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Survival Probability and Intensity Derived from Credit Default Swaps

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Survival Probability and Intensity Derived from Credit Default Swaps

A Directed Research Project

Submitted to the Faculty of the

WORCESTER POLYTECHNIC INSTITUTE

in partial fulfillment of the requirements for the

Professional Degree of Master of Science

in

Financial Mathematics

by

Yi Lan

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Approved:

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Professor Marcel Blais, Advisor

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Professor Bogdan Vernescu, Head of Department

Abstract

This project discusses the intensity and survival probability derived from Credit Default Swaps (CDS). We utilize two models, the reduced intensity model and the Shift Square Root Diffusion (SSRD) model. In the reduced intensity model, we assume a deterministic intensity and implement a computer simulation to derive the company's survival probability and intensity from the CDS market quotes. In the SSRD model, the interest rate and intensity are both stochastic and correlated. We discuss the impaction of correlation on the interest rate and intensity. We also conduct a Monte Carlo simulation to determine the dynamics of stochastic interest rate and intensity.

Acknowledgements

I would like to express my gratitude to all those who gave me the support to complete this project. At the very first, I would like to thank my supervisor, Prof. Marcel Blais, who guided me throughout all my research on this project. And thanks to Min Huang. Without his help I would not been able to access to lots of useful financial market data to finish my research.

Table of Contents

List of Tables

List of Charts

1 Introduction

In 1997 one team from JP Morgan Chase invented the credit default swap (CDS). A CDS is a contract between two counterparties. It was designed to shift the risk to a third party ensuring protection against default. Default occurs when a company fails to make payments owed to some entity. The buyer of the CDS makes a series of payments to the seller and in exchange receives a certain cash amount if a credit instrument defaults. CDS can be used for hedging, speculation, and arbitrage. The spread of a CDS is the annual amount that the protection buyer pays to the protection seller over the length of the contract. As shown in figure 1.1, a CDS is purchased at time t_a and regular premium payments are made at times $t_{a+1}, t_{a+2}, t_{a+3}$ ... If no default occurs, then the buyer continues paying premiums at t_i, and so on until the end of the contract at time t_b .

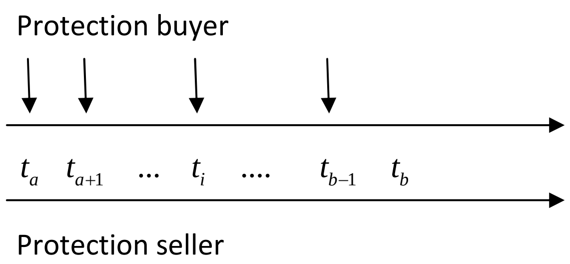

Figure 1.1. Schematic of CDS if no default occurs

However, if default happens at time t_i, the protection seller pays the buyer for the loss, and the buyer stops paying premiums, as illustrated in figure 1.2.

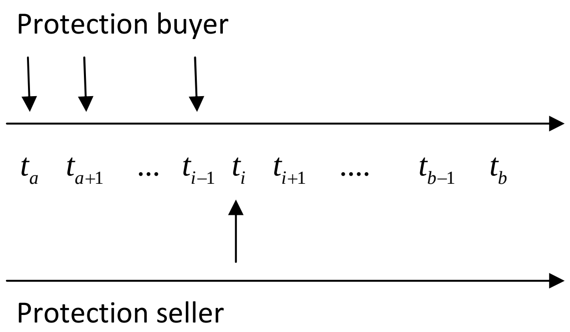

Figure 1.2. Schematic of CDS when default happens

2 CDS Payoffs [1]

Based on the default time of a company and the corresponding protection payment date, the payoff of a CDS is divided into two parts: the premium leg and the protection leg. We define a set of parameters before our analysis. First a timeline with equally spaced intervals is created, as shown in figure 2.1, and the interval widths are $\alpha_i = T_i - T_{i-1}$, $i \in [a,b]$. We also define $D(t,T_i) = \frac{B(t)}{B(T)}$ as the discount factor, where $B(t) = e^{\int_t^T r_u du}$ is the bank-account numeraire, r is the instantaneous short interest rate, and τ is the first default time of company. We also define $T_{\beta(\tau)}$ as the first T_i that follows the first default time τ, L_{GD} as the protection payment when default happens, R as the premium payment in exchange for the protection against the default probability, and $1_{\{T_a < \tau < T_b\}}$ as an indicator function.

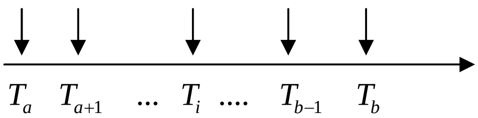

Figure 2.1. Timeline of the payoff

2.1 Running CDS

For a Running CDS (RCDS), the protection payment rate R is exchanged at specific times or when default happens in exchange for a single protection payment. The amount L_{GD} is paid when default happens.

The premium leg is given by:

$$D(t,\tau)(\tau - T_{\beta(\tau)-1}) R 1_{\{T_a < \tau < T_b\}} + \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R 1_{\{\tau \geq T_i\}} .$$

The protection leg is given by:

$$-1_{\{T_a < \tau \leq T_b\}} D(t,\tau) L_{GD} .$$

Therefore the discounted payoff for a RCDS is given by:

$$\Pi_{RCDS_{a,b}} = D(t,\tau)(\tau - T_{\beta(\tau)-1}) R 1_{\{T_a < \tau < T_b\}} + \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R 1_{\{\tau \geq T_i\}} - 1_{\{T_a < \tau \leq T_b\}} D(t,\tau) L_{GD} . \quad (2.1)$$

2.2 Postponed Payoffs Running CDS (PRCDS)

In this case, the protection payment L_{GD} is paid at the first T_i after the default time, i.e. $T_{\beta(\tau)}$. The discounted payoff at T_a is given by:

$$\Pi_{PRCDS_{a,b}} = \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R 1_{\{\tau \geq T_i\}} - \sum_{i=a+1}^{b} 1_{\{T_{i-1} < \tau \leq T_i\}} D(t,T_i) L_{GD} \ . \tag{2.2}$$

2.3 Postponed Payoffs Running CDS 2

There is another postponed payment form in the CDS. In this case, one more R payment is made as compared to the postponed payoffs running CDS. The discounted payoff at the contract initial time is given by:

$$\Pi_{PR2CDS_{a,b}} = \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R 1_{\{\tau > T_{i-1}\}} - \sum_{i=a+1}^{b} 1_{\{T_{i-1} < \tau \leq T_i\}} D(t,T_i) L_{GD} \ . \tag{2.3}$$

2.4 CDS Forward Rates

The CDS forward rate $R_{a,b}(t)$ is defined as that value of R that makes the value of the discounted CDS payoff equal to zero at time t, which is determined by:

$$CDS_{a,b}(t, R_{a,b}(t), L_{GD}) = E\left[\Pi(t) \middle| G_t\right] = 0 .$$

In the above equation, $\Pi(t)$ is the discount CDS payoff at time t, formulated in section 2 by (2.1), (2.2) or (2.3). $G_t = F_t \vee \sigma(\{\tau < u\}, u \leq t)$[1] denotes the information on the default free market up to time t and the exact default time if default happens. To simplify the computation, it is better to switch the filtration to the default-free market by using the following equation:

$$CDS_{a,b}(t, R_{a,b}(t), L_{GD}) = E\left[\Pi(t) \middle| G_t\right] = \frac{1_{\{\tau > t\}}}{Q\{\tau > t | F_t\}} E\left[\Pi(t) \middle| F_t\right] . \tag{2.4}$$

---

[1] $\sigma(\{\tau < u\}, u \leq t)$ denotes the sigma-algebra of the default time before time t.

3 Poisson Process

In this project, we use a Poisson process to describe the default time of a company. The default time can be viewed as the first jump of a Poisson process. Based on the nature of the intensity function, the Poisson process can be classified as time homogeneous Poisson process, time inhomogeneous Poisson process, and a Cox process. We will recall some important facts about these processes in the following text.

3.1 Time Homogeneous Poisson Process [1]

A time homogeneous Poisson process is defined as process with stationary independent increments and initial value of zero.

The time between two consecutive jumps are independently and identically distributed as an exponential random variable with mean $\frac{1}{\gamma}$, where γ is constant in time. If we define the default time τ as the first jump and $Q\{\}$ denotes the probability of an event, then:

$$Q\{\tau \in [t, t+dt) \mid \tau \geq t\} = \gamma dt . \tag{3.1}$$

Thus, the probability of default happening in dt interval knowing that default has not taken place so far is γdt. The survival probability to time t therefore is given by:

$$Q\{\tau > t\} = \exp(-\gamma t) . \tag{3.2}$$

Also, the probability of defaulting time between time s and t is:

$$Q\{s < \tau \leq t\} = \exp(-\gamma s) - \exp(-\gamma t) . \tag{3.3}$$

3.2 Time Inhomogeneous Poisson Process [1]

In this case we consider that the intensity $\gamma(t)$ is deterministic and time varying. We assume the intensity to be positive and piecewise continuous function in time. We define:

$$\Gamma(t) = \int_0^t \gamma(u) du . \tag{3.4}$$

By inverting the function, we can obtain the default time τ by using a standard exponential random variable ξ:

$$\tau = \Gamma^{-1}(\xi).$$

The probability of default occurring in the next dt time interval is:

$$Q\{\tau \in [t, t+dt) \mid \tau \geq t\} = \gamma(t)dt. \tag{3.5}$$

We can easily get the survival probability up to time t:

$$Q\{\tau > t\} = \exp(-\Gamma(t)) = \exp(-\int_0^t \gamma(s)ds). \tag{3.6}$$

Similarly, the survival probability between time t and s is:

$$Q\{s < \tau \leq t\} = \exp(-\Gamma(s)) - \exp(-\Gamma(t)). \tag{3.7}$$

3.3 Cox Process [1]

Different from the previous two processes, the Cox process assumes a time varying and stochastic intensity λ_t. We can still get a similar formula in the computation of the survival probability.

