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ON THE SOLUTION OF LONG'S EQUATION WITH SHEAR*

MAYER HUMI†

Abstract. Long's equation describes two dimensional stratified flow over terrain. Its numerical solutions under various approximations were investigated by many authors under the assumption that the base flow field is without shear. Special attention was paid to the properties of the gravity waves that are predicted to be generated as a result. In this paper we address, analytically, the nature and properties of these solutions when shear is present and derive some constraints on the possible generation of gravity waves under these circumstances.

Key words. gravity waves, Long's equation, shear

AMS subject classifications. 76B60, 76E05, 76E30, 86A10

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1. Introduction. Long's equation [1, 2, 3, 4] models the flow of stratified incompressible fluid (in the Boussinesq approximation) in two dimensions over terrain. When the base state of the flow (that is, the unperturbed flow field far upstream) is without shear, the numerical solutions (in the form of steady lee waves) of this equation in various settings and approximations were studied by many authors [5, 6, 7, 8, 9, 10, 11, 12, 13]. The most common approximation in these studies was to set Brunt–Väisälä frequency to a constant or a step function over the computational domain. Moreover, the values of the parameters $\beta$ and $\mu$ which appear in this equation were set to zero. In this (singular) limit the nonlinear terms and one of the leading second order derivatives in the equation drop out and the equation reduces to that of a linear harmonic oscillator over a two dimensional domain. Careful studies [8] showed that these approximations are justified unless wave breaking is present in the solution [9].

Long's equation also provides the theoretical framework for the analysis of experimental data [15, 16, 17] under the assumption of shearless base flow. (An assumption which, in general, is not supported by the data.) An extensive list of references appears in [18, 19, 20].

An analytic approach to the study of this equation and its solutions was initiated recently by the author [14]. We showed that for a base flow without shear and under rather mild restrictions the nonlinear terms in the equation can be simplified. We also identified the "slow variable" that controls the nonlinear oscillations in this equation and, using phase averaging approximation, derived a formula for the attenuation of the stream function perturbation with height. This result is generically related to the presence of the nonlinear terms in Long's equation.

The objective of this paper is to study the nature of the solutions to Long's equation when shear is present in the base flow and Brunt–Väisälä frequency is a continuous function of height. Using conditions which depend solely on the base flow and Brunt–Väisälä frequency we characterize the qualitative nature of the perturbations from the base flow and how their amplitude varies with height. These results

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are independent of the actual detailed description of the terrain that caused these perturbations. Furthermore we derive conditions under which these perturbations are not oscillatory; i.e., no gravity waves are generated by the flow. To the best of our knowledge this issue was never considered in the literature before (in the context of Long's equation).

The plan of the paper is as follows: In section 2 we present a short review of the derivation of Long's equation and the solution of its linearized version. In section 3 we derive constraints on the solutions of this equation in a general setting and in particular in the presence of shear. In section 4 solutions to this equation with different shear profiles are studied explicitly. In section 5 we carry out simulations of Long's equation for shearless and shear base flows. We end in section 6 with summary and conclusions.

2. Long's equation: A short review. In two dimensions \((x,z)\) the flow of a steady inviscid and incompressible stratified fluid (in the Boussinesq approximation) is modeled by the following equations:

\[
\begin{align*}
(2.1) & \quad u_x + w_z = 0, \\
(2.2) & \quad u\rho_x + w\rho_z = 0, \\
(2.3) & \quad \rho(uu_x + wu_z) = -p_z, \\
(2.4) & \quad \rho(ww_x + wu_z) = -p_z - \rho g,
\end{align*}
\]

where subscripts indicate differentiation with respect to the indicated variable, \(\mathbf{u} = (u, w)\) is the fluid velocity, \(\rho\) is its density, \(p\) is the pressure, and \(g\) is the acceleration of gravity.

