9-14-2008

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September 14, 2008

*Research supported in part by OTKA Grant No. K68322.

†Research supported in part by the National Science Foundation under Grant No. DMS-0456401, by OTKA Grant No. K68322 and by a Janos Bolyai Research Scholarship.

‡Research supported in part by the Ellentuck Fund.

Abstract

Here improving on our earlier results we prove that there exists an n_0 such that for $n \geq n_0$, in every 2-coloring of the edges of $K_n^{(4)}$ there is a monochromatic Hamiltonian 3-tight Berge cycle. This proves the $c = 2$, $t = 3$, $r = 4$ special case of a conjecture from [5].

1 Introduction

$V(G)$ and $E(G)$ denote the vertex-set and the edge-set of the graph G. (A, B, E) denotes a bipartite graph $G = (V, E)$, where $V = A + B$, and $E \subset A \times B$. For a graph G and a subset U of its vertices, $G|_U$ is the restriction of G to U. $N(v)$ is the set of neighbors of $v \in V$. Hence the size of $N(v)$ is $|N(v)| = deg(v) = deg_G(v)$, the degree of v. $\delta(G)$ stands for the minimum and $\Delta(G)$ for the maximum degree in G. When A, B are subsets of $V(G)$, we denote by $e(A, B)$ the number of edges of G with one endpoint in A and the other in B. In particular, we write $deg(v, U) = e(\{v\}, U)$ for the number of edges from v to U. A graph G_n on n vertices is γ-*dense* if it has at least $\gamma\binom{n}{2}$ edges. A bipartite graph $G(k, l)$ is γ-dense if it contains at least γkl edges.

Let $\mathcal{H}$ be an r-uniform hypergraph (a family of some r-element subsets of a set). The *shadow graph* of $\mathcal{H}$ is defined as the graph $\Gamma(\mathcal{H})$ on the same vertex set, where two vertices are adjacent if they are covered by at least one edge of $\mathcal{H}$. A coloring of the edges of an r-uniform hypergraph $\mathcal{H}$, $r \geq 2$, induces a multicoloring on the edges of the shadow graph $\Gamma(\mathcal{H})$ in a natural way; every edge e of $\Gamma(\mathcal{H})$ receives the color of all hyperedges containing e. We shall denote by $c(x, y)$ the color set of the edge xy in $\Gamma(\mathcal{H})$. A subgraph of $\Gamma(\mathcal{H})$ is *monochromatic* if the color sets of its edges have a nonempty intersection. Let $K_n^{(r)}$ denote the complete r-uniform hypergraph on n vertices.

In any r-uniform hypergraph $\mathcal{H}$ for $2 \leq t \leq r$ we define an r-uniform t-tight Berge-cycle of length ℓ, denoted by $C_\ell^{(r,t)}$, as a sequence of distinct vertices $v_1, v_2, \ldots, v_\ell$, such that for each set $(v_i, v_{i+1}, \ldots, v_{i+t-1})$ of t consecutive vertices on the cycle, there is an edge e_i of $\mathcal{H}$ that contains these t vertices and the edges e_i are all distinct for $i, 1 \leq i \leq \ell$ where $\ell + j \equiv j$. This notion was introduced in [5] and for $t = 2$ we get ordinary Berge-cycles ([1]) and for $t = r$ we get the tight cycle (see e.g. [14] or [20]). A Berge-cycle of length n in a hypergraph of n vertices is called a Hamiltonian Berge-cycle. It is important to keep in mind that, in contrast to the case $r = t = 2$, for $r > t \geq 2$ a Berge-cycle $C_\ell^{(r,t)}$, is not determined uniquely, it is considered as an arbitrary choice from many possible cycles with the same triple of parameters.

In this paper, continuing investigations from [5], [8], [10], [11] and [12], we study long Berge-cycles in hypergraphs. In [5] (by generalizing an earlier conjecture from [8]) the following conjecture was formulated.