The cumulated intensity to time t can be expressed as:

$$\Lambda(t) = \int_0^t \lambda_s ds. \tag{3.8}$$

The probability that the company will default in the next dt interval is:

$$Q\left\{\tau \in [t, t+dt) \middle| \tau \geq t, F_t\right\} = \lambda_t dt, \tag{3.9}$$

where F_t contains the default free market information up to time t shown in section 2.4.

The probability that the default time τ of the company is greater than t is:

$$Q\left\{\tau \geq t\right\} = Q\left\{\Lambda(\tau) \geq \Lambda(t)\right\} = Q\left\{\Lambda(\tau) \geq \int_0^t \lambda(s)ds\right\} = E\left[Q\left\{\Lambda(\tau) \geq \int_0^t \lambda(s)ds\right\}\middle| F_t\right]. \tag{3.10}$$

The cumulated intensity at default time $\Lambda(\tau) = \xi$ is an exponential random variable which is independent of F_t, thus

$$E\left[Q\left\{\Lambda(\tau) \geq \int_0^t \lambda(s)ds\right\}\middle| F_t\right] = E\left[Q\left\{\xi \geq \int_0^t \lambda(s)ds\right\}\right] = E\left[e^{-\int_0^t \lambda(s)ds}\right]. \quad (3.11)$$

4 Reduced Intensity Model

There are several models that deal with CDS to explore the intensity and survival probability in detail, such as structural models and the reduced form models. They both are used to model credit risk. Structural models are based on the complete knowledge of a detailed information set. They were developed by Black, Scholes, and Merton [1]. In contrast, the reduced form model invented by Jarrow and Turnbull is based on the information set available to the market [2]. In this chapter, we mainly focus on the reduced form intensity model to determine the intensity and the survival probability by using the CDS forward rates market quotes.

4.1 Assumption [1]

In the reduced form model, we assume the default time is the time-inhomogeneous Poisson process, which means the intensity $\gamma(t)$ is deterministic and piecewise constant in time:

$\gamma(t) = \gamma_i \quad for \quad t \in [T_{i-1}, T_i)$,

and the cumulated intensity function at time t is:

$$\Gamma(t) = \int_0^t \gamma(u)du = \sum_{i=1}^{n} \gamma_i dt .$$

In this intensity model, we also assume that the interest rates and the default time are independent. We are only concerned about the value of the running CDS at the contract initial time. As for the postponed running CDS, we can derive the formula in similar manner.

4.2 Methodology [1]

To calculate the CDS value, we need to apply the filter change equation (2.4) for the purpose of simplification.

$$CDS_{a,b}(t,R_{a,b}(t),L_{GD})=\frac{1_{\{\tau>t\}}}{Q\{\tau>t|F_t\}}E\left[\Pi_{RCDS}(t)\middle|F_t\right]$$

$$=\frac{1_{\{\tau>t\}}}{Q\{\tau>t|F_t\}}E\left[D(t,\tau)(\tau-T_{\beta(\tau)-1})R_{a,b}1_{\{T_a<\tau<T_b\}}+\sum_{i=a+1}^{b}D(t,T_i)\alpha_i R_{a,b}1_{\{\tau\geq T_i\}}-1_{\{T_a<\tau\leq T_b\}}D(t,\tau)L_{GD}\middle|F_t\right]$$

$$=\frac{1_{\{\tau>t\}}}{Q\{\tau>t|F_t\}}\left\{R_{a,b}E\left[D(t,\tau)(\tau-T_{\beta(\tau)-1})1_{\{T_a<\tau<T_b\}}\middle|F_t\right]+\sum_{i=a+1}^{b}\alpha_i R_{a,b}E\left[D(t,T_i)1_{\{\tau\geq T_i\}}\middle|F_t\right]\right.$$
$$\left.-L_{GD}E\left[1_{\{T_a<\tau\leq T_b\}}D(t,\tau)\middle|F_t\right]\right\}$$

(4.1)

$\Pi_{RCDS}(t)$ is the discounted CDS payoff at time t defined in equation (2.1).

Considering the situation at initial time 0,

$$CDS_{a,b}(0,R_{a,b},L_{GD})$$
$$=\frac{1_{\{\tau>0\}}}{Q\{\tau>0\}}\left\{R_{a,b}E\left[D(0,\tau)(\tau-T_{\beta(\tau)-1})1_{\{T_a<\tau<T_b\}}\right]+\sum_{i=a+1}^{b}\alpha_i R_{a,b}\,E\left[D(0,T_i)1_{\{\tau\geq T_i\}}\right]\right.$$
$$\left.-L_{GD}E\left[1_{\{T_a<\tau\leq T_b\}}D(0,\tau)\right]\right\} \quad (4.2)$$

Since we assume independence between the default time and the interest rate,

$$E\left[D(0,T_i)1_{\{\tau\geq T_i\}}\right]=E\left[D(0,T_i)\right]E\left[1_{\{\tau\geq T_i\}}\right], \quad (4.3)$$

and the protection leg term of the running CDS formula (4.1) is:

$$-L_{GD}E[1_{\{T_a<\tau<T_b\}}D(0,\tau)]=-L_{GD}E[\int_{t=0}^{\infty}1_{\{T_a<t\leq T_b\}}D(0,t)1_{\{\tau\in[t,t+dt)\}}]$$
$$=-L_{GD}\int_{t=T_a}^{T_b}E[D(0,t)]E[1_{\{\tau\in[t,t+dt)\}}]=-L_{GD}\int_{t=T_a}^{T_b}P(0,t)Q(\tau\in[t,t+dt)) \quad (4.4)$$

Applying equation (3.5) and (3.6), we obtain:

$$Q\{\tau\in[t,t+dt)\}=Q\{\tau\in[t,t+dt)|\tau\geq t\}Q\{\tau\geq t\}=\gamma(t)dt\exp(-\int_0^t\gamma(s)ds). \quad (4.5)$$

Thus, put (4.5) back into (4.4), we can rewrite the equation (4.4) as:

$$-L_{GD}\int_{t=T_a}^{T_b}P(0,u)\gamma(u)\exp(-\int_0^u\gamma(s)ds)du. \quad (4.6)$$

By assuming the piecewise constant intensity rate $\gamma : \gamma(t) = \gamma_i$ for $t \in [T_{i-1}, T_i)$, the above formula (4.6) becomes:

$$-L_{GD} \sum_{i=a+1}^{b} \gamma_i \int_{T_{i-1}}^{T_i} P(0,u) \exp(-\Gamma_{i-1} - \gamma_i (u - T_{i-1})) du \, , \tag{4.7}$$

where $\Gamma_i = \int_{T_a}^{T_i} \gamma(s) ds = \sum_{i=a+1}^{j} \gamma_i \alpha_i$ is defined as the cumulative intensity.

The premium leg of the running CDS in formula (4.1) is:

$$\begin{aligned} & R_{a,b} E[D(0,\tau)](\tau - T_{\beta(\tau)-1}) E[1_{\{T_a < \tau < T_b\}}] + \sum_{i=a+1}^{b} \alpha_i R_{a,b} E[D(0,T_i)] E[1_{\{\tau \geq T_i\}}] \\ & = R_{a,b} \int_{T_a}^{T_b} P(0,t)(t - T_{\beta(t)-1}) Q\{\tau \in [t, t+dt)\} + \sum_{i=a+1}^{b} \alpha_i R_{a,b} P(0,T_i) Q\{\tau \geq T_i\} \end{aligned} \tag{4.8}$$

Using the discretization of γ as the defined piecewise constant γ_i defined above, (4.8) becomes:

$$R_{a,b} \sum_{i=a+1}^{b} \gamma_i \int_{T_{i-1}}^{T_i} P(0,u)(u - T_{i-1}) \exp(-\Gamma_{i-1} - \gamma_i (u - T_{i-1})) du + R_{a,b} \sum_{i=a+1}^{b} P(0,T_i) \alpha_i \exp(-\Gamma_i) \, . \tag{4.9}$$

Therefore we obtain a discretized scheme of a running CDS payoff at time 0 in the reduced intensity model:

$$\begin{aligned} & CDS_{a,b}(0, R, L_{GD,} \Gamma) \\ & = R \sum_{i=a+1}^{b} \gamma_i \int_{T_{i-1}}^{T_i} P(0,u)(u - T_{i-1}) \exp(-\Gamma_{i-1} - \gamma_i (u - T_{i-1})) du + R \sum_{i=a+1}^{b} P(0,T_i) \alpha_i \exp(-\Gamma_i) \\ & \quad - L_{GD} \sum_{i=a+1}^{b} \gamma_i \int_{T_{i-1}}^{T_i} P(0,t) \exp(-\Gamma_{i-1} - \gamma_i (u - T_{i-1})) du \end{aligned} \tag{4.10}$$

5 Shifted Square Root Diffusion (SSRD) Model

In this chapter, we consider the situation with a stochastic intensity and a stochastic interest rate. This model was proposed by Brigo and Alfonsi in 2003[1]. The Cox-Ingersoll-Ross model is applied in SSRD to describe the dynamics of interest rate and intensity.

5.1 Assumption

Since the intensity is stochastic in this model, the default time of a company can be viewed as the first jump of a Cox process. We denote the stochastic intensity as λ_t and the stochastic interest rate as r_t .

5.2 CIR++Short Rate Model [1]

We can write the short rate as the sum of two parts, a deterministic function φ and a Markovian process x_t^α :

$$r_t = x_t^\alpha + \varphi(t,\alpha) . \tag{5.1}$$

According to the CIR model, the dynamics of short rate x_t^α can be written as:

$$dx_t^\alpha = k(\theta - x_t^\alpha)dt + \sigma\sqrt{x_t^\alpha}\,dW_t , \tag{5.2}$$

where α is the parameter vector $\alpha = (k,\theta,\sigma,x_0^\alpha)$.

The zero coupon bond price derived from the CIR model is:

$$P^{CIR}(t,T,x_t^\alpha,\alpha) = E\left[e^{-\int_t^T x(s)ds} \middle| F_t \right] = A(t,T,\alpha)\exp(-B(t,T,\alpha))x_t) , \tag{5.3}$$

where:

$$A(t,T,\alpha) = \left[\frac{2h\exp\{(k+h)(T-t)/2\}}{2h+(k+h)(\exp\{(T-t)h\}-1)} \right]^{2k\theta/\sigma^2} ,$$

$$B(t,T,\alpha) = \frac{2(\exp\{(T-t)h\}-1)}{2h+(k+h)(\exp\{(T-t)h\}-1)} ,$$

And $h = \sqrt{k^2 + 2\sigma^2}$.

