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Luca Capogna
University of Arkansas

Giovanna Citti
University of Bologna

Maria Manfredini
University of Bologna

REGULARITY OF NON-CHARACTERISTIC MINIMAL GRAPHS IN THE HEISENBERG GROUP $\mathbb{H}^1$

LUCA CAPOGNA, GIOVANNA CITTI, AND MARIA MANFREDINI

Abstract. Minimal surfaces in the sub-Riemannian Heisenberg group can be constructed by means of a Riemannian approximation scheme, as limit of Riemannian minimal surfaces. We study the regularity of Lipschitz, non-characteristic minimal surfaces which arise as such limits. Our main results are a-priori estimates on the solutions of the approximating Riemannian PDE and the ensuing C^∞ regularity of the sub-Riemannian minimal surface along its Legendrian foliation.

1. Introduction

The first Heisenberg group $\mathbb{H}^1$ is a Lie group with a 3-dimensional Lie algebra $h = V^1 \oplus V^2$ such that $\dim(V^1) = 2, \dim(V^2) = 1$, $[V^1, V^1] = V^2$ and $[h, V^2] = 0$. Let $\mathcal{S}, \mathcal{X}, \mathcal{Y} \in h$ be any basis such that $[\mathcal{S}, \mathcal{X}] = \mathcal{Y} \in V^2$. By assigning a left-invariant Riemannian metric g_0 on the *horizontal* sub-bundle $H\mathbb{H}^1$ given by the V^1 layer, we obtain a sub-Riemannian space $(\mathbb{H}^1, g_0)$. We choose $\mathcal{S}, \mathcal{X}$ such that they are orthonormal with respect to g_0. The corresponding control metric d_0 (the *Carnot-Caratheodory metric* [32]) is easily shown to be well defined. We extend g_0 to a (left-invariant) Riemannian metric g_1 on the full tangent bundle of h requiring that V^2 and V^1 are orthogonal in this extension. The *dilated* metrics g_ε, $\varepsilon > 0$ are defined so that $\mathcal{S}, \mathcal{X}, \varepsilon\mathcal{Y}$ are orthonormal. We define d_ε to be the corresponding distance function. We define *polarized* coordinates (x_1, x_2, x_3) in $\mathbb{H}^1$ by identifying the triplet with the point $\exp(x_3\mathcal{S})\exp(x_1\mathcal{X} + x_2\mathcal{Y})$. The Baker-Campbell-Hausdorff formula yields $\mathcal{S} = \partial_3$, $\mathcal{X} = \partial_1 + x_3\partial_2$ and $\mathcal{Y} = \partial_2$.

If $M \subset \mathbb{H}^1$ is a C^1 surface, then $p \in M$ is called *characteristic* if both $\mathcal{S}, \mathcal{X}$ are tangent to M at p. An intrinsic graph (see [24, 25]) is a (non-characteristic) graph of the form

$$M = \{x_3 = u(x_1, x_2) | \, (x_1, x_2) \in \Omega \subset \mathbb{R}^2\}, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is an open set. An analogue of the classical implicit function theorem [25, 14, 1] shows that any surface $\{f = 0\}$ with $\mathcal{S}f, \mathcal{X}f \in C(\mathbb{H}^1)$ can be represented as an intrisic graph, in a neighborhood of any of its non-characteristic points.

The flow associated to the line bundle of tangent directions which are also horizontal foliate the complement of the characteristic locus of the surface. This is called *Legendrian foliation* in the literature. Note that the horizontal tangent bundle of an intrisic graph (1.1) is spanned by the single vector field $T = Xu\mathcal{S} + \mathcal{X}|_{u((x_1,x_2),x_1,x_2)}$. We note that the projection of this vector field on $T\Omega$ yields the vector field in Ω,

$$X_{1,u} = \partial_1 + u\partial_2.$$

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Minimal surfaces. Several equivalent notions of *horizontal mean curvature* H_0 for a C^2 surface $M \subset \mathbb{H}^1$ (outside characteristic points) have been given in the literature. To quote a few: H_0 can be defined in terms of the first variation of the area functional [20, 29, 9, 36, 40, 31, 7], as horizontal divergence of the horizontal unit normal or as limit of the mean curvatures H_ε in the Riemannian metrics g_ε as $\varepsilon \to 0$. It is also well known (see for example [9, 29, 18]) that H_0 coincides with the curvature of the projection of the Legendrian leaves on the Horizontal plane.

A C^2 non characteristic surface $M \subset \mathbb{H}^1$ is called *minimal* if it satisfies $H_0 = 0$ identically. In particular for a C^2 intrinsic graph, a direct computation yields that the PDE can be written in terms of the vector $X_{1,u}$ as follows

$$H_0 = X_{1,u}\left(\frac{X_{1,u}u}{\sqrt{1+|X_{1,u}u|^2}}\right) = 0. \tag{1.2}$$

A deep result of Ambrosio, Serra-Cassano and Vittone [1] shows that such PDE continues to hold below the C^2 threshold in a suitably weak sense.

Generalized solutions and Riemannian approximants. Because the horizontal mean curvature arises as first variation of the sub-Riemannian perimeter, minimal surfaces are critical points of the perimeter. As such these objects can be interpreted in weak sense, far below the threshold of C^2 smoothness (see [1], [26], [33],[34],[9],[11],[35]). As an example, starting from the family of shears $x_2 - x_1x_3 + g(x_3)$ (see [34]) one can obtain the intrinsic graph $x_3 = u(x_1, x_2) = \frac{x_2}{x_1 - sgn\,(x_2)}$ defined in $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 1\}$ which is minimal in the sense that it is foliated by horizontal lifts of segments, it is locally Lipschitz (with respect to the Euclidean metric) but clearly not C^1 smooth.

In [33] and in [11], the authors prove existence of (respectively $W^{1,p}$ and Lipschitz) minimal surfaces using the Riemannian approximation scheme: as $\varepsilon \to 0$

$$(\mathbb{H}^1, d_\varepsilon) \to (\mathbb{H}^1, d_0),$$

in the Gromov-Hausdorff topology (see [7, Section 2.4] for a detailed description).

The approach to existence of solutions in these papers is based on a-priori estimates for the minimizers of the approximating Riemannian g_ε perimeter functionals [33] and on solutions of the "*Riemannian*" regularized versions of (1.2) [11]. In adapting the approximation scheme to the intrisic graphs setting we note that the minimal surface PDE for the metric g_ε corresponding to intrinsic graphs (1.1)

$$L_\varepsilon u = \sum_{i=1}^{2} X_{i,u}\left(\frac{X_{i,u}u}{\sqrt{1+|\nabla_\varepsilon u|^2}}\right) = 0, \text{ in } \Omega \subset \mathbb{R}^2 \tag{1.3}$$

where

$$X_{2,u} = \varepsilon\partial_2 \quad \nabla_\varepsilon = (X_{1,u}, X_{2,u}),$$

is a natural elliptic regularization of the PDE (1.2). As such, it is more amenable to establishing a-priori higher regularity estimates. The difficulty of course resides in obtaining estimates which are uniform in the parameter ε as $\varepsilon \to 0$.

These observations lead us to the definition of the class of minimal surfaces we want to investigate

Definition 1.1. *We say that a function $u \in Lip(\Omega)$ is a vanishing viscosity solution of the equation (1.2) if there exists a sequence of positive numbers ε_j with $\varepsilon_j \to 0$ when $j \to \infty$, and a sequence (u_j) in $C^\infty(\Omega)$ such that:*

(i) $\sum_{i=1}^2 X_{i,u_j}\left(\frac{X_{i,u_j}u_j}{\sqrt{1+|\nabla_{\varepsilon_j}u_j|^2}}\right) = 0.$ *in Ω for all $j \in N$.*

(ii) The sequence (u_j) is bounded in $Lip(\Omega)$ and uniformly convergent on subcompacts of Ω to u.

Remark 1.2. *Existence of vanishing viscosity solutions in the case of $t-$graphs, i.e. graphs of the form $x_2 = g(s, x_1)$, has been proved in* [11, Theorem A and Theorem 4.5]. *In the same paper the authors establish that such solutions are perimeter minimizers and address uniqueness. Assuming C^1 convergence of the approximating solutions, outside the characteristic sets the $t-$graphs solutions in* [11], *both the approximating and the limit solutions, can be represented as intrisic graphs and hence yield vanishing viscosity intrinsic graphs.*

Regularity results. Given the examples of non-smooth minimal surfaces mentioned above, the question arises as which kind of regularity can one expect. This problem has beeen recently addressed in a series of papers [34], [11], [5], [10] and [3]. Valuable insights into the problem of regularity are also provided in the works [34], [11], and [35] in the form of examples of non-smooth minimal surfaces. The regularity properties of the implicit function in the implicit function theorem quoted earlier provides an interesting insight into this problem and indicates that one should look for regularity only in the direction of the Legendrian foliation. Indeed we prove

Theorem 1.3 (Regularity)**.** *If u is a vanishing viscosity solution of* (1.2)*, then for all $\alpha \in (0,1)$ and $K \subset\subset \Omega \subset \mathbb{R}^2$,*

$$u \in C^{1,\alpha}(K) \tag{1.4}$$

and for all $k \in \mathbb{N}$ and $p > 1$

$$X_{1,u}^k u \in W_{loc}^{1,p}(\Omega). \tag{1.5}$$

Here $C^{1,\alpha}(K)$ and $W_{loc}^{1,p}(\Omega)$ denote the spaces of functions with Hölder continuous Euclidean gradient and the classical Sobolev Space. In particular one has $X_{1,u}^k u \in C_{loc}^{\alpha}(\Omega)$ (the Euclidean Hölder space) for all $\alpha \in (0,1)$ and hence $X_{1,u}^2 u = 0$, holds pointwise everywhere.

Remark 1.4. *To better understand the notion of intrinsic regularity we return to the non-smooth minimal graph $u(x_1, x_2) = \frac{x_2}{x_1 - sgn\,(x_2)}$ described earlier. Although this function is not C^1 in the Euclidean sense, observe that $X_1 u = 0$ for every $x_1, x_2 \in \Omega$ Hence, this is an example of a minimal surface which is not smooth but which can be differentiated indefinitely in the direction of the Legendrian foliation.*

Remark 1.5. *The regularity theory for intrinsic minimal surfaces in $\mathbb{H}^n$ with $n > 1$ is quite different. In the recent paper* [5] *we show that any Lipschitz continuous vanishing viscosity minimal intrinsic graph in $\mathbb{H}^n$, $n > 1$ (defined through the Riemannian approximation scheme) is smooth. The main reason is that in higher dimension the horizontal tangent bundle generates as a Lie algebra the full tangent bundle, while this does not happen in the $n = 1$ case.*

As a consequence of the regularity theorem we can prove that the Sobolev weak derivatives of vanishing viscosity solutions agree with Lie derivatives along the leaves of the Legendrian foliation. Hence, we obtain that vanishing viscosity solutions actually satisfy (1.2) everywhere pointwise. This result immediately yields a *rigidity* of the Legendrian foliation.

Corollary 1.6 (Lie differentiability and Legendrian foliation)**.** *Let $x_3 = u(x_1, x_2)$, $(x_1, x_2) \in \Omega$ be a Lipschitz continuous vanishing viscosity minimal graph. The flow of the vector $X_{1,u}$ yields a foliation of the domain Ω by polynomial curves γ of degree two. For every fixed $x_0 \in \Omega$ denote by γ the unique leaf passing through that fixed point. The function u is differentiable at x_0 in the Lie sense along γ and the equation* (1.2) *reduces to*

$$\frac{d^2}{dt^2}(u(\gamma(t))) = 0.$$

Comparison with other regularity results. We describe the relation between our results in Theorem 1.3 and Corollary 1.6 and the regularity results in [10] and in [3]. In [10] Cheng, Hwang and Yang prove that any C^1 weak solution of the prescribed (continuous) horizontal mean curvature PDE, has C^2 smooth Legendrian foliation outside of the characteristic set. In [3], Bigolin and Serra Cassano study the regularity of minimal intrisic graphs (1.1) where (1.2) is interpreted in a weak sense (i.e. *broad** solutions defined in [3, Definition 3.1]) and prove (among other results) that Lipschitz regularity of the intrisic gradient $X_{1,u}u$ implies the Euclidean Lipschitz regularity of the function u. In the present paper we require only Lipschitz continuity of u and prove higher order intrinsic differentiability than either [3], and [10]. On the other hand, we only deal with the case $H_0 = 0$ and with those solutions which are limits of Riemannian minimal graphs. In this sense our results are more specialized than the ones in the other two papers.

Applications of Theorem 1.3. Invoking the implicit function theorem, we can apply Theorem 1.3 to study the regularity away from the characteristic locus of the Lipschitz perimeter minimizers found in [11]. Since the results in that paper apply to $t - graph$ and not intrinsic graphs we need some extra assumptions on the convergence of the approximating solution to be able to invoke our intrinsic graphs regularity. Here and in the following ∇_E denotes the Euclidean gradient in $\mathbb{R}^2$, and (z_1, z_2, z_3) are the exponential coordinates $\exp(z_1\mathcal{S} + z_2\mathcal{X} + z_3\mathcal{Y}) = \exp(x_3\mathcal{S})\exp(x_1\mathcal{X} + x_2\mathcal{Y})$.