Conjecture 1.1. *For any fixed $2 \leq c, t \leq r$ satisfying $c + t \leq r + 1$ and sufficiently large n, if we color the edges of $K_n^{(r)}$ with c colors, then there is a monochromatic Hamiltonian t-tight Berge-cycle.*

In [5] it was proved that if the conjecture is true it is best possible, since for any values of $2 \leq c, t \leq r$ satisfying $c+t > r+1$ the statement is not true. The conjecture can easily be proved for $c = t = 2$ and $r = 3$, see [8]. The asymptotic form of the conjecture was proved for $c = 3$, $t = 2$ and $r = 4$ in [8] and for every r and $c = r-1$, $t = 2$ in [11] - in both papers the Regularity Lemma [21] was used. In this paper we prove the conjecture in a *sharp* form for the first non-trivial special case: $c = 2$, $t = 3$ and $r = 4$ and thus providing more evidence to the truth of the conjecture in general.

Theorem 1.2. *There exists an n_0 such that for $n \geq n_0$, in every 2-coloring of the edges of $K_n^{(4)}$ there is a monochromatic Hamiltonian 3-tight Berge-cycle.*

This improves a result of [12] where under the same assumptions we could only find a monochromatic 3-tight Berge-cycle of length at least $n - 10$. It also improves a result from [5] where we did manage to find a Hamiltonian monochromatic 3-tight Berge-cycle but only in 2-colorings of the edges of the complete 5-uniform hypergraph. In the proof we combine the proof method of the weaker statement from [12] with stability arguments discussed in the next section.

2 A stability version of the Gerencsér-Gyárfás Theorem

For graphs $G_1, G_2, \ldots, G_r$, the Ramsey number $R(G_1, G_2, \ldots, G_r)$ is the smallest positive integer n such that if the edges of a complete graph K_n are partitioned into r disjoint color classes giving r graphs $H_1, H_2, \ldots, H_r$, then at least one H_i $(1 \leq i \leq r)$ has a subgraph isomorphic to G_i. The existence of such a positive integer is guaranteed by Ramsey's classical result [19]. The number $R(G_1, G_2, \ldots, G_r)$ is called the Ramsey number for the graphs $G_1, G_2, \ldots, G_r$. There is very little known about $R(G_1, G_2, \ldots, G_r)$ even for very special graphs (see eg. [7] or [18]). For $r = 2$ a theorem of Gerencsér and Gyárfás [6] states that

$$R(P_n, P_n) = \left\lfloor \frac{3n-2}{2} \right\rfloor.$$

In the proof we will use a stability version of the Gerencsér-Gyárfás Theorem that we proved recently in [13]. For this purpose we need to define a relaxed version of the extremal coloring in this theorem. We work with 2-edge *multicolorings* (G_1, G_2) of a graph G. Here multicoloring means that the edges can receive more than one color, i.e. the graphs G_i are not necessarily edge disjoint. The subgraph colored with color i *only* is denoted by G_i^*, i.e.

$$G_1^* = G_1 \setminus G_2, G_2^* = G_2 \setminus G_1.$$

Extremal Coloring 1 (with parameter α): There exists a partition $V(G) = A \cup B$ such that

- $|A| \geq (1-\alpha)\frac{2|V(G)|}{3}$, $|B| \geq (1-\alpha)\frac{|V(G)|}{3}$.
- The graph $G_1^*|_A$ is $(1-\alpha)$-dense and the bipartite graph $G_2^*|_{A\times B}$ is $(1-\alpha)$-dense, where say G_1 is red and G_2 is blue. (Note that we have no restriction on the coloring inside the smaller set.)

Then the following stability version of the Gerencsér-Gyárfás Theorem from [13] claims that we can either find a monochromatic path substantially longer than $2n/3$, or the coloring is close to the extremal coloring.

Lemma 2.1. *For every $\alpha > 0$ there exist positive reals η, c_1 ($0 < \eta \ll \alpha \ll 1$ where $\ll$ means sufficiently smaller) and a positive integer n_0 such that for every $n \geq n_0$ the following holds: if the edges of the complete graph K_n are 2-multicolored then we have one of the following two cases.*

- *Case 1: K_n contains a monochromatic path P of length at least $(\frac{2}{3}+\eta)n$. Furthermore, in the process of finding P, for each vertex of the path P we have at least $c_1 \log n$ choices.*
- *Case 2: This is an Extremal Coloring 1 (EC1) with parameter α.*

Surprisingly, as far as we know, this natural question has not been studied, despite the fact that stability versions for some classical density (see [2]) and Ramsey-type results (see [9] and [15]) are known.

Lemma 2.1 (and thus Theorem 1.2) can also be proved from the Regularity Lemma, however, in [13] we used a more elementary approach using only the Kővári-Sós-Turán bound [16].

