10-2008

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December 5, 2007

Abstract

In this paper we study multipartite Ramsey numbers for odd cycles. We formulate the following conjecture: Let $n \geq 5$ be an arbitrary positive odd integer, then in any two-coloring of the edges of the complete 5-partite graph $K((n-1)/2, (n-1)/2, (n-1)/2, (n-1)/2, 1)$ there is a monochromatic C_n,

*2000 Mathematics Subject Classification: 05C55, 05C38.

[†]Research supported in part by the National Science Foundation under Grant No. DMS-0456401.

a cycle of length n. This roughly says that the Ramsey number for C_n (i.e. $2n-1$) will not change (somewhat surprisingly) if four large "holes" are allowed. Note that this would be best possible as the statement is not true if we delete from K_{2n-1} the edges within a set of size $(n+1)/2$. We prove an approximate version of the above conjecture.

1 Introduction

1.1 Ramsey numbers for odd cycles

If G_1 and G_2 are graphs, then the Ramsey number $R(G_1, G_2)$ is the smallest positive integer r such that if the edges of a complete graph K_r are partitioned into 2 disjoint color classes giving graphs H_1 and H_2, then one of the subgraphs H_i ($i = 1, 2$) has a subgraph isomorphic to G_i. The existence of such a positive integer is guaranteed by Ramsey's original paper [17]. The number $R(G_1, G_2)$ is called the Ramsey number for the graphs G_1 and G_2. The determination of these numbers has turned out to be remarkably difficult in certain cases (see eg. [5] or [16] for results and problems). In this paper we consider the case when each G_i is a cycle C_n on n vertices, where n is odd. A theorem obtained independently by Rosta [18] and Faudree and Schelp [2] (see also a new simple proof in [8]) states that for any $n \geq 5$ odd positive integer

$$R(C_n, C_n) = 2n - 1. \tag{1}$$

Recently there has been some interest to see what happens to the Ramsey numbers when we allow fixed edge deletions from the complete graph K_r, in particular if we delete complete subgraphs from K_r. One result of this type appeared in [7], where we gave a tripartite version of the Gerencsér-Gyárfás Theorem, i.e. we showed that the Ramsey number for a path is about the same when two-colorings of a complete graph or a balanced complete tripartite graph are considered. Another result of this type appeared in [15], where it was shown for any odd $n \geq 5$ that if we delete the edges of a complete subgraph of order $(n-1)/2$ from the complete graph of order $2n-1$ and we two-color the rest, we can still guarantee a monochromatic C_n.

In this paper along these lines we consider a multipartite version of (1). We formulate the following conjecture.

Conjecture 1. *Let $n \geq 5$ be an arbitrary positive odd integer, then in any two-coloring of the edges of the complete 5-partite graph $K(\frac{n-1}{2}, \frac{n-1}{2}, \frac{n-1}{2}, \frac{n-1}{2}, 1)$, there is a monochromatic C_n.*

Again this roughly says that the Ramsey number for C_n will not change (somewhat surprisingly) if four large "holes" are allowed. Note that this would be best possible as the statement is already not true if we have an independent set of size $\frac{n+1}{2}$ (so even one hole of size $\frac{n+1}{2}$ is not allowed). Indeed, let us remove all the edges spanned by the set A, where $|A| = \frac{n+1}{2}$. Divide the vertices $V \setminus A$ into two sets B and C with $|B| = \frac{n-1}{2}$ and $|C| = n - 1$. Let the first color be all the edges within B, within C and between A and B. The second color is the remaining edges. Then it is easy to see that there is no monochromatic C_n.

Conjecture 1 holds for $n = 5$, but is open in general. It is the purpose of this paper to give an approximate result which gives further evidence to the truth of this conjecture. More precisely we prove the following theorem.