Corollary 1.7. *Let $O \subset \mathbb{R}^2$ be a strictly convex, smooth open set, $\phi \in C^{2,\alpha}(\bar{O})$ and for each $(z_1, z_2) \in O$ denote by $(z_1, z_2)^* = (z_2, -z_1)$. Consider the family*

$$\{g_\varepsilon(z_1, z_2)\}_\varepsilon \quad \sup_O |g_\varepsilon| + \sup_O |\nabla_E g_\varepsilon| \le C \quad \textit{(uniformly in } \varepsilon),$$

of smooth solutions of the approximating minimal surface PDE

$$div\left(\frac{\nabla_E g_\varepsilon + (z_1, z_2)^*}{\sqrt{\varepsilon^2 + |\nabla_E g_\varepsilon + (z_1, z_2)^*|}}\right) = 0 \ \textit{in } O \quad \textit{and } g_\varepsilon = \phi \quad \textit{in } \partial O$$

found in [11, Theorem 4.5]. *If for $p_0 = (p_0^1, p_0^2) \in O$, $a > 0$ and for every $\varepsilon > 0$ we have $|\partial_{z_1} g_\varepsilon(p_0)| > a > 0$ (or any other partial derivative is non-vanishing at p_0 uniformly in ε) then there is a sequence $\varepsilon_k \to 0$ such that the Lipschitz perimeter minimizer $g = \lim_{\varepsilon_k \to 0} g_{\varepsilon_k}$ satisfies $g \in C^{1,\alpha}$ and is infinitely many times differentiable in the direction of the Legendrian foliation of $z_3 = g(z_1, z_2)$, in a neighborhood of the point p_0.*

Proof. The implicit function theorem implies that the level set of

$$z_3 - g_\varepsilon(z_1, z_2)$$

can be written as smooth intrinsic graphs $x_3 = u_\varepsilon(x_1, x_2)$ in a neighborhood Ω of $(p_0^1, p_0^2, g(p_0))$. The Lipschitz bounds on g_ε (proved in [11, Propositions 4.2-4]) yield uniform Lipschitz bounds on u_ε, thus allowing to apply Theorem 1.3 and conclude the proof. □

From this result one may conclude immediately the following

Corollary 1.8. *Let $z_3 = g(z_1, z_2)$, $(z_1, z_2) \in \Omega \subset \mathbb{R}^2$ be a C^1 minimal graph which is the C^1 limit of Riemannian minimal graphs as in* [11, Theorem 4.5]. *In the neighborhood of any non-characteristic point $g \in C^{1,\alpha}$ and is infinitely many times differentiable along the Legendrian foliation.*

Theorem 1.3 can be also used to rule out minimal intrinsic graphs which do not arise as limits of Riemannian minimal graphs. For instance, the example exhibited above $x_3 = u(x_1, x_2) = \frac{x_2}{x_1 - sgn\,(x_2)}$ defined in $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 1\}$ cannot be a vanishing viscosity minimal graph as it lacks the $C^{1,\alpha}$ regularity from Theorem 1.3.

Sketch of the proof and final remarks. We now turn to a description of the techniques used in the proof of Theorem 1.3. Since u is a vanishing viscosity solution, this theorem is proved by means of a priori estimates (uniform in the parameter ε as it decreases to zero) for each element of the approximating sequence u_j of solutions of (1.3).

The PDE (1.2), has a structure similar to the Levi equation in $\mathbb{R}^3$. In fact, the Levi equation can be represented as Riemannian approximation of a sum of squares of vector fields

$$Z_u^2 + W_u u^2 + \varepsilon^2 \partial_2^2 u = 0,$$

for suitable non linear vector fields Z_u, W_u depending on the solution u. Regularity results for Lipschitz continuous viscosity solutions were established in [13] and [16]. The techniques used in these papers are based on a modification of the Moser iteration, along with representation formula and uniform estimates on the fundamental solution. In [16], [19] the authors address properties of the analogue of the Legendrian foliation for a Levi flat graph.

The cited work on the Levi equations provides a coarse outline and a strategy for the proof of Theorem 1.3. However, the equation (1.2) presents additional difficulties (lack of a background sub-Laplacian, worse nonlinearity) with respect to the Levi equation, and the adaptation of the known techniques is very non trivial and requires new ideas.

The first step in the proof involves the study of a *linearization* (of sorts) of (1.3) (for simplicity we will refer to the approximating functions u_j simply as u)

$$M_{\varepsilon,u} z = \sum_{i=1}^{2} X_{i,u}\left(\frac{X_{i,u} z}{\sqrt{1+|\nabla_\varepsilon u|^2}}\right) = 0, \text{ in } \Omega \subset \mathbb{R}^2. \tag{1.6}$$

There are two main difficulties in establishing a-priori estimates, uniform in ε, for this PDE:

- The first problem is due to the fact that the coefficients of the equation involve the function u which although smooth satisfies a-priori bounds which are uniform in ε only for the Lipschitz norm. To deal with this obstacle we operate a *freezing* argument, substituting the function u with an analogue of its first order Taylor polynomial, and then carefully study the remainder terms. The regularity of rough coefficients degenerate elliptic PDE has been studied by many authors, see for instance the monograph [39] for a survey of the literature and new, ground-breaking techniques.
- The second difficulty stems from the fact that, although (1.6) is elliptic, its coerciveness degenerates as $\varepsilon \to 0$. Now, the approximation of degenerate elliptic operators with elliptic *regularization* is a well known and widely used trick. For instance in [30], the sub-Laplacian $\mathcal{L}u = \sum_{i=1}^m X_i^2 u$ associated to a system of Hörmander vector fields is approximated by $\mathcal{L}_\varepsilon u = \mathcal{L}u + \varepsilon \Delta u$. While it is true that the ellipticity of $\mathcal{L}_\varepsilon$ degenerates as $\varepsilon \to 0$, and hence the constants involved in elliptic estimates degenerate as well, the operator $\mathcal{L}_\varepsilon$ satisfies *sub-elliptic* estimates, which are uniform in ε. In our case however, the left-hand side of equation (1.6) approximates not a sub-Laplacian but the operator $X_{1,u}(X_{1,u} z/\sqrt{1+(X_{1,u}u)^2})$ which is not sub-elliptic. To solve this problem, and obtain the regularity in L^p of the derivatives of z, we introduce a completely new ad-hoc lifting process, inspired in part by Rothschild and Stein's techniques in [38]. The vector fields $X_{1,u}, X_{2,u}$ are lifted to a three-dimensional space $\Omega \times (-1,1)$ by adding a new variable s and horizontal vector field ∂_s. The lifted vectors are $\widetilde{X}_1 = \partial_{x_1} + (u(x) + s^2)\partial_{x_2}$, $\widetilde{X}_2 = \varepsilon \partial_{x_2}$, and $\widetilde{X}_3 = \partial_s$ and the set $\{\widetilde{X}_1, \widetilde{X}_3\}$ form a step-three bracket generating system whose commutators yield the direction of degeneracy ∂_{x_2} of (1.3). At this point we operate a freezing argument and approximate the operator $M_{\varepsilon,u}$ with higher dimensional Hörmander type sub-Laplacians (namely (3.16) and (3.17), built from the frozen, lifted vector fields. In this way we extablish a priori $W^{2,p}$ estimates uniform in ε (Theorem 3.1), which will be the starting point of the regularity proof.

In the proof of Theorem 1.3 we will switch back and forth from representations of (1.3) (and its differentiated versions) in divergence and in non-divergence form. The former works best to deal with higher regularity, via Caccioppoli estimates. The second is tailor-made for the freezing technique and the $W^{2,p}$ estimates.

An important ingredient in the proof is the recent result in [15], establishing uniform estimates on the fundamental solutions of Riemannian regularizations of sub-Laplacians (see the statement in Theorem 3.3).

For other aspects of minimal surfaces in the Heisenberg group, including classification and Bernstein-type results, see [8, 27, 2, 22, 21, 37, 36, 35]. These works also contain more comprehensive lists of references.

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2. Preliminaries

In this section we will always assume that u and f are fixed smooth functions defined on an open set Ω of $\mathbb{R}^2$,and that u is a solution of the PDE $L_\varepsilon u = f$ in Ω. In particular we remark that

$$||u||_{L^\infty(\Omega)} + ||\nabla_\varepsilon u||_{L^\infty(\Omega)} + ||\partial_2 u||_{L^\infty(\Omega)} < \infty, \tag{2.1}$$

and we set

$$M = ||u||_{L^\infty(\Omega)} + ||\nabla_\varepsilon u||_{L^\infty(\Omega)} + ||\partial_2 u||_{L^\infty(\Omega)} \tag{2.2}$$

where for any function ϕ defined on Ω we have let $\nabla_\varepsilon \phi = (X_{1,u}\phi, X_{2,u}\phi)$.

We will use the notation $W^{1,p}_\varepsilon(\Omega)$, $p > 1$ to denote the Sobolev space corresponding to the norm $||\phi||_{W^{1,p}_\varepsilon(\Omega)} = ||\phi||_{L^p(\Omega)} + ||\nabla_\varepsilon\phi||_{L^p(\Omega)}$. For simplicity, unless we want to stress the dependence on u, we will simply write X_1, X_2 instead of $X_{1,u}, X_{2,u}$. We will denote by $W^{k,p}_0(\Omega)$ the space of $L^p(\Omega)$ functions ϕ such that $X_1^l\phi \in L^p(\Omega)$ for all $1 \le l \le k$.

We recall that, under assumption (2.1), the following result holds, (see [5])

Proposition 2.1. *Let u be a solution of equation (1.3) satisfying (2.1). For every compact set $K \subset\subset \Omega$ then there exist a real number α and a constant C, only dependent on the bounds on the constant M in (2.2) and on the choice of the compact set K such that*

$$||u||_{W^{2,2}_\varepsilon(K)} + ||\partial_2 u||_{W^{1,2}_\varepsilon(K)} + ||u||_{C^{1,\alpha}_u(K)} \le C.$$

2.1. An interpolation inequality.

Proposition 2.2. *For every $p \ge 3$, there exists a constant C, dependent on p, and the constant M in (2.2) such that for every function $z \in C^\infty(\Omega)$ and for every $\phi \in C^\infty_0(\Omega)$, and every $\delta > 0$*

$$\int |X_i z|^{p+1/2}\phi^{2p} \le \frac{C}{\delta}\int \Big(|z|^{4p+2}\phi^{2p} + |z|^{(2p+1)/2}|X_i\phi|^{(2p+1)/2} + |z|^{(2p+1)/2}\phi^{2p}\Big)$$

$$+\delta\int |\nabla_\varepsilon(|X_i z|^{(p-1)/2})|^2\phi^{2p},$$

where i can be either 1 or 2.

Proof. This is a slight variant of [13, Prop. 4.2]. We have

$$\int |X_i z|^{p+1/2}\phi^{2p} = \int X_i z|X_i z|^{p-1/2}\mathrm{sign}(X_i z)\phi^{2p} =$$

(integrating by parts and using the fact that $X_1^* = -X_1 - \partial_2 u$ and $X_2^* = -X_2$)

$$= -\int \delta_{i1}\partial_2 u\, z|X_i z|^{p-1/2}\mathrm{sign}(X_i z)\phi^{2p} - (p-1/2)\int z X_i^2 z|X_i z|^{p-3/2}\phi^{2p}$$

$$- 2p\int z|X_i z|^{p-1/2}\mathrm{sign}(X_i z)\phi^{2p-1}X_i\phi \le \tag{2.3}$$

(where δ_{ij} denotes the Kroeneker's delta, by Hölder inequality)

$$\le \frac{C}{\delta}\int z^{(2p+1)/2}(\phi^{2p} + |X_i\phi|^{(2p+1)/2}) + \frac{C}{\delta}\int z^{4p+2}\phi^{2p} +$$

$$+\delta\int |X_i z|^{p+1/2}\phi^{2p} + \delta\int |\nabla_\varepsilon(|X_i z|^{(p-1)/2})|^2\phi^{2p},$$

choosing $\delta > 0$ sufficiently small we conclude the proof. $\square$

A slight modification of the previous proposition, is the following:

Proposition 2.3. *For every $p \ge 3$, for every function $z \in C^\infty(\Omega)$ there exists a constant C, dependent on p, the constant M in (2.2) such that and for every $\phi \in C_0^\infty(\Omega)$, and every $\delta > 0$*

$$\int |X_i z|^{p+1}\phi^{2p} \le$$

$$\le C\left(\int\left(z^{p+1}\phi^{2p} + z^2|X_i z|^{p-1}\phi^{2p-2}|X_i\phi|^2\right) + \int |X_i^2 z|^2|X_i z|^{p-3}|z|^2\phi^{2p}\right),$$

where i can be either 1 or 2.

Proof. We have

$$\int |X_i z|^{p+1}\phi^{2p} = \int X_i z|X_i z|^p\mathrm{sign}(X_i z)\phi^{2p} =$$

(integrating by parts and using the fact that $X_1^* = -X_1 - \partial_2 u$ and $X_2^* = -X_2$)

$$= -\int \delta_{i1}\partial_2 u\, z\,|X_i z|^p\mathrm{sign}(X_i z)\phi^{2p} - p\int z X_i^2 z|X_i z|^{p-1}\phi^{2p}$$

$$- 2p\int z|X_i z|^p\mathrm{sign}(X_i z)\phi^{2p-1}X_i\phi \le \tag{2.4}$$

(by Hölder inequality)

$$\le \frac{C}{\delta}\int\left(z^{p+1}\phi^{2p} + z^2|Xz|^{p-1}\phi^{2p-2}|X_i\phi|^2\right) + \delta\int |X_i z|^{p+1}\phi^{2p} + \frac{C}{\delta}\int |z|^2|X_i^2 z|^2|X_i z|^{p-3}\phi^{2p},$$

choosing δ sufficiently small we obtain the desired inequality. $\square$

2.2. **The horizontal mean curvature as a divergence form operator.** We now prove that if u is a smooth solution of equation (1.3) then its derivatives $\partial_2 u$ and $X_k u$ are solution of a similar mean curvature equation with different right hand side (see also [5, Lemma 3.1]). Differentiating the equation $L_\varepsilon u = 0$ with respect to X_k one obtains

Lemma 2.4. *If u is a smooth solution of $L_\varepsilon u = 0$ then $z = X_k u$ with $k \leq 2$ is a solution of the equation*

$$X_i\left(\frac{a_{ij}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}} X_j z\right) = -[X_k, X_i]\left(\frac{X_i u}{\sqrt{1+|\nabla_\varepsilon u|^2}}\right) - X_i\left(\frac{a_{ij}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}}[X_k, X_j]u\right), \tag{2.5}$$

where a_{ij} are defined as

$$a_{ij} : R^2 \to R \quad a_{ij}(p) = \delta_{ij} - \frac{p_i p_j}{1+|p|^2}. \tag{2.6}$$

Lemma 2.5. *If u is a smooth solution of $L_\varepsilon u = 0$ then $v = \partial_2 u$ is a solution of the equation*

$$X_i\Big(\frac{a_{ij}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}} X_j v\Big) = -\frac{a_{11}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}} v^3 - \frac{a_{1j}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}} v X_j v - X_i\Big(\frac{a_{i1}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}} v^2\Big) \tag{2.7}$$

where a_{ij} are defined in (2.6).