For the sake of completeness we will give a sketch of the proof of Lemma 2.1 in Section 4. Actually it turns out that the proof of the lemma is somewhat easier in this particular application.

3 Outline of the proof of Theorem 1.2

As in Lemma 2.1 we will use the following main parameters

$$0 < \eta \ll \alpha \ll 1,$$

and the constant $c_2 = 25$. We shall assume that n is sufficiently large.

We will follow the same rough outline as in [12]. Indeed, suppose that a 2-coloring c is given on the edges of $\mathcal{K} = K_n^{(4)}$. Let V be the vertex set of $\mathcal{K}$ and observe that c defines a 2-multicoloring on the complete 3-uniform hypergraph $\mathcal{T}$ with vertex set V by coloring a triple T with the colors of the edges of $\mathcal{K}$ containing T. We say that $T \in \mathcal{T}$ is *good in color i* if T is contained in at least two edges of $\mathcal{K}$ of color i ($i = 1, 2$). Let G be the shadow graph of $\mathcal{K}$. The following easy lemma is from [12].

Lemma 3.1. *Every edge $xy \in E(G)$ is in at least $n-4$ good triples of the same color.*

Proof. Consider an edge xy in G. Coloring c induces a 2-coloring c' on the pairs of $W = V \setminus \{x, y\}$. Applying a result of Bollobás and Gyárfás, [3], there exists a subgraph H with at least $|W| - 2 = n - 4$ vertices such that H is 2-connected and monochromatic under c', say in color 1. In particular, every vertex of H has degree at least two in color 1. Thus, for every vertex z of H, $\{x, y, z\}$ is a good triple in color 1. $\square$

Using Lemma 3.1, we can define a 2-multicoloring c^* on the shadow graph $G = \Gamma(\mathcal{K})$ by coloring $xy \in E(G)$ with the color(s) of the (at least $n - 4$) good triples containing xy. We apply the stability version of the Gerencsér-Gyárfás Theorem (Lemma 2.1) for this 2-multicoloring of G. In Section 4 we will sketch the proof of Lemma 2.1 in this particular application. Case 2, i.e. Extremal Coloring 1 (EC1) is handled in Section 7. Assuming that we have the non-extremal case, Case 1, we can find in G a monochromatic path P (say in red) of length $l \geq (2/3+\eta)n$. From now on in the non-extremal case we work in the color red. Label the edges of P by $e_j = \{p_j, p_{j+1}\}$, $j = 1, 2, \ldots, l - 1$. From Lemma 2.1 it also follows that we can guarantee that all the triples $\{p_j, p_{j+1}, p_{j+2}\}$, $j = 1, 2, \ldots, l-2$ are good in red. Indeed, when we select p_{j+2} we select a vertex from the available $c_1 \log n$ choices that forms a good triple with $\{p_j, p_{j+1}\}$. Since only two vertices are forbidden we still have plenty to choose from.

We plan to splice in the remaining vertices in $V(G) \setminus V(P)$ into (most of) the edges $e_{2j} = \{p_{2j}, p_{2j+1}\}$. For this purpose we make sure that if we plan to splice in the vertex $v \in V(G) \setminus V(P)$ into the edge e_{2j}, then all 3 triples $\{p_{2j-1}, p_{2j}, v\}$, $\{p_{2j}, v, p_{2j+1}\}$ and $\{v, p_{2j+1}, p_{2j+2}\}$ are good in red. This guarantees that we will be able to make this into a 3-tight Berge-cycle later.