5.3 CIR++ Intensity Model [1]

By using an approach similar to that in section 5.2, we can separate the intensity into a deterministic function and a stochastic process,

$$\lambda_t = y_t^{\beta} + \phi(t,\beta)\,, \tag{5.4}$$

and the dynamics of y_t^{β} can be written as:

$$dy_t^{\beta} = \kappa(\mu - y_t^{\beta})dt + \nu\sqrt{y_t^{\beta}}\,dZ_t\,, \tag{5.5}$$

where β is the parameter vector $\beta = (\kappa, \mu, \upsilon, y_0^{\beta})$.

If the correlation between the short rate and intensity is ρ , then the two stochastic processes have the following relationship:

$dW_t dZ_t = \rho dt$.

5.4 Methodology [1]

We compute the discounted CDS payoff at time t in the SSRD model:

$$CDS_{a,b}(t, R_{a,b}(t), L_{GD}) = \frac{1_{\{\tau>t\}}}{Q\{\tau > t | F_t\}} E\left[\Pi_{RCDS}(t) \middle| F_t\right]$$

$$= \frac{1_{\{\tau>t\}}}{Q\{\tau > t | F_t\}} E\left[D(t,\tau)(\tau - T_{\beta(\tau)-1})R_{a,b}1_{\{T_a<\tau<T_b\}} + \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R_{a,b} 1_{\{\tau \ge T_i\}} - 1_{\{T_a<\tau\le T_b\}} D(t,\tau) L_{GD} \middle| F_t \right]$$

The premium leg of the above formula can be written as,

$$\frac{1_{\{\tau>t\}}}{Q\{\tau > t | F_t\}} E\left[D(t,\tau)(\tau - T_{\beta(\tau)-1})R_{a,b}1_{\{T_a<\tau<T_b\}} + \sum_{i=a+1}^{b} D(t,T_i)\alpha_i R_{a,b} 1_{\{\tau \ge T_i\}} \middle| F_t \right]$$

$$= \frac{1_{\{\tau>t\}} R_{a,b}}{E[\exp(-\int_0^t \lambda_s ds) | F_t]} E\left[\int_{T_a}^{T_b} D(t,s)(s - T_{\beta(s)-1})1_{\{\tau\in[s,s+ds)\}} + \sum_{i=a+1}^{b} D(t,T_i)\alpha_i 1_{\{\tau \ge T_i\}} \middle| F_t \right]$$

$$= \frac{1_{\{\tau>t\}} R_{a,b}}{E[\exp(-\int_0^t \lambda_s ds) | F_t]} E\left[E\left[\int_{T_a}^{T_b} D(t,s)(s - T_{\beta(s)-1})1_{\{\tau\in[s,s+ds)\}} + \sum_{i=a+1}^{b} D(t,T_i)\alpha_i 1_{\{\tau \ge T_i\}} \middle| F_{T_b} \right] \middle| F_t \right]$$

$$=\frac{1_{\{\tau>t\}}R_{a,b}}{E[\exp(-\int_0^t \lambda_s ds)\big|F_t]}\left\{E\left[\int_{T_a}^{T_b} D(t,s)(s-T_{\beta(s)-1})E\left[1_{\{\tau\in[s,s+ds)\}}\big|F_{T_b}\right]\Big|F_t\right]+E\left[\sum_{i=a+1}^{b} D(t,T_i)\alpha_i E\left[1_{\{\tau\geq T_i\}}\big|F_{T_b}\right]\Big|F_t\right]\right\}$$

$$=\frac{1_{\{\tau>t\}}R_{a,b}}{E[\exp(-\int_0^t \lambda_s ds)\big|F_t]}\left\{E\left[\int_{T_a}^{T_b} D(t,s)(s-T_{\beta(s)-1})Q\{\tau\in[s,s+ds)\big|F_{T_b}\}\Big|F_t\right]+E\left[\sum_{i=a+1}^{b} D(t,T_i)\alpha_i\, Q\{\tau\geq T_i\big|F_{T_b}\}\Big|F_t\right]\right\}$$

$$=\frac{1_{\{\tau>t\}}R_{a,b}}{E[\exp(-\int_0^t \lambda_s ds)\big|F_t]}\left\{E\left[\int_{T_a}^{T_b} D(t,s)(s-T_{\beta(s)-1})\exp(-\int_0^s \lambda_u du)\lambda_s ds\Big|F_t\right]+E\left[\sum_{i=a+1}^{b} D(t,T_i)\alpha_i \exp(-\int_0^{T_i} \lambda_u du)\Big|F_t\right]\right\}$$

$$=1_{\{\tau>t\}}R_{a,b}\left\{\int_{T_a}^{T_b}(s-T_{\beta(s)-1})E\left[\exp(-\int_t^s r_u du)\exp(-\int_t^s \lambda_u du)\lambda_s ds\Big|F_t\right]+\sum_{i=a+1}^{b}\alpha_i E\left[\exp(-\int_t^{T_i} r_u du)\exp(-\int_t^{T_i}\lambda_u du)\Big|F_t\right]\right\}$$

$$=1_{\{\tau>t\}}R_{a,b}\left\{\int_{T_a}^{T_b}(s-T_{\beta(s)-1})E\left[\exp(-\int_t^s (r_u+\lambda_u)du)\lambda_s ds\Big|F_t\right]+\sum_{i=a+1}^{b}\alpha_i E\left[\exp(-\int_t^{T_i}(r_u+\lambda_u)du)\Big|F_t\right]\right\}$$

Similarly, we can also calculate the protection leg,

$$\frac{1_{\{\tau>t\}}}{Q\{\tau>t\big|F_t\}}E\left[1_{\{T_a<\tau\leq T_b\}}D(t,\tau)L_{GD}\big|F_t\right]=\frac{1_{\{\tau>t\}}}{E\left[\exp(-\int_0^t \lambda_s ds)\big|F_t\right]}E\left[\int_{T_a}^{T_b} D(t,s)L_{GD}1_{\{\tau\in[s,s+ds)\}}\Big|F_t\right]$$

$$=\frac{1_{\{\tau>t\}}}{E\left[\exp(-\int_0^t \lambda_s ds)\big|F_t\right]}E\left[E\left[\int_{T_a}^{T_b} D(t,s)L_{GD}1_{\{\tau\in[s,s+ds)\}}\Big|F_{T_b}\right]\Big|F_t\right]$$

$$=\frac{1_{\{\tau>t\}}}{E\left[\exp(-\int_0^t \lambda_s ds)\big|F_t\right]}L_{GD}E\left[\int_{T_a}^{T_b} D(t,s)E\left[1_{\{\tau\in[s,s+ds)\}}\big|F_{T_b}\right]\Big|F_t\right]$$

$$=\frac{1_{\{\tau>t\}}}{E\left[\exp(-\int_0^t \lambda_s ds)\big|F_t\right]}L_{GD}E\left[\int_{T_a}^{T_b} D(t,s)Q\{\tau\in[s,s+ds)\big|F_{T_b}\}\Big|F_t\right]$$

$$=\frac{1_{\{\tau>t\}}}{E\left[\exp(-\int_0^t \lambda_s ds)\big|F_t\right]}L_{GD}E\left[\int_{T_a}^{T_b} D(t,s)\exp(-\int_0^s \lambda_u du)\lambda_s ds\Big|F_t\right]$$

$$=\frac{1_{\{\tau>t\}}}{E\left[\exp(-\int_0^t \lambda_s ds)\big|F_t\right]}L_{GD}E\left[\int_{T_a}^{T_b}\exp(-\int_t^s r_u du)\exp(-\int_0^s \lambda_u du)\lambda_s ds\Big|F_t\right]$$

$$=1_{\{\tau>t\}}L_{GD}\int_{T_a}^{T_b}E\left[\exp(-\int_t^s (r_u+\lambda_u)du)\Big|F_t\right]ds \tag{5.6}$$

Therefore we can rewrite the CDS term as a function of interest rate r_t and intensity λ_t:

$$
\begin{aligned}
&CDS_{a,b}(t,R_{a,b}(t),L_{GD})\\
&=1_{\{\tau>t\}}R_{a,b}\left\{\int_{T_a}^{T_b}(s-T_{\beta(s)-1})E\left[\exp(-\int_t^s(r_u+\lambda_u)du)\lambda_s\middle|F_t\right]ds+\sum_{i=a+1}^{b}\alpha_iE\left[\exp(-\int_t^{T_i}(r_u+\lambda_u)du)\middle|F_t\right]\right\}\\
&\quad-1_{\{\tau>t\}}L_{GD}\int_{T_a}^{T_b}E\left[\exp(-\int_t^s(r_u+\lambda_u)du)\lambda_s\middle|F_t\right]ds\\
&=1_{\{\tau>t\}}\left\{R_{a,b}\int_{T_a}^{T_b}(s-T_{\beta(s)-1})E\left[\exp(-\int_t^s(r_u+\lambda_u)du)\lambda_s\middle|F_t\right]ds+R_{a,b}\sum_{i=a+1}^{b}\alpha_iE\left[\exp(-\int_t^{T_i}(r_u+\lambda_u)du)\middle|F_t\right]\right.\\
&\qquad\left.-L_{GD}\int_{T_a}^{T_b}E\left[\exp(-\int_t^s(r_u+\lambda_u)du)\lambda_s\middle|F_t\right]ds\right\}
\end{aligned}
$$

(5.7)

5.5 Lack of Correlation Case in SSRD Model

Under the uncorrelated assumption, we can separate the expectations on the interest rate and the intensity in (5.7), thus

$$
\begin{aligned}
&CDS_{a,b}(t,R_{a,b}(t),L_{GD})\\
&=1_{\{\tau>t\}}\left\{R_{a,b}\int_{T_a}^{T_b}(s-T_{\beta(s)-1})E\left[\exp(-\int_t^s r_udu)\middle|F_t\right]E\left[\exp(-\int_t^s\lambda_udu)\lambda_s\middle|F_t\right]ds\right.\\
&\qquad+R_{a,b}\sum_{i=a+1}^{b}\alpha_iE\left[\exp(-\int_t^{T_i}r_udu)\middle|F_t\right]E\left[\exp(-\int_t^{T_i}\lambda_udu)\middle|F_t\right]\\
&\qquad\left.-L_{GD}\int_{T_a}^{T_b}E\left[\exp(-\int_t^s r_udu)\middle|F_t\right]E\left[\exp(-\int_t^s\lambda_udu)\lambda_s\middle|F_t\right]ds\right\}
\end{aligned} \quad (5.8)
$$

We know that $E\left[\exp(-\int_t^s r_udu)\middle|F_t\right]=P^{mkt}(t,s)$, (5.9)

and from (5.4) in the CIR++ intensity model we know that:

$$
\begin{aligned}
&E\left[\exp(-\int_t^s\lambda_udu)\middle|F_t\right]=E\left[\exp(-\int_t^s(y_u^\beta+\phi(u,\beta))du)\middle|F_t\right]\\
&=\exp(-\int_t^s\phi(u,\beta)du)E\left[\exp(-\int_t^s y_u^\beta du)\middle|F_t\right]=\exp(-\int_t^s\phi(u,\beta)du)P^{CIR}(t,s,y_t,\beta)
\end{aligned} \quad (5.10)
$$

Next, to simplify the above formula, let us first review some facts in the CIR++ model.