We can nondimensionalize these equations by introducing

\[
\begin{align*}
\tilde{x} = \frac{x}{L}, \quad \tilde{z} = \frac{N_0}{U_0} z, \quad \tilde{u} = \frac{u}{U_0}, \quad \tilde{w} = \frac{L N_0}{U_0} w, \\
\tilde{\rho} = \frac{\rho}{\rho_0}, \quad \tilde{p} = \frac{N_0}{g U_0 \rho_0} p,
\end{align*}
\]

where \(L\) represents a characteristic length and \(U_0, \rho_0\) represent, respectively, the free stream velocity and density. \(N_0\) is the characteristic Brunt–Väisälä frequency

\[
(2.6) \quad N_0^2 = -\frac{g}{\rho_0} \frac{d\rho_0}{dz}.
\]

In these new variables, (2.1)–(2.4) take the following form (for brevity we drop the bars):

\[
\begin{align*}
(2.7) & \quad u_x + w_z = 0, \\
(2.8) & \quad u\rho_x + w\rho_z = 0, \\
(2.9) & \quad \beta \rho(uu_x + wu_z) = -p_z, \\
(2.10) & \quad \beta \rho(ww_x + wu_z) = -\mu^{-2}(p_z + \rho),
\end{align*}
\]
where

\begin{align}
\beta &= \frac{N_0 U_0}{g}, \\
\mu &= \frac{U_0}{N_0 L}.
\end{align}

\(\beta\) is the Boussinesq parameter [13] which controls stratification effects (assuming \(U_0 \neq 0\), and \(\mu\) is the long wave parameter which controls dispersive effects (or the deviation from the hydrostatic approximation). In the limit \(\mu = 0\) the hydrostatic approximation is fully satisfied [20].

In view of (2.7) we can introduce a stream function \(\psi\) so that

\begin{align}
(2.13) \quad u &= \psi_z, \quad w = -\psi_x.
\end{align}

From (2.8) and (2.13) we infer that \(\rho = \rho(\psi)\) and (after some algebra) derive the following equation for \(\psi\) [13]:

\begin{align}
(2.14) \quad \psi_{zz} + \mu^2 \psi_{xx} - N^2(\psi) \left[ z - \frac{\beta}{2} (\psi^2 + \mu^2 \psi^2) \right] = G(\psi),
\end{align}

where

\begin{align}
(2.15) \quad N^2(\psi) &= \frac{\partial \psi}{\partial \rho},
\end{align}

is the nondimensional Brunt–Väisälä frequency. \(G(\psi)\) is some unknown function which is determined from the base flow, which henceforth we assume to be a function of \(z\) only. To carry out this determination we consider (2.14) at \(x = -\infty\) and express the left-hand side of this equation in terms of \(\psi\) only (assuming that disturbances do not propagate far upstream [19]). Equation (2.14) is referred to as Long’s equation.

For example, if we let

\begin{align}
(2.16) \quad \psi(-\infty, z) &= z,
\end{align}

i.e., consider a shearless base flow with \(u(-\infty, z) = 1\), then

\begin{align}
(2.17) \quad G(\psi) &= -N^2(\psi) \left( \psi + \frac{\beta}{2} \right),
\end{align}

and (2.14) becomes

\begin{align}
(2.18) \quad \psi_{zz} + \mu^2 \psi_{xx} - N^2(\psi) \left[ z - \psi + \frac{\beta}{2} (\psi^2 + \mu^2 \psi^2 - 1) \right] = 0.
\end{align}

However, it is evident that different profiles for the base flow at \(x = -\infty\) will lead to different forms of \(G(\psi)\) (for examples, see section 4).

For a general base flow in an unbounded domain over topography with shape \(f(x)\) and maximum height \(H\) the following boundary conditions are imposed on \(\psi\):

\begin{align}
(2.19) \quad \psi(-\infty, z) &= \psi^0(z), \\
(2.20) \quad \psi(x, \epsilon f(x)) &= \text{constant}, \quad \epsilon = \frac{HN_0}{U_0},
\end{align}
where the constant in (2.20) is (usually) set to zero. As to the boundary condition on 
\( \psi(\infty, z) \) we observe that Long’s equation contains no dissipation terms and therefore
only radiation boundary conditions can be imposed in this limit. Similarly at \( z = \infty \)
it is customary to impose (following [7]) radiation boundary conditions. For the
perturbation from the shearless base flow

\[(2.21) \quad \phi = \psi - z,\]

(2.18) becomes

\[(2.22) \quad \phi_{zz} - \alpha^2 \phi_z^2 + \mu^2 (\phi_{xx} - \alpha^2 \phi_z^2) - N^2(\phi)(\beta \phi_z - \phi) = 0,\]

where

\[(2.23) \quad \alpha^2 = \frac{N^2(\psi)\beta}{2}.\]