However, as in [12], there could be a small (constant) number of exceptional vertices in $V(G) \setminus V(P)$ that simply cannot be spliced in into *any* of the edges e_{2j}. In order to avoid this technical difficulty we do the following. First we build an initial red path P' that has length c_2. This determines a small number of exceptional vertices in $V(G) \setminus V(P')$ that cannot be spliced in into P'. For each such exceptional vertex v we make sure artificially that we will be able to splice it in. Indeed, we define a *v-absorbing bridge* $\{p_1, p_2, p_3, p_4\}$ (in red) in the following way. The edges $\{p_1, p_2\}$, $\{p_2, p_3\}$ and $\{p_3, p_4\}$ are all red (under c^*) in G and either all 3 triples $\{p_1, p_2, v\}$, $\{p_2, v, p_3\}$ and $\{v, p_3, p_4\}$ are good in red (type 1 bridge), or otherwise there exists a vertex $w \notin \{v, p_1, p_2, p_3, p_4\}$ such that $\{p_1, p_2, v, w\}$, $\{p_2, v, p_3, w\}$ and $\{v, p_3, p_4, w\}$ are all red edges of $\mathcal{K}$ (type 2 bridge). Note that in the second case the triples $\{p_1, p_2, v\}$, $\{p_2, v, p_3\}$ and $\{v, p_3, p_4\}$ might not be good in red, as w might be the only vertex that can be added to them. However, this definition will imply that in both cases v can be spliced in into the edge $\{p_2, p_3\}$. We will call the coloring an Extremal Coloring 2 (EC2) if the following statement is *not* true: For *every* vertex $v \in V(G)$ and for *both* colors there are at least $\sqrt{\alpha} n^4$ v-absorbing bridges (with the same w if they are type 2). Extremal Coloring 2 (EC2) is handled later in Section 6.

Assuming that EC2 does not hold we connect P' and these red absorbing bridges for the exceptional vertices into a path P'' that still has a constant length. Then we extend this to a red cycle C' that has length at least $(2/3+\eta)n$ and that contains P'' as a subpath. Now we are able to splice in all the remaining vertices into the cycle C' and thus resulting in a red Hamiltonian 3-tight Berge-cycle.

4 Sketch of the proof of Lemma 2.1

We follow a similar outline as in applications of the Regularity Lemma. However, a regular pair is replaced with a complete balanced bipartite graph $K(t,t)$ with $t \geq c \log n$ for some constant c (thus the size of the pair is somewhat smaller but this is still good enough for our purposes). Then a monochromatic connected matching in the reduced graph (the usual tool in these types of proofs using the Regularity Lemma) is replaced with a monochromatic cover consisting of vertex disjoint complete balanced bipartite graphs $K_i(t_i,t_i)$, $1 \leq i \leq s$ such that $t_i \geq c \log n$ for every $1 \leq i \leq s$ for some constant c. Let us call a cover like this a monochromatic complete balanced bipartite graph cover. The size of this cover is the total number of vertices in the union of these complete bipartite graphs. In the general proof of Lemma 2.1 in [13] it was also important to make these complete bipartite graphs connected in a certain sense; in this particular application the connection will be automatic as we shall see.

Then Lemma 2.1 follows from the following lemma.

Lemma 4.1. *For every $\alpha > 0$ there exist a positive real η ($0 < \eta \ll \alpha \ll 1$ where $\ll$ means sufficiently smaller) and a positive integer n_0 such that for every $n \geq n_0$ the following holds: if the edges of the complete graph K_n are 2-multicolored then we have one of the following two cases.*

- *Case 1: K_n contains a monochromatic complete balanced bipartite graph cover of size at least $(\frac{2}{3} + 2\eta)n$.*
- *Case 2: This is an Extremal Coloring 1 (EC1) with parameter α.*

Indeed, let us assume that we have Case 1 in this Lemma. It is easy to connect these monochromatic (say red) complete balanced bipartite graphs $K_i(t_i,t_i)$ into one red path. Indeed, if p_1 and p_2 are the last two vertices on the subpath corresponding to a complete bipartite graph, and p_3 and p_4 are the first two vertices on the subpath corresponding to the next complete bipartite graph, we just have to make sure that both triples $\{p_1,p_2,p_3\}$ and $\{p_2,p_3,p_4\}$ are good in red. But we can easily achieve this as for each red edge we have at most 2 exceptional vertices. Then of course the connecting pair $\{p_2,p_3\}$ is not necessarily a red edge in G so we get a somewhat weaker statement than Lemma 2.1 but this is just as good for this application. We just have to make sure that we are not splicing in any vertices into this $\{p_1,p_2,p_3,p_4\}$ section of the path, but this does not create any difficulties.

To prove Lemma 4.1 we proceed as follows. Let us assume that the majority of the edges are red. Then we repeatedly apply in the red subgraph the Kővári-Sós-Turán theorem [16] to find complete balanced bipartite graphs $K_i(t,t), 1 \leq i \leq s$ with $t \geq c \log n$ until we can. Denote the resulting red complete balanced bipartite graph cover by M_1. If this red cover M_1 has size $|M_1| \geq (2/3 + 2\eta)n$, then we are done, we have Case 1 in Lemma 4.1. Otherwise we can show that we can either increase the size of this red cover by a constant fraction, or we can find directly a monochromatic complete bipartite graph cover of size at least $(2/3 + 2\eta)n$ unless we are in the Extremal Coloring 1 (Case 2), as desired.