Theorem 1. *For all $0 < \eta < 1/2$ there exists an $n_0 = n_0(\eta)$ with the following properties. For any $n \geq n_0$ positive odd integer, in any two-coloring of the edges of the complete 5-partite graph of order $(2+\eta)n$ with 5 parts of size $g(1), g(2), g(3), g(4)$ and $g(5)$, where we have*

$$n/2 \geq g(1) \geq g(2) \geq g(3) \geq g(4) \geq g(5) \geq \eta n,$$

there is a monochromatic C_n.

We note that recently there has been some interest in multipartite versions of classical results, see e.g. the result of Magyar and Martin [14], a tripartite version of the Corrádi-Hajnal Theorem, or our result in [7], a tripartite version of the Gerencsér-Gyárfás Theorem.

In the proof of Theorem 1 the notion of an **odd connected matching** plays a central role; this is a matching M in a graph G such that all edges of M are in the same non-bipartite connected component of G, such a component is called an **odd component**. This is related to the concept of a connected matching that was introduced by Łuczak [13] and applied e.g. in [3], [6] and [7].

Sections 2 and 3 provide our main tools including the Regularity Lemma. Then in Section 4 we prove our main lemma (Lemma 6) which states that in any two-coloring of a $(1-\varepsilon)$-dense 5-partite graph with the right parameters there is a sufficiently large monochromatic odd connected matching. Finally in Section 5 we show how Lemma 6 implies Theorem 1.

1.2 Notation and definitions

For basic graph concepts see the monograph of Bollobás [1]. Disjoint union of sets will be sometimes denoted by $+$. $V(G)$ and $E(G)$ denote the vertex-set and the edge-set of the graph G. Usually G_n is a graph with n vertices, $G(n_1, \ldots, n_k)$ is a k-partite graph with classes containing $n_1, \ldots, n_k$ vertices. (A, B, E) denotes a bipartite graph $G = (V, E)$, where $V = A + B$, and $E \subset A \times B$. K_n is the complete graph on n vertices, $K(n_1, \ldots, n_k)$ is the complete k-partite graph with classes containing $n_1, \ldots, n_k$ vertices, P_n (C_n) is the path (cycle) with n vertices. For a graph G and a subset U of its vertices, $G|_U$ is the restriction to U of G. $\Gamma(v)$ is the set of neighbors of $v \in V$. Hence the size of $\Gamma(v)$ is $|\Gamma(v)| = deg(v) = deg_G(v)$, the degree of v. $\delta(G)$ stands for the minimum, and $\Delta(G)$ for the maximum degree in G. For a vertex $v \in V$ and set $U \subset V - \{v\}$, we write $deg(v, U)$ for the number of edges from v to U. A graph G_n is γ-**dense** if it has at least $\gamma\binom{n}{2}$ edges. The (A, B, E) bipartite graph is γ-**dense** if it has at least $\gamma|A||B|$ edges. The $G(n_1, \ldots, n_k)$ k-partite graph is γ-**dense** if all the bipartite graphs between two classes are γ-dense. When A, B are disjoint subsets of $V(G)$, we denote by $e_G(A, B)$ the number of edges of G with one endpoint in A and the other in B. For non-empty A and B,

$$d_G(A, B) = \frac{e_G(A, B)}{|A||B|}$$

is the **density** of the graph between A and B.

Definition 1. *The bipartite graph $G = (A, B, E)$ is (ε, G)-**regular** if*

$$X \subset A,\ Y \subset B,\ |X| > \varepsilon|A|,\ |Y| > \varepsilon|B| \quad \textit{imply} \quad |d_G(X, Y) - d_G(A, B)| < \varepsilon,$$

otherwise it is ε-irregular.

2 The Regularity Lemma

In the proof a two-color version of the Regularity Lemma plays a central role.