Proof. Differentiating the PDE we obtain

$$\begin{aligned}
X_i\Big(\frac{a_{ij}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}} X_j v\Big) &= -[\partial_2, X_i]\Big(\frac{X_i u}{\sqrt{1+|\nabla_\varepsilon u|^2}}\Big) - X_i\Big(\frac{a_{ij}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}}[\partial_2, X_j]u\Big) \\
&= -[\partial_2, X_1]\Big(\frac{X_1 u}{\sqrt{1+|\nabla_\varepsilon u|^2}}\Big) - X_i\Big(\frac{a_{i1}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}}[\partial_2, X_1]u\Big) \\
&= -v\partial_2\Big(\frac{X_1 u}{\sqrt{1+|\nabla_\varepsilon u|^2}}\Big) - X_i\Big(\frac{a_{i1}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}} v^2\Big).
\end{aligned}$$

□

Let us consider the linear equation satisfied by the components of the gradient of u:

$$M_\varepsilon z = X_i\Big(\frac{a_{ij}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}} X_j z\Big).$$

If z is a smooth solution of equation

$$M_\varepsilon z = f, \tag{2.8}$$

then its intrinsic derivatives $X_i z$ are still solutions of the the same equation with a different right-hand side.

Lemma 2.6. *If z is a smooth solution of (2.8) then $s_1 = X_1 z$ is a solution of the equation*

$$M_\varepsilon s_1 = X_1 f + X_i\left(\frac{a_{i2}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}}\partial_2 u X_2 z\right) - \tag{2.9}$$
$$-X_i\left(X_1\Big(\frac{a_{ij}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}}\Big)X_j z\right) + \partial_2 u X_2\left(\frac{a_{2j}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}}X_j z\right).$$

An analogous computation ensures that

Lemma 2.7. *If z is a solution of (2.8) then $s_2 = X_2 z$ is a solution of the equation*

$$M_\varepsilon s_2 = X_2 f - X_i\left(\frac{a_{i1}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}}\partial_2 u X_2 z\right) - \tag{2.10}$$
$$-X_i\left(X_2\Big(\frac{a_{ij}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}}\Big)X_j z\right) - \partial_2 u X_2\left(\frac{a_{1j}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}}X_j z\right).$$

3. The horizontal mean curvature as a non-divergence form operator: $W^{2,p}_{\mathrm{LOC}}$ a priori estimates.

The operator L_ε defined in (1.3) can be represented in non-divergence form

$$N_\varepsilon u = \sum_{i,j=1}^{2} a_{ij}(\nabla_\varepsilon u) X_i X_j u, \tag{3.1}$$

where a_{ij} are defined in (2.6).

Following the approach in the papers [13, 16] we *linearize* the operator N_ε in the following way: While the coefficients of the vector fields X_i depend on a fixed function u, they will be applied to an arbitrary function z, sufficiently regular. The associated linear non divergence form operator is

$$N_{\varepsilon,u} z = \sum_{i,j=1}^{2} a_{ij}(\nabla_\varepsilon u) X_{i,u} X_{j,u} z, \tag{3.2}$$

where the coefficients a_{ij} are defined in (2.6).

The main result of this section is the following

Theorem 3.1. *Let us assume that z is a classical solution of $N_{\varepsilon,u} z = 0$.*

(i) Let us assume that $\alpha \in]0,1[$, $p > 10/3$ and for every $K \subset\subset \Omega$ there exists a constant C such that

$$||u||_{C^{1,\alpha}(K)} + ||\partial_2 z||_{L^p(K)} + ||\partial_2 X_u z||_{L^2(K)} + ||\nabla^2_\varepsilon z||_{L^2(K)} \le C.$$

Then for any compact set $K_1 \subset\subset K$, there exists a constant C_1 only dependent on K, C, and on the constant in 2.2 such that

$$||z||_{W^{2,10/3}_\varepsilon(K_1)} \le C_1.$$

(ii) If, in addition to the previous conditions, there exists a constant $\widetilde{C}$ such that

$$||\partial_2 X_u z||_{L^4(K)} \le \widetilde{C},$$

with $\alpha \ge 1/4$, then for every $p > 1$ there exists a constant C_1 only dependent on C and $\widetilde{C}$ and p such that

$$||z||_{W^{2,p}_\varepsilon(K_1)} \le C_1.$$

3.1. **Lifting and freezing.** The operator N_ε is an elliptic (Riemannian) approximation of the sub-Riemannian mean curvature operator in the right-hand side of (1.2). Its linearization $N_{\varepsilon,u}$ can be interpreted an uniformly elliptic operator, with least eigenvalue depending on ε. It is well known that this approximating operator has a fundamental solution, but its estimates strongly depend on ε. In order to obtain estimates uniform in ε we further approximate it with an Hörmander type operator, a sum of squares of vector fields, which has a similar behavior in the direction X_1, but for which the direction ∂_2 is the direction of one of the commutators (a step-three commutator!). The idea is to use a new version of the famous Rothschild and Stein lifting theorem, only partially inspired to the procedure in [38]. In order to deal with the non-smoothness of u, we will also operate a freezing: roughly speaking we approximate the coefficients of the vector field X_1 with their first order Taylor polynomials.

Lifting. The first step is to lift the vector fields to a higher dimensional space through the introduction of a new variable s. The points of the extension space will be denoted $(x, s) \in \Omega \times (-1, 1) \subset R^3$, with $x = (x_1, x_2)$. The lifted vector fields are defined as follows

$$\widetilde{X}_1 = \partial_{x_1} + (u(x) + s^2)\partial_{x_2}, \ \widetilde{X}_2 = \varepsilon\partial_{x_2}, \text{ and } \widetilde{X}_3 = \partial_s. \tag{3.3}$$

The $C^{1,\alpha}$ distribution $\{\widetilde{X}_1, \widetilde{X}_3\}$ is a step 3, bracket generating distribution since

$$[\widetilde{X}_1, \widetilde{X}_3] = -2s\partial_{x_2} \text{ and } [\widetilde{X}_3, [\widetilde{X}_1, \widetilde{X}_3]] = -2\partial_{x_2}. \tag{3.4}$$

The associated homogeneous dimension is $Q = 5$. If $x_0 = ((x_0)_1, (x_0)_2) \in \Omega$ is a fixed point, then for all $x \in \Omega$, $s \in \mathbb{R}$ one can define exponential coordinates $(\widetilde{e}_1(x, s), \widetilde{e}_2(x, s), \widetilde{e}_3(x, s))$, based at $(x_0, 0)$, via the formula

$$(x, s) = \exp_{(x_0,0)}(\widetilde{e}_1(x, s)\widetilde{X}_1 + \widetilde{e}_2(x, s)\widetilde{X}_2 + \widetilde{e}_3(x, s)\widetilde{X}_3). \tag{3.5}$$

Here for any Lipschitz vector field Z in $\mathbb{R}^3$ we denote by $\exp_{(x_0,0)}(Z)$ the point $\gamma(1)$ where γ is a curve such that $\gamma(0) = (x_0, 0)$ and $\gamma'(s) = Z(\gamma(s))$. The exponential coordinates can be explicitly computed yielding

$$\widetilde{e}_1(x, s) = (x - x_0)_1, \ \varepsilon\widetilde{e}_2(x, s) = (x - x_0)_2 - (x - x_0)_1\Big(\int_0^1 u(\gamma(\tau))d\tau - \frac{s^2}{3}\Big), \text{ and } \ \widetilde{e}_3(x, s) = s.$$

Note that if $d_E(x, x_0) = \sqrt{(x - x_0)_1^2 + (x - x_0)_2^2}$ is the Euclidean distance in Ω then for x sufficiently close to x_0 and for a certain constant $C > 0$ (both depending on the $C^{1,\alpha}$ norm of u) one has that

$$C^{-1}d_E(x, x_0) \le \sqrt{\widetilde{e}_1(x, 0)^2 + \varepsilon^2\widetilde{e}_2(x, 0)^2} \le Cd_E(x, x_0).$$

Next, we define an analogue of the first order Taylor polynomial of u as

$$P^1_{x_0}u\ (x) = u(x_0) + e_1(x)\widetilde{X}_1u(x_0, 0) + e_2(x)\widetilde{X}_2u(x_0, 0), \tag{3.6}$$

where $e_1(x) = \widetilde{e}_1(x, 0)$ and

$$\varepsilon e_2(x) = (x - x_0)_2 - (x - x_0)_1u(x_0). \tag{3.7}$$

We remark explicitly that

$$\big|\varepsilon(\widetilde{e}_2(x, 0) - e_2(x))\widetilde{X}_2u(x_0, 0)\big| = \Big|\Big((x - x_0)_1\Big(\int_0^1 [u(\gamma(\tau)) - u(x_0)] \ d\tau\Big)\partial_2u(x_0)\Big| \tag{3.8}$$

Freezing. At this point we introduce an appropriate *freezing* of the vector fields by defining

$$X_{1,x_0} = \partial_{x_1} + (P^1_{x_0}u(x) + s^2)\partial_{x_2},\ X_{2,x_0} = \varepsilon\partial_{x_2} \text{ and } X_{3,x_0} = \partial_s. \tag{3.9}$$

Observe that $\{X_{1,x_0}, X_{3,x_0}\}$ is a distribution of smooth vector fields satisfying Hörmander's finite rank hypothesis with step three. We denote by $d_{x_0}(\cdot,\cdot)$ the corresponding Carnot-Caratheodory distance and remark that the homogeneous dimension of the space is 5. We also need the Riemannian distance function $d_{x_0,\varepsilon}(\cdot,\cdot)$ defined as the control distance associated to $\{X_{1,x_0}, X_{2,x_0}, X_{3,x_0}\}$. Define exponential coordinates $(\bar{e}_1, \bar{e}_2, \bar{e}_3)$ in a neighborhood of x_0 through the formula

$$(x,s) = \exp_{(x_0,0)}(\bar{e}_1(x,s)X_{1,x_0} + \bar{e}_2(x,s)X_{2,x_0} + \bar{e}_3(x,s)X_{3,x_0}).$$

Note that

$$\bar{e}_1(x,0) = (x-x_0)_1,\ \bar{e}_2(x,s) = \frac{1}{\varepsilon}\bigg((x-x_0)_2 - (x-x_0)_1\Big(\int_0^1 P^1_{x_0}u(\gamma(\tau))d\tau - \frac{s^2}{3}\Big)\bigg) \text{ and } \bar{e}_3(x,0) = 0.$$

It is well known (see for instance the discussion in [7, Section 2.4] that $(\mathbb{R}^3, d_{x_0,\varepsilon})$ converges in the Gromov-Hausdorff sense to $(\mathbb{R}^3, d_{x_0})$. In particular one has that for each fixed x and s, then $d_{x_0,\varepsilon}((x,s),(x_0,0)) \to d_{x_0}((x,s),(x_0,0))$ as $\varepsilon \to 0$. Moreover the volume of the balls $B_\varepsilon((x_0,0),R)$ in the $d_{x_0,\varepsilon}$ metric converges to the volume of the limit Carnot-Caratheodory balls, i.e. $|B_\varepsilon((x_0,0),R)| \to |B_0((x_0,0),R)|$ as $\varepsilon \to 0$. In particular, for $\varepsilon_0 > \varepsilon > 0$ sufficiently small there exists a constant $C > 0$ depending only on ε_0 such that

$$|B_\varepsilon((x_0,0),R)| \geq CR^5. \tag{3.10}$$

All this can also be seen explicitly in our special setting by observing that there exists a constant $C > 0$ such that for x near x_0, one has (see [32])

$$\begin{aligned} C^{-1}d_{x_0,\varepsilon}((x,s),(x_0,0)) &\leq \sqrt{\bar{e}_1^2(x,s) + \min(\bar{e}_2^2(x,s),\ (\varepsilon\bar{e}_2(x,s))^{2/3}\ ) + \bar{e}_3^2(x,s)} \leq Cd_{x_0,\varepsilon}((x,s),(x_0,0)). \\ C^{-1}d_{x_0}((x,s),(x_0,0)) &\leq \bigg(\bar{e}_1^6(x,s) + (\varepsilon\bar{e}_2)^2(x,s) + \bar{e}_3^6(x,s)\bigg)^{1/6} \leq Cd_{x_0}((x,s),(x_0,0)). \end{aligned}$$

Recall also that for x and x_0 sufficiently close, $\varepsilon_0 > \varepsilon > 0$ sufficiently small there exist positive constants C_1, C_2 depending only on ε_0 and on the $C^{1,\alpha}$ norm of u such that

$$d_E((x,s),(x_0,0)) \leq C_1 d_{x_0,\varepsilon}((x,s),(x_0,0)) \leq C_2 d_{x_0}((x,s),(x_0,0)). \tag{3.11}$$

Since $|\bar{e}_2(x,s)| \to \infty$ as $\varepsilon \to 0$ one has that for a fixed (x,s) then $\lim_{\varepsilon\to 0} d_{x_0,\varepsilon}((x,s),(x_0,0)) \approx d_{x_0}((x,s),(x_0,0))$.

In the following we will denote by $d_{x_0,\varepsilon}(x,x_0)$ the quantity $d_{x_0,\varepsilon}((x,0),(x_0,0))$.