Let $K_i(t,t) = (V_1^i, V_2^i), 1 \leq i \leq s$. Denote

$$V_1 = \cup_{i=1}^s V_1^i,\ V_2 = \cup_{i=1}^s V_2^i \text{ and } V_3 = W \setminus (V_1 \cup V_2).$$

Since in V_3 we cannot pick another red complete balanced bipartite subgraph $K(t,t)$, V_3 is almost complete in the blue-only subgraph. Next let us look at the bipartite graphs (V_1, V_3) and (V_2, V_3). We can show that either one of them is almost complete in blue-only or we can increase our red cover M_1. We continue in this fashion; we collect more and more information about the structure of the coloring until we arrive at the Extremal Coloring 1 (Case 2). For details consult [13].

5 The non-extremal case

Assume in this case that we do not have Extremal Colorings 1 or 2. Following the outline above first we build an initial red path P' in G that has length $c_2 = 25$. Label the edges of P' by $e_j = \{p_j, p_{j+1}\}$, $j = 1, 2, \dots, 24$. P' determines a small number of exceptional vertices in $V(G) \setminus V(P')$ in the following way. As indicated above for a vertex $v \in V(G) \setminus V(P')$ and for an edge $e_{2j} = \{p_{2j}, p_{2j+1}\}$ of P' we say that v can be spliced in into e_{2j} if all 3 triples $\{p_{2j-1}, p_{2j}, v\}$, $\{p_{2j}, v, p_{2j+1}\}$ and $\{v, p_{2j+1}, p_{2j+2}\}$ are good in red. A vertex $v \in V(G) \setminus V(P')$ is *exceptional* if it can be spliced in into at most 6 edges e_{2j} of P'. We claim that the number of these exceptional vertices in $V(G) \setminus V(P')$ is at most 12. Indeed, for each fixed edge e_{2j} of P', $1 \leq j \leq 12$, there could be only at most 6 vertices of $V(G) \setminus V(P')$ that cannot be spliced in into e_{2j} since for each of the pairs $\{p_{2j-1}, p_{2j}\}$, $\{p_{2j}, p_{2j+1}\}$ and $\{p_{2j+1}, p_{2j+2}\}$ there could be at most 2 exceptional vertices. Then, as usual, we define an auxiliary bipartite graph G_b between the edges e_{2j} and the vertices $v \in V(G) \setminus V(P')$ where we put an edge between e_{2j} and v, if v cannot be spliced in into e_{2j}. By the above G_b has at most $6 \cdot 12 = 72$ edges. Then indeed the number of exceptional vertices is at most 12, since otherwise the number of edges of this bipartite graph would be more than $12 \cdot 6 = 72$, a contradiction. Note that the degree of all non-exceptional vertices of $V(G) \setminus V(P')$ in $\overline{G_b}$ is at least 6, i.e. each non-exceptional vertex can be spliced in into at least 6 edges e_{2j} of P'; a fact that will be important later.

For the at most 12 exceptional vertices we will find vertex disjoint absorbing bridges in red where they will be spliced in. The fact that we are not in EC2 makes this possible. Indeed, we do the following for the exceptional vertices. Denote the exceptional vertices with $v_1, v_2, \dots, v_{12}$ (we may assume that there are exactly 12 such vertices by taking arbitrary vertices from $V(G) \setminus V(P')$. We find vertex disjoint v_i-absorbing bridges $P_i = \{p_1^i, p_2^i, p_3^i, p_4^i\}$ for $1 \leq i \leq 12$ such that the following are true (to make sure that the paths can be connected and that this new path can be a part of a 3 -tight Berge-cycle).

- The triples $\{p_{c_2-1}, p_{c_2}, p_1^1\}$ and $\{p_{c_2}, p_1^1, p_2^1\}$ are good in red. This allows us to connect P' and the bridge P_1.
- The triples $\{p_3^i, p_4^i, p_1^{i+1}\}$ and $\{p_4^i, p_1^{i+1}, p_2^{i+1}\}$ are good in red for $1 \le i \le 11$. This allows us to connect the bridges P_i and P_{i+1}.
- If P_i is a type 2 bridge with the 4th vertex w_i, then the vertices p_4^{i-1} (or p_{c_2} if $i = 1$) and p_1^{i+1} are not equal to w_i.