Lemma 1 (Regularity Lemma [19]). *For every positive ε and positive integer m there are positive integers M and n_0 such that for $n \geq n_0$ the following holds. For all graphs G_1 and G_2 with $V(G_1) = V(G_2) = V$, $|V| = n$, there is a partition of V into $l + 1$ classes (clusters)*

$$V = V_0 + V_1 + V_2 + \ldots + V_l$$

such that

- $m \leq l \leq M$
- $|V_1| = |V_2| = \ldots = |V_l|$
- $|V_0| < \varepsilon n$
- *apart from at most $\varepsilon\binom{l}{2}$ exceptional pairs, the pairs $\{V_i, V_j\}$ are (ε, G_s)-regular for $s = 1, 2$.*

For an extensive survey on different variants of the Regularity Lemma see [11]. Note also that if we apply the Regularity Lemma for a multipartite graph G with big enough partite classes we can guarantee that for each cluster that is not V_0, all vertices of the cluster belong to the same partite class of G (see eg. [14]).

We will also use the following simple property of (ε, G)-regular pairs.

Lemma 2. *Let G be a bipartite graph with bipartition $V(G) = V_1 \cup V_2$ such that $|V_1| = |V_2| = m \geq 45$. Furthermore, let $e_G(V_1, V_2) \geq m^2/4$ and the pair $\{V_1, V_2\}$ be (ε, G)-regular for $0 < \varepsilon < 0.01$. Then for every $l, 1 \leq l \leq m - 5\varepsilon m$ and for every pair of vertices $v' \in V_1, v'' \in V_2$, where $deg(v'), deg(v'') \geq m/5$, G contains a path of length $2l+1$ connecting v' and v''.*

This lemma is used by Łuczak in [13]. Lemma 2 (with somewhat weaker parameters) also follows from the much stronger Blow-up Lemma (see [9] and [10]).

3 Further graph theory tools

A set M of pairwise disjoint edges of a graph G is called a matching. The size $|M|$ of a maximum matching is the matching number, $\nu(G)$. A key notion in our approach is the notion of an odd connected matching. A matching M is an odd connected matching in G if all edges of M are in the same non-bipartite connected component of G, such a component is called an odd component. For a multipartite graph G, we shall work with its *multipartite complement*, $\overline{G}$, defined as the graph we obtain from the usual complement after deleting all edges within the partite classes. The next lemmas collect some simple properties of multipartite graphs of high density.

Lemma 3. *Assume $\Delta(\overline{G_n}) < \sqrt{\varepsilon}n$ and $H = [A, B]$ is a bipartite subgraph of G_n with $2\sqrt{\varepsilon}n < |A| \leq |B|$. Then H is a connected subgraph of G_n and contains a matching of size at least $|A| - \sqrt{\varepsilon}n$. Moreover, if only $2\sqrt{\varepsilon}n < |B|$ and $A \neq \emptyset$ is assumed then there is a subgraph H' which is connected and covers A and all but at most $\sqrt{\varepsilon}n$ vertices of B.*

Proof: Two vertices in A (B) have a common neighbor in B (A). Also if $a \in A, b \in B$ then any neighbor of a and b have a common neighbor in A. Thus H is a connected subgraph. Moreover any maximum matching M misses fewer than $\sqrt{\varepsilon}n$ vertices of A. The statement about H' follows by fixing a vertex $a \in A$ and H' is obtained by deleting from B the vertices nonadjacent to A. $\square$

Lemma 4. *Assume that G is an r-partite graph with N vertices such that $r \geq 3$, $\Delta(\overline{G}) < \rho N$ where $\rho < \frac{1}{r(r-1)}$. Suppose that the largest partite class of G has at most as many vertices as the sum of the orders of the other color classes. Then G has a matching covering at least $(1 - 2\rho)N$ vertices of G.*

Proof: Assume that $V_1, \ldots V_r$ are the vertex classes of G in nondecreasing order. Continue with the following procedure until the number of vertices U uncovered in $V_1 \cup \ldots \cup V_{r-1}$ is smaller than the number of vertices in V_r. Note that by the assumption of the lemma, the stopping condition is not present at the beginning.