In the Cox process we have

$$Q(\tau > t) = E\left[e^{-\int_0^t \lambda(s)ds} \right]. \tag{5.11}$$

Substituting the intensity formula into (5.11), we obtain:

$$Q(\tau > t) = E\left[e^{-\int_0^t \lambda(s)ds} \right] = E\left[e^{-\int_0^t y_s^\beta + \phi(t,\beta)ds} \right] = e^{-\int_0^t \phi(t,\beta)ds} E\left[e^{-\int_0^t y_s^\beta ds} \right]. \tag{5.12}$$

Meanwhile, we have the data from market quotes that:

$$Q(\tau > t)_{mkt} = e^{-\Gamma_{mkt}(t)}. \tag{5.13}$$

To calibrate the market data into the model, we equate (5.12) and (5.13). The deterministic function $\phi(t,\beta)$ in the CIR++ intensity model can be derived as:

$$\Psi(t,\beta) = \int_0^t \phi(s,\beta)ds = \Gamma_{mkt} + \ln\left(E\left[e^{-\int_0^t y_s^\beta ds} \right] \right) = \Gamma_{mkt}(t) + \ln P^{CIR}(0,t,y_0,\beta). \tag{5.14}$$

Through (5.14), we can simplify the expression of (5.10) as below:

$$\begin{aligned}
&E\left[\exp(-\int_t^s \lambda_u du) \middle| F_t \right] = \exp(-\int_t^s \phi(u,\beta)du) P^{CIR}(t,s,y_t,\beta) \\
&= \exp\left(\Gamma_{mkt}(t) + \ln P^{CIR}(0,t,y_0,\beta) - \Gamma_{mkt}(s) - \ln P^{CIR}(0,s,y_0,\beta) \right) P^{CIR}(t,s,y_t,\beta) \\
&= \exp\left(-\left(\Gamma_{mkt}(s) - \Gamma_{mkt}(t) \right) \right) \frac{P^{CIR}(0,t,y_0,\beta)}{P^{CIR}(0,s,y_0,\beta)} P^{CIR}(t,s,y_t,\beta) \\
&= \exp\left(-\left(\Gamma_{mkt}(s) - \Gamma_{mkt}(t) \right) \right)
\end{aligned} \tag{5.15}$$

Also,

$$\begin{aligned}
&E\left[\exp(-\int_t^s \lambda_u du)\lambda_s \middle| F_t \right] = E\left[\frac{d}{ds}\left(\exp(-\int_t^s \lambda_u du) \right) \middle| F_t \right] = \frac{d}{ds} E\left[\exp(-\int_t^s \lambda_u du) \right] \\
&= \frac{d}{ds}\left(\exp\left(-\left(\Gamma_{mkt}(s) - \Gamma_{mkt}(t) \right) \right) \right) \\
&= \gamma_{mkt}(s) \exp\left(-\left(\Gamma_{mkt}(s) - \Gamma_{mkt}(t) \right) \right)
\end{aligned} \tag{5.16}$$

After putting (5.9), (5.15) and (5.16) into (5.8), a discrete computation scheme of the CDS payoff is obtained:

$$CDS_{a,b}(t,R_{a,b}(t),L_{GD})$$
$$=1_{\{\tau>t\}}\left\{R_{a,b}\int_{T_a}^{T_b}(s-T_{\beta(s)-1})P^{mkt}(t,s)\gamma_{mkt}(s)\exp\left(-\left(\Gamma_{mkt}(s)-\Gamma_{mkt}(t)\right)\right)ds\right.$$
$$\left.+R_{a,b}\sum_{i=a+1}^{b}\alpha_i P^{mkt}(t,T_i)\exp\left(-\left(\Gamma_{mkt}(T_i)-\Gamma_{mkt}(t)\right)\right)-L_{GD}\int_{T_a}^{T_b}P^{mkt}(t,T_i)\gamma_{mkt}(s)\exp\left(-\left(\Gamma_{mkt}(s)-\Gamma_{mkt}(t)\right)\right)ds\right\}$$
(5.17)

Comparing the CDS payoff formula from the deterministic intensity model (4.10) with the one from the stochastic intensity model (5.17), we find that the two expressions are consistent, i.e. under the no correlation condition, the assumption of deterministic intensity and stochastic intensity lead to exactly the same result [1].

5.5.1 Numerical Scheme [1]

In the previous chapter we have derived the running CDS formula for a piecewise constant intensity in an integral form. Here we develop it into a discrete form which is convenient for computer simulation.

$$CDS_{a,b}(0,R,L_{GD})$$
$$=R\sum_{i=a+1}^{b}\gamma_i\int_{T_{i-1}}^{T_i}(u-T_{i-1})P(0,u)\exp(-\Gamma_{i-1}-\gamma_i(u-T_{i-1}))du+R\sum_{i=a+1}^{b}P(0,T_i)\alpha_i\exp(-\Gamma_i)$$
$$-L_{GD}\sum_{i=a+1}^{b}\gamma_i\int_{T_{i-1}}^{T_i}P(0,t)\exp(-\Gamma_{i-1}-\gamma_i(u-T_{i-1}))du$$
$$=R\sum_{i=a+1}^{b}\gamma_i(T_i-T_{i-1})P(0,T_i)\exp(-\Gamma_{i-1}-\gamma_i(T_i-T_{i-1}))(T_i-T_{i-1})+R\sum_{i=a+1}^{b}P(0,T_i)\alpha_i\exp(-\Gamma_i)$$
$$-L_{GD}\sum_{i=a+1}^{b}\gamma_i P(0,T_i)\exp(-\Gamma_{i-1}-\gamma_i(T_i-T_{i-1}))(T_i-T_{i-1})$$
$$=R\sum_{i=a+1}^{b}\gamma_i P(0,T_i)\exp(-\Gamma_i)\alpha_i^2+R\sum_{i=a+1}^{b}P(0,T_i)\alpha_i\exp(-\Gamma_i)-L_{GD}\sum_{i=a+1}^{b}\gamma_i P(0,T_i)\exp(-\Gamma_i)\alpha_i$$
(5.18)

where $\alpha_i=T_i-T_{i-1}$, and $\Gamma_i=\int_0^{T_i}\gamma(s)ds=\sum_{k=1}^{i}(T_i-T_{i-1})\gamma_i$.

As we mentioned in section 2.4, the CDS forward rate $R_{a,b}(0)$ makes the discounted payoff formula (5.18) equal to zero at $t=0$. Therefore, to get the intensities, we can plug the market quotes of the CDS forward rate for different maturities into (5.18) and solve for γ_i.

For example, we assume the starting time of the CDS contract is at $T_a = 0$, and let the end of the contract T_b be 1 year, 2 years, 3 years… etc. We also set up the discrete time T_i quarterly. For the market quotes $R_{0,1}(0)$, we can get the first year intensities $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ by solving the equation:

$$CDS_{0,1y}(0, R_{0,1y}^{MKT}, L_{GD}; \gamma_1 = \gamma_2 = \gamma_3 = \gamma_4) = 0 . \quad (5.19)$$

For the second year, by using the market CDS data and the first year intensities derived from (5.19), we can solve for the intensities of the second year:

$$CDS_{0,2y}(0, R_{0,2y}^{MKT}, L_{GD}, \gamma_1, \gamma_2, \gamma_3, \gamma_4; \gamma_5 = \gamma_6 = \gamma_7 = \gamma_8) = 0 . \quad (5.20)$$

Therefore, for the n[th] year intensities, we can just plug in the CDS forward rates and solve the equations iteratively.

5.5.2 Simulation Results

Below we present some numerical examples, based on historical IBM, and Dell CDS data[3]-[6].