Indeed, from the fact that we have at least $\sqrt{\alpha}n^4$ v_i-absorbing bridges for each $1 \le i \le 12$ (since we are not in EC2) we can find vertex disjoint $\{p_2^i, p_3^i\}$ in such a way that we have at least $\sqrt{\alpha}n/4$ available choices for both p_1^i and p_4^i. Then clearly we can pick p_1^i and p_4^i such that the above properties hold.

Thus indeed we can connect $P', P_1, P_2, \ldots, P_{12}$ into one path. Splice in the vertices $v_1, v_2, \ldots, v_{12}$ into their bridges between p_2^i and p_3^i. Denote the resulting path by P''. For technical reasons let us "leave open" the endpoints of this path. This P'' has the following properties. Any triple of consecutive three vertices on P'' is good in red if it does not contain any of the vertices $v_i, 1 \le i \le 12$, or if it does contain a vertex v_i with a type 1 bridge. For the consecutive triples T that contain a vertex v_i with a type 2 bridge with the 4th vertex w_i, the corresponding 4-edge of $\mathcal{K}$ containing T will be $T \cup \{w_i\}$. The above construction guarantees that there will not be any repetitions of these 4-edges and thus indeed P'' can be a part of a 3-tight Berge-cycle. Note that the length of P'' is still a constant $(25 + 60 = 85)$.

Using the fact that P'' has length 85, we can still apply Lemma 2.1 to find in G a red path $Q = \{q_1, q_2, \ldots, q_l\}$, $f_i = \{q_i, q_{i+1}\}$, $l \ge (2/3 + \eta)n$ that is vertex disjoint from P''. Indeed, we mark the vertices in P'' as forbidden vertices, and by Lemma 2.1 we still have at least $c_1 \log n - 85 \ge c_1 \log n/2$ available choices for each vertex of Q (using that n is sufficiently large). Furthermore, as in P', we can also guarantee that any triple of consecutive three vertices on Q is good in red and that we can connect the endpoints of P'' and Q similarly as above. Thus we get a cycle $C' = P'' \cup Q$. Consider the bipartite graph $\overline{G_b}$ between the remaining vertices in $V(G) \setminus V(C')$ and the set of edges

$$E = \{e_{2j} \mid 2 \le 2j \le c_2 - 1\} \bigcup \{f_{2i} \mid 2 \le 2i \le l - 1\},$$

where we put en edge between a vertex $v \in V(G) \setminus V(C')$ and an edge e_{2j} or f_{2i} if the vertex can be spliced in into the edge.

Claim 1. *There is a perfect matching M in $\overline{G_b}$ from $V(G) \setminus V(C')$.*

Indeed, we have to check Hall's condition, i.e. for every $S \subset V(G) \setminus V(C')$ we need $|N_{\overline{G_b}}(S)| \ge |S|$. For $|S| \le 6$, this is true as

$$|N_{\overline{G_b}}(S)| \ge deg(v) \ge 6 \ge |S|,$$

for an arbitrary $v \in S$. However, for $|S| \ge 7$ we have

$$|N_{\overline{G_b}}(S)| = |E| \ge (1/3 + \eta/2)n \ge |S|, \tag{1}$$

as desired (since for each $e \in E$ we can have at most 6 exceptional vertices that cannot be spliced in into e).

We splice in the vertices of $V(G) \setminus V(C')$ into the edges where they are matched under M. Now we finish the proof of the non-extremal case by claiming that the Hamiltonian cycle C that we get after splicing in the vertices of $V(G) \setminus V(C')$ is indeed a red 3-tight Berge-cycle. Indeed, every triple of three consecutive vertices on C that does not contain a vertex v_i with a type 2 bridge is good in red. For the triples containing a vertex v_i with a type 2 bridge we already found the distinct red 4-edges of $\mathcal{K}$ containing them (by adding the corresponding w_i to the triple). For the other triples, since they are good in red, there are at least two red 4-edges of $\mathcal{K}$ available to cover them. However, no edge of $\mathcal{K}$ can cover more than two of these triples of C. Thus, by Hall's theorem again, there is a matching from these triples of C to the set of red edges of $\mathcal{K}$ containing them, and thus resulting in a red Hamiltonian 3-tight Berge-cycle finishing the proof in the non-extremal case.