Lemma 3.2. *If $u \in C^{1,\alpha}_E(\Omega)$ there exists constant $\varepsilon_0, C > 0$ and a neighborhood U of x_0 depending only on the $C^{1,\alpha}_E$ norm of u such that for all x in a sufficiently small neighborhood of x_0, and for all $\varepsilon_0 > \varepsilon > 0$,*

$$|u(x) - P^1_{x_0}u(x)| \leq Cd^{1+\alpha}_{x_0,\varepsilon}(x,x_0). \tag{3.12}$$

Proof. Fix the points x and x_0 and define the $C^{1,1}$ planar curve

$$\gamma(t) = \exp_{x_0}\bigg(t\big(\bar{e}_1(x,0)X_{1,x_0} + \bar{e}_2(x,0)X_{2,x_0}\big)\bigg),$$

so that $\gamma(1) = x$ and $\gamma(0) = x_0$. From the mean-value theorem, for all $t \in (0,1)$ we can find $\widetilde{t} \in (0,t)$ such that

$$u(\gamma(t)) - u(x_0) = \bar{e}_1(x,0)X_{1,x_0}(\gamma(\widetilde{t})) + \bar{e}_2(x,0)X_{2,x_0}(\gamma(\widetilde{t})).$$

Hence for all $t \in (0,1)$ one has

$$\begin{aligned}
u(\gamma(t)) - P^1_{x_0}u(\gamma(t)) &= u(\gamma(t)) - u(x_0) = e_1(\gamma(t))\widetilde{X}_1 u(x_0,0) - e_2(\gamma(t))\widetilde{X}_2 u(x_0,0) \\
&= \sum_{i=1}^{2}\Big(\bar{e}_i(x,0)X_{i,x_0}u(\gamma(\widetilde{t})) - e_i(\gamma(t))\widetilde{X}_i u(x_0,0)\Big) \\
&= (x-x_0)_1(u(x) - P^1_{x_0}u(x))\partial_{x_2}u(x) + \bar{e}_2(x,0)\big(X_{2,x_0}u(\gamma(\widetilde{t})) - X_{2,x_0}u(x_0,0)\big) \\
&\quad + \bar{e}_2(x,0)\big(X_{2,x_0}u(x_0,0) - \widetilde{X}_{2,x_0}u(x_0,0)\big) + \big(\bar{e}_2(x,0) - e_2(\gamma(t))\big)\widetilde{X}_2 u(x_0,0). \qquad (3.13)
\end{aligned}$$

Next, note that

$$\begin{aligned}
|\bar{e}_2(x,0) - e_2(\gamma(t))| = \quad & \frac{1}{\varepsilon}\Big| - (x-x_0)_1\int_0^1 P^1_{x_0}u(\gamma(\tau))d\tau - \frac{s^2}{3} + (\gamma(t)-x_0)_1 u(x_0)\Big| \\
&\leq \frac{1}{\varepsilon}\Big|(x-x_0)_1\int_0^1 (P^1_{x_0}u(\gamma(\tau)) - u(\gamma(\tau))d\tau + \frac{s^2}{3}\Big| \\
&\quad + \frac{1}{\varepsilon}\Big|\int_0^1 \big((\gamma(t)-x_0)_1 u(x_0) - (x-x_0)_1 u(\gamma(\tau))\big)d\tau\Big| \\
&\leq \frac{C}{\varepsilon}d_E^{1+\alpha}(x,x_0) + \frac{1}{\varepsilon}\Big|(x-x_0)_1\int_0^1 (P^1_{x_0}u(\gamma(\tau)) - u(\gamma(\tau))\ )d\tau\Big|. \qquad (3.14)
\end{aligned}$$

From the latter, from (3.13) and from observing that $X_{2,x_0} = \widetilde{X}_{2,x_0}$, and

$$|\bar{e}_2(x,0)\big(X_{2,x_0}u(\gamma(\widetilde{t})) - X_{2,x_0}u(x_0,0)\big)| \leq C d_E^{\alpha}(x,x_0)|\varepsilon\bar{e}_2(x,0)|$$

we obtain

$$\begin{aligned}
(3.15)\quad & |u(\gamma(t)) - P^1_{x_0}u(\gamma(t))| \\
&\leq C(d_E^{1+\alpha}(x,x_0) + |\varepsilon\bar{e}_2(x,0)|d_E^{\alpha}(x,x_0) + |(x-x_0)_1|\int_0^1 |P^1_{x_0}u(\gamma(\tau)) - u(\gamma(\tau))|d\tau.
\end{aligned}$$

For x sufficiently close to x_0 we have

$$|\varepsilon\bar{e}(x,0)| \leq C d_{x_0,\varepsilon}(x,x_0) \text{ and } |(x-x_0)|_1 \leq \frac{1}{2}.$$

Integrating (3.15) in the t variable from 0 to 1 and using the latter we obtain

$$\int_0^1 |P^1_{x_0}u(\gamma(\tau)) - u(\gamma(\tau))|d\tau \leq C d^{1+\alpha}_{x_0,\varepsilon}(x,x_0).$$

Substituting this estimate back in (3.15) we conclude the proof. □

The frozen operators. The freezing process described earlier allows to introduce *'frozen'* sub-Laplacians operators N_{ε,x_0} formally defined as N_ε, but in terms of the smooth vector fields X^{ε}_{i,x_0} instead of the original non-smooth vector fields X_i. Consider the operators

$$N_{\varepsilon,x_0}z = \sum_{i,j=1}^{3} a_{ij}(\nabla_\varepsilon u(x_0))X_{i,x_0}X_{j,x_0}z, \qquad (3.16)$$

where a_{ij} are defined in (2.6), and

$$N_{x_0} z = \sum_{\substack{i,j=1\\ i\neq 2, j\neq 2}}^{3} a_{ij}(\nabla_\varepsilon u(x_0)) X_{i,x_0} X_{j,x_0} z, \tag{3.17}$$

N_{x_0} is an uniformly subelliptic operator with C^∞ coefficients, and N_{ε,x_0} can be considered as its elliptic regularization, with coefficients dependent on ε. The linear theory yields that both N_{x_0} and N_{ε,x_0} have fundamental solutions Γ_{x_0} and Γ_{ε,x_0} respectively (see [28], [32] and [4]). Since both Γ_{x_0} and $\Gamma_{\varepsilon,x_0}(\zeta,\xi)$ depend on many variables, the notation

$$X_{i,x_0}(\zeta_1)\Gamma_{\varepsilon,x_0}(\,\cdot\,,\xi)$$

shall denote the X_{i,x_0}-derivative of $\Gamma_{\varepsilon,x_0}(\zeta,\xi)$ with respect to the variable ζ, evaluated at the point ζ_1.

Precise estimates for the fundamental solution Γ_{x_0} have been established in [32] and [4], while in [15] it is proved that the fundamental solution Γ_{ε,x_0} of N_{ε,x_0} locally satisfies the same estimates as the limit kernel Γ_{x_0}, with *choice of constants independent of* ε. These results can be summarized as follows;

Theorem 3.3. *([15]) Let $z_0 \in \Omega$. For every compact set $K \subset \Omega \times (-1,1)$, for every $k \in \mathbb{N}$ and for every multi-index $I = (i_1, ..., i_k)$ with $i_j \in \{1,2,3\}$, there exist two positive constants C, C_p independent of ε, such that*

$$|\nabla^I_{\varepsilon,x_0}(\xi)\Gamma_{\varepsilon,x_0}(\cdot,\zeta)| \leq C_k \frac{d^{2-k}_{x_0,\varepsilon}(\xi,\zeta)}{|B_\varepsilon(\xi, d_{x_0,\varepsilon}(\xi,\zeta))|}, \tag{3.18}$$

for every $\xi, \zeta \in K$ with $\xi \neq \zeta$, where $B_\varepsilon(\xi, r)$ denotes the ball with center ξ and radius r in the distance $d_{x_0,\varepsilon}$, and $\nabla^I_{\varepsilon,x_0} = X_{i_1,x_0}...X_{i_n,x_0}$ denotes derivatives of order $|I| = k$ along the frozen vector fields X_{i,x_0}. If $k = 0$ one intends that no derivative are applied on Γ_{ε,x_0}.

Remark 3.4. *Let u be as in Lemma 3.2. Notice that if we set*

$$\mathcal{R} = X_{1,u} - X_{1,x_0} = (u(x) - P^1_{x_0}u(x) - s^2)\partial_{x_2},$$

then since ∂_{x_2} has order three, and in view of Lemma 3.2, the operator $\mathcal{R}$ has order $2-\alpha$ at the point x_0 (in the sense of [23]). Consequently, the estimates in (3.18) do not continue to hold if the derivatives along the frozen vector fields $\nabla^I_{\varepsilon x_0}$, $|I| = 2$, are substituted by derivatives along $X^\varepsilon_{i,u}$ evaluated at the point $\xi = x_0$.

Since the measure of the Ball is doubling, with doubling constant independent of ε, then the space $(\Omega\times(-1,1), d_{x_0,\varepsilon}, dx)$ is a space of homogenous type and the the following version of the fractional integration theorem holds, (see for instance [6])

Proposition 3.5. *If K is a fixed compact set $K \subset\subset \Omega \times (-1,1)$, if $f \in L^q(\Omega \times (-1,1))$ supported in K, and for each $(x,s) \in \Omega \times (-1,1)$ we set*

$$I_p(f)(x,s) = \int \frac{d^p_{x_0,\varepsilon}((x,s),(\zeta,\sigma))}{|B_\varepsilon((x,s), d_{x_0,\varepsilon}((x,s),(\zeta,\sigma))|} f(\zeta,\sigma)d\zeta d\sigma \tag{3.19}$$

then there exists a constant C, depending on K, r, q but independent of ε such that,

$$||I_p(f)||_{L^r(\Omega\times(-1,1))} \leq C||f||_{L^q(\Omega\times(-1,1))},$$

and where $5 - pq > 0$ and $r = \frac{5q}{5-pq}$.

Corollary 3.6. *Let $f \in C^\infty_0(\Omega)$ and extend it to a function on $\Omega \times (-1,1)$ by setting $f(x,s) := f(x)$. Let $\mathcal{K} : (\Omega \times (-1,1))^2 \to \mathbb{R}$ be a kernel satisfying*

$$|\mathcal{K}((x,s),(\zeta,\sigma))| \leq C \frac{d^p_{x_0,\varepsilon}((x,s),(\zeta,\sigma))}{|B_\varepsilon((x,s), d_{x_0,\varepsilon}((x,s),(\zeta,\sigma))|}.$$

Set $u(x,s)$ to be defined as

$$u(x,s) = \int \mathcal{K}((x,s),(\zeta,\sigma)) f(\xi) d\zeta d\sigma.$$

If $u(x,s) = u(x)$ for all values of s then

$$||u||_{L^r(\Omega)} \leq C||f||_{L^q(\Omega)},$$

with r and q as in the previous proposition.

As a consequence of the definition of fundamental solution one has the following representation formula:

Proposition 3.7. *Let u be a fixed smooth function defined in Ω, let $N_{\varepsilon,u}$ be as in (3.2), and let z be a classical solution of $N_{\varepsilon,u}z = g \in C^\infty(\Omega)$. Extend both u and z to be functions defined on $\Omega \times (-1,1)$ by letting them be constant along the s variable. For any $\phi \in C_0^\infty(\Omega \times (-1,1))$, $\xi \in \Omega$ and $s \in (-1,1)$, the product $z(\xi)\phi(\xi,s)$ can be represented as*

$$\begin{aligned}
z(\xi)\phi(\xi,s) = & \int_{\Omega\times(-1,1)} \Gamma((\xi,s),(\zeta,\sigma)) \left(z\, N_{\varepsilon x_0}\phi + \sum_{ij=1}^{2} \bar{a}_{ij}(x_0)\Big(X_{i,z_0} z X_{j,z_0}\phi + X_{j,z_0} z X_{i,z_0}\phi \Big) \right) d\zeta \\
& + \int_{\Omega\times(-1,1)} \Gamma_{\varepsilon,x_0}((\xi,s),(\zeta,\sigma))\; g(\zeta)\, \phi(\zeta,\sigma) d\zeta d\sigma + \\
& + \sum_{ij=1}^{2} \int_{\Omega\times(-1,1)} \Gamma_{\varepsilon,x_0}((\xi,s),(\zeta,\sigma)) \Big(\bar{a}_{ij}(x_0) - \bar{a}_{ij}(\zeta) \Big) X_{i,u} X_{j,u} z(\zeta)\, \phi(\zeta) d\zeta d\sigma \\
& - \sum_{j=1}^{2} \bar{a}_{1j}(x_0) \int_{\Omega\times(-1,1)} \Gamma_{\varepsilon,x_0}((\xi,s),(\zeta,\sigma)) \Big(u(\zeta) - P^1_{x_0} u(\zeta) - \sigma^2 \Big) \partial_2 X_{j,u} z(\zeta)\, \phi(\zeta,\sigma) d\zeta d\sigma \\
& + \sum_{i=1}^{2} \bar{a}_{i1}(x_0) \int_{\Omega\times(-1,1)} X_{i,x_0}(\xi,s) \Gamma_{\varepsilon,x_0}(\cdot,(\zeta,\sigma)) \Big(u(\zeta) - P^1_{x_0} u(\zeta) - \sigma^2 \Big) \partial_2 z(\zeta)\, \phi(\zeta,\sigma) d\zeta d\sigma \\
& + \sum_{i=1}^{2} \bar{a}_{i1}(x_0) \int_{\Omega\times(-1,1)} \Gamma_{\varepsilon,x_0}((\xi,s),(\zeta,\sigma)) \Big(u(\zeta) - P^1_{x_0} u(\zeta) - \sigma^2 \Big) \partial_2 z(\zeta)\, X_{i,x_0}\phi(\zeta,\sigma) d\zeta d\sigma.
\end{aligned} \tag{3.20}$$

