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ARBITRAGE-FREE MODELS IN MARKETS WITH TRANSACTION COSTS

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Abstract

In the paper [7], Guasoni studies financial markets which are subject to proportional transaction costs. The standard martingale framework of stochastic finance is not applicable in these markets, since the transaction costs force trading strategies to have bounded variation, while continuous-time martingale strategies have infinite transaction cost. The main question that arises out of [7] is whether it is possible to give a convenient condition to guarantee that a trading strategy has no arbitrage. Such a condition was proposed and studied in [6] and [1], the so-called stickiness property, whereby an asset's price is never certain to exit a ball within a predetermined finite time. In this paper, we define the multidimensional extension of the stickiness property, to handle arbitrage-free conditions for markets with multiple assets and proportional transaction costs. We show that this condition is sufficient for a multi-asset model to be free of arbitrage. We also show that d-dimensional fractional Brownian models are jointly sticky, and we establish a time-change result for joint stickiness.

1 Introduction

In [7], a market with multiple risky assets and proportional transaction costs were studied. In the setting of [7], the market contains one risk free asset, used as a numeraire and hence assumed identically equal to 1, and d risky assets, given by an $\mathbb{R}^d-$valued process $Y_t = (Y_t^1, Y_t^2, \cdots, Y_t^d)$

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that is càdlàg (right- continuous with left-limits), adapted, and quasi-left continuous (i.e., $Y^i_\tau = Y^i_{\tau-}, 1 \leq i \leq d$ for all predictable stopping times τ). Transaction costs are proportional and each unit of numeraire traded in the risky assets generates a transaction cost of k units that are charged to the riskless asset account.

Trading strategies are given by adapted, left-continuous, $\mathbb{R}^d-$valued processes $\theta = (\theta^1_t, \theta^2_t, \cdots, \theta^d_t)$ that are of finite variation and satisfy the following admissibility condition:

$$V_t(\theta) = \sum_{i=1}^{d} \int_0^t \theta^i_s dY^i_s - \sum_{i=1}^{d} (\int_0^t kY^i_s d|D\theta^i|_s + k|\theta^i_t|Y^i_t) \geq -M \text{ a.s.} \tag{1}$$

for some determistic $M > 0$ and all $t \geq 0$. Here $D\theta^i$ is the derivative of θ^i_t in the sense of distribution, and $|D\theta^i|_t$ is the total variation measure associated to $D\theta^i$ in $[0, t]$. In (1), the term $\sum_{i=1}^d \int_0^t kY^i_s d|D\theta^i|_s$ corresponds to the cost of trading and $\sum_{i=1}^d k|\theta^i_t|Y^i_t$ corresponds to the liquidation cost at the end of trading.

Definition 1. *An admissible trading strategy θ is an arbitrage strategy if $V_t(\theta) \geq 0$ and $P(V_t(\theta) > 0) > 0$ for some $t > 0$.*

Remark 1. *Due to Proposition 2.5 of [7] and the quasi-left continuity assumption on the price processes, left-continuity of the trading strategies θ can be relaxed to predictablity.*

In the case when there is only one risky asset, the model (1) reduces to

$$V_t(\theta) = \int_0^t \theta_s dY_s - k \int_0^t Y_s d|D\theta|_s - kY_t|\theta_t|. \tag{2}$$

This model was studied in the recent papers [8, 1]. In [8], the notion of stickiness (see definition 2.9 of [8] and also Proposition 1 of [1]) was introduced as a sufficient for no-arbitrage in the model (2). It was also shown that a large class of Markov processes and models with full support in the Wiener space are sticky. In [1] stickiness was further studied and other classes of sticky processes were provided. In this note, we introduce a condition, which we call joint stickiness, and show that it is sufficient for no-arbitrage in the model (1), see proposition 1. Then we show joint stickiness remains unchanged under composition with continuous functions, see proposition 2. As an example, we show the joint sticky property for independent fractional Brownian motions with possibily different Hurst parameters, see Proposition 3. Lastly, we show a time change result on joint stickiness and provide non-semimartingale joint sticky processes by using time change, see Proposition 4 and corollaries thereafter.

2 Main Results

Let $X_t = (X^1_t, X^2_t, \cdots, X^d_t)$ be a càdlàg process adapted to the filtration $\mathbb{F} = (\mathscr{F}_t)_{t \in [0,T]}$. For any $\mathbb{F}$ stopping time $\tau \leq T$, let $A^{\tau,\epsilon}_i = \{sup_{t \in [\tau,T]} |X^i_t - X^i_\tau| < \epsilon\}$ for any $\epsilon > 0$.