6 Extremal Coloring 2

For technical reasons we treat first Extremal Coloring 2. In fact, this can be reduced to the non-extremal case. Let us assume that we have an Extremal Coloring 2. By the definition there must exist a color (say red) and a vertex v_r, such that we cannot find at least $\sqrt{\alpha}n^4$ v_r-absorbing bridges in red. In this case we will show that either we can find a Hamiltonian 3-tight Berge-cycle in blue or we can find sufficiently many v_r-absorbing bridges in red after all with a somewhat weaker definition of a bridge, that is just as good.

We will show first that we may assume that the blue edges form a $(1-\alpha^{1/10})$-dense subgraph in G. Indeed, if the density of the red edges is at most $\alpha^{1/10}$, then this is immediate. Otherwise, consider the set of red edges and mark those red edges e for which v_r is not among the at most 2 exceptional vertices, i.e. for which (e, v_r) forms a good triple in red. If the density of the marked red edges is at least $\alpha^{1/10}$, then we could clearly find at least $\sqrt{\alpha}n^4$ paths of length 3 consisting of marked red edges. However, these paths are v_r-absorbing bridges in red, a contradiction with our assumption. Indeed, one may take a subgraph of the marked red edges where the minimum degree is at least half of the original average degree (see e.g. Proposition 1.2.2 in [4]), and then use a greedy procedure and the fact that $\alpha \ll 1$.

Thus we may assume that this is not the case, the density of the marked red edges is less than $\alpha^{1/10}$. Next we will show that we may assume that all unmarked red edges are blue as well in this 2-multicoloring. Let us take an unmarked red edge f. By definition, the triple (f, v_r) is not a good triple in red, so apart from at most one edge all 4-edges of $\mathcal{K}$ containing the triple are blue. In other words f is contained in at least $(n-4)$ blue triples. It seems as this is a slightly weaker condition than being blue in G, as these $(n-4)$ blue triples might not be good in blue. On the other hand, it is always the same vertex (namely v_r) that we have to add to each of these triples to get a blue

4-edge of $\mathcal{K}$, and this property is just as good for building a 3-tight Berge-cycle and that is our ultimate goal. Let us call these edges *weak blue* edges, since they are almost as good as blue edges. Then every unmarked red edge of G is weak blue and thus the density of the blue edges (blue or weak blue) is at least $(1-\alpha^{1/10})$, as claimed.

Thus certainly in this case in blue (or weak blue) we can find a monochromatic path much longer than $(2/3+\eta)n$. Next we will show that we may assume that in blue we have sufficiently many absorbing bridges for every vertex, and thus we are in EC2 only because of the red color. Then we can proceed similarly in blue, as in the non-extremal case in red. Indeed, having weak blue edges instead of blue edges is not going to create any difficulties since we can always choose v_r as the 4th vertex of the blue 4-edge containing a triple of three consecutive vertices with a weak blue edge. This finishes the proof in this case.

Thus to finish let us assume that we do not have sufficiently many absorbing bridges for every vertex in blue, i.e. there exists a vertex v_b such that we cannot find at least $\sqrt{\alpha}n^4$ v-absorbing bridges in blue. Similarly as above (with the colors playing the opposite roles) we may assume that the density of red edges (red or weak red, where here for the weak red edges we always have to add v_b as the 4th vertex) is at least $(1-\alpha^{1/10})$. Thus at least $(1-2\alpha^{1/10})$-portion of all the edges are both red and blue. Consider all those 4-edges of $\mathcal{K}$ that we get when we add $\{v_r, v_b\}$ (if $v_r = v_b$, we add an arbitrary other vertex) to these edges and the majority color induced by these edges. If this color is red, then we can find many (certainly much more than $\sqrt{\alpha}n^4$) v_r-absorbing type 2 bridges in red where the vertex w in the definition of the type 2 bridge can be chosen as v_b. We might have to use weak red edges on these red bridges instead of just red edges, but they are just as good for building bridges. We just have to make sure that v_b is never used on these bridges. Thus we have sufficiently many v_r-absorbing bridges in red after all. If the majority color is blue then we have sufficiently many v_r-absorbing type 2 bridges in blue, as desired.