Since \( \psi \) is set to zero at the bottom topography, the corresponding (approximate)
boundary condition on \( \phi \) for small \( \epsilon \) will be

\[(2.24) \quad \phi(x, 0) = -\epsilon f(x).\]

Thus for small amplitude topography the boundary condition can be applied at \( z = 0. \)

In the limits \( \beta = 0, \mu = 0 \) and when \( N(\psi) \) is a constant whose value over the
domain is \( N, \) (2.22) reduces to a linear equation

\[(2.25) \quad \phi_{zz} + N^2 \phi = 0.\]

We observe that the limit \( \beta = 0 \) can be obtained by letting either \( U_0 \to 0 \) or \( N_0 \to 0. \)
In the following we assume that this limit is obtained as \( U_0 \to 0 \) (so that stratification
persists in this limit). The general solution of (2.25) is

\[(2.26) \quad \phi(x, z) = p(x) \cos(Nz) + q(x) \sin(Nz),\]

where the functions \( p(x), q(x) \) have to be chosen so that the boundary conditions
derived from (2.19), (2.20) and the radiation boundary conditions are satisfied. These
boundary conditions lead in general to an integral equation for \( p(x) \) and \( q(x) :\)

\[(2.27) \quad q(x) \cos(\epsilon Nf(x)) + H[q(x)] \sin(\epsilon Nf(x)) = -\epsilon f(x),\]

where \( H[q(x)] \) is the Hilbert transform of \( q(x). \) This equation has to be solved nu-
merically [6, 7].

It is clear from the form of the general solution given by (2.26) that it represents
a wave in the \( z \)-direction, and the properties of this wave (under varied physical
conditions) were investigated by the authors mentioned in section 1. It should be
observed, however, that (2.25) is a “singular limit” of Long’s equation, as one of the
leading second order derivatives drops when \( \mu = 0 \) and the nonlinear terms drop
when \( \beta = 0. \) This approximation and its limitations were considered numerically and
analytically [6, 7, 14] and were found to be justified under the assumption that the
base flow is shearless. It is used in the actual analysis of atmospheric data [16, 17, 18].
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3. Properties of solutions to Long's equation with shear. In this section we address the nature and properties of the perturbation from the base flow when shear is present. To simplify our notation and treatment we set \( \mu = 1 \) since we can always scale \( x \) as \( \tilde{x} = x/\mu \) (and drop the bars on \( \tilde{x} \)). We also assume that

\[
\lim_{x \to -\infty} \psi(x, z) = \psi^0(z).
\]

Long's equation with shear is then

\[
(\psi_{xx} - \alpha^2 \psi_x^2) + (\psi_{xx} - \alpha^2 \psi_x^2) - N^2(\psi)z = G(\psi),
\]

and in the limit \( x \to -\infty \), \( G(\psi) \) must satisfy

\[
G(\psi^0) = \psi_{xx}^0 - N^2(\psi^0) \left[ z + \frac{\beta}{2}(\psi_x^0)^2 \right].
\]

To treat the perturbation from the base state we write

\[
\psi(x, z) = \psi^0(x) + \phi(x, z).
\]

Substituting this in (3.2) and linearizing using (3.3) we obtain

\[
\nabla^2 \phi - 2\alpha^2 \psi_x^0 \phi_z - (N^2)'(\psi^0)z\phi - G'(\psi^0)\phi = 0,
\]

where primes denote differentiation with respect to \( \psi \).

Since \( \psi^0 \) is a function of \( z \) only, this equation is separable and we can deduce the properties of its solution by applying separation of variables. Introducing

\[
\phi(x, z) = \chi(x)\eta(z),
\]

we obtain

\[
\chi(x)\eta_{xx} + \lambda \chi(x) = 0
\]

and

\[
\eta(z)_{xx} - 2\alpha^2 \psi_x^0 \eta_z - [\lambda + (N^2)'(\psi^0)z + G'(\psi^0)]\eta = 0,
\]

where \( \lambda \) is the separation of variables constant. Equation (3.8) can be rewritten as

\[
\frac{d}{dz} \left( e^{-2\alpha^2 \psi^0} \frac{d}{dz} \eta(z) \right) - H(z)\eta = 0,
\]

where

\[
H(z) = e^{-2\alpha^2 \psi^0} \left[ \lambda + (N^2)'(\psi^0)z + G'(\psi^0) \right].
\]

Equations (3.9)-(3.10) demonstrate that the properties of the perturbation as a function \( z \) depend only on \( \lambda \) (i.e., the wave number in the \( x \) direction) and the initial state of the flow.