(a) IBM CDS Calibration, Oct. 28th, 2008

Recovery Rate=40%

Maturity Tb(yr)	Maturity (date)	R(0,Tb)
0.5	2009-4-28	39.1
1	2009-10-28	47.327
2	2010-10-28	54.669
3	2011-10-28	63.894
4	2012-10-28	72.652
5	2013-10-28	77.16
7	2015-10-28	77.472
10	2018-10-28	79.439

Table 5.1. Maturity dates & corresponding CDS quotes in bps of IBM on Oct. 28th, 2008

Date	Intensity	Survival Probability
2009-4-28	0.0065	0.9967
2009-10-28	0.0093	0.9921
2010-10-28	0.0104	0.9819
2011-10-28	0.0139	0.9683
2012-10-28	0.0169	0.9521
2013-10-28	0.0163	0.9367
2015-10-28	0.0131	0.9124
2018-10-28	0.0144	0.8739

Table 5.2. Calibration with piecewise constant intensity of IBM on Oct. 28th, 2008

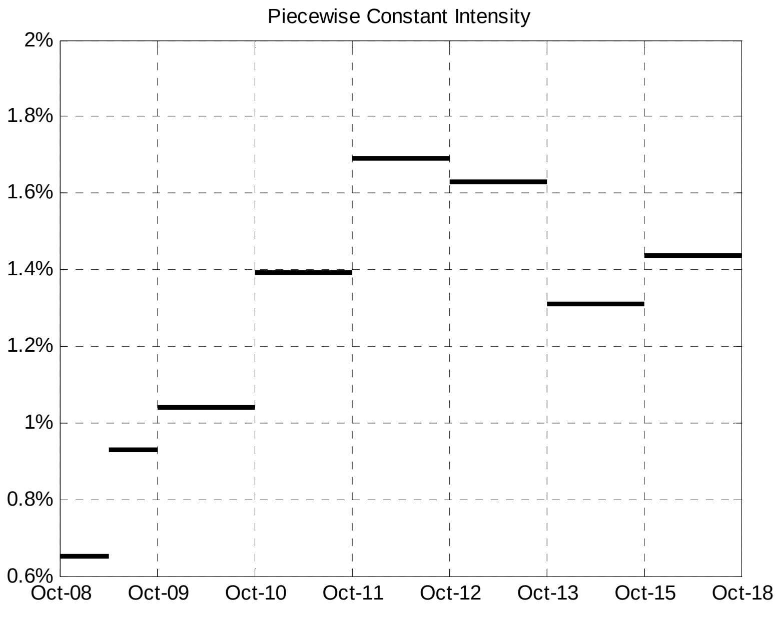

Figure 5.1. Piecewise constant intensity γ calibrated on CDS quotes of IBM, Oct. 28th, 2008

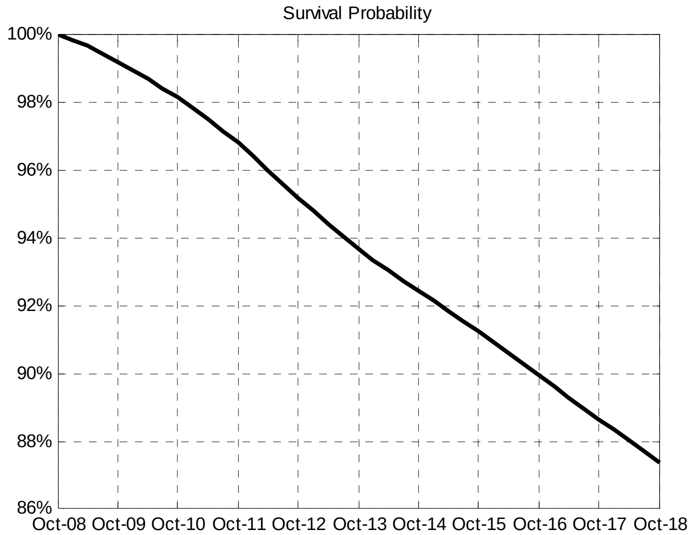

Figure 5.2. Survival probability from calibration on CDS quotes of IBM, Oct. 28th, 2008

Table 5.1 gives the CDS forward market rates of IBM for different maturities (0.5 year to 10 years) on Oct. 2008. We calibrate these data into the discrete equation (5.18) to calculate the intensity and default probability within 10 years. Table 5.2 gives the corresponding numerical results of the intensity and default probabilities. Then we plot the simulation result of the intensities and survival probabilities data in Figure 5.1 and Figure 5.2.

(b) IBM CDS Calibration, Dec. 12ed, 2011

Recovery Rate=40%

Maturity Tb(yr)	Maturity (date)	R(0,Tb)
1	2012-12-12	18.73
2	2013-12-12	24.00
3	2014-12-12	32.00
4	2015-12-12	39.00
5	2016-12-12	45.17
7	2018-12-12	56.66
10	2021-12-12	67.67

Table 5.3. Maturity dates & corresponding CDS quotes in bps of IBM on Dec. 12nd, 2011

Date	Intensity	Survival Probability
2012-12-12	0.0031	99.69%
2013-12-12	0.0049	99.20%
2014-12-12	0.0081	98.40%
2015-12-12	0.0102	97.40%
2016-12-12	0.0120	96.24%
2018-12-12	0.0150	93.39%
2021-12-12	0.0168	88.80%

Table 5.4. Calibration with piecewise constant intensity of IBM on Dec. 12nd, 2011

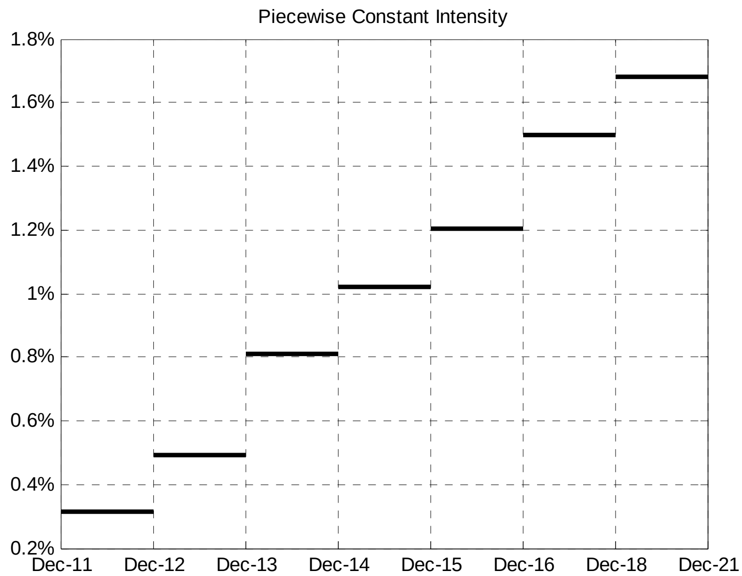

Figure 5.3. Piecewise constant intensity γ calibrated on CDS quotes of IBM on Dec. 12nd, 2011

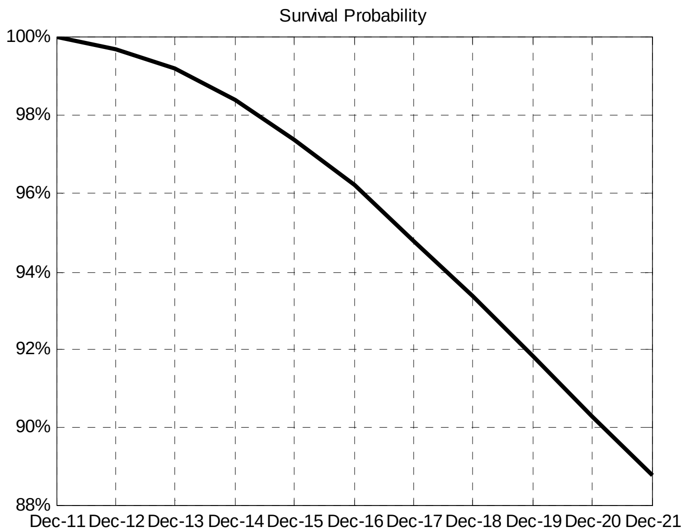

Figure 5.4: Survival probability from calibration on CDS quotes of IBM on Dec.12nd, 2011

Table 5.3 gives the CDS forward market rates of IBM for different maturities (1 year to 10 years) on Dec. 2011. Table 5.2 gives the corresponding numerical result of the intensity and default probabilities. We also show the intensity and probability curve in Figure 5.3 and 5.4.

(c) Dell CDS Calibration, Aug 22nd, 2008

Recovery Rate=40%

Maturity Tb(yr)	Maturity (date)	R(0,Tb)
0.5	2009-2-22	30.9
1	2009-8-22	36.345
2	2010-8-22	44.44
3	2011-8-22	56.817
4	2012-8-22	65.173
5	2013-8-22	72.996
7	2015-8-22	76.434
10	2018-8-22	80.395

Table 5.5. Maturity dates & corresponding CDS quotes in bps of Dell on Aug. 22nd, 2008

Date	Intensity	Survival Probability
2009-2-22	0.0051	99.74%
2009-8-22	0.0070	99.39%
2010-8-22	0.0088	98.52%
2011-8-22	0.0139	97.16%
2012-8-22	0.0155	95.67%
2013-8-22	0.0182	93.95%
2015-8-22	0.0145	91.26%
2018-8-22	0.0155	87.10%

Table 5.6. Calibration with piecewise constant intensity of Dell on Aug. 22nd, 2008

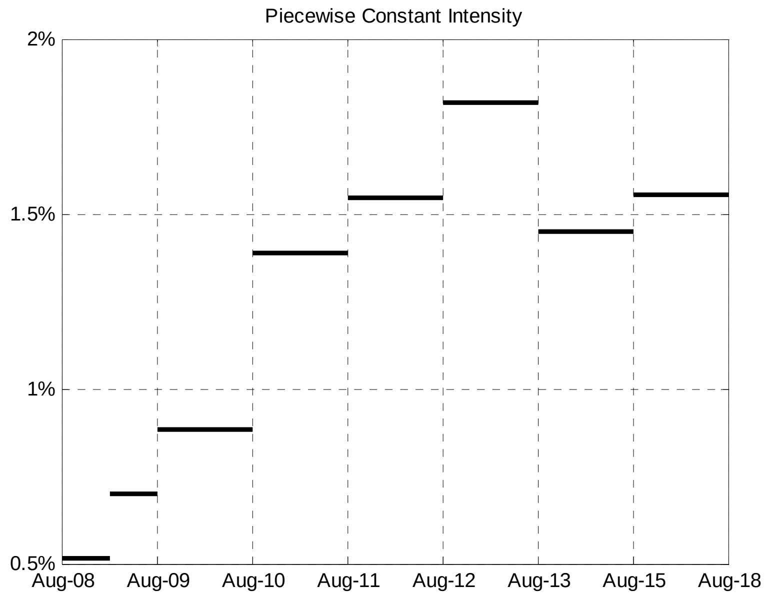

Figure 5.5. Piecewise constant intensity γ calibrated on CDS quotes of Dell on Aug. 22nd, 2008

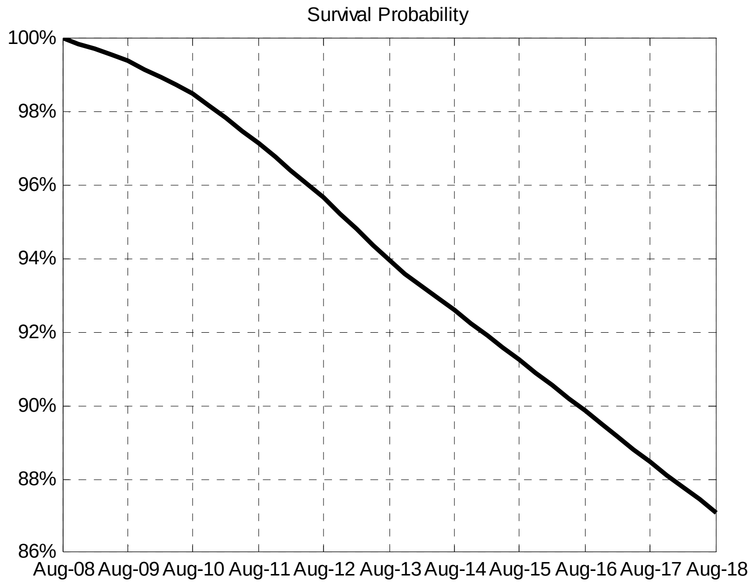

Figure 5.6. Survival probability from calibration on CDS quotes of Dell on Aug. 22nd, 2008

Similar to the calibration of IBM, Table 5.5 gives the CDS forward market rates of Dell for different maturities (0.5 year to 10 years) on Aug. 2008. Table 5.6 shows the corresponding intensity and default probabilities. Figure 5.5 and 5.6 lines all the data in time.