We can repeat the same argument for blue as well if blue also violates the condition of having sufficiently many bridges. Thus in summary we can claim that either we can find a monochromatic Hamiltonian 3-tight Berge-cycle or we can assume that we have sufficiently many bridges for every vertex in both colors.

7 Extremal Coloring 1

Assume finally that we have an Extremal Coloring 1. Thus there exists a partition $V(G) = A \cup B$ such that

- $|A| \geq (1-\alpha)\frac{2|V(G)|}{3}$, $|B| \geq (1-\alpha)\frac{|V(G)|}{3}$.
- The graph $G_1^*|_A$ is $(1-\alpha)$-dense and the bipartite graph $G_2^*|_{A\times B}$ is $(1-\alpha)$-dense, where say G_1 is red and G_2 is blue.

The main idea is the same as in the non-extremal case; either in red or in blue we have to find a long enough monochromatic cycle in G and then we splice in the remaining vertices into roughly every other edge on the cycle. In light of the previous section we may assume that we have sufficiently many bridges for every vertex in both colors, so this is not going to be a problem. However, here we might not be able to find a long enough monochromatic cycle since we are in EC1.

First we will redistribute certain exceptional vertices from A and B. A vertex $u \in A$ is *exceptional* if its red-only degree in A is significantly less than $|A|$, i.e. we have

$$deg_{G_1^*}(u, A) < (1 - \alpha)|A|, \tag{2}$$

From the density condition in $G_1^*|_A$, it follows that the number of these exceptional vertices in A is at most $\alpha|A|$. If in (2) we have the stronger inequality

$$deg_{G_1^*}(u, A) < \sqrt{\alpha}|A|,$$

then we move u from A to B, since indeed now we have

$$deg_{G_2}(u, A) > (1 - \sqrt{\alpha})|A|.$$

Similarly, a vertex $v \in B$ is *exceptional* if its blue-only degree in A is significantly less than $|A|$, i.e. we have

$$deg_{G_2^*}(v, A) < (1 - \alpha)|A|, \tag{3}$$

From the density condition in $G_2^*|_{A \times B}$, it follows again that the number of these exceptional vertices in B is at most $\alpha|B|$. If in (3) we have the stronger inequality

$$deg_{G_2^*}(v, A) < \sqrt{\alpha}|A|,$$

then we move v from B to A, since now we have

$$deg_{G_1}(v, A) > (1 - \sqrt{\alpha})|A|.$$

For simplicity let us denote the resulting sets still by A and B. We distinguish two cases.

Case 1: $|B| \leq \lfloor \frac{n}{3} \rceil$. In this case we will find a red Hamiltonian 3-tight Berge-cycle. We proceed exactly as in the non-extremal case, but we have to be slightly more careful because of the sharp size conditions. We will build P'' consisting of P' and the absorbing bridges for the exceptional vertices as in the non-extremal case. However, here we also make sure that the connecting edges between the subpaths are also red edges (it is not hard to see from the degree conditions that this is possible). Furthermore, we can also see from the degree conditions (e.g. using Pósa's condition, see [2] or [17]) that $C' = P'' \cup Q$ may cover all vertices in A. The set of edges E where we can splice in the remaining vertices includes now literally every second edge on C', so it has size $|E| \geq \lfloor \frac{|C'|}{2} \rfloor \geq \lfloor \frac{n}{3} \rfloor$. Then corresponding to (1) we still have

$$|N_{\overline{G_b}}(S)| = |E| \geq \lfloor \frac{n}{3} \rfloor \geq |S|, \tag{4}$$

and thus we can still splice in every remaining vertex of $V(G) \setminus V(C')$ resulting in a red Hamiltonian 3-tight Berge-cycle.

Case 2: $|B| > \lfloor \frac{n}{3} \rceil$. In this case we will find a blue Hamiltonian 3-tight Berge-cycle. Now we build $C' = P'' \cup Q$ in the blue almost-complete bipartite graph between A and B in such a way that we cover all vertices of B with C'. Then (4) is true again, and we can splice in every remaining vertex of $V(G) \setminus V(C')$ resulting in a blue Hamiltonian 3-tight Berge-cycle. This finishes the proof of Theorem 1.2. □

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