To obtain further information about the properties of \( \eta \) we observe that \( e^{-2\alpha^2 \psi^0} \geq 0 \) and it is possible to apply to (3.9) the comparison theorem of Strum and Picone [21]. A direct application of these theorems leads to the following result.
Assume that on the interval \([a, z]\)

\begin{align}
0 < m & \leq e^{-2\alpha^2 \psi^0} \leq M, \\
k & \leq H(z) \leq K.
\end{align}

Then the following hold:

1. If \(0 < k\), the solution of (3.9) is not oscillatory (no waves on the interval \([a, z]\)).
2. If \(k < 0\) and

\[\frac{-\pi^2}{(z - a)^2} < \frac{k}{M},\]

then the solution of (3.9) is not oscillatory (no waves).
3. A sufficient condition for (3.9) to have an oscillatory solution with \(n\) zeros is that

\[\frac{K}{M} \leq \frac{-n^2 \pi^2}{(z - a)^2}.\]

That is, the wavenumber of the wave will increase as \(K\) becomes more negative. We observe also that the estimate for \(k\) depends on the value of \(\lambda\). As \(0 < \lambda\) increases, this estimate for the lower bound of \(H(z)\) will increase and when \(k\) satisfies the inequality (3.13), the solution for \(\eta\) will become nonoscillatory (that is, the wave is trapped). This demonstrates the “coupling” between the horizontal wavenumber of oscillations and the nature of the solution in the vertical direction.

To obtain further insight into the nature of the solution for \(\eta(z)\) we multiply (3.8) by \(\eta\) and integrate over \([0, z]\). Using integration by parts we obtain

\[
[\eta^0 \alpha^2 \psi_z^2 \eta^2]_0^z = \int_0^z \left\{ (\eta')^2 + [\alpha^2 \psi_z^0 + F(z)] \eta^2 \right\} dz,
\]

where

\[
F(z) = \frac{1}{2} [\lambda + (N^2)'(\psi^0) z + G'(\psi^0)].
\]

Assuming that \(\eta(0) = 0\) (i.e., the amplitude of the perturbation at ground level is 0), (3.15) can be written as

\[
\frac{1}{2} \frac{d\eta^2}{dz}(z) = \alpha^2 \psi_z^0(z) \eta^2(z) + \int_0^z (\eta')^2 dz + \int_0^z [\alpha^2 \psi_z^0 + F(z)] \eta^2 dz.
\]

Hence we conclude that the amplitude of the perturbation will increase with height if

\[
\psi_z^0(z) > 0, \quad \alpha^2 \psi_z^0 + F(z) > 0
\]

on the interval \([0, z]\).

To proceed we now invoke the (modified) Poincaré inequality on the interval \([0, z]\). This inequality states that if \(\eta\) is smooth enough and \(\eta(0) = 0\), then

\[
\int_0^z (\eta')^2(s) ds \geq \frac{\pi^2}{4z^2} \int_0^z \eta^2(s) ds.
\]
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(For the proof of this inequality see the appendix.)

To apply this inequality we rewrite (3.15) (assuming \( \eta(0) = 0 \)) as

\[
\frac{1}{2} \frac{d\eta^2}{dz}(z) \leq \alpha^2 \psi_0^0(z) \eta^2(z) + \int_0^z (\eta')^2 dz + \max[\alpha^2 \psi_0^0 + F(z)] \int_0^z \eta^2 dz.
\]

Using the Poincaré inequality to estimate the integral of \( \eta^2 \) yields

\[
\frac{1}{2} \frac{d\eta^2}{dz}(z) \leq \alpha^2 \psi_0^0(z) \eta^2(z) + \left\{ 1 + \frac{4z^2}{\pi^2} \max[\alpha^2 \psi_0^0 + F(z)] \right\} \int_0^z (\eta')^2 dz.
\]

Hence if

\[
\psi_0^0 < 0, \quad \left\{ 1 + \frac{4z^2}{\pi^2} \max[\alpha^2 \psi_0^0 + F(z)] \right\} < 0
\]
on the interval \([0, z]\), then the perturbation \( \eta \) will decay with height.

We observe that the conditions (3.18) and (3.22) depend only on the properties of the base flow and the variation of \( N^2 \) with height.