In order to simplify notations we have set

$$\bar{a}_{ij}(\zeta) = a_{ij}(\nabla_\varepsilon u(\zeta)).$$

Proof. In view of the definition of fundamental solution, we have

(3.21)
$$\begin{aligned}
z(\xi)\phi(\xi,s) &= \int_{\Omega\times(-1,1)} \Gamma_{\varepsilon,x_0}((\xi,s),(\zeta,\sigma))N_{\varepsilon x_0}(z\phi)(\zeta,\sigma)d\zeta d\sigma \\
&= \int_{\Omega\times(-1,1)} \Gamma_{\varepsilon,x_0}((\xi,s),(\zeta,\sigma))\left(z\,N_{\varepsilon x_0}\phi + \sum_{ij=1}^{2}\bar a_{ij}(x_0)\Big(X_{i,x_0}uX_{j,x_0}\phi + X_{j,x_0}zX_{i,x_0}\phi\Big)\right)d\zeta d\sigma \\
&+ \int_{\Omega\times(-1,1)} \Gamma_{\varepsilon,x_0}((\xi,s),(\zeta,\sigma))N_{\varepsilon u}z(\zeta)\,\phi(\zeta,\sigma)d\zeta d\sigma \\
&+ \int_{\Omega\times(-1,1)} \Gamma_{\varepsilon,x_0}((\xi,s),(\zeta,\sigma))(N_{\varepsilon x_0}-N_{\varepsilon u})z(\zeta,\sigma)\,\phi(\zeta,\sigma)d\zeta d\sigma.
\end{aligned}$$

Let us now compute the difference between the operator $N_{\varepsilon u}$ and its frozen counterpart. We will emphasize the presence of the variable σ in those terms where the coefficients of the vector fields involve that variable.

$$\begin{aligned}
\Big((N_{\varepsilon x_0}-N_{\varepsilon u})z\Big)(\zeta,\sigma) &= \sum_{ij=1}^{2}\Big(\bar a_{ij}(x_0)-\bar a_{ij}(\zeta)\Big)X_{i,u}X_{j,u}z(\zeta)- \\
&- \sum_{ij=1}^{2}\bar a_{ij}(z_0)\Big(X_{i,u}X_{j,u}-X_{i,x_0}X_{j,x_0}\Big)z(\zeta,\sigma) = \\
&= \sum_{ij=1}^{2}\Big(\bar a_{ij}(x_0)-\bar a_{ij}(\zeta)\Big)X_{i,u}X_{j,u}z(\zeta)- \\
&- \sum_{ij=1}^{2}\bar a_{ij}(x_0)\Big((X_{i,u}-X_{i,x_0})X_{j,u}+X_{i,x_0}(X_{j,u}-X_{j,x_0})\Big)z(\zeta,\sigma) \\
&= \sum_{ij=1}^{2}\Big(\bar a_{ij}(x_0)-\bar a_{ij}(\zeta)\Big)X_{i,u}X_{j,u}z(\zeta)- \qquad (3.22)\\
&- \sum_{ij=1}^{2}\bar a_{ij}(x_0)\Big(\delta_{i1}(u(\zeta)-P^1_{z_0}u(\zeta)-\sigma^2)\partial_2X_{j,u}+X_{i,x_0}(\delta_{j,1}(u(\zeta)-P^1_{x_0}u(\zeta)-\sigma^2)\partial_2)\Big)z(\zeta) \\
&= \sum_{ij=1}^{2}\Big(\bar a_{ij}(x_0)-\bar a_{ij}(\zeta)\Big)X_{i,u}X_{j,u}z(\zeta) \\
&- \sum_{i=1}^{2}\bar a_{i1}(x_0)(u(\zeta)-P^1_{x_0}u(\zeta)-\sigma^2)\partial_2X_{i,u}z(\zeta) \\
&- \sum_{i=1}^{2}\bar a^{\varepsilon}_{i1}(x_0)X_{i,x_0}\Big((u(\zeta)-P^1_{x_0}u(\zeta)-\sigma^2)\partial_2\Big)z(\zeta).
\end{aligned}$$

The integral of the last term in (3.22) becomes

$$\sum_{i=1}^{2}\bar a_{i1}(x_0)\int_{\Omega\times(-1,1)}\Gamma_{\varepsilon,x_0}((\xi,s),(\zeta,\sigma))X_{i,x_0}\Big((u(\zeta)-P^1_{x_0}u(\zeta)-\sigma^2)\partial_2\Big)z(\zeta)\phi(\zeta,\sigma)d\zeta d\sigma \qquad (3.23)$$

(integrating by part)

$$
\begin{aligned}
&= -\sum_{i=1}^{2} \bar{a}_{i1}(x_0) \int_{\Omega\times(-1,1)} X_{i,x_0}(\xi,s)\Gamma_{\varepsilon,x_0}(\cdot,(\zeta,\sigma))\,(u(\zeta)-P^1_{x_0}u(\zeta)-\sigma^2)\partial_2 z(\zeta)\,\phi(\zeta,\sigma)d\zeta d\sigma \\
&\quad -\sum_{i=1}^{2} \bar{a}_{i1}(x_0) \int_{\Omega\times(-1,1)} \Gamma_{\varepsilon,x_0}((\xi,s),(\zeta,\sigma))\,(u(\zeta)-P^1_{x_0}u(\zeta)-\sigma^2)\partial_2 z(\zeta)\,X_{i,x_0}\phi(\zeta,\sigma)d\zeta d\sigma
\end{aligned} \tag{3.24}
$$

Inserting all terms in the preceeding formula we conclude the proof. $\square$

Lemma 3.8. *Let u and $x_0 \in \Omega$ be as above. There exists a neighborhood U of $(x_0, 0)$, possibly depending on ε, such that for all $(\xi, s) \in U$ one has*

$$
\begin{aligned}
&\left|\Big(X_{k,x_0}X_{l,x_0}(\xi,s) - X_{k,x_0}X_{l,x_0}(x_0,0)\Big)\nabla_{x_0,\varepsilon}\Gamma_{\varepsilon,x_0}(\cdot,(\zeta,\sigma))\right| \\
&\qquad \leq d_{x_0}((\xi,s),(x_0,0))\left(\left|\nabla^I_{x_0,\varepsilon}\Gamma_{\varepsilon,x_0}((\xi,s),(\zeta,\sigma))\right| + \left|\nabla^I_{x_0,\varepsilon}\Gamma_{\varepsilon,x_0}((x_0,0),(\zeta,\sigma))\right|\right), \text{ with } |I| = 4.
\end{aligned} \tag{3.25}
$$

Proof. The proof follows from the mean value principle: Set $L = d_{x_0}((\xi, s), (x_0, 0))$. For every $\delta > 0$ consider a horizontal curve $\gamma : [0, L + \delta] \to \Omega \times (-1, 1)$, parametrized by arc-length and joining (ξ, s) to $(x_0, 0)$. For every $f \in C^1$ one has

$$
|f(\xi,s) - f(x_0,0)| \leq \int_0^L |\frac{d}{dt} f(\gamma(t))|dt = \int_0^L |\langle \nabla f(\gamma(t)), \gamma'(t)\rangle|dt \leq (L+\delta)\sup_{\gamma} |\nabla_{x_0,\varepsilon} f|.
$$

There exists neighborhood $(x_0, 0) \in U$ depending on the C^1 norm of f for which we have $|\nabla_{x_0,\varepsilon} f(\zeta, \sigma)| \leq 2(|\nabla_{x_0,\varepsilon} f(x_0, 0)| + |\nabla_{x_0,\varepsilon} f(\xi, s)|)$ for all $(\zeta, \sigma) \in U$. The lemma now follows from choosing

$$
f = X_{k,x_0}X_{l,x_0}X_{i,x_0}\Gamma_{\varepsilon,x_0}(\cdot,(\zeta,\sigma))
$$

and observing that the smoothness of Γ_{ε,x_0} depends on $\varepsilon > 0$. $\square$

Lemma 3.9. *Let z be a smooth solution of $N_{\varepsilon,u}z = 0$. For any $s \in (-1,1)$ and $x_0 \in \Omega$, one can represent the 'frozen' second order horizontal derivatives of z at x_0 as*

(3.26)

$$X_{k,x_0}X_{l,x_0}(z(x_0)\phi(x_0,s)) =$$

$$= \int_{\Omega\times(-1,1)} X_{k,x_0}X_{l,x_0}(x_0,s)\Gamma_{\varepsilon,x_0}(\cdot,(\zeta,\sigma)) \left(z\, N_{\varepsilon x_0}\phi + \sum_{ij=1}^{2} \bar{a}_{ij}(x_0)\Big(X_{i,z_0}zX_{j,z_0}\phi + X_{j,z_0}zX_{i,z_0}\phi\Big)\right) d\zeta$$

$$+ \sum_{ij=1}^{2} \int_{\Omega\times(-1,1)} X_{k,x_0}X_{l,x_0}(x_0,s)\Gamma_{\varepsilon,x_0}(\cdot,(\zeta,\sigma))\Big(\bar{a}_{ij}(x_0) - \bar{a}_{ij}(\zeta)\Big)X_{i,u}X_{j,u}z(\zeta)\,\phi(\zeta,\sigma)d\zeta d\sigma$$

$$- \sum_{j=1}^{2} \bar{a}_{1j}(x_0) \int_{\Omega\times(-1,1)} X_{k,x_0}X_{l,x_0}(x_0,s)\Gamma_{\varepsilon,x_0}(\cdot,(\zeta,\sigma))\Big(u(\zeta) - P^1_{x_0}u(\zeta) - \sigma^2\Big)\partial_2 X_{j,u}z(\zeta)\,\phi(\zeta,\sigma)d\zeta d\sigma$$

$$+ \sum_{i=1}^{2} \bar{a}_{i1}(x_0) \int_{\Omega\times(-1,1)} X_{k,x_0}X_{l,x_0}X_{i,x_0}(x_0,s)\Gamma_{\varepsilon x_0}(\cdot,(\zeta,\sigma))\Big(u(\zeta) - P^1_{x_0}u(\zeta) - \sigma^2\Big)\partial_2 z(\zeta)\,\phi(\zeta,\sigma)d\zeta d\sigma$$

$$+ \sum_{i=1}^{2} \bar{a}_{i1}(x_0) \int_{\Omega\times(-1,1)} X_{k,x_0}X_{l,x_0}(x_0,s)\Gamma_{\varepsilon,x_0}(\cdot,(\zeta,\sigma))\Big(u(\zeta) - P^1_{x_0}u(\zeta) - \sigma^2\Big)\partial_2 z(\zeta)\, X_{i,x_0}\phi(\zeta,s)d\zeta d\sigma.$$

Proof. Since the proof is similar to that of [17, Proposition 3.9], we only sketch the argument for the *most singular* term in the representation formula, i.e.

(3.27)

$$I((\xi,s),(x_0,0)) = \sum_{i=1}^{2} \bar{a}_{i1}(x_0) \int_{\Omega\times(-1,1)} X_{i,x_0}(\xi,s)\Gamma_{\varepsilon x_0}(\cdot,(\zeta,\sigma))\Big(u(\zeta) - P^1_{x_0}u(\zeta) - \sigma^2\Big)\partial_2 z(\zeta)\,\phi(\zeta,s)d\zeta d\sigma.$$

We want to show that

$$X_{k,x_0}X_{l,x_0}(x_0,0)I(\cdot,(x_0,0)) = I^{(2)}, \tag{3.28}$$

where

(3.29)

$$I^{(2)}(x_0) = \sum_{i=1}^{2} \bar{a}_{i1}(x_0) \int_{\Omega\times(-1,1)} X_{k,x_0}X_{l,x_0}(x_0,s)X_{i,x_0}\Gamma_{\varepsilon x_0}(\cdot,(\zeta,\sigma))\Big(u(\zeta) - P^1_{x_0}u(\zeta) - \sigma^2\Big)\partial_2 z(\zeta)\,\phi(\zeta,s)d\zeta d\sigma$$

Note that the latter is well defined in view of the estimates (3.12) and (3.18). To show (3.28) we consider a family of smooth test functions $\chi_{x_0,\varepsilon}((\xi,s),(\zeta,\sigma))$, satisfying for some choice of $C > 0$ and for small $\varepsilon > 0$, (i) $0 \le \chi_{x_0,\varepsilon}((\xi,s),(\zeta,\sigma)) \le 1$, (ii) $\chi_{x_0,\varepsilon}(((\xi,s),(\zeta,\sigma)) = 0$ if $d_{x_0,\varepsilon}((\xi,s),(\zeta,\sigma)) \le 2C\varepsilon$, (iii) $\chi_{x_0,\varepsilon}((\xi,s),(\zeta,\sigma)) = 1$ if $d_{x_0,\varepsilon}((\xi,s),(\zeta,\sigma)) \ge 4C\varepsilon$, (iv) $|\nabla^I_{\varepsilon,x_0}\chi_{x_0,\varepsilon}(((\xi,s),(\zeta,\sigma))| \le C\varepsilon^{-|I|}$ for all multi-indeces I, for some choice of $C > 0$ and for small $\varepsilon > 0$. For the existence of such function see [12] and note that the construction argument in that paper uses only the estimates on the fundamental solution. Define the smooth approximation

$$I_\varepsilon((\xi,s),(x_0,0)) = \sum_{i=1}^{2} \bar{a}_{i1}(x_0) \int_{\Omega\times(-1,1)} X_{i,x_0}(\xi,s)\Gamma_{\varepsilon x_0}(\cdot,(\zeta,\sigma)) \cdot \Big(u(\zeta) - P^1_{x_0}u(\zeta) - \sigma^2\Big)\ \chi_{x_0,\varepsilon}((\xi,s),(\zeta,\sigma))\ \partial_2 z(\zeta)\,\phi(\zeta,s)d\zeta d\sigma. \tag{3.30}$$