Definition 2. *We say that $X_t = (X^1_t, X^2_t, \cdots, X^d_t)$ is jointly sticky with respect to $\mathbb{F}$ if*

$$P[\cap_{i=1}^d A^{\tau,\epsilon}_i | \mathscr{F}_\tau] > 0 \ a.s. \tag{3}$$

for any $\mathbb{F}$ stopping time $\tau \leq T$, and any $\epsilon > 0$.

In the following proposition, we show that joint stickiness implies no arbitrage in the model (1).

Proposition 1. *Let $X = (X_t^1, X_t^2, \cdots, X_t^d)$ be a jointly sticky, adapted, and càdlàg process . Then, the market $(1, e^{X_t^0}, e^{X_t^1}, \cdots, e^{X_t^d})$ does not admit arbitrage with propotional transaction costs k for any $k > 0$.*

Proof. Fix $k > 0$. Assume $\theta_s = (\theta_s^1, \theta_s^2, \cdots, \theta_s^d)$ is an arbitrage strategy. Then there is $t \in [0, T]$ such that $V_t(\theta) \geq 0$ and $P(V_t(\theta) > 0) > 0$. Let $\tau = \inf\{s \geq 0 : \theta_s^i \neq 0, i = 1, 2, \cdots, d\} \wedge t$. If $\tau = t$ almost surely, then the left-continuity of the paths and the definition of τ implies $\theta_s = 0$ on $[0, t]$ for almost all ω, thus $V_t(\theta) = 0$ almost surely and this contradicts with the assumption $P(V_t(\theta) > 0) > 0$. Therefore we assume that the event $A = \{\tau < t\}$ has positive probability. Let $Y_s^i = e^{X_s^i}, \tilde{Y}_s^i = Y_{\tau \wedge s}^i$, and $Z_s^i = Y_s^i - \tilde{Y}_s^i$ for all $1 \leq i \leq d$ and all $s \in [0, t]$. We can write (1) as following

$$V_s(\theta) = \sum_{i=1}^{d} \int_{\tau}^{s} \theta_\mu^i d\tilde{Y}_\mu^i + \sum_{i=1}^{d} \int_{\tau}^{s} \theta_\mu^i dZ_\mu^i - k \sum_{i=1}^{d} (\int_{\tau}^{s} Y_\mu^i d|D\theta^i|_\mu + |\theta_s^i| Y_s^i). \tag{4}$$

on A for any $s \in [\tau, t]$. Observe that $\sum_{i=1}^{d} \int_{\tau}^{s} \theta_\mu^i d\tilde{Y}_\mu^i = 0$ and

$$\sum_{i=1}^{d} \int_{\tau}^{s} \theta_\mu^i dZ_\mu^i = \sum_{i=1}^{d} Z_s^i \theta_s^i - \sum_{i=1}^{d} \int_{\tau}^{s} Z_\mu^i dD\theta_\mu^i.$$

on A for any $s \in [\tau, t]$. Thus (4) becomes

$$V_s(\theta) = \sum_{i=1}^{d} (Z_s^i \theta_s^i - k|\theta_s^i| Y_s^i) - \sum_{i=1}^{d} (\int_{\tau}^{s} Z_\mu^i dD\theta_\mu^i + k \int_{\tau}^{s} Y_\mu^i d|D\theta^i|_\mu) \tag{5}$$

Let $A_i^{\tau,\epsilon} = \{sup_{s \in [\tau,t]} |X_s^i - X_\tau^i| < \epsilon\}$ for any $\epsilon > 0$. Since $X_s = (X_s^1, X_s^2, \cdots, X_s^d)$ is jointly sticky, the event $A_1^\epsilon = A \cap (\cap_{i=1}^d A_i^{\tau,\epsilon})$ has postive probability for any $\epsilon > 0$. Observe that on A_1^ϵ, $|Z_s^i| \leq (e^\epsilon - 1) Y_s^i$ for all $s \in [\tau, t]$ and for each $1 \leq i \leq d$. Therefore on A_1^ϵ, we have

$$|Z_s^i \theta_s^i| \leq (e^\epsilon - 1) |\theta_s^i| Y_s^i \tag{6}$$

and

$$|\int_{\tau}^{s} Z_\mu^i dD\theta_\mu^i| \leq (e^\epsilon - 1) \int_{\tau}^{s} Y_\mu^i d|D\theta^i|_\mu \tag{7}$$

for all $s \in [\tau, t]$. From (5) (6), and (7) we conclude that on A_1^ϵ

$$V_s(\theta) \leq (e^\epsilon - 1 - k) \sum_{i=1}^{d} [|\theta_s^i| Y_s^i + \int_{\tau}^{s} Y_\mu^i d|D\theta^i|_\mu], \ \ \forall s \in [\tau, t] \tag{8}$$