3.1. Some special cases.

1. \( \psi_0^0(z) = z \) and \( N = \) constant.

This is essentially the only case that has been treated in the literature on Long's equation. It represents a shearless base state with constant Brunt–Väisälä frequency.

In this case \( G(\psi) \) is given by (2.17), and (3.8) reduces to

\[
\eta_{zz} - 2\alpha^2 \eta_z + (N^2 - \lambda) \eta = 0,
\]
whose solution is \([14]\)

\[
\eta = Ae^{\alpha^2 z} \cos(mz + \gamma),
\]

where \( m = \sqrt{(N^2 - \lambda)} \) and \( \gamma \) is a constant.

2. \( \alpha = 0 \).

When \( \alpha \) is very small we can neglect the second term in (3.8) which reduces then to

\[
\eta_{zz} = 2F(z)\eta.
\]

Introducing

\[
v = \frac{\eta_z}{\eta} = \frac{d\ln|\eta|}{dz},
\]

this equation becomes

\[
v_z + v^2 = 2F(z).
\]

Hence \( v_z = 2F(z) - v^2 \leq 2F(z) \). We can conclude therefore that if \( F(z) < 0 \), then \( v \) and hence \( \frac{d\ln|\eta|}{dz} \) are decreasing with height; i.e., the perturbation is being dissipated. On the other hand, if \( F(z) > 0 \), then \( v \) is increasing when \( 2F(z) - v^2 \) is positive and decreasing when this quantity is negative, and therefore there will be oscillations in the amplitude of \( \eta \).
4. Some examples with shear. In this section we consider some examples whose base flow is not shearless and derive explicitly the corresponding equations for the perturbations. We use then analytic methods to explore the properties of the solutions to these equations.

4.1. $\psi^0 = z^2$, $N = \text{constant}$. For this base flow $u = z$; that is, $u$ increases linearly with height. Using (3.3) we find that

\begin{equation}
G(\psi) = 2 - N^2(\psi)[\psi^{1/2} + 2\beta \psi]
\end{equation}

and Long’s equation (3.2) for $\psi$ (with $\mu = 1$) becomes

\begin{equation}
(\psi_{zz} - \alpha^2 \psi^2_z) + (\psi_{xx} - \alpha^2 \psi^2_x) - N^2(\psi)z = 2 - N^2(\psi)[\psi^{1/2} + 2\beta \psi].
\end{equation}

To derive an equation for a perturbation from the base flow, we set $\psi = z^2 + \phi(x, z)$. Substituting this in (4.2) and linearizing, we obtain

\begin{equation}
\nabla^2 \phi - 4\alpha^2 z \frac{\partial \phi}{\partial z} + \left( \frac{N^2}{2z} + 4\alpha^2 \right) \phi = 0.
\end{equation}

This equation is separable, and we can consider three types of solutions:
1. $\phi(x, z) = e^{-kz} \eta(z)$ with $k > 0$.
2. $\phi(x, z) = \sin(kx) \eta(z)$ with $k > 0$.
3. $\phi(x, z) = \eta(z)$ (that is, a vertical perturbation, $k = 0$).

In all three cases we obtain for $\eta(z)$ the following equation:

\begin{equation}
\eta'' - 4\alpha^2 z \eta' + \left( 4\alpha^2 \pm k^2 + \frac{N^2}{2z} \right) \eta = 0.
\end{equation}

The solution of this equation is given by Heun biconfluent functions [22]. To explore analytically when the solution $\eta(z)$ is oscillatory, we rewrite (4.4) in the form

\begin{equation}
[e^{-2\alpha^2 z^2} \eta']' + \left( 4\alpha^2 \pm k^2 + \frac{N^2}{2z} \right) e^{-2\alpha^2 z^2} \eta = 0.
\end{equation}

This equation has the same form as (3.9), and therefore we can apply the oscillation theorems of Strum and Picone. In fact, since $0 < e^{-2\alpha^2 z^2} \leq 1$, oscillations can occur only if $(4\alpha^2 \pm k^2 + \frac{N^2}{2z})$ is positive enough. (See (3.11), (3.12), and (3.14) and note that there is a minus sign in front of $H(z)$ in (3.9).) This can happen for proper values of $N^2$ and $\alpha^2$ in cases 1 and 3 above. It can also happen when $k$ is small in the second case, but $4\alpha^2 + \frac{N^2}{2z}$ is large enough. Furthermore, one can show numerically that in all cases the amplitude of the perturbation grows with height. Thus the perturbation will always feed on the energy of the base flow. (However, it should be kept in mind that the amplitude of the perturbation cannot grow indefinitely. Once it violates the assumptions made to derive Long’s equation and the approximations that led to (4.4), this solution becomes invalid.)