Once we establish the bounds

$$\Big| I_\varepsilon((\xi,s),(x_0,0)) - I((\xi,s),(x_0,0))\Big| \le C\varepsilon^{2+\alpha}$$

$$\Big| X_{k,x_0}X_{l,x_0}(\xi,s) I_\varepsilon(\cdot,(x_0,0)) - I^{(2)}(x_0)\Big| \le C\varepsilon^{\alpha}, \tag{3.31}$$

for[1] $d_{x_0,\varepsilon}((\xi,s),(x_0,0)\le \varepsilon$, then invoking [17, Proposition 3.2] we immediately conclude (3.28). To prove the first estimate in (3.31) we first observe that for $\varepsilon > 0$ sufficiently small

$$\begin{aligned}(3.32)\quad & \Big| I_\varepsilon((\xi,s),(x_0,0)) - I((\xi,s),(x_0,0))\Big| \\ & \le C \int_{d_{x_0,\varepsilon}((\zeta,\sigma),(\xi,s)\le 4C\varepsilon} \Big(1-\chi_{x_0,\varepsilon}((\xi,s),(\zeta,\sigma))\Big) \frac{d_{x_0,\varepsilon}((\xi,s),(\zeta,\sigma))}{|B_\varepsilon(\xi, d_{x_0,\varepsilon}((\xi,s),(\zeta,\sigma))|}\cdot \\ & \qquad \cdot d_{x_0,\varepsilon}((\zeta,\sigma),(x_0,0))^{1+\alpha} |\partial_2 z(\zeta)\,\phi(\zeta,s)| d\zeta d\sigma.\end{aligned}$$

Since

$$d_{x_0,\varepsilon}((\zeta,\sigma),(x_0,0)) \le C(d_{x_0,\varepsilon}((\zeta,\sigma),(\xi,s)) + d_{x_0,\varepsilon}((x_0,0),(\xi,s)) \le C(d_{x_0,\varepsilon}((\zeta,\sigma),(\xi,s)) + \varepsilon),$$

then we conclude

$$\Big| I_\varepsilon((\xi,s),(x_0,0)) - I((\xi,s),(x_0,0))\Big| \le C \int_{d_{x_0,\varepsilon}((\zeta,\sigma),(\xi,s)\le 4C\varepsilon} \frac{d_{x_0,\varepsilon}((\xi,s),(\zeta,\sigma))^{2+\alpha}}{|B_\varepsilon(\xi, d_{x_0,\varepsilon}((\xi,s),(\zeta,\sigma))|} \le C\varepsilon^{2+\alpha}.$$

Next we turn to the second estimate in (3.31). Observe that

$$\Big| X_{k,x_0}X_{l,x_0}(\xi) I_\varepsilon((\xi,s),(x_0,0)) - I^{(2)}(x_0)\Big| \le |A_1| + |A_2| + |A_3|,$$

where

$$\begin{aligned}(3.33)\quad A_1 = \int\limits_{\Omega\times(-1,1)} \Big| & \Big(X_{k,x_0}X_{l,x_0}(\xi,s) - X_{k,x_0}X_{l,x_0}(x_0,s)\Big) \nabla_{x_0,\varepsilon}\Gamma(\cdot,(\zeta,\sigma))\cdot \\ & \cdot \Big(u(\zeta) - P^1_{x_0}u(\zeta) - \sigma^2\Big)\ \chi_{x_0,\varepsilon}((\xi,s),(\zeta,\sigma))\ \partial_2 z(\zeta)\,\phi(\zeta,\sigma)\Big| d\zeta d\sigma,\end{aligned}$$

$$\begin{aligned}(3.34)\quad A_2 = \int\limits_{\Omega\times(-1,1)} \Big| & X_{k,x_0}X_{l,x_0}(x_0,s) \nabla_{x_0,\varepsilon}\Gamma(\cdot,(\zeta,\sigma))\cdot \\ & \cdot \Big(u(\zeta) - P^1_{x_0}u(\zeta) - \sigma^2\Big)\ \Big(1-\chi_{x_0,\varepsilon}((\xi,s),(\zeta,\sigma))\Big)\ \partial_2 z(\zeta)\,\phi(\zeta,\sigma)\Big| d\zeta d\sigma,\end{aligned}$$

[1] In view of the dependence of the neighborhood U from ε in Lemma 3.8, possibly one needs to work with a smaller scale $d_{x_0,\varepsilon}((\xi,s),(x_0,0)\le o(\varepsilon)$ depending on the C^3 norm of Γ_{ε,x_0}

and

$$(3.35)\quad A_3 = \int_{\Omega\times(-1,1)} \Big| X_{l,x_0}\nabla_{x_0,\varepsilon}(\xi,s)\Gamma(\cdot,(\zeta,\sigma))\cdot$$
$$\cdot\Big(u(\zeta)-P^1_{x_0}u(\zeta)-\sigma^2\Big)\ X_{k,x_0}(\xi)\chi_{x_0,\varepsilon}((\xi,s),(\zeta,\sigma))\ \partial_2 z(\zeta)\,\phi(\zeta,\sigma)\Big| d\zeta d\sigma$$
$$+\int_{\Omega\times(-1,1)} \Big|\nabla_{x_0,\varepsilon}(\xi,s)\Gamma(\cdot,(\zeta,\sigma))\cdot$$
$$\cdot\Big(u(\zeta)-P^1_{x_0}u(\zeta)-\sigma^2\Big)\ X_{l,x_0}X_{k,x_0}(\xi,s)\chi_{x_0,\varepsilon}(\cdot,(\zeta,\sigma))\ \partial_2 z(\zeta)\,\phi(\zeta,\sigma)\Big| d\zeta d\sigma$$

Invoking Lemma 3.8 one can complete the proof arguing as in [17, page 734]. As usual, we examine in detail only the integral A_1 which contains the most singular integrand. Note that if $\zeta\in\mathrm{supp}\chi_{x_0,\varepsilon}((\xi,s),\cdot)$ then $d_{x_0,\varepsilon}((\xi,s),(\zeta,\sigma))\ge 2C\varepsilon$, consequently

$$d_{x_0}((x_0,0),(\zeta,\sigma))\ge \frac{1}{C}d_{x_0,\varepsilon}((\xi,s),(\zeta,\sigma))-d_{x_0,\varepsilon}((\xi,s),(x_0,0))\ge\varepsilon.$$

In view of Lemma 3.8, (3.3) and Lemma 3.2 we have

$$(3.36)\quad |A_1|\le C\int_{d_{x_0}((x_0,0),(\zeta,\sigma))\ge\varepsilon} d_{x_0}((\xi,s),(x_0,0))d^{1+\alpha}_{x_0}((x_0,0),(\zeta,\sigma))\phi(\zeta)\frac{d_{x_0}((x_0,0),(\zeta,\sigma))^{-2}}{|B((x_0,0),d_{x_0}((x_0,0),(\zeta,\sigma)))|}d\zeta d\sigma$$
$$+C\int_{d_{x_0}((\xi,s),(\zeta,\sigma))\ge\varepsilon} d_{x_0}((\xi,s),(x_0,0))d^{1+\alpha}_{x_0}((x_0,0),(\zeta,\sigma))\phi(\zeta)\frac{d_{x_0}((\xi,s),(\zeta,\sigma))^{-2}}{|B((\xi,s),d_{x_0}((\xi,s),(\zeta,\sigma)))|}d\zeta d\sigma\le C\varepsilon^\alpha.$$

□

Using the representation formula above, the fractional integration result in Proposition 3.5 and Corollary 3.6 we finally can proceed to the proof of the main result of the section:

Proof of Theorem 3.1. We will prove (i) only. The proof of (ii) follows along a similar argument. Using the representation formula (3.26) one can represent the second horizontal derivatives $X_{k,x_0}X_{l,x_0}(z(x_0)\phi(x_0,s)$ of z at any point $(x_0,s)\in\Omega\times(-1,1)$ through integral operator with kernels of the form

$$X_{k,x_0}X_{l,x_0}X_{i,x_0}(x_0,s)\Gamma_{\varepsilon x_0}(\cdot,(\zeta,\sigma))\Big(u(\zeta)-P^1_{x_0}u(\zeta)-\sigma^2\Big)$$

$$X_{k,x_0}X_{l,x_0}(x_0,s)\Gamma_{\varepsilon,x_0}(\cdot,(\zeta,\sigma))\Big(\bar a_{ij}(x_0)-\bar a_{ij}(\zeta)\Big)$$

and

$$X_{k,x_0}X_{l,x_0}(x_0,s)\Gamma_{\varepsilon,x_0}(\cdot,(\zeta,\sigma))\Big(u(\zeta)-P^1_{x_0}u(\zeta)-\sigma^2\Big)$$

To establish the non-singular character of such kernels one needs to invoke the estimates on the derivatives of the fundamental solution of the frozen operator $\Gamma_{\varepsilon x_0}$ in Theorem 3.3. To estimate the $L^{10/3}$ norm of each term in the right-hand side of (3.26) one uses the fractional integral estimates in Corollary 3.6. The 'worst' possible term is the one corresponding to three derivatives on Γ, i.e.

$$\sum_{i=1}^{2} \bar a_{i1}(x_0) \int_{\Omega\times(-1,1)} X_{k,x_0} X_{l,x_0} X_{i,x_0}(x_0,s)\Gamma_{\varepsilon x_0}(\cdot,(\zeta,\sigma))\Big(u(\zeta) - P^1_{x_0}u(\zeta) - \sigma^2\Big)\partial_2 z(\zeta)\,\phi(\zeta) d\zeta d\sigma \tag{3.37}$$

$$= \int \mathcal{K}((x_0,s),(\zeta,\sigma))\partial_2 z(\zeta)\,\phi(\zeta) d\zeta d\sigma,$$

with (in view of Remark 3.4)

$$|\mathcal{K}((x_0,s),(\zeta,\sigma))| \le C \frac{d^\alpha_{x_0,\varepsilon}((x_0,s),(\zeta,\sigma))}{|B_\varepsilon((x,s), d_{x_0,\varepsilon}((x_0,s),(\zeta,\sigma))|}.$$

Note that the expression in (3.37) does not depend on s. Moreover since from the assumptions one has $p > 10/3 >$, and hence $p > 50/(15+10\alpha)$ then Corollary 3.6 yields immediately that the integral in (3.37) is in L^r with

$$r = \frac{5p}{5-p\alpha} > \frac{10}{3}.$$

The rest of the terms in the right-hand side of (3.26) are estimated similarly.

At this point we have proved that the function $x_0 \to X_{k,x_0}X_{l,x_0}z(x_0)$ is in $L^{\frac{10}{3}}(K)$. Next we observe that

$$X_{k,u}X_{l,u}(x_0)z - X_{k,x_0}X_{l,x_0}(x_0,0)z = \begin{cases} 0 & \text{if } l=2 \text{ and } k=1,2 \\ u(x_0)\partial_{x_2}u(x_0)\partial_{x_2}z(x_0) & \text{if } l=1 \text{ and } k=1 \\ 0 & \text{if } l=1 \text{ and } k=2 \end{cases}.$$

In view of the hypothesis $\partial_{x_2}z \in L^p(K)$ then one finally concludes

$$X_{k,u}X_{l,u}z - X_{k,x_0}X_{l,x_0}(\cdot,0)z \in L^p(K)$$

with $p > 10/3$. □

4. Cacciopoli type inequalities: $W^{m,p}_{\varepsilon,\mathrm{LOC}}$ a priori estimates.

In this section we prove a *a priori* estimate, in the Sobolev spaces $W^{m,p}_{\varepsilon,\mathrm{loc}}(\Omega)$ for solutions z of equation (2.8), under the assumption that u is smooth and satisfies (2.2).

The starting point is a Caccioppoli type inequality for derivatives of solution of (2.8) in the directions X_i. By Lemmas 2.6 and 2.7 these derivatives solve the same equation (2.8), with a different second member, hence we will focus on this PDE.

Lemma 4.1. *Assume that $f_0 \in L^1_{loc}(\Omega)$, and that $z \in W^{2,2}_{\varepsilon,loc}(\Omega) \cap W^{1,3}_{1,loc}(\Omega)$ is a solution of equation (2.8) For every $p \ge 3$ there exist constants $C_1 = C_1(p,M)$ with the constant M as in (2.2) and independent of ε and z such that for every non-negative $\phi \in C^\infty_0(\Omega)$, we have*

$$\int |\nabla_\varepsilon(|z|^{(p-1)/2})|^2\phi^2 \le C_1\left(\int |z|^{p-1}\big(\phi^2 + |\nabla_\varepsilon\phi|^2\big) - \int f|z|^{p-3}z\phi^2\right).$$

Proof. Let us multiply both members of equation (2.8) by $|z|^{p-3}z\phi^2$, and integrate. We obtain

$$\int f|z|^{p-3}z\phi^2 = \int X_i\Big(\frac{a_{ij}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}}X_j z\Big)|z|^{p-3}z\phi^2 = \tag{4.1}$$

(since $X_1^* = -X_1 - \partial_2 u$, $X_2^* = -X_2$)

$$= -\int \partial_2 u\delta_{i1}\frac{a_{ij}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}}X_j z|z|^{p-3}z\phi^2 -$$

$$-(p-2)\int \frac{a_{ij}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}} X_j z X_i z |z|^{p-3}\phi^2 - 2\int \frac{a_{ij}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}} X_j z |z|^{p-3} z\phi X_i\phi.$$

This obviously implies that there exists a constant $C>0$ such that

$$\frac{4(p-2)}{(p-1)^2}\int |\nabla_\varepsilon(|z|^{(p-1)/2})|^2\phi^2 \le C\int |z|^{p-1}(|\nabla_\varepsilon\phi|^2+\phi^2) - \int f|z|^{p-3}z\phi^2, \tag{4.2}$$

concluding the proof. □

Lemma 4.2. *Let $p\ge 3$ be fixed and u be a function satisfying the bound (2.1). Let $f\in C^\infty(\Omega)$, and let z be a smooth solution of equation (2.8). There exist two constants C and $\widetilde{C}$ which depend on p and the constant M in (2.2) but are independent of ε and z such for every $\phi\in C_0^\infty(\Omega)$, $\phi>0$,*

$$\int |\nabla_\varepsilon(|\nabla_\varepsilon z|^{(p-1)/2})|^2\phi^{2p} \le C\int \big(|\nabla_\varepsilon\phi|^2+\phi^2\big)^p + \tag{4.3}$$

$$+\int |\nabla_\varepsilon z|^{p+1/2}\phi^{2p} + \int |X_2(\partial_2 u)|^p\phi^{2p} + \int |f|^{(2p+1)/7}\phi^{2p} + \int |f|^{(2p+1)/5}(|\nabla_\varepsilon\phi|+\phi)^{(2p+1)/5}\phi^{(8p-1)/5} +$$

$$+\int |\nabla_\varepsilon^2 u||\nabla_\varepsilon z|^{p-1}\phi^{2p} + \int |\nabla_\varepsilon^2 u|^2|\nabla_\varepsilon z|^{p-1}\phi^{2p} + \int |\nabla_\varepsilon^2 u||\nabla_\varepsilon z|^{p-1}\phi^{2p-1}|\nabla_\varepsilon\phi|.$$