(d) Dell CDS Calibration, Dec. 12nd, 2011

Recovery Rate=40%

Maturity Tb(yr)	Maturity (date)	R(0,Tb)
1	2012-12-12	67.50
2	2013-12-12	84.50
3	2014-12-12	100.50
4	2015-12-12	120.00
5	2016-12-12	137.57
7	2018-12-12	161.52
10	2021-12-12	178.66

Table 5.7. Maturity dates & corresponding CDS quotes in bps of Dell on Dec. 12nd, 2011

Date	Intensity	Survival Probability
2012-12-12	0.0114	99.87%
2013-12-12	0.0173	97.18%
2014-12-12	0.0228	94.99%
2015-12-12	0.0313	92.07%
2016-12-12	0.0371	88.71%
2018-12-12	0.0404	81.83%
2021-12-12	0.0405	72.46%

Table 5.8. Calibration with piecewise constant intensity of Dell on Dec. 12nd, 2011

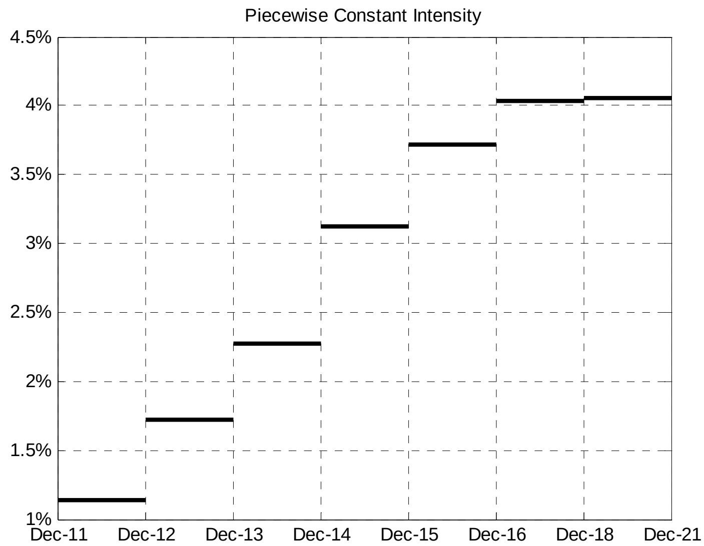

Figure 5.7. Piecewise constant intensity γ calibrated on CDS quotes of Dell on Dec.12nd, 2011

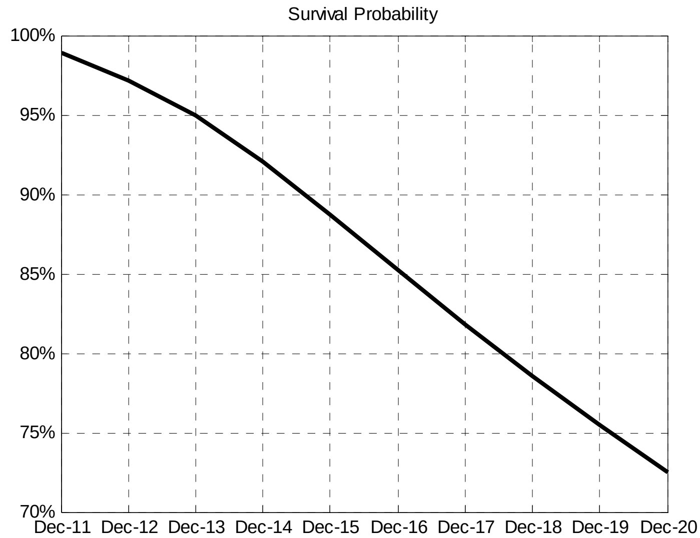

Figure 5.8. Survival probability from calibration on CDS quotes of Dell, on Dec. 12nd, 2011

Table 5.7 gives the CDS forward market rates of Dell for different maturities (1 year to 10 years) on Dec. 2011. Table 5.8 shows the corresponding intensity and default probabilities. Figure 5.7and 5.8 show the change of intensity and default probabilities in time.

Based on the simulation results, we can analyze the company information for IBM and Dell in different time periods separately. For IBM we find that the intensities did not change dramatically within the 10 year period. The survival probabilities derived from 2008 and 2011 have a similar curve, which also means that the company is showing a steady performance and status. For Dell, the 10-year survival probability in 2008 is 87.10%;however, the 10-year survival probability drops to 72.45% in 2011. By comparing the data in 2008 and 2011, it demonstrates a decline for Dell in business performance.

5.6 Correlation Case in SSRD Model

In this section, we consider a more general case, in which we can put a correlation factor between the short rate and stochastic intensity. We will discuss the effects on the stochastic short rate and intensity processes in the different correlations scenarios.

To simulate the dynamics of the short rate and stochastic intensity, we first need to derive the parameters in the short rate CIR++ model and the intensity CIR++ model separately. The parameter $\alpha = (k, \theta, \sigma, x_0^{\alpha})$ in the CIR++ short rate model can be derived from the market quotes of the interest rate products, such as caps, floors and zero coupon bonds. Similarly, the parameters $\beta = (\kappa, \mu, \upsilon, y_0^\beta)$ in the stochastic intensity model can be obtained by the market price of the credit default swap products. In this project we will not demonstrate a detailed procedure and scheme on the parameter derivation. Instead, we just use the data given in the Brigo and Mercurio's book. [1]

Recovery Rate=25%

Maturity Tb(yr)	Maturity (dates)	R(0,Tb)
1	2004-12-20	1450
3	2006-12-20	1200
5	2008-12-20	940
7	2010-12-20	850
10	2013-12-20	850

Table 5.9. Maturity dates & corresponding CDS quotes in bps of Parmalat on Dec. 8th, 2003

According to the calibration to the Parmalat CDS data and Cap prices, Alfonsi gets the below parameters to describe the dynamics of short rate and intensities in the CIR++ model on Dec. 8th, 2003[1]:

$$\alpha : k = 0.528950, \theta = 0.0319904, \sigma = 0.130035, x_0 = 8.32349 \times 10^{-5}$$
$$\beta : \kappa = 0.583307, \mu = 0.0149846, \upsilon = 0.0479776, y_0 = 0.192973$$

We will use the above data to simulate the correlated short rate and stochastic intensities and conduct the comparison and analysis in the following sections.

5.6.1 Discretization Scheme of Short Rate and Intensities

We first discretize the dynamics of x and y in the CIR++ model by using the parameter vectors α and β. Brigo and Alfonsi have proposed an implicit Euler schemes and derived the corresponding explicit schemes for the process in 2003[1].

The implicit Euler schemes are:

$$x^{\alpha}_{t_{i+1}} = x^{\alpha}_{t_i} + (k\theta - \frac{\sigma^2}{2} - kx^{\alpha}_{t_{i+1}})(t_{i+1} - t_i) + \sigma\sqrt{x^{\alpha}_{t_{i+1}}}(W_{t_{i+1}} - W_{t_i})$$
$$y^{\beta}_{t_{i+1}} = y^{\beta}_{t_i} + (\kappa\mu - \frac{\upsilon^2}{2} - \kappa y^{\beta}_{t_{i+1}})(t_{i+1} - t_i) + \upsilon\sqrt{y^{\beta}_{t_{i+1}}}(Z_{t_{i+1}} - Z_{t_i}) \quad (5.20)$$
$$dWdZ = \rho dt$$

To obtain an explicit expression, we need to solve for the above equations:

$$x^{\alpha}_{t_{i+1}} = \left(\frac{\sigma(W_{t_{i+1}} - W_{t_i}) + \sqrt{\Delta_{t_i}}}{2(1 + k(t_{i+1} - t_i))} \right)^2$$
$$where \quad (5.21)$$
$$\Delta_{t_i} = \sigma^2 (W_{t_{i+1}} - W_{t_i})^2 + 4(x^{\alpha}_{t_i} + (k\theta - \frac{\sigma^2}{2})(t_{i+1} - t_i))(1 + k(t_{i+1} - t_i))$$

Similarly,

$$y^{\beta}_{t_{i+1}} = \left(\frac{\upsilon(Z_{t_{i+1}} - Z_{t_i}) + \sqrt{\Delta^*_{t_i}}}{2(1 + \kappa(t_{i+1} - t_i))} \right)^2$$
$$where \quad (5.22)$$
$$\Delta^*_{t_i} = \upsilon^2 (Z_{t_{i+1}} - Z_{t_i})^2 + 4(y^{\beta}_{t_i} + (\kappa\mu - \frac{\upsilon^2}{2})(t_{i+1} - t_i))(1 + \kappa(t_{i+1} - t_i))$$

We know the standard Brownian motion W_t has a normal distribution $N(0,t)$, and the increments $W_t - W_s$ have a normal distribution $N(0,t-s)$. To get two correlated standard Brownian motions, we need to generate two independent standard Brownian motion paths W_t and V_t first, and then let $Z_t = \rho W_t + \sqrt{1-\rho^2} V_t$. Thus the two Brownian motions W_t and Z_t have a correlation of ρ.