4.2. $G(\psi) = 0$, $N = \text{constant}$. In this case, instead of choosing the base flow and deriving $G(\psi)$ using (3.3), we make the ansatz that $G(\psi) = 0$ and compute the corresponding base flow. With this assumption (3.3) becomes

\begin{equation}
\psi_{zz}^0 - N^2\left[ z + \frac{\beta}{2}(\psi^0_z)^2 \right] = 0.
\end{equation}
Introducing \( y = \psi_0 \), we obtain a Ricatti equation for \( y(z) \):

\[
y' - \alpha^2 y^2 - N^2 z = 0.
\]

This equation can be linearized by the transformation

\[
y(z) = -\frac{1}{\alpha^2} \frac{v(z)'}{v(z)},
\]

which leads to

\[
v(z)'' + \alpha^2 N^2 z v(z) = 0.
\]

This can be identified as a Bessel equation whose solution is

\[
v(z) = C \sqrt{z} J_{1/3} \left( \frac{2}{3} N \alpha z^{3/2} \right),
\]

and hence

\[
\psi^0(z) = -\frac{1}{\alpha^2} \ln(|v(z)|).
\]

The resulting Long’s equation for this base flow is (3.2) with \( G(\psi) = 0 \). This equation can be linearized by the transformation \( \xi = e^{-\alpha^2 \psi} \), and we obtain

\[
\nabla^2 \xi + N^2 \alpha^2 z \xi = 0.
\]

This equation (for the full flow) is separable, and the nature of the solution will depend on the separation constant. We distinguish three cases:

1. \( \xi(x, z) = \xi(z) \).

   In this case \( \xi \) satisfies (4.9) and hence \( \xi \) is given by (4.10).

2. \( \xi(x, z) = e^{-nz} \eta(z), n > 0 \).

   This lead to

\[
\eta'' + [N^2 \alpha^2 z + n^2] \eta = 0.
\]

3. \( \xi(x, z) = \sin(nz) \eta(z), n > 0 \).

   The equation for \( \eta \) becomes

\[
\eta'' + [N^2 \alpha^2 z - n^2] \eta = 0.
\]

The solution to equations (4.13), (4.14) is given in terms of Airy functions. However, the qualitative nature of the solution of these equations can be deduced from the comparison theorems of Strum and Picone [21]. (In this case the function \( e^{-2\alpha^2 \psi^0} \) in (3.9) is replaced by 1, and hence \( m = M = 1 \) in (3.11).) Thus for the second case we have \( [N^2 \alpha^2 z + n^2] > 0 \) and therefore the solution for \( \eta \) will be oscillatory. Moreover, the wavenumber of the oscillations will increase with height. On the other hand, for the third case the solution will be nonoscillatory for small \( z \) but may become oscillatory with height, viz. when \( [N^2 \alpha^2 z - n^2] \) becomes positive.
4.3. $\psi^0 = -\cos(az)$, $N = \text{constant}$. For this base flow $u = a \sin(az)$; that is, $u$ oscillates with the height.

In this case we obtain for $G(\psi)$ the expression

\begin{equation}
G(\psi) = -a^2 \psi - N^2 \left[ \frac{\pi - \arccos(\psi)}{a} + \frac{\beta a^2}{2} (1 - \psi^2) \right].
\end{equation}

The linearized equation for the perturbation $\phi(x, z)$ from the base flow is

\begin{equation}
\nabla^2 \phi - 2a^2 \sin(az) \frac{\partial \phi}{\partial z} + \left[ a^2 + 2a^2 \cos(az) - \frac{N^2}{a \sin(az)} \right] \phi = 0.
\end{equation}

For a solution of the form $\phi(x, z) = e^{-kx} \eta(z)$ or $\phi(x, z) = \sin(kx) \eta(z)$ this yields

\begin{equation}
\left[ e^{2a^2 \cos(az) \eta} \right]' + e^{2a^2 \cos(az)} \left[ a^2 \pm k^2 + 2a^2 a^2 \cos(az) - \frac{N^2}{a \sin(az)} \right] \eta = 0.
\end{equation}

Oscillations in the solution of this equation will occur whenever the expression in the square brackets of the last term is positive. Since $\sin(az)$ takes both positive and negative values and $\frac{N^2}{a \sin(az)}$ is dominant when $az \approx 0$ or $az \approx \pi$, the solution will exhibit different qualitative behavior in different regions (i.e., oscillatory in some and nonoscillatory in others).