Take the edges of a largest matching from V_1 to V_2, one by one. Take the edges of a largest matching from the unmatched vertices of V_2 to V_3, one by one, etc. The last step is to take a largest matching from the unmatched vertices of V_{r-2} to V_{r-1}, one by one.

This procedure covers two vertices at a time and by Lemma 3 leaves at most ρN vertices uncovered in each of the first $r-1$ partite classes. If the stopping condition never occurs then

$$(r-1)\rho N \geq |V_r| \geq \frac{N}{r},$$

contradicting the assumption of the lemma. Thus the procedure eventually stops - notice that we have at least $|V_r| - 2$ uncovered vertices at this time because each step covered two points. Now we finish by taking a largest matching from the set U to V_r. At this point by Lemma 3 we leave at most ρN uncovered points in V_r and also in $V(G) \setminus V_r$, proving the lemma. $\square$

Lemma 5. *Let $0 < \varepsilon, \eta < 1/2$. Assume that $G = G(g(1), g(2), g(3), g(4), g(5))$ is a $(1-\varepsilon)$-dense 5-partite graph on $(2+\eta)n = g(1)+g(2)+g(3)+g(4)+g(5)$ vertices, where we have*

$$n/2 \geq g(1) \geq g(2) \geq g(3) \geq g(4) \geq g(5) \geq \eta n.$$

Then G has a 5-partite subgraph $H = H(h(1), h(2), h(3), h(4), h(5))$ with $h(i) \geq (1 - \frac{5\sqrt{\varepsilon}}{8\eta^2})g(i)$ for all $1 \leq i \leq 5$ such that $\Delta(\overline{H}) < 4\sqrt{\varepsilon}g(1)$.

Proof: If G has $p(i)$ vertices in the same partite class of $g(i)$ vertices with degree at least $4\sqrt{\varepsilon}g(i)$ in $\overline{G}$, then $\overline{G}$ has at least $p(i)4\sqrt{\varepsilon}g(i)$ edges. Therefore $p(i)4\sqrt{\varepsilon}g(i) \leq 10\varepsilon g(1)^2$, implying $p(i) \leq \frac{5}{2}\sqrt{\varepsilon}\frac{g(1)^2}{g(i)}$. Removing these $p(i)$ vertices from the partite class with $g(i)$ vertices for each $1 \leq i \leq 5$, the remaining vertices induce the subgraph H. Clearly $\Delta(\overline{H}) < 4\sqrt{\varepsilon}g(1)$. We also have

$$h(i) \geq g(i) - \frac{5}{2}\sqrt{\varepsilon}\frac{g(1)^2}{g(i)} = \left(1 - \frac{5}{2}\sqrt{\varepsilon}\frac{g(1)^2}{g(i)^2}\right) g(i) \geq \left(1 - \frac{5\sqrt{\varepsilon}}{8\eta^2}\right) g(i). \qquad \square$$

4 Large monochromatic odd connected matchings in dense 5-partite graphs

In our main lemma we show that we can find large monochromatic odd connected matchings in dense 5-partite graphs.

Lemma 6. *For all* $0 < \varepsilon \leq \frac{\eta^6}{100^2}$, $0 < \eta < \frac{1}{2}$, *there exists an* $n_0 = n_0(\varepsilon, \eta)$ *with the following properties. For any* $n \geq n_0$ *positive integer, assume that* $G = G(g(1), g(2), g(3), g(4), g(5))$ *is a* $(1-\varepsilon)$*-dense 5-partite graph on* $(2+\eta)n = g(1)+g(2)+g(3)+g(4)+g(5)$ *vertices, where we have*

$$n/2 \geq g(1) \geq g(2) \geq g(3) \geq g(4) \geq g(5) \geq \eta n.$$

Then for each two-coloring of G *there is a monochromatic odd connected matching covering at least* n *vertices.*

Proof: Let us apply Lemma 5 first for the 5-partite graph G to find a 5-partite subgraph $H = H(h(1), h(2), h(3), h(4), h(5))$ of G with $h(i) \geq (1 - \frac{5\sqrt{\varepsilon}}{8\eta^2})g(i)$ for all $1 \leq i \leq 5$ such that $\Delta(\overline{H}) < 4\sqrt{\varepsilon}g(1) \leq 2\sqrt{\varepsilon}n$. Thus each vertex in any partite set of H is adjacent to almost all (all but $2\sqrt{\varepsilon}n$) vertices in the remaining four partite sets of H. This is used throughout without special mention and observe that with $\rho = 2\sqrt{\varepsilon}$ any subgraph F of H with at least n vertices satisfies the condition $\Delta(\overline{F}) < \rho n \leq \rho|V(F)|$ in Lemma 4. Here we use the assumption $\rho = 2\sqrt{\varepsilon} < \frac{1}{5\times 4} = \frac{1}{20}$.

We consider only the two-coloring (red/blue) of $E(H)$ induced by the coloring of $E(G)$. In this two-coloring of H select a maximal monochromatic (say red) connected odd component C. In fact we need only that the component we select is not proper part of another odd component in the other color. It is easy to check that there exists monochromatic odd components since H is a 5-chromatic graph.

For $1 \leq i \leq 5$, let V_i denote the partite sets of H, so $|V_i| = h(i)$, let $X_i = V_i \setminus V(C), Y_i = V(C) \cap V_i$. Call an X_i *large* if $|X_i| > 4\sqrt{\varepsilon}n$.

We will distinguish two cases:

Case 1: At least two of the X_i-s are large.

By Lemma 3 (since all edges of H from a large X_i to C are blue) the blue subgraph of H is a spanning connected subgraph. Moreover, any blue edge between two Y_i-s would make the blue subgraph odd - this contradicts the choice of C. Similarly, any blue edge between a pair of large X_i-s would contradict the choice of C. Thus C and the union of the large X_i-s both are almost complete red partite graphs. The larger, denote it by F, have at least

$$\frac{1}{2}\left(2 + \eta - (2+\eta)\frac{5\sqrt{\varepsilon}}{8\eta^2} - 12\sqrt{\varepsilon}\right)n = (1+\alpha)n$$

vertices. The condition $\varepsilon \leq \frac{\eta^6}{100^2}$ ensures that

$$(1 - 2\rho)(1 + \alpha) \geq 1, \tag{2}$$

in particular α is positive here. Since each partite class of F has at most $\frac{n}{2}$ vertices, the assumption of Lemma 4 about the size of the largest partite class holds and it is also ensured that F is r-partite with $r \geq 3$. Thus the red matching M obtained by Lemma 4 is odd and M covers at least $(1-2\rho)(1+\alpha)n \geq n$ vertices because of (2).

Case 2: At most one X_i - say X_1 (if there is one) - is large.

If there is no blue edge in the subgraph induced by $S = V(C) \setminus V_1$ then we can apply Lemma 4 to the red subgraph of H induced by S and we have a red odd matching in C covering almost all points of S - this is much larger than we need since $|S| > \frac{3n}{2}$. Thus we may assume that there is at least one blue edge in S. If X_1 is large then this ensures - since all edges of H from X_1 to S are blue - a blue odd component C_1 of H covering X_1 and all vertices of S. Let Z denote the part of V_1 that is not covered by C_1. Since all edges of H from Z to $V(H) \setminus V_1$ are red (from the definition of C_1) we can cover Z by a red matching M_R. Similarly we can cover X_1 with a blue matching M_B that is disjoint from M_R. One can easily complete the matching $M_R \cup M_B$ to a matching M covering almost all vertices of $V(H)$ (here we can apply Lemma 4 again for the 2-colored partite graph spanned by $V(H) \setminus (V(M_R) \cup V(M_B))$. Selecting the majority color from the edges of M we have the required monochromatic large odd matching.

If X_1 is not large then C covers almost all vertices of H. Select the largest red matching M in C. If M covers less than n vertices then - since all edges of H in the subgraph induced by $T = V(C) \setminus V(M)$ are blue - Lemma 4 ensures again that we have a blue matching covering almost all vertices of the odd component induced by T. This matching is large enough. $\square$

5 Proof of Theorem 1

We will assume that n is a sufficiently large odd natural number. Let $0 < \eta < 1/2$ be arbitrary and choose

$$\varepsilon = \frac{(\frac{\eta}{2})^6}{100^2}. \tag{3}$$

Let G be the complete 5-partite graph of order $(2+\eta)n$ with 5 parts of size $g(1), g(2), g(3), g(4)$ and $g(5)$, where we have

$$n/2 \geq g(1) \geq g(2) \geq g(3) \geq g(4) \geq g(5) \geq \eta n.$$

We need to show that each 2-edge coloring of G leads to a monochromatic C_n. Consider a 2-edge coloring (G_1, G_2) of G. Let V_i denote the partite classes, so $|V_i| = g(i)$. Apply the two-color 5-partite version of the Regularity Lemma (Lemma 1), with ε as in (3) and (by using the remark after the lemma) we can get a partition for $i = 1, \ldots, 5$ of $V_i = V_i^0 + V_i^1 + \ldots + V_i^{l_i}$, where $|V_i^j| = m$, $1 \leq j \leq l_i$, $1 \leq i \leq 5$ and $|V_i^0| < \varepsilon n$, $1 \leq i \leq 5$. We define the following **reduced graph** G^r: The vertices of G^r are p_i^j, $1 \leq j \leq l_i$, $1 \leq i \leq 5$, and we have an edge between vertices $p_{i_1}^{j_1}$ and $p_{i_2}^{j_2}$, $1 \leq j_1 \leq l_{i_1}$,

$1 \le j_2 \le l_{i_2}$, $1 \le i_1, i_2 \le 5$, $i_1 \ne i_2$, if the pair $\{V_{i_1}^{j_1}, V_{i_2}^{j_2}\}$ is (ε, G_s)-regular for $s = 1, 2$. Thus we have a one-to-one correspondence $f : p_i^j \to V_i^j$ between the vertices of G^r and the non-exceptional clusters of the partition. Then G^r is a $(1-\varepsilon)$-dense 5-partite graph on $l = l_1 + \ldots + l_5$ vertices, where note again that l is a constant (it does not depend on n). Define a 2-edge coloring (G_1^r, G_2^r) of G^r in the following way. The color of the edge between the clusters $V_{i_1}^{j_1}$ and $V_{i_2}^{j_2}$ is the majority color in the pair $\{V_{i_1}^{j_1}, V_{i_2}^{j_2}\}$. Let

$$l' = \frac{l}{4+\eta} \tag{4}$$

(assume for simplicity that this is an integer).

Using (3), (4) and Lemma 6 with $\eta/2$ instead of η implies that in such a 2-coloring of G^r we can find a monochromatic odd connected matching $M = \{e_1, e_2, \ldots, e_{l'}\}$ covering $2l'$ vertices of G^r. Assume that M is in G_1^r. Thus we have

$$\left| \cup_{i=1}^{l'} \cup_{p \in e_i} f(p) \right| \ge \frac{2+\eta}{2+\frac{\eta}{2}} (1-\varepsilon) n \ge (1+\frac{\eta}{8}) n, \tag{5}$$

i.e. the total number of vertices of G in the clusters covered by M is significantly more than n, a fact that will be important later. Furthermore, define $f(e_i) = (C_1^i, C_2^i)$ for $1 \le i \le l'$ where C_1^i, C_2^i are the clusters assigned to the end points of e_i. In the remainder from this odd connected matching M in G_1^r we will construct a cycle C_n in G_1.

Since M is a connected matching in G_1^r we can find a connecting path P_i^r in G_1^r from $f^{-1}(C_2^i)$ to $f^{-1}(C_1^{i+1})$ for every $1 \le i \le l'-1$. Note that these paths in G_1^r may not be internally vertex disjoint. The last connecting path $P_{l'}^r$ in G_1^r from $f^{-1}(C_2^{l'})$ to $f^{-1}(C_1^1)$ will be used to guarantee the right parity. Since M is an odd connected matching in G_1^r we can find an odd cycle C in the component of G_1^r containing M. For the construction of the last connecting path $P_{l'}^r$ let us take first an arbitrary connecting path P' in G_1^r from $f^{-1}(C_2^{l'})$ to an arbitrary cluster C' on the cycle C, and then another connecting path P'' in G_1^r from another cluster C'' on the cycle C to $f^{-1}(C_1^1)$. On the cycle C there are two paths, Q' and Q'', of different parity connecting C' and C''. As a first try construct the last connecting path $P_{l'}^r$ as (P', Q', P''). Now compute the parity of the length of the closed trail that consists of all the connecting paths $P_i^r, 1 \le i \le l'$ and where we connect each C_1^i and C_2^i with an arbitrary path of odd length. If this parity is even, then we change the construction (and thus the parity) of the last connecting path $P_{l'}^r$ to (P', Q'', P''). Thus we may assume that this parity is always odd and note that it remains odd in the remainder when we "blow-up" parts of the path.

From these paths P_i^r in G_1^r we can construct vertex disjoint connecting paths P_i in G_1 connecting a typical vertex v_2^i of C_2^i to a typical vertex v_1^{i+1} of C_1^{i+1}. More precisely we construct P_1 with the following simple greedy strategy. Denote $P_1^r = (p_1, \ldots, p_t), 2 \leq t \leq l$, where according to the definition $f(p_1) = C_2^1$ and $f(p_t) = C_1^2$. Let the first vertex u_1 $(= v_2^1)$ of P_1 be a vertex $u_1 \in C_2^1$ for which $deg_{G_1}(u_1, f(p_2)) \geq m/4$ and $deg_{G_1}(u_1, C_1^1) \geq m/4$. By ε-regularity most of the vertices satisfy this in C_2^1. The second vertex u_2 of P_1 is a vertex $u_2 \in (f(p_2) \cap N_{G_1}(u_1))$ for which $deg_{G_1}(u_2, f(p_3)) \geq m/4$. Again by regularity most vertices satisfy this in $f(p_2) \cap N_{G_1}(u_1)$. The third vertex u_3 of P_1 is a vertex $u_3 \in (f(p_3) \cap N_{G_1}(u_2))$ for which $deg_{G_1}(u_3, f(p_4)) \geq m/4$. We continue in this fashion, finally the last vertex u_t $(= v_1^2)$ of P_1 is a vertex $u_t \in (f(p_t) \cap N_{G_1}(u_{t-1}))$ for which $deg_{G_1}(u_t, C_2^2) \geq m/4$.

Then we move on to the next connecting path P_2. Here we follow the same greedy procedure, we pick the next vertex from the next cluster in P_2^r. However, if the cluster has occurred already on the paths P_1^r or P_2^r, then we just have to make sure that we pick a vertex that has not been used on P_1 or P_2.

We continue in this fashion and construct the vertex disjoint connecting paths P_i in G_1, $1 \leq i \leq l'$. These will be parts of the final cycle C_n in G_1. We remove the internal vertices of these paths from G_1. Note that the total number of removed vertices, denoted by C, is still a constant of size at most $l^2 \ll n$, if n is sufficiently large. By doing this we may create some discrepancies in the cardinalities of the clusters of this odd connected matching. We remove at most l^2 vertices from each cluster of the matching to assure that now we have the same number of vertices left in each cluster of the matching. Assume without loss of generality that $\lfloor \frac{n-C}{l'} \rfloor$ is odd (otherwise take $\lfloor \frac{n-C}{l'} \rfloor - 1$). By applying Lemma 2 for $1 \leq i \leq l' - 1$, find a path of length $\lfloor \frac{n-C}{l'} \rfloor$ in $G_1|_{f(e_i)}$ connecting v_1^i and v_2^i. Indeed, (3) and (5) imply that the conditions of Lemma 2 are satisfied, since in each $f(e_i)$ we still have $(1 + \frac{\eta}{9})\frac{n}{l'}$ vertices available after the removals if n is sufficiently large. Finally apply Lemma 2 one more time to find a path of the right length in $G_1|_{f(e_{l'})}$ connecting v_1^i and v_2^i so that the overall length of the cycle is exactly n. This completes the proof of Theorem 1. □