Proof. Since z is a solution of equation (2.8) then by Lemma 2.6 $s_1 = X_1 z$ satisfies equation

$$M_\varepsilon s_1 = X_1(f) + X_i\left(\frac{a_{i2}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}}\partial_2 u X_2 z\right) -$$

$$-X_i\left(X_1\Big(\frac{a_{ij}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}}\Big)X_j z\right) + \partial_2 u X_2\left(\frac{a_{2j}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}}X_j z\right)$$

Using Lemma 4.1 we deduce

$$\int |\nabla_\varepsilon(|s_1|^{(p-1)/2})|^2\phi^{2p} \le C_1\int |s_1|^{p-1}\big(|\nabla_\varepsilon\phi|^2+\phi^2\big)\phi^{2p-2} - \tag{4.4}$$

$$-\int |s_1|^{p-3}s_1 X_i\left(\frac{a_{i2}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}}\partial_2 u X_2 z\right)\phi^{2p} +$$

$$+\int |s_1|^{p-3}s_1 X_i\left(X_1\Big(\frac{a_{ij}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}}\Big)X_j z\right)\phi^{2p}$$

$$-\int |s_1|^{p-3}s_1\partial_2 u X_2\left(\frac{a_{2j}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}}X_j z\right)\phi^{2p} - \int |s_1|^{p-3}s_1 X_1 f\phi^{2p} =$$

(integrating by part all terms in the right hand side)

$$= C_1\int |s_1|^{p-1}\big(|\nabla_\varepsilon\phi|^2+\phi^2\big)\phi^{2p-2} + \int |s_1|^{p-3}s_1\frac{a_{12}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}}(\partial_2 u)^2 X_2 z\phi^{2p} +$$

$$+(p-2)\int |s_1|^{p-3}X_i s_1\frac{a_{i2}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}}\partial_2 u X_2 z\phi^{2p} +$$

$$+2p\int |s_1|^{p-3}s_1\frac{a_{i2}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}}\partial_2 u X_2 z\phi^{2p-1}X_i\phi +$$

$$-\int |s_1|^{p-3} s_1 \partial_2 u X_1\Big(\frac{a_{1j}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}}\Big) X_j z\phi^{2p}$$
$$-(p-2)\int |s_1|^{p-3} X_i s_1 X_1\Big(\frac{a_{ij}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}}\Big) X_j z\phi^{2p}$$
$$-2p\int |s_1|^{p-3} s_1 X_1\Big(\frac{a_{ij}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}}\Big) X_j z\phi^{2p-1}X_i\phi$$
$$+(p-2)\int |s_1|^{p-3} X_2 s_1 \partial_2 u \frac{a_{2j}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}} X_j z\phi^{2p}+$$
$$+\int |s_1|^{p-3} s_1 X_2 \partial_2 u \frac{a_{2j}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}} X_j z\phi^{2p}+$$
$$+2p\int |s_1|^{p-3} s_1 \partial_2 u \frac{a_{2j}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}} X_j z\phi^{2p-1}X_2\phi+$$
$$+\int |s_1|^{p-3} s_1 \partial_2 u f\phi^{2p} + 2p\int |s_1|^{p-3} s_1 f X_1\phi\phi^{2p-1}$$
$$+(p-2)\int |s_1|^{p-3} X_1 s_1 \mathrm{sign}(s_1) f\phi^{2p} \le$$
$$\le C_1\int |s_1|^{p-1}\big(|\nabla_\varepsilon\phi|^2+\phi^2\big)\phi^{2p-2} + \frac{C}{\delta}\int |X_2 z|^{p-1}\phi^{2p} + \delta\int |\nabla_\varepsilon s_1|^2 |s_1|^{p-3}\phi^{2p}+$$
$$+\int |\nabla^2_\varepsilon u||\nabla_\varepsilon z|^{p-1}\phi^{2p} + \frac{C}{\delta}\int |\nabla^2_\varepsilon u|^2|\nabla_\varepsilon z|^{p-1}\phi^{2p} + \int |\nabla^2_\varepsilon u||\nabla_\varepsilon z|^{p-1}\phi^{2p-1}|\nabla_\varepsilon\phi|+$$
$$+C\int |X_2(\partial_2 u)|^p\phi^{2p} + \int |\nabla_\varepsilon z|^p\phi^{2p} + \frac{C}{\delta}\int |s_1|^{p+1/2}\phi^{2p}+$$
$$+\frac{C}{\delta}\int |f|^{(2p+1)/7}\phi^{2p} + \int |f|^{(2p+1)/5}(|\nabla_\varepsilon\phi|+\phi)^{(2p+1)/5}\phi^{(8p-1)/5} \le$$
$$\le C_1\int \big(|\nabla_\varepsilon\phi|^2+\phi^2\big)^p + \frac{C}{\delta}\int |\nabla_\varepsilon z|^p\phi^{2p} + \frac{C}{\delta}\int |s_1|^{p+1/2}\phi^{2p}+$$
$$+\int |X_2(\partial_2 u)|^p\phi^{2p} + \delta\int |s_1|^{p-3}|\nabla_\varepsilon s_1|^2\phi^{2p}+$$
$$+\int |\nabla^2_\varepsilon u||\nabla_\varepsilon z|^{p-1}\phi^{2p} + \frac{C}{\delta}\int |\nabla^2_\varepsilon u|^2|\nabla_\varepsilon z|^{p-1}\phi^{2p} + \int |\nabla^2_\varepsilon u||\nabla_\varepsilon z|^{p-1}\phi^{2p-1}|\nabla_\varepsilon\phi|+$$
$$+\frac{C}{\delta}\int |f|^{(2p+1)/7}\phi^{2p} + \int |f|^{(2p+1)/5}(|\nabla_\varepsilon\phi|+\phi)^{(2p+1)/5}\phi^{(8p-1)/5}$$

It follows that

$$\int |\nabla_\varepsilon(|s_1|^{(p-1)/2})|^2\phi^{2p} \le C\Big(\int \big(|\nabla_\varepsilon\phi|^2+\phi^2\big)^p+ \tag{4.5}$$
$$+\int |\nabla_\varepsilon z|^{p+1/2}\phi^{2p} + \int |X_2(\partial_2 u)|^p\phi^{2p} + \int |f|^{(2p+1)/7}\phi^{2p} + \int |f|^{(2p+1)/5}(|\nabla_\varepsilon\phi|+\phi)^{(2p+1)/5}\phi^{(8p-1)/5}$$
$$+\int |\nabla^2_\varepsilon u||\nabla_\varepsilon z|^{p-1}\phi^{2p} + \int |\nabla^2_\varepsilon u|^2|\nabla_\varepsilon z|^{p-1}\phi^{2p} + \int |\nabla^2_\varepsilon u||\nabla_\varepsilon z|^{p-1}\phi^{2p-1}|\nabla_\varepsilon\phi|\Big).$$

An analogous estimate holds for $s_2 = X_2 z$, i.e.

$$\int |\nabla_\varepsilon(|s_2|^{(p-1)/2})|^2\phi^{2p} \le C\Big(\int \big(|\nabla_\varepsilon\phi|^2+\phi^2\big)^p+ \tag{4.6}$$

$$+\int|\nabla_\varepsilon z|^{p+1/2}\phi^{2p}+\int|X_2(\partial_2 u)|^p\phi^{2p}+\int|f|^{(2p+1)/7}\phi^{2p}+\int|f|^{(2p+1)/5}(|\nabla_\varepsilon\phi|+\phi)^{(2p+1)/5}\phi^{(8p-1)/5}$$
$$+\int|\nabla^2_\varepsilon u||\nabla_\varepsilon z|^{p-1}\phi^{2p}+\int|\nabla^2_\varepsilon u|^2|\nabla_\varepsilon z|^{p-1}\phi^{2p}+\int|\nabla^2_\varepsilon u||\nabla_\varepsilon z|^{p-1}\phi^{2p-1}|\nabla_\varepsilon\phi|\Big).$$

The conclusion follows immediately.

□

Theorem 4.3. *Let $p\ge 3$ be fixed and u be a function satisfying the bound (2.1). Let $f\in C^\infty(\Omega)$, and let z be a smooth solution of equation (2.8). There exists a constant C, which depends on p and the constant M in (2.2) but is independent of ε and z such that for every $\phi\in C^\infty_0(\Omega)$, $\phi>0$,*

$$\int|\nabla_\varepsilon z|^{p+1/2}\phi^{2p}+\int|\nabla_\varepsilon\big(|\nabla_\varepsilon z|^{(p-1)/2}\big)|^2\phi^{2p}\le$$
$$\le C\Big(\int|z|^{4p+2}\phi^{2p}+\int\big(\phi^2+|\nabla_\varepsilon\phi|^2\big)^p+\int|\nabla_\varepsilon(\partial_2 u)|^p\phi^{2p}+$$
$$+\int\Big(|f|^{(2p+1)/7}\phi^{2p}+|f|^{(2p+1)/5}(|\nabla_\varepsilon\phi|+\phi)^{(2p+1)/5}\phi^{(8p-1)/5}\Big)+\int|\nabla^2_\varepsilon u|^{2p}\phi^{2p}\Big).$$

Proof. By Proposition 2.2, and Lemma 4.2 calling $s_1=X_1z$ we have

$$\int|s_1|^{p+1/2}\phi^{2p}\le\frac{C}{\delta}\int\Big(|z|^{4p+2}\phi^{2p}+|z|^{(2p+1)/2}\phi^{2p}+|z|^{(2p+1)/2}|\nabla_e\phi|^{(2p+1)/2}\Big)+$$
$$+\int\big(|\nabla_\varepsilon\phi|^2+\phi^2\big)^p+$$
$$+\delta\int(|\nabla_\varepsilon z|^{p+1/2}+|X_2(\partial_2 u)|^p)\phi^{2p}+\int(|\nabla^2_\varepsilon u|^{2p}+|\nabla^2_\varepsilon u|^p)\phi^{2p}+$$
$$+C\int\Big(|f|^{(2p+1)/7}\phi^{2p}+|f|^{(2p+1)/5}(|\nabla_\varepsilon\phi|+\phi)^{(2p+1)/5}\phi^{(8p-1)/5}\Big).$$

To estimate $s_2=X_2z$ we argue in the same way and obtain

$$\int|s_2|^{p+1/2}\phi^{2p}\le\frac{C}{\delta}\int\Big(|z|^{4p+2}\phi^{2p}+|z|^{(2p+1)/2}\phi^{2p}+|z|^{(2p+1)/2}|\nabla_e\phi|^{(2p+1)/2}\Big)+$$
$$+\int\big(|\nabla_\varepsilon\phi|^2+\phi^2\big)^p+$$
$$+\delta\int(|\nabla_\varepsilon z|^{p+1/2}+|X_2(\partial_2 u)|^p)\phi^{2p}+\int(|\nabla^2_\varepsilon u|^{2p}+|\nabla^2_\varepsilon u|^p)\phi^{2p}+$$
$$+C\int\Big(|f|^{(2p+1)/7}\phi^{2p}+|f|^{(2p+1)/5}(|\nabla_\varepsilon\phi|+\phi)^{(2p+1)/5}\phi^{(8p-1)/5}\Big).$$

Hence, if δ is sufficiently small

$$\int\Big(|s_1|^{p+1/2}+|s_2|^{p+1/2}\Big)\phi^{2p}\le\frac{C}{\delta}\int\Big(|z|^{4p+2}\phi^{2p}+|z|^{(2p+1)/2}\phi^{2p}+|z|^{(2p+1)/2}|\nabla_\varepsilon\phi|^{(2p+1)/2}\Big)+$$
$$+\int\big(|\nabla_\varepsilon\phi|^2+\phi^2\big)^p+\int|\nabla_e(\partial_2 u)|^p\phi^{2p}+\int(|\nabla^2_\varepsilon u|^{2p}+|\nabla^2_\varepsilon u|^p)\phi^{2p}+$$
$$+C\int\Big(|f|^{(2p+1)/7}\phi^{2p}+|f|^{(2p+1)/5}(|\nabla_\varepsilon\phi|+\phi)^{(2p+1)/5}\phi^{(8p-1)/5}\Big).$$

The conclusion follows from the latter, (4.5) and (4.6) and the Hölder inequality

$$\int|\nabla_\varepsilon u|^p\phi^{2p}\le\int|\nabla_\varepsilon u|^{2p}\phi^{2p}+\int\phi^{2p}.$$

□

Next we iterate once the previous result

Theorem 4.4. *Let $p \geq 3$ be fixed and u be a function satisfying the bound (2.1). Consider a function $f \in C^\infty(\Omega)$, and z a smooth solution of equation (2.8). Let Ω_1, Ω_2 so that $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$. There exists a constant C, which depends on p, on Ω_i, and on the constant M in (2.2), but is independent of ε or z such that*

$$||z||^{p+1/2}_{W^{2,p+1/2}_\varepsilon(\Omega_1)} + \sum_{|I|=2} |||\nabla^I_\varepsilon z|^{(p-1)/2}||^2_{W^{1,2}_\varepsilon(\Omega_1)} \leq$$

$$\leq C\Big(||f||^{(2p+1)/5}_{W^{1,(2p+1)/5}_\varepsilon(\Omega_2)} + ||v||^{4p+2}_{W^{1,4p+2}_\varepsilon(\Omega_2)} +$$

$$+||u||^{4p+2}_{W^{2,4p+2}_\varepsilon(\Omega_2)} + ||u||^{2p/3}_{W^{3,2p/3}_\varepsilon(\Omega_2)} + ||z||^{4p+2}_{W^{1,4p+2}_\varepsilon(\Omega_2)} + ||z||^{2p/3}_{W^{2,2p/3}_\varepsilon(\Omega_2)} + 1\Big).$$

Here I is a multi-index and $\nabla^I_\varepsilon z$ denotes derivatives of order $|I|$ along $X_{i,u}$, $i = 1, 2$.