5. **Numerical simulations for shear flow.** In previous sections we discussed from an analytical point of view the impact of shear on the generation of gravity waves using first order perturbation expansion. To elicit more insight on this issue we compare numerically in this section the solutions of Long's equation with and without shear over the same topography and with the same values of the geophysical parameters (viz. $\mu$, $\beta$, and $N^2$). Without loss of generality we set $\mu = 1$ in the following (see remark at the beginning of section 3).

The equation for a perturbation from a shearless base flow (without approximations) is given by (2.22). Similarly the exact equation for the perturbation from pure shear flow (see section 4.1) is given by

\begin{equation}
\nabla^2 \phi - \alpha^2 \left[ 4z \phi_z + \phi_z^2 + \phi_x^2 - 4 \phi \right] + N^2 \sqrt{z^2 + \phi - z} = 0.
\end{equation}

To simplify (2.22) and (5.1) we introduce

\begin{equation}
\eta = e^{-\alpha^2 \phi}, \quad \alpha \neq 0,
\end{equation}

and observe that

\begin{equation}
-\frac{1}{\alpha^2 \eta} \eta_{zz} = \phi_{zz} - \alpha^2 \phi_z^2.
\end{equation}

Substituting this result for the second order derivatives of $x$ and $z$, (2.22) and (5.1) transform, respectively, to

\begin{align}
\nabla^2 \eta - 2 \alpha^2 \eta_z + N^2 \eta \ln \eta & = 0, \\
\nabla^2 \eta + \alpha^2 \left\{ -4z \eta_z + \eta \left[ 4 \ln(\eta) + N^2 \left( z - \sqrt{\frac{\alpha^2 z^2 - \ln(\eta)}{\alpha^2}} \right) \right] \right\} & = 0.
\end{align}
These equations are linear in the derivatives of $\eta$ and nonlinear only in terms which contain $\eta$ itself. This simplifies the numerical algorithm for their solution.

To solve (5.4), (5.5) we implement Newton’s iteration scheme. To this end we define $F(\eta)$ as the left-hand side (of each) of these equations and take its Frechet derivative, i.e., compute

\begin{equation}
F(\eta_0 + \delta) \cong F(\eta_0) + L(\eta_0)\delta + O(\delta^2),
\end{equation}

where $L(\eta_0)$ is a linear operator. A short computation using (5.4) yields

\begin{equation}
L_1(\eta_0) = \nabla^2 + N^2 \left[ (1 + \ln \eta_0) - \beta \frac{\partial}{\partial z} \right].
\end{equation}

Similarly, for (5.5) we obtain

\begin{equation}
L_2(\eta_0) = \nabla^2 + \alpha^2 \left[ -4z \frac{\partial}{\partial z} + 4(1 + \ln \eta_0) + N^2 \left( z - \sqrt{z^2 - \frac{\ln(\eta_0)}{\alpha^2}} + \frac{1}{2\alpha^2\sqrt{z^2 - \frac{\ln(\eta_0)}{\alpha^2}}} \right) \right].
\end{equation}

To use Newton’s iteration scheme to solve (5.4), (5.5) we now let $F(\eta_0 + \delta) = 0$ in (5.6) with $\delta = \eta_{m+1} - \eta_{m}$ (where the index $m$ denotes the iteration number). This leads, respectively, to the following iteration schemes for the solution of these equations:

\begin{equation}
L_1(\eta_m)\eta_{m+1} = N^2\eta_m,
\end{equation}

\begin{equation}
L_2(\eta_m)\eta_{m+1} = \left[ 4\alpha^2 + \frac{N^2}{2\sqrt{z^2 - \frac{\ln(\eta_m)}{\alpha^2}}} \right] \eta_m.
\end{equation}