Note that $\sum_{i=1}^{d} |\theta_t^i| Y_t^i + \int_{\tau}^{t} Y_s^i d|D\theta^i|_s > 0$ almost surely on A (this follows from the definitions of A and τ). Therefore from (7) it follows that $V_t(\theta) < 0$ on $A_1^\epsilon \subset A$ whenever $\epsilon < ln(1 + k)$. This contradicts with the assumption $P(V_t(\theta) \geq 0) = 1$, since $P(A_1^\epsilon) > 0$ for all $\epsilon > 0$. This shows that θ can not be an arbitrage strategy. This completes the proof. □ □

Example 1. *Let $L_t^1, L_t^2, \cdots, L_t^d$ be a sequence of independent Lévy processes in $[0,T]$ with respect to the filtration $\mathbb{F}$. Then $L=(L_t^1, L_t^2, \cdots, L_t^d)$ is jointly sticky with respect to $\mathbb{F}$. To see this, let τ be any stopping time of $\mathbb{F}$. Let $A_i^{\tau,\epsilon}=\{\sup_{t\in[0,T-\tau]}|L_{\tau+t}^i - L_\tau^i| < \epsilon\}$ for each $1\le i\le d$ and for any $\epsilon>0$. Then we have*

$$P(\cap_{i=1}^d A_i^{\tau,\epsilon}|\mathscr{F}_\tau) = P(\cap_{i=1}^d A_i^{\tau,\epsilon}) = \prod_{i=1}^d P(A_i^{\tau,\epsilon}) > 0.$$

The first equality above follows from the independence of $L_{\tau+t}^i - L_\tau^i$ with $\mathscr{F}_\tau$ for each $1\le i\le d$, the second equality follows from the independence assumption on $L_t^i, i=1,2,\cdots,d$, and the last inequality follows from stickiness of lévy processes (the stickiness of Lévy processes was shown in [8]).

Proposition 2. *Let $X_t=(X_t^1, X_t^2, \cdots, X_t^d)$ be a jointly sticky process with respect to the filtration $\mathbb{F}$. Let $\{f_1, f_2, \cdots, f_d\}$ be a family of real valued continuous functions on $\mathbb{R}^d$. Let $Y_t^i = f_i(X_t^1, X_t^2, \cdots, X_t^d)$ for each $i\in\{1,2,\cdots,d\}$. Then the process $Y=(Y_t^1, Y_t^2, \cdots, Y_t^d)$ is also jointly sticky with respect to $\mathbb{F}$.*

Proof. Fix any $\epsilon>0$. For any stopping time $\tau\le T$, let $B_i^{\tau,\epsilon}=\{sup_{t\in[\tau,T]}|Y_t^i - Y_\tau^i| < \epsilon\}$ for each $i\in\{1,2,\cdots,d\}$. We need to show

$$P[\cap_i^d B_i^{\tau,\epsilon}|\mathscr{F}_\tau] > 0 \text{ a.s.} \tag{9}$$

and this is equivalent to showing $P[A\cap(\cap_i^d B_i^{\tau,\epsilon})]>0$ for any $A\in\mathscr{F}_\tau$ with $P(A)>0$. Fix $A\in\mathscr{F}_\tau$ with $P(A)>0$, and let $M>0$ be such that the event $A_0=A\cap\{-M\le X_\tau^i\le M, 1\le i\le d\}$ has positive probability. Note that $A_0\in\mathscr{F}_\tau$. The set $O=[-M-1, M+1]\times[-M-1, M+1]\times\cdots\times[-M-1, M+1]$ is a closed bounded set in $\mathbb{R}^d$. Since $f_1, f_2, \cdots, f_d$ are continuous on $\mathbb{R}^d$, they are uniformly continuous on O. Therefore, there is a $\delta_0>0$, such that for each $1\le i\le d$, $|f_i(x)-f_i(y)|<\epsilon$ as long as $x,y\in O$ and $||x-y||<\delta_0$. Let $\delta_1 = min(1,\delta_0)$ and let $A_i^{\tau,\delta_1}=\{sup_{t\in[\tau,T]}|X_t^i - X_\tau^i|<\delta_1\}$. Since X_t is jointly sticky, the set $A_1 = A_0\cap(\cap_{i=1}^d A_i^{\tau,\delta_1})$ has positive probability. On A_1, we have $X_\tau\in O$, $X_t\in O$, and $||X_t - X_\tau||\le\delta_1\le\delta_0$ for all $t\in[\tau,T]$. Therefore $A_1\subset\cap_i^d B_i^{\tau,\epsilon}$. Since $A_1\subset A$, we have $P[A\cap(\cap_i^d B_i^{\tau,\epsilon})] > P(A_1) > 0$. This completes the proof. □ □

Example 2. *Let $B=(B_t^1, B_t^2, \cdots, B_t^d)$ be $d-$dimensional Brownian motion. Then the process $X=(|B_t^1|^{\frac{1}{3}}, |B_t^1+B_t^2|^{\frac{1}{3}}, \cdots, |B_t^1+B_t^d|^{\frac{1}{3}})$ is not a semimartingale; see Theorem 72 on page 221 of [10]. However, X is jointly sticky thanks to Proposition 2 and Example 1.*

The following corollary extends the Proposition 1 in [1].

Corollary 1. *If the process $X_t=(X_t^1, X_t^2, \cdots, X_t^d)$ is jointly sticky, then for any real valued continuous function $g:\mathbb{R}^d\to\mathbb{R}$, the process $Y_t = g(X_t^1, X_t^2, \cdots, X_t^d)$ is sticky.*

In the following Proposition shows that any finite sequence of independent fractional Brownian motions with possibily different Hurst parameters is jointly sticky.

Proposition 3. *Let $B_t^{H_i}=\int_{-\infty}^t[(t-s)^{H_i-\frac{1}{2}} - 1_{\{s\le 0\}}(-s)^{H_i-\frac{1}{2}}]dB_t^{(i)}, i=1,2,\cdots,d$ be a sequence of independent fractional Brownian motions in $[0,T]$ with respective Hurst parameters $H_1, H_2, \cdots, H_d\in(0,1)$. Then for any deterministic continuous functions $f_1, f_2, \cdots, f_d$ on $[0,T]$, the process $B=(B_t^{H_1}+f_1(t), B_t^{H_2}+f_2(t), \cdots, B_t^{H_d}+f_d(t))$ is jointly sticky .*

Proof. Let

$$\Omega = \{\omega \in C(\mathbb{R}) : \omega(0) = 0 \text{ and } \forall t \in \mathbb{R}, \lim_{s\to t} \frac{\omega(t)-\omega(s)}{\sqrt{|t-s|log(\frac{1}{|t-s|})}} = 0\},$$

$\mathscr{B}$ the $\sigma-$algebra of subsets of Ω that is generated by the cylinder sets and P the Wiener measure on $(\Omega, \mathscr{B})$. Let $(\Omega^{(i)}, \mathscr{B}^{(i)}, P^{(i)}), i = 1, 2, \cdots, d$ be d copies of $(\Omega, \mathscr{B}, P)$. With slight abuse of notation, we denote by P the $d-$dimensional Wiener measure $P^{(1)} \times P^{(2)} \times \cdots \times P^{(d)}$ on $(\Omega^d, \mathscr{B}^d)$, where $(\Omega^d, \mathscr{B}^d)$ is the product space of $(\Omega^{(i)}, \mathscr{B}^{(i)}), i = 1, 2, \cdots, d$. Without loss of generality, we assume that for each $1 \le i \le d$, $B_t^{H_i}$ is defined on $(\Omega^d, \mathscr{B}^d, P)$ by the improper Riemann-Stieltjes integrals

$$B_t^{H_i}(\omega) = \int_{-\infty}^{t} [(t-s)^{H_i - \frac{1}{2}} - 1_{\{s\le 0\}}(-s)^{H_i-\frac{1}{2}}] d\omega^{(i)}(s), \ t \ge 0. \tag{10}$$

where $\omega = (\omega^{(1)}, \omega^{(2)}, \cdots, \omega^{(d)}) \in \Omega^d$ (see the proof of Theorem 4.3 of [3]). Let $\mathbb{F}^B = (\mathscr{F}_t^B)_{t\in[0,T]}$ be the filtration given by

$$\mathscr{F}_t^B = \vee_{i=1}^d \sigma\{B_s^{H_i} : 0 \le s \le t\}.$$

Then $B_t^{H_1}, B_t^{H_2}, \cdots, B_t^{H_d}$ are independent fractional Brownian motions in the filtered probability space $(\Omega^d, \mathscr{B}^d, \mathbb{F}^B = (\mathscr{F}_t^B)_{t\in[0,T]}, P)$. Let $\mathscr{F}_t^{\Omega^d} =$

$$\vee_{i=1}^d \sigma\{\{\omega \in \Omega^d : \omega^{(i)}(s_j) \le a_j, j = 1, 2, \cdots n\} : -\infty < s_j \le t, a_j \in \mathbb{R}, n \in N\}$$

Then $(\omega^{(1)}(t), \omega^{(2)}(t), \cdots, \omega^{(d)}(t))$ is $d-$dimensional Brownian motion in the filtered probability space $(\Omega^d, \mathscr{B}^d, \mathbb{F}^{\Omega^d} = (\mathscr{F}_t^{\Omega^d})_{t\in[0,T]}, P)$. It is clear that $\mathscr{F}_t^B \subset \mathscr{F}_t^{\Omega^d}, t \ge 0$, therefore $\mathbb{F}^B$ stopping times are also $\mathbb{F}^{\Omega^d}$ stopping times.

Now, let τ be any stopping time of $\mathbb{F}^B$ and let

$$A_i^{\tau,\epsilon} = \{sup_{t\in[0,T-\tau]} |B_{\tau+t}^{H_i} - B_\tau^{H_i} + f_i(\tau+t) - f_i(\tau)| < \epsilon\}$$

for each $1 \le i \le d$ and for any $\epsilon > 0$. To show the jointly stickiness of B, we need to show

$$P(\cap_{i=1}^d A_i^{\tau,\epsilon} | \mathscr{F}_\tau^B) > 0 \text{ a.s.}$$

However, since $\mathscr{F}_\tau^B \subset \mathscr{F}_\tau^{\Omega^d}$, it is sufficient to show

$$P(\cap_{i=1}^d A_i^{\tau,\epsilon} | \mathscr{F}_\tau^{\Omega^d}) > 0 \text{ a.s.} \tag{11}$$

We divide the proof of (11) into two steps.

(A) For each $\omega(s) \in \Omega^d$ set

$$\pi_1^{(i)}\omega(s) = \omega^{(i)}(s) 1_{(-\infty,\tau(\omega)]}(s), s \in \mathbb{R},$$

$$\pi_2^{(i)}\omega(s) = \omega^{(i)}(\tau(\omega)+s) - \omega^{(i)}(\tau(\omega)), s \ge 0,$$

for all $1 \le i \le d$. For each $1 \le i \le d$, let

$$\Omega_1^{(i)} = \{\pi_1^{(i)}\omega : \omega \in \Omega^d\}, \quad \Omega_2^{(i)} = \{\pi_2^{(i)}\omega : \omega \in \Omega^d\}$$

and let $\mathscr{B}_1^{(i)}$ and $\mathscr{B}_2^{(i)}$ be the $\sigma-$algebras generated by the cylinder sets of $\Omega_1^{(i)}$ and $\Omega_2^{(i)}$ respectively. Also let $\Omega_i = \Omega_i^{(1)} \times \Omega_i^{(2)} \times \cdots \times \Omega_i^{(d)}, i = 1,2$ and let $\mathscr{B}_i = \mathscr{B}_i^{(1)} \times \mathscr{B}_i^{(2)} \times \cdots \times \mathscr{B}_i^{(d)}, i = 1,2$. It is clear that $\pi_1^{(i)} : \Omega^d \to \Omega_1^{(i)}$ is $\mathscr{F}_\tau^{\Omega^d}$ measurable for each $1 \le i \le d$, hence the map $\pi_1 : \Omega^d \to \Omega_1$ given by $\pi_1\omega = (\pi_1^{(1)}\omega, \pi_1^{(2)}\omega, \cdots, \pi_1^{(d)}\omega)$ is $\mathscr{F}_\tau^{\Omega^d}$ measurable (for notational simplicity we write $\pi_1\omega := \omega_1 = (\omega_1^{(1)}, \omega_1^{(2)}, \cdots, \omega_1^{(d)}))$. Also it follows from Theorem 6.16 of [13] that $\pi_2\omega := (\pi_2^{(1)}\omega(s), \pi_2^{(2)}\omega(s), \cdots, \pi_2^{(d)}\omega(s))$ is $d-$dimensional Brownian motion independent from $\mathscr{F}_\tau^{\Omega^d}$. Define a map $\tau' : \Omega_1 \to \mathbb{R}$ by $\tau'(\omega_1) := \tau(\omega)$, where $\omega \in \Omega^d$ is such that $\omega_1 = \pi_1\omega$ (note that if $\omega', \omega \in \Omega^d$ and $\pi_1\omega' = \pi_1\omega$, then $\tau(\omega) = \tau(\omega')$, since τ is $\mathscr{F}_\tau^{\Omega^d}$ measurable). Then for each $\omega \in \Omega^d$, we can write

$$\begin{aligned}&(B^{H_i}_{\tau+t} - B^{H_i}_{\tau})(\omega) = \\ &\int_{-\infty}^{\tau'(\pi_1\omega)} [(\tau'(\pi_1\omega) + t - s)^{H_i - \frac{1}{2}} - (\tau'(\pi_1\omega) - s)^{H_i + \frac{1}{2}}] d\pi_1^{(i)}\omega(s) \\ &+ \int_0^t (t-s)^{H_i - \frac{1}{2}} d\pi_2^{(i)}\omega(s)\end{aligned} \tag{12}$$

Note that $\tau'(\pi_1(\omega)) = \tau(\omega)$, therefore $\tau'(\pi_1(\cdot))$ is $\mathscr{F}_\tau^{\Omega^d}$ measurable.

(B) Let $A_i^{\tau,\epsilon}$ be as in (2) for each $1 \le i \le d$. For each $\omega_1 = (\omega_1^{(1)}, \omega_1^{(2)}, \cdots, \omega_1^{(d)})$ in Ω_1, define

$$\begin{aligned}h_t^i(\omega_1) := &\int_{-\infty}^{\tau'(\omega_1)} [(\tau'(\omega_1) + t - s)^{H_i - \frac{1}{2}} - (\tau'(\omega_1) - s)^{H_i + \frac{1}{2}}] d\omega_1^{(i)}(s) \\ &+ f(\tau'(\omega_1) + t) - f(\tau'(\omega_1)),\end{aligned}$$

and for each $\omega_2 = (\omega_2^{(1)}, \omega_2^{(2)}, \cdots, \omega_2^{(d)}) \in \Omega_2$ define

$$H_t^i(\omega_2) := \int_0^t (t-s)^{H_i - \frac{1}{2}} d\omega_2^{(i)}(s)$$

Then from (12) and the definition of τ', it follows that

$$[B^{H_i}_{\tau+t} - B^{H_i}_{\tau} + f_i(\tau + t) - f_i(\tau)](\omega) = h_t^i(\pi_1\omega) + H_t^i(\pi_2\omega), t \ge 0. \tag{13}$$

Define

$$C^i := \{(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 : \sup_{t \in [0, T - \tau'(\omega_1)]} |h_t^i(\omega_1) + H_t^i(\omega_2)| < \epsilon\}$$

for each $1 \le i \le d$. Since $h_t^i + H_t^i$ is continuous process in $\Omega_1 \times \Omega_2$, C^i is $\mathscr{B}_1 \times \mathscr{B}_2$ measurable for each $1 \le i \le d$. Since π_1 is $\mathscr{F}_\tau^{\Omega^d}$ measurable and π_2 is independent from $\mathscr{F}_\tau^{\Omega^d}$, from Proposition A.2.5 of [4], for almost every $\omega \in \Omega^d$ we have

$$E[1_{\cap_{i=1}^d C^i}(\pi_1, \pi_2) | \mathscr{F}_\tau^{\Omega^d}](\omega) = \phi(\pi_1\omega) \tag{14}$$

where $\phi(\omega_1) = E1_{\cap_{i=1}^d C^i}(\omega_1, \pi_2)$. From (13) and the definitions of C^i and $A_i^{\tau,\epsilon}$, it is clear that $1_{C^i}(\pi_1\omega, \pi_2\omega) = 1_{A_i^{\tau,\epsilon}}(\omega)$ for each $1 \le i \le d$ and $\omega \in \Omega^d$. Therefore

$$E[1_{\cap_{i=1}^d A_i^{\tau,\epsilon}} | \mathscr{F}_\tau^{\Omega^d}](\omega) = E[1_{\cap_{i=1}^d C^i}(\pi_1, \pi_2) | \mathscr{F}_\tau^{\Omega^d}](\omega) = \phi(\pi_1\omega).$$

for each $\omega \in \Omega^d$. In the following, we will show that $\phi(\omega_1) > 0$ for each $\omega_1 \in \Omega_1$. To see this, note that the random variables $1_{C^i}(\omega_1, \pi_1), i = 1, 2, \cdots, d$ are independent for each fixed $\omega_1 \in \Omega_1$ (this follows from the independence of the Brownian motions $\pi_2^{(i)}\omega, i = 1, 2, \cdots, d$ and the definitions of H_t^i). Therefore, we have $\phi(\omega_1) = E1_{C^1}(\omega_1, \pi_2)E1_{C^2}(\omega_1, \pi_2)\cdots E1_{C^d}(\omega_1, \pi_2)$. Let

$$B_i^\epsilon(\omega_1) = \{\omega \in \Omega^d : (\omega_1, \pi_i\omega) \in C^i\}$$

for each $1 \leq i \leq n$. Then, we have $1_{C^i}(\omega_1, \pi_i) = 1_{B_i^\epsilon(\omega_1)}$ for each $1 \leq i \leq d$. This shows that $\phi(\omega_1) = P(B_1^\epsilon) \times P(B_2^\epsilon) \times \cdots \times P(B_d^\epsilon)$. Therefore, it is sufficient to show $P(B_i^\epsilon) > 0$ for each $1 \leq i \leq d$. Note that

$$B_i^\epsilon(\omega_1) = \{\omega \in \Omega^d : \sup_{t\in[0,T-\tau'(\omega_1)]} |h_t^i(\omega_1) + \int_0^t (t-s)^{H_i - \frac{1}{2}} d\pi_i\omega < \epsilon\}.$$

If $\tau'(\omega_1) = 0$, then $B_i^\epsilon(\omega_1) = \Omega^d$, so $P(B_i^\epsilon(\omega_1)) > 0$. If $\tau'(\omega_1) > 0$, then since $\pi_i(\omega)$ is a Brownian motion and $h_t^i(\omega_1)$ is a deterministic continuous function for each ω_1, from the results in [6, 8, 11], it follows that $h_t^i(\omega_1) + \int_0^t (t-s)^{H_i - \frac{1}{2}} d\pi_i\omega$ has full support in $C[0, \tau'(\omega_1)]$. This, in turn, implies that $B_i^\epsilon(\omega_1)$ has positive probability for each i. Therefore $\phi(\omega_1) > 0$ for each $\omega_1 \in \Omega_1$. Now, the result follows from (14).This completes the proof. □

□

In the following Proposition we show a time change result on joint stickiness.

Proposition 4. *Let $X_t = (X_t^1, X_t^2, \cdots, X_t^d)$ be a continuous process adapted to the filtration $\mathbb{F}$. Let V_t be a nondecreasing continuous process such that for each t, V_t is $\mathbb{F}$ stopping time. Then we have the following*

(i) X is jointly sticky with respect to $\mathbb{F}$ if and only if for any stopping time $\tau \leq T$ of $\mathbb{F}$ and any $\delta > 0$, the stopping time $\tau_1 = \inf\{t \geq \tau : |X_t^i - X_\tau^i| \geq \delta, 1 \leq i \leq d\} \wedge T$ satisfies $P(\tau_1 = T|\mathscr{F}_\tau) > 0$ a.s.

(ii) If X is jointly sticky with respect to $\mathbb{F}$, then the time changed process $Y_t = X_{V_t\wedge T} = (X^1_{V_t\wedge T}, X^2_{V_t\wedge T}, \cdots, X^d_{V_t\wedge T})$ is jointly sticky with respect to the filtration $\mathbb{G} = (\mathscr{G}_t)_{t\in[0,T]}$, where $\mathscr{G}_t = \mathscr{F}_{V_t\wedge T}$.

Proof. Proof of (i): Assume X is jointly sticky. To show $P(\tau_1 = T|\mathscr{F}_\tau) > 0$, we need to show $P(A \cap \{\tau_1 = T\}) > 0$ for any $A \in \mathscr{F}_\tau$ with $P(A) > 0$. Let $A_i^{\tau,\frac{\delta}{2}} = \{sup_{t\in[\tau,T]}|X_t^i - X_\tau^i| < \frac{\delta}{2}\}$. Since X is jointly sticky, the event $A_1 = A \cap (\cap_{i=1}^d A_i^{\tau,\frac{\delta}{2}})$ has positive probability. On A_1 we clearly have $\tau_1 = T$ and so $P(A \cap \{\tau_1 = T\}) > 0$. To show the other direction, let $\tau \leq T$ be any stopping time and $A \in \mathscr{F}_\tau$ be any event with $P(A) > 0$. For any $\epsilon > 0$, let $A_i^{\tau,\epsilon} = \{sup_{t\in[\tau,T]}|X_t^i - X_\tau^i| < \epsilon\}$ for each i. We need to show the event $A_1 = A \cap (\cap_i^d A_i^{\tau,\epsilon})$ has positive probability. Let $\tau_1 = \inf\{t \geq \tau : |X_t^i - X_\tau^i| \geq \frac{\epsilon}{2}, 1 \leq i \leq d\} \wedge T$. Since $P(\tau_1 = T|\mathscr{F}_\tau) > 0$ almost surely, the event $A \cap \{\tau_1 = T\}$ has positive probability. By the definition of τ_1 we have $A \cap \{\tau_1 = T\} \subset A_1$ and therefore $P(A_1) > 0$. This completes the proof.

Proof of (ii): Denote $Y_t^i = X^i_{V_t\wedge T}$ for each i. Let $\tau \leq T$ be any stopping time of $\mathbb{G}$. For any $\delta > 0$, let $\tau_1 = \inf\{t \geq \tau : |Y_t^i - Y_\tau^i| \geq \delta, 1 \leq i \leq d\} \wedge T$. Due to part (i) above, we only need to show $P(\tau_1 = T|\mathscr{G}_\tau) > 0$ almost surely. This is equivalent to showing that for any $A \in \mathscr{G}_\tau$ with

$P(A) > 0$, $P(A \cap \{\tau_1 = T\}) > 0$. To see this, let $\tau_0 = \inf\{t \geq \tau : |Y_t^i - Y_\tau^i| \geq \frac{\delta}{4}, 1 \leq i \leq d\} \wedge T$. Let $\tau^A = \tau 1_A + T 1_{\Omega/A}$ and $\tau_0^A = \tau_0 1_A + T 1_{\Omega/A}$, then both of τ^A and τ_0^A are $\mathbb{G}$ stopping times. Since $\tau^A < \tau_0^A$ on A, there exists a deterministic number k such that $A_1 = \{\tau^A < k < \tau_0^A\}$ has positive probability. Note that $A_1 \subset A$ and $A_1 \in \mathscr{G}_k$. Since $\tau_0 > k > \tau$ on A_1, by the definition of τ_0, for each $1 \leq i \leq d$ we have

$$sup_{t \in [\tau, k]} |Y_t^i - Y_\tau^i| \leq \frac{\delta}{4} \tag{15}$$

on A_1. Let $\theta = \inf\{t \geq V_k \wedge T : |X_t^i - X_{V_k \wedge T}^i| \geq \frac{\delta}{4}, 1 \leq i \leq d\} \wedge T$. Since X is jointly sticky, the event $A_2 = A_1 \cap \{\theta = T\}$ has positive probability. Since $\theta = T$ on A_2, for each $1 \leq i \leq d$ we have

$$\sup_{t \in [k, T]} |X_{V_t \wedge T}^i - X_{V_k \wedge T}^i| \leq \frac{\delta}{4} \tag{16}$$

on A_2. From (15) and (16), for each $1 \leq i \leq d$ we have

$$sup_{t \in [\tau, T]} |Y_t^i + Y_\tau^i| \leq sup_{t \in [\tau, k]} |Y_t^i - Y_\tau^i| + \sup_{t \in [k, T]} |Y_t^i - Y_k^i| < \delta.$$

on A_2. This shows that $\tau_1 = T$ on A_2. This completes the proof. □ □

Example 3. *Let $B_t^{H_1}, B_t^{H_2}, \cdots, B_t^{H_d}$ be a sequence of independent fractional Brownian motions with respect to the filtration $\mathbb{F} = (\mathscr{F}_t)_{t \geq 0}$. Let ν_t be any bounded time change. Then the process $X_t = (B_{\nu_t}^{H_1}, B_{\nu_t}^{H_2}, \cdots, B_{\nu_t}^{H_d})$ is jointly sticky with respect to the filtration $(\mathscr{F}_{\nu_t})_{t \in [0,T]}$. To see this, let M be such that $\nu_t \leq M$ almost surely for all $t \in [0, T]$. From Proposition 3, $B = (B_t^{H_1}, B_t^{H_2}, \cdots, B_t^{H_d})$ is jointly sticky for the filtration $(\mathscr{F}_t)_{t \in [0,M]}$. Then, from part (ii) of Proposition (4), we conclude X is jointly sticky with respect to $(\mathscr{F}_{\nu_t})_{t \in [0,T]}$.*

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