References

[1] B. Bollobás, *Extremal Graph Theory*, Academic Press, London (1978).

[2] R.J. Faudree, R.H. Schelp, All Ramsey numbers for cycles in graphs, *Discrete Mathematics*, 8 (1974), pp. 313-329.

[3] A. Figaj, T. Łuczak, The Ramsey number for a triple of long even cycles, *Journal of Combinatorial Theory, Ser. B*, 97 (2007), pp. 584-596.

[4] L. Gerencsér, A. Gyárfás, On Ramsey-type problems, *Ann. Univ. Sci. Budapest Eötvös Sect. Math.* 10 (1967), pp. 167-170.

[5] R.L. Graham, B.L. Rothschild, J.H. Spencer, *Ramsey Theory*, John Wiley & Sons, 1990.

[6] A. Gyárfás, M. Ruszinkó, G. N. Sárközy, E. Szemerédi, Three-color Ramsey numbers for paths, *Combinatorica*, 27 (2007), pp. 35-69.

[7] A. Gyárfás, M. Ruszinkó, G. N. Sárközy, E. Szemerédi, Tripartite Ramsey numbers for paths, *Journal of Graph Theory*, 55 (2007), pp. 164-174.

[8] G. Károlyi, V. Rosta, Generalized and geometric Ramsey numbers for cycles, *Theoretical Computer Science*, 263 (2001), pp. 87-98.

[9] J. Komlós, G. N. Sárközy and E. Szemerédi, Blow-up Lemma, *Combinatorica* 17 (1997), pp. 109-123.

[10] J. Komlós, G. N. Sárközy and E. Szemerédi, An algorithmic version of the Blow-up Lemma, *Random Structures and Algorithms* 12 (1998), pp. 297-312.

[11] J. Komlós, M. Simonovits, Szemerédi's Regularity Lemma and its applications in graph theory, in *Combinatorics, Paul Erdős is Eighty* (D. Miklós, V.T. Sós, and T. Szőnyi, Eds.), Bolyai Society Math. Studies, Vol.2, pp. 295-352, Budapest, 1996.

[12] L. Lovász, M. D. Plummer, Matching Theory, Joint edition of North-Holland and Akadémiai Kiadó, 1986.

[13] T. Łuczak, $R(C_n, C_n, C_n) \leq (4 + o(1))n$, *Journal of Combinatorial Theory, Ser. B* 75 (1999), pp. 174-187.

[14] C. Magyar, R.R. Martin, Tripartite version of the Corrádi-Hajnal theorem, *Discrete Mathematics* 254 (2002), pp. 289-308.

[15] V. Nikiforov, R.H. Schelp, Cycles and stability, to appear in *Journal of Combinatorial Theory, Ser. B.*

[16] S.P. Radziszowski, Small Ramsey numbers, *Electronic Journal of Combinatorics*, (2002) DS1.

[17] F. P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc., 2nd Ser.*, 30 (1930), pp. 264-286.

[18] V. Rosta, On a Ramsey type problem of J.A. Bondy and P. Erdős, I & II, *Journal of Combinatorial Theory, Ser. B*, 15 (1973), pp. 94-120.

[19] E. Szemerédi, Regular partitions of graphs, Colloques Internationaux C.N.R.S. N$^{\underline{o}}$ 260 - *Problèmes Combinatoires et Théorie des Graphes*, Orsay (1976), 399-401.