To solve these equations over a finite two dimensional domain $[-a, a] \times [0, b]$ with bottom topography, we used central finite differences with a grid of 151 x 101 points. The (approximate) boundary conditions which were imposed on $\eta$ in (5.9), (5.10), respectively, were

\begin{align}
\eta(-a, z) &= 1, \quad \eta(a, z) = 1, \quad \eta(x, 0) = e^{\alpha^2 f(x)}, \\
\eta(-a, z) &= 1, \quad \eta(a, z) = 1, \quad \eta(x, b) = 1, \quad \eta(x, 0) = e^{\alpha^2 f^2(x)}.
\end{align}

To mimic radiation boundary conditions and avoid reflection of the outgoing wave we used “sponge boundaries” at $x = a$ and $z = b$ (as is done in the NCAR/MM5 mesoscale model [23] and others). The following values of the parameters were used, respectively, in these simulations:

\begin{equation}
\epsilon = 0.35, \quad N = 1, \quad \beta = 4.10^{-3}
\end{equation}

with topography shape function

\begin{equation}
f(x) = \frac{1}{(1 + x^2)^{3/2}}.
\end{equation}

The convergence criterion for the iterations was $\max |\eta_{m+1} - \eta_m| \leq 10^{-10}$. Figures 1 and 2 compare the results obtained for the perturbation $\phi(x, z)$ by using (5.9) and (5.10). We see that Figure 1 (for the shearless base flow) displays a clear pattern of gravity waves. On the other hand, Figure 2 shows that the perturbation from the shear flow feeds on the energy of the base flow and creates a vortex high above the topography.
Fig. 1. Contour plot of $\alpha^2 \phi$ using (5.9) (shearless base flow).

Fig. 2. Contour plot of $\alpha^2 \phi$ using (5.10) (shear base flow).
6. Summary and conclusions. We derived in this paper some criteria for the excitation of gravity waves by a flow over topography using Long’s equation. These criteria depend on only the nature of the base flow and the variation of $N^2$ with height. From an operational point of view these criteria will be useful both experimentally and theoretically. Currently the experimental practice is to ignore the shear in the base flow and attempt to deduce the quantitative attributes of the gravity waves using the shearless Long’s equation. This procedure can be refined now by taking this important feature into account. Our analysis also shows that no simulation of Long’s equation over actual topography is needed to determine the qualitative nature of the perturbation that is generated by the topography.

We also demonstrated that in some cases this perturbation will be damped by the shear, while in other cases the perturbation will grow, feeding on the energy that is present in the base flow.

Appendix: Poincaré inequality.

Theorem A.1. Let $u(x)$ be a bounded differentiable function on $[0,a]$ with $u(0) = 0$; then

\begin{equation}
\int_0^a [u'(x)]^2 dx \geq \frac{\pi^2}{4a^2} \int_0^a u^2(x) dx.
\end{equation}

Proof. To prove this inequality we introduce

\begin{equation}
h(x) = \frac{\pi}{2a} \tan \left( \frac{\pi(x-a)}{2a} \right) \tag{A.2} \end{equation}

This function satisfies $h(a) = 0$ and the differential equation

\begin{equation}
h' - h^2 = \frac{\pi^2}{4a^2} \tag{A.3} \end{equation}

We now consider the integral

\begin{equation}
\int_0^a [uh + u']^2 dx \geq 0, \tag{A.4}
\end{equation}

\begin{equation}
\int_0^a [uh + u']^2 dx = \int_0^a u^2 h^2 dx + \int_0^a (u')^2 dx + 2 \int_0^a uh' dx \geq 0, \tag{A.5}
\end{equation}

but

\begin{align}
\int_0^a uuh' dx &= \frac{u^2 h}{2} \bigg|_0^a - \frac{1}{2} \int_0^a u^2 h' dx \\
&= \frac{1}{2} [u^2(a)h(a) - u^2(0)h(0)] - \frac{1}{2} \int_0^a u^2 h' dx \\
&= -\frac{1}{2} \int_0^a u^2 h' dx. \tag{A.6}
\end{align}

Hence from (A.5),

\begin{equation}
\int_0^a (u')^2 dx \geq \int_0^a u^2 h' dx - \int_0^a u^2 h^2 dx = \int_0^a u^2 (h' - h^2) dx = \frac{\pi^2}{4a^2} \int_0^a u^2 dx, \tag{A.7}
\end{equation}

which proves the theorem. □
REFERENCES