To construct the path in figure 5.9 and 5.10 below, we first generated two correlated standard Brownian motions by the method mentioned in the above paragraph. Then we apply the numerical formula in (5.21) and (5.22) to discretize each time step of x_t and y_t. Here we let the time step to be a quarter of a year, which is 0.25 in the simulation.

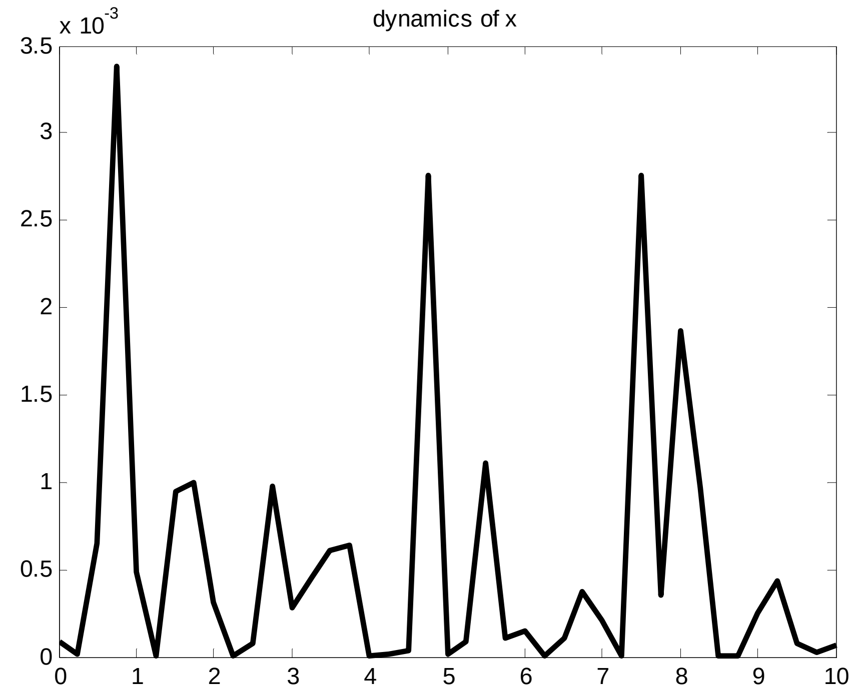

Figure 5.9. The dynamics of x in short rate model from Parmalat CDS data on Dec. 8th, 2003

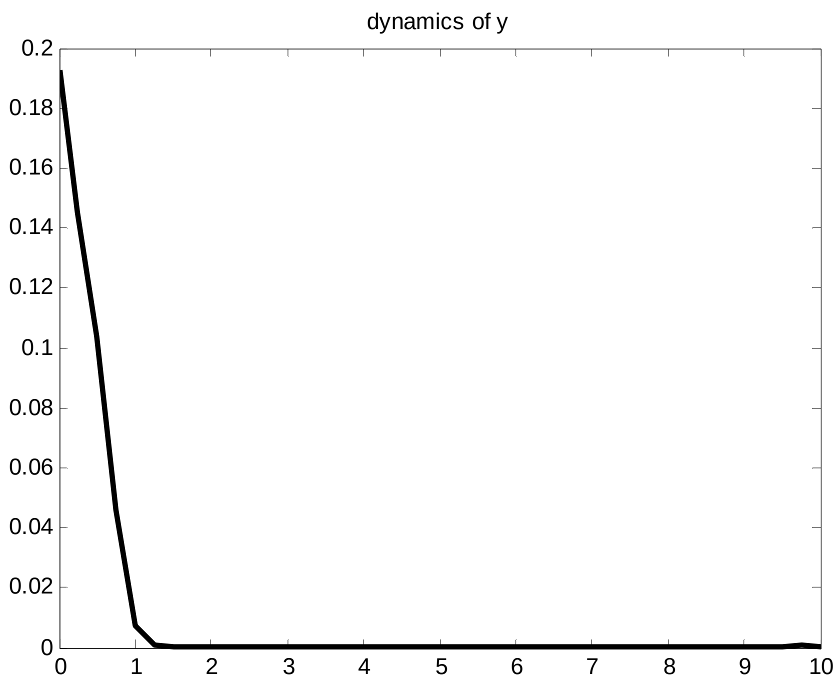

Figure 5.10. The dynamics of y in intensity model from Parmalat CDS data on Dec. 8th, 2003

In the next step we need to derive the deterministic functions $\varphi(t,\alpha)$ and $\phi(t,\beta)$ in equation (5.1) and (5.4). [1]

For $\varphi(t,\alpha)$ in the CIR++ model, we have:

$$\varphi^{CIR}(t,\alpha) = f^{mkt}(0,t) - f^{CIR}(0,t,\alpha)$$
$$where:$$
$$f^{mkt}(0,t) = -\frac{\partial P^{mkt}(0,t)}{\partial t} = -\frac{P^{mkt}(0,t_{i+1}) - P^{mkt}(0,t_i)}{t_{i+1} - t_i} \tag{5.23}$$
$$f^{CIR}(0,t,\alpha) = 2k\theta \frac{\exp\{th\}-1}{2h+(k+h)(\exp\{th\}-1)} + x_0 \frac{4h^2\exp\{th\}}{\left[2h+(k+h)(\exp\{th\}-1)\right]^2}$$
$$h = \sqrt{k^2 + 2\sigma^2}$$

Therefore, we can obtain $\varphi(t,\alpha)$ by substituting the market data of the zero coupon bonds for different maturities and $\alpha = (k,\theta,\sigma,x_0^{\alpha})$ into (5.23).

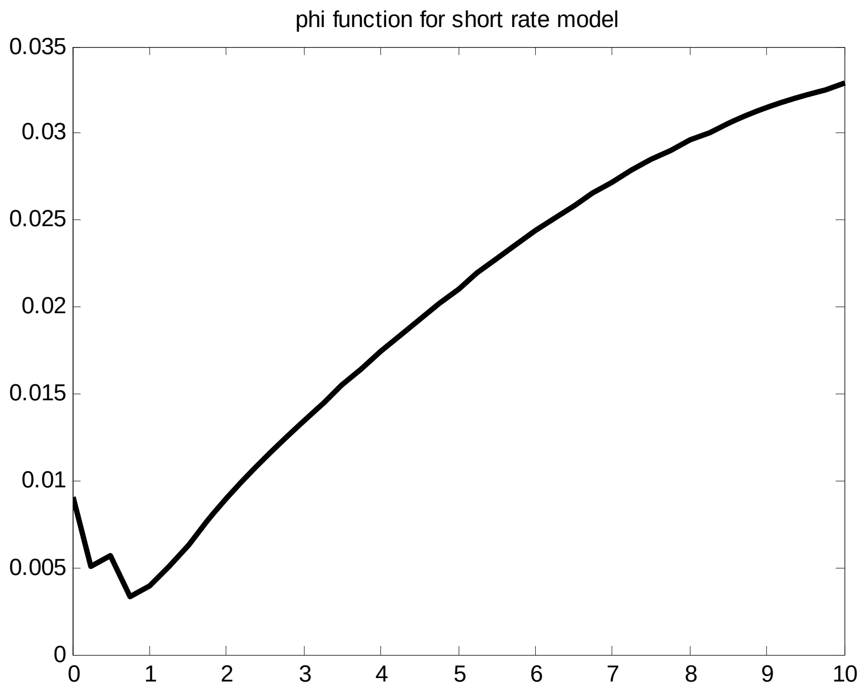

Figure 5.11. The deterministic function in the CIR++ short rate model

As for $\phi(t,\beta)$, because of its deterministic feature, we can set up an equation for $\phi(t,\beta)$ under the condition of zero correlation. The most straight forward approach is to implement equation (5.14) into the discretization scheme:

$$\sum_{i=a+1}^{b} \phi(t_i,\beta)\alpha_i = \sum_{i=a+1}^{b} \gamma_i + \ln\left(P^{CIR}(0,t,y_0,\beta)\right)$$

where $P^{CIR}(0,t,y_t^{\beta},\beta)$ is the bond price derived from the CIR model with parameter vector β:

$$P^{CIR}(0,t,y_t^{\beta},\beta) = A^*(0,t,\beta)\exp(-B^*(0,t,\beta))y_t)$$

with:

$$A^*(0,t,\beta) = \left[\frac{2h^*\exp\left\{(\kappa+h^*)t/2\right\}}{2h^*+(\kappa+h^*)(\exp\left\{th^*\right\}-1)}\right]^{2\kappa\mu/\upsilon^2}$$

$$B^*(0,t,\beta) = \frac{2(\exp\left\{th^*\right\}-1)}{2h^*+(\kappa+h^*)(\exp\left\{th^*\right\}-1)}$$

and $h^* = \sqrt{\kappa^2+2\upsilon^2}$.

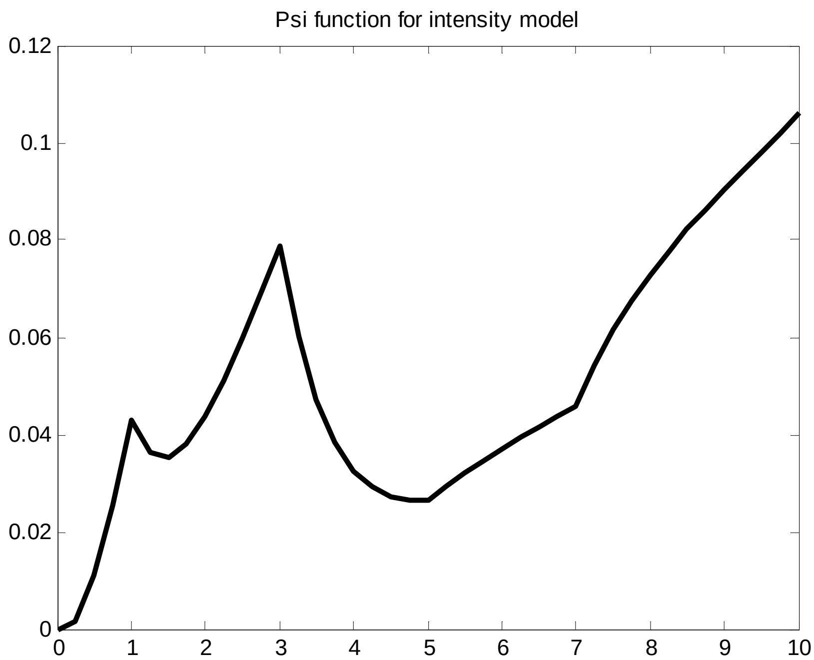

Figure 5.12. The deterministic function in the CIR++ intensity model

5.6.2Correlation Effect on Interest Rate and Stochastic Intensity

Here we choose different ρ to explore the effects of the correlation on the interest rate and stochastic intensity. We consider three cases:

1. Interest rate and stochastic intensity have no correlation (0).

2. Interest rate and stochastic intensity have a negative correlation (-1).

3. Interest rate and stochastic intensity have a positive correlation (1).

As mentioned in the 5.6.1, we generate two correlated standard Brownian motions path first. We fix one Brownian motion path and change the other path by varying the correlation coefficient between them. Then we can observe the changes in the dynamics of y as shown in the below figure.