Proof. If z is a solution of (2.8) then, by Lemma 2.7 the function $s_2 = X_2 z$ is a solution of the equation:

$$M_\varepsilon s_2 = \widetilde{f}_0$$

where

$$\widetilde{f}_0 = X_2 f - X_i\left(\frac{a_{i1}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}}\partial_2 u X_2 z\right) -$$

$$-X_i\left(X_2\Big(\frac{a_{ij}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}}\Big)X_j z\right) - \partial_2 u X_2\left(\frac{a_{1j}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}}X_j z\right).$$

Let us choose Ω_3 such that $\Omega_1 \subset\subset \Omega_3 \subset\subset \Omega_2$. By Theorem 4.3 there exists a constant C independent of ε such that

$$||s_2||^{p+1/2}_{W^{1,p+1/2}_\varepsilon(\Omega_3)} + ||\ |\nabla_\varepsilon s_2|^{(p-1)/2}||^2_{W^{1,2}_\varepsilon(\Omega_3)} \leq \tag{4.7}$$

$$\leq C\Big(||\widetilde{f}_0||^{(2p+1)/5}_{L^{(2p+1)/5}(\Omega_2)} + ||v||^p_{W^{1,p}_\varepsilon(\Omega_2)} + ||u||^{2p}_{W^{2,2p}_\varepsilon(\Omega_2)} + ||s_2||^{4p+2}_{L^{4p+2}(\Omega_2)} + 1\Big).$$

We note that for three fixed functions f, g, h,

$$||fgh||^{(2p+1)/5}_{L^{(2p+1)/5}(\Omega_2)} \leq$$

$$\leq ||f||^{2p/3}_{L^{2p/3}(\Omega_2)} + ||g||^{4p(2p+1)/4p-3}_{L^{4p(2p+1)/4p-3}(\Omega_2)} + ||h||^{4p(2p+1)/4p-3}_{L^{4p(2p+1)/4p-3}(\Omega_2)} \leq$$

$$\leq ||f||^{2p/3}_{L^{2p/3}(\Omega_2)} + ||g||^{4p+2}_{L^{4p+2}(\Omega_2)} + ||h||^{4p+2}_{L^{4p+2}(\Omega_2)} + C.$$

If follows that

$$||\widetilde{f}_0||^{(2p+1)/5}_{L^{(2p+1)/5}(\Omega_2)} \leq$$

$$\leq ||f||^{(2p+1)/5}_{W^{1,(2p+1)/5}(\Omega_2)} + ||\nabla^2_\varepsilon u v \nabla_\varepsilon z||^{(2p+1)/5}_{L^{(2p+1)/5}(\Omega_2)} + ||\nabla_\varepsilon v \nabla_\varepsilon z||^{(2p+1)/5}_{L^{(2p+1)/5}(\Omega_2)} +$$

$$+||(\nabla^2_\varepsilon u)^2 \nabla_\varepsilon z||^{(2p+1)/5}_{L^{(2p+1)/5}(\Omega_2)} + ||(1+\nabla^2_\varepsilon u) v \nabla^2_\varepsilon z||^{(2p+1)/5}_{L^{(2p+1)/5}(\Omega_2)} +$$

$$+||\nabla^3_\varepsilon u \nabla_\varepsilon z||^{(2p+1)/5}_{L^{(2p+1)/5}(\Omega_2)} + ||\nabla^2_\varepsilon u \nabla^2_\varepsilon z||^{(2p+1)/5}_{L^{(2p+1)/5}(\Omega_2)} \leq$$

$$\leq C\Big(||f||^{(2p+1)/5}_{W^{1,(2p+1)/5}_\varepsilon(\Omega_2)} + ||v||^{4p+2}_{W^{1,4p+2}_\varepsilon(\Omega_2)} +$$

$$+||u||^{4p+2}_{W^{2,4p+2}_\varepsilon(\Omega_2)} + ||u||^{2p/3}_{W^{3,2p/3}_\varepsilon(\Omega_2)} + ||z||^{4p+2}_{W^{1,4p+2}_\varepsilon(\Omega_2)} + ||z||^{2p/3}_{W^{2,2p/3}_\varepsilon(\Omega_2)} + 1\Big).$$

Arguing in the same way with the function $s_1 = X_1 z$ we conclude the proof. $\square$

Iterations of Theorem 4.3 yield the following

Theorem 4.5. *Let $p \geq 3$, $m \geq 1$ be a fixed positive integer and u be a function satisfying the bound (2.1). Assume that $f \in C^\infty(\Omega)$, and let z be a smooth solution of equation (2.8) in Ω. If $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$ then there exists a constant C which depends on p, Ω_i and on M in (2.2), but is independent of ε or z such that the solution satisfies the following estimate*

$$||z||^{p+1/2}_{W^{m+1,p+1/2}_\varepsilon(\Omega_1)} + \sum_{|I|=m+1} |||\nabla^I_\varepsilon z|^{(p-1)/2}||^2_{W^{1,2}_\varepsilon(\Omega_1)} \leq$$
$$\leq C\Big(||f||^{(2p+1)/5}_{W^{m,(2p+1)/5}_\varepsilon(\Omega_2)} + ||v||^{4p+2}_{W^{m,4p+2}_\varepsilon(\Omega_2)} +$$
$$+||u||^{4p+2}_{W^{m+1,4p+2}_\varepsilon(\Omega_2)} + ||u||^{2p/3}_{W^{m+2,2p/3}_\varepsilon(\Omega_2)} + ||z||^{4p+2}_{W^{m,4p+2}_\varepsilon(\Omega_2)} + ||z||^{2p/3}_{W^{m+1,2p/3}_\varepsilon(\Omega_2)} + 1\Big).$$

5. A priori estimates for the non-linear approximating PDE

We now return to the equation $L_\varepsilon u = 0$. Let u be a smooth solution satisfying (2.1). In view of Proposition 2.1 and Theorem 3.1 (i) we have the following statement: for every open set $\Omega_1 \subset\subset \Omega$ there exists a positive constant constant C which depends on Ω_1 and on M in (2.2), but is independent of ε such that

$$||u||_{W^{2,10/3}_\varepsilon(\Omega_1)} + ||\partial_2 u||_{W^{1,2}_\varepsilon(\Omega_1)} + ||u||_{C^{1,\alpha}_E(\Omega_1)} \leq C. \tag{5.1}$$

Our first step is the higher integrability of the Hessian of u. The proof rests on the estimates obtainted from the freezing technique in Theorem 3.1 and from a new Euclidean[2] Caccipoli inequality (5.8).

Lemma 5.1. *Let u be a smooth solution of*

$$L_\varepsilon u = 0,$$

in $\Omega \subset \mathbb{R}^2$ satifying (2.1) *and denote $v = \partial_2 u$. For every open set $\Omega_1 \subset\subset \Omega$, for every $p \geq 1$ there exists a positive constant C which depends on Ω_1, p, and on M in (2.2), but is independent of ε such that*

$$||u||^p_{W^{2,p}_\varepsilon(\Omega_1)} + ||\nabla_\varepsilon v||^4_{L^4(\Omega_1)} \leq C.$$

Proof. In view of Lemma 2.5 the function $v = \partial_2 u$ satisfies the equation:

$$X_i\Big(\frac{a_{ij}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}} X_j v\Big) = f,$$

with

$$f = -\frac{a_{11}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}} v^3 - 3\frac{a_{1j}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}} v X_j v - X_i\Big(\frac{a_{i1}(\nabla_\varepsilon u)}{\sqrt{1+|\nabla_\varepsilon u|^2}}\Big) v^2$$

Hence, applying Lemma 4.2 one has

$$\int |\nabla_\varepsilon(|\nabla_\varepsilon v|^{(p-1)/2})|^2 \phi^{2p} \leq C\Big(\int \big(|\nabla_\varepsilon \phi|^2 + \phi^2\big)^p + \tag{5.2}$$
$$+\int (|\nabla_\varepsilon v|^{p+1/2})\phi^{2p} + \int |X_2(\partial_2 u)|^p \phi^{2p} + \int |f|^{(2p+1)/7}\phi^{2p} +$$
$$+\int |f|^{(2p+1)/5}(|\nabla_\varepsilon \phi| + \phi)^{(2p+1)/5}\phi^{(8p-1)/5} +$$
$$+\int |\nabla^2_\varepsilon u||\nabla_\varepsilon v|^{p-1}\phi^{2p} + \int |\nabla^2_\varepsilon u|^2 |\nabla_\varepsilon v|^{p-1}\phi^{2p} + \int |\nabla^2_\varepsilon u||\nabla_\varepsilon v|^{p-1}\phi^{2p-1}|\nabla_\varepsilon \phi|\Big).$$

There exist positive constants $C_1 = C_1(|\nabla_\varepsilon \phi|, \phi, M)$ and $C_2 = C_2(M)$ such that for $p = 3$ we obtain

[2]rather than subelliptic

$$\int |\nabla_\varepsilon^2 v|^2\phi^6 \le C_1 + C_2\Big(\int |\nabla_\varepsilon v|^{3+1/2}\phi^6 + \int (1+|\nabla_\varepsilon v| + |\nabla_\varepsilon^2 u|)^{7/5}\phi^{23/5}(|\nabla_\varepsilon\phi| + \phi)^{7/5} + \\ + \int |\nabla_\varepsilon^2 u||\nabla_\varepsilon v|^2\phi^6 + \int |\nabla_\varepsilon^2 u|^2|\nabla_\varepsilon v|^2\phi^6 + \int |\nabla_\varepsilon^2 u||\nabla_\varepsilon v|^2\phi^5|\nabla_\varepsilon\phi|\Big). \tag{5.3}$$

It follows that

$$\int |\nabla_\varepsilon^2 v|^2\phi^6 \le \frac{C_2}{\delta}\int |\nabla_\varepsilon^2 u|^4\phi^6 + \delta\int |\nabla_\varepsilon v|^4\phi^6 + \frac{C_1}{\delta}. \tag{5.4}$$

Analogously, if we set $z = X_1 u$, or $z = X_2 u$, using Lemma 2.4 and arguing as above we have

$$\int |\nabla_\varepsilon^2 z|^2\phi^6 \le \frac{C_2}{\delta}\int |\nabla_\varepsilon^2 u|^4\phi^6 + \frac{C_1}{\delta} + C_2\int |\nabla_\varepsilon v|^3\phi^6 \tag{5.5}$$

Using Lemma 2.3, (5.4) and (5.1), we obtain immediately

$$\int |\nabla_\varepsilon v|^4\phi^6 \le C_1 + C_2\int |\nabla_\varepsilon^2 v|^2\phi^6 \le C_1 + \frac{C_2}{\delta}\int |\nabla_\varepsilon^2 u|^4\phi^6 + \delta\int |\nabla_\varepsilon v|^4\phi^6$$

Hence

$$\int |\nabla_\varepsilon v|^4\phi^6 \le C_1 + C_2\int |\nabla_\varepsilon^2 u|^4\phi^6 \tag{5.6}$$

Consequently, from the latter and (5.5) we deduce that

$$\int |\nabla_\varepsilon^2 z|^4\phi^6 \le C_1 + C_2\int |\nabla_\varepsilon^2 u|^4\phi^6 \tag{5.7}$$

Next, from the intrinsic Cacciopoli inequalities (5.6) and (5.7) we deduce an Euclidean Cacciopoli inequality: Note that

$$|\nabla_E X_1 z| \le |X_1^2 z| + C_2|\partial_2 X_1 z| \le |X_1^2 z| + C_2|v\partial_2 z| + C_2|X_1\partial_2 z| \le$$

(since $\partial_2 z = \partial_2 X_1 u = v^2 + X_1 v$)

$$|\nabla_\varepsilon^2 z| + C_2|\nabla_\varepsilon^2 v| + C_2|\nabla_\varepsilon v| + C_2.$$

From the latter and (5.6) and (5.7) we infer

$$\int |\nabla_E\nabla_\varepsilon z|^2\phi^6 \le C_2\Big(\int |\nabla_\varepsilon^2 v|^2\phi^6 + \int |\nabla_\varepsilon^2 z|^2\phi^6 + 1\Big) \le C_2\int |\nabla_\varepsilon z|^4\phi^6 + C_1 \tag{5.8}$$

Now we can apply the standard Euclidean Sobolev inequality in $\mathbb{R}^2$ and obtain

$$\Big(\int (|\nabla_\varepsilon z|\phi^3)^6\Big)^{1/3} \le C_2\int |\nabla_E(\nabla_\varepsilon z\phi^3)|^2 \le C_2\int |\nabla_\varepsilon z|^4\phi^6 + C_1 \le$$

(using Hölder inequality)

$$\le C_2\Big(\int (|\nabla_\varepsilon z|\phi^3)^6\Big)^{1/3}\Big(\int_{supp(\phi)} |\nabla_\varepsilon z|^3\Big)^{2/3} + C_1.$$

By (5.1) and the fact that $|\nabla_\varepsilon z| \le |\nabla_\varepsilon^2 u|$, we already know that $|\nabla_\varepsilon z| \in L^3_{loc}$. In fact

$$\Big(\int_{supp(\phi)} |\nabla_\varepsilon z|^3\Big)^{2/3} \le \Big(\int_{supp(\phi)} |\nabla_\varepsilon z|^{10/3}\Big)^{3/5}|supp(\phi)|^{1/15}.$$

Recall that C_2 doen not depend on $|\nabla_\varepsilon\phi|$. If we choose the support of ϕ sufficiently small, we can assume that the integral $\int_{supp(\phi)} |\nabla_\varepsilon z|^3$ is arbitrