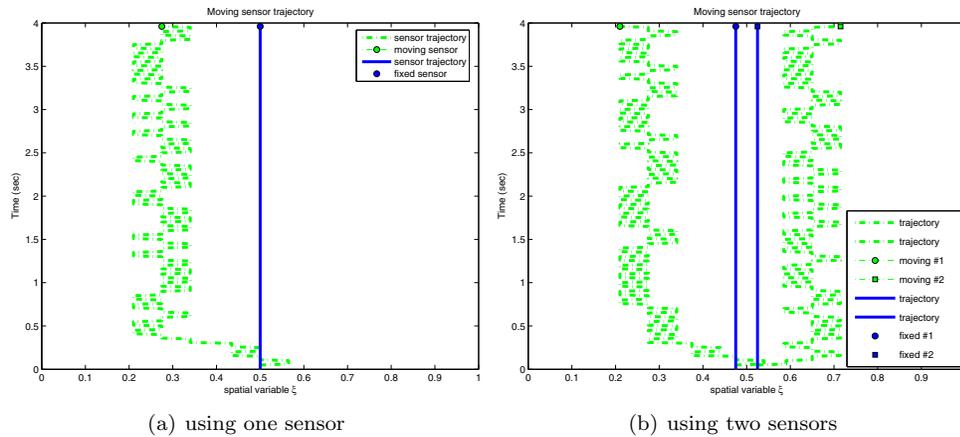


FIG. 4.4. Case 1. Sensor trajectory: moving (dashed lines), fixed (solid lines).

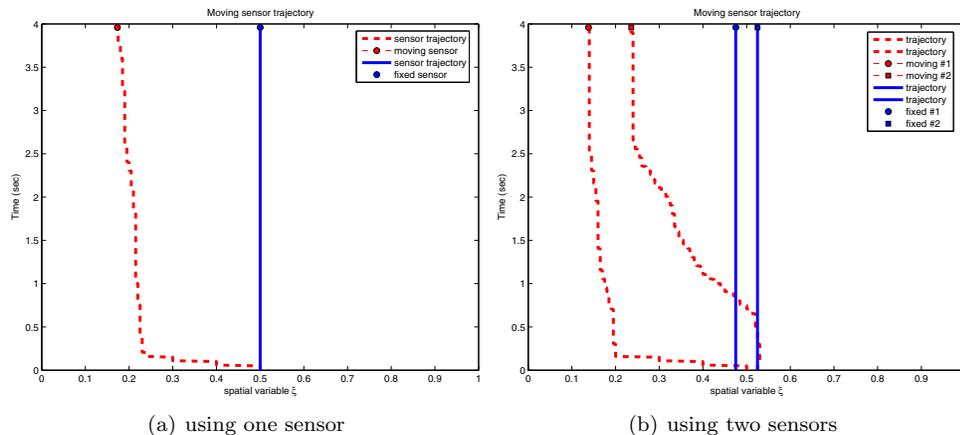


FIG. 4.5. Case 2. Sensor trajectory: moving (dotted lines), fixed (solid lines).

global estimation error) tends to give better results than Case 1 (based on the localized measurement error). The sensor trajectories for both cases are presented in Figures 4.4 and 4.5. The initial position for the single sensor case was chosen as  $\xi_1^s(0) = 0.5\ell$ , and initial conditions for the two-sensor case were chosen as  $\xi_1^s(0) = 0.475\ell$  and  $\xi_2^s(0) = 0.525\ell$ . By examination of the initial condition, and hence the initial condition of the estimation error, it is observed that the guidance policies send the sensor(s) to the region of largest spatial error.

These results clearly validate the basic premise of this paper. Namely, a set of mobile sensors moving according to either one of two guidance policies proposed in this paper will perform better than a set of static sensors located at the initial locations of the mobile sensors.

**5. Summary and concluding remarks.** In this paper we have considered the problem of controlling a network of fully connected, sensor-equipped vehicles to estimate a spatially distributed process described by a linear PDE. The process was assumed to be driven by a zero mean Gaussian noise and the goal was to improve

state estimation via the use of spatially distributed mobile sensors. By utilizing the resulting state estimation error, two guidance policies for the mobile sensors were proposed. The first guidance policy seeks to have each agent minimize the infinity norm of the state estimation error over the sensory domain of the associated sensor. Implicitly imbedded into the sensor guidance policy was a velocity requirement in the sense of moving a given sensor to the spatial location within the domain that had the largest deviation of the estimation error. Such a guidance policy rendered the error system a hybrid one having system operators that generate an exponentially stable  $C_0$  semigroup and forcing terms that satisfied an  $L_2$  bound.

The second guidance policy seeks to have each agent minimize the  $L_2$  norm of the global estimation error over the entire domain  $\mathcal{D}$ . A Lyapunov-based argument was used to show that the  $L_2$  state estimation error associated with the nominal process monotonically decreases until the error is zero within the ranges of all sensors in the network. Simulation studies implementing and comparing the two proposed control policies were provided. The simulations show that moving the sensors according to the proposed control laws is advantageous to not moving the sensors at all. While both the static and mobile cases eventually achieve zero estimation error, the mobile sensors converge faster than the static network.

Both methods required that the filter gain be equal to the adjoint of the measurement operator. Such a choice significantly simplifies the observer gain design and, more importantly, minimizes the computational requirements required when one solves an optimal filter problem via the solution to associated filter differential Riccati equations.

While the current goal was to estimate the process state efficiently, the more interesting case of utilizing mobile sensors would be to detect spatiotemporally varying disturbances and moving sources that may represent contamination or intrusion. Preliminary work on such a case that utilizes the above methods within the abstract theory of infinite dimensional systems has recently been considered in [26] for simple detection of a moving source within a two-dimensional spatial domain and in [27, 28] for the integrated state estimation, intrusion detection, and containment. While the proposed framework easily allows for two- and three-dimensional diffusion-advection processes governed by elliptic operators, a major challenge results in the numerical implementation as the dimension of the finite dimensional representation of the process increases polynomially. However, in this case efficient model-reduction schemes that are based on Karhunen–Loève expansions may be incorporated in order to allow for real-time feasibility. Such a task is currently being undertaken by the first author.

In the current work, vehicle cooperation is in the sense that spatial information is shared between full-connected (communicationwise) vehicles. This sharing of information allows for the coordination of the motion in order to achieve satisfactory estimates of the field over a given domain. There are two open questions that remain to be addressed regarding increased vehicle cooperation. The first involves relaxing the communication full-connectedness assumption. Methods recently developed by the second author (see [89]) that guarantee satisfactory domain coverage under arbitrary intermittent communication structures, with decentralized decision making, can be applied to the two strategies developed in this paper, especially the second strategy. Other communication considerations, such as time delays, fading channels, partially connected and dynamic communication structures, will also be the focus of future work by the authors, as well as distributed processes governed by nonlinear dynamics. However for the abstract framework considered here, collision avoidance

takes additional importance as one must restrict the motion of the mobile sensors within the set of admissible sensor locations in order to guarantee observability.

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