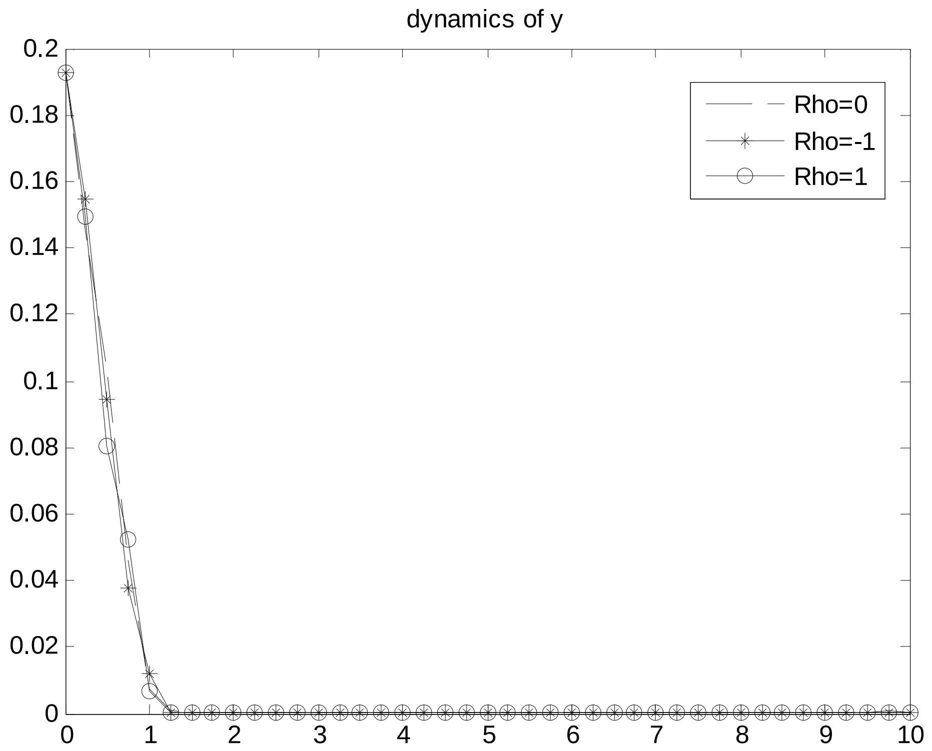

Figure 5.13. Comparison of the intensity with different correlation to interest rate

From the simulation results in Figure 5.13, we find out that the three curves are quite close to each other, especially after 1 year. Therefore we can come to the conclusion that the correlation does not affect the dynamics of y much. In such case, we can ignore the correlation between the interest rate and intensity when calibrating them into the CDS.

5.6.3 Monte Carlo Simulation

Monte Carlo methods are stochastic techniques based on the use of random numbers and probability statistics to investigate problems. In this project we implemented Monte Carlo methods to simulate the dynamics of the interest rate and stochastic intensity.

In the simulation we use the same Parmalat CDS data on Dec. 08th, 2003[1] and the corresponding derived parameter vectors α and β shown in section 5.6. A correlation of 0.3 is used in this simulation. We generate 10,000 sample paths to implement the Monte Carlo method.

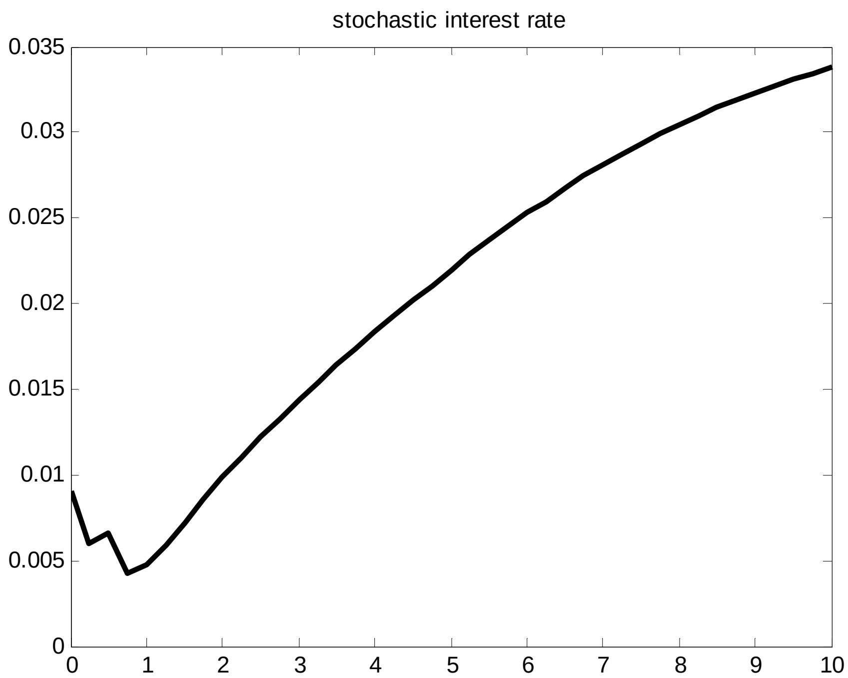

Figure 5.14. Monte Carlo simulation of interest rate in SSRD model

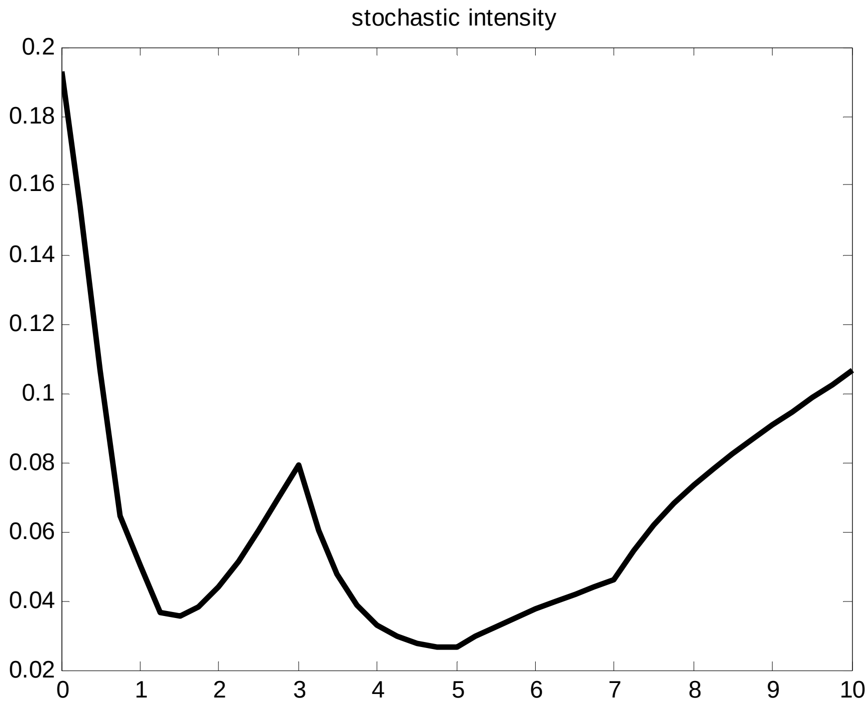

Figure 5.15. Monte Carlo simulation of stochastic intensity in SSRD model

From the simulation results, we discover that the stochastic interest rate has some oscillation during the first year, then displays a steady increase in the following years. As for the stochastic intensity, the peak shows in the first year, which means that the Parmalat company faces a default crisis during the first year. The intensity is oscillating but smaller after year 1. This means that the company comes into a relatively steady situation compared to the first year.

6 Conclusion

In this paper we introduced the basic concept of CDS, and explained the premium payment, protection payment and the relationship between the two. Then we presented a reduced form (Intensity) model based on the time-inhomogeneous Poisson process. We used the market instrument CDS and bonds prices to infer the implied default probabilities from market quotes. A computation scheme was developed to calculate the corresponding intensity and survival probability from the market data for several companies. Then we investigated the features of the SSRD model. We showed that when there is no correlation between the interest rate and stochastic intensities, the SSRD model have the same results with the reduced intensity model. We also simulated the dynamics of the interest rate and stochastic intensity using the Monte Carlo method.

7 Reference and Data

1. Damiano Brigo and Fabio Mercurio: Interest Rate Models - Theory and Practice with Smile, Inflation and Credit (Springer Finance, 2006).

2. R. Jarrow and S. Turnbull, “Pricing Options of Financial Securities Subject to Default Risk”, Journal of Finance, 50 (1995): 53-86.

3. “Credit Default Swap Price,” International Business Machines Corporation (IBM US), October 28, 2008, BLOOMBERG (accessed May 20, 2009).

4. “Credit Default Swap Price,” International Business Machines Corporation (IBM US), December 12, 2011, BLOOMBERG (accessed December 12, 2011).

5. “Credit Default Swap Price,” Dell Inc. (DELL US), August 22, 2008, BLOOMBERG (accessed May 20, 2009).

6. “Credit Default Swap Price,” Dell Inc. (DELL US), December 12, 2011, BLOOMBERG (accessed December 12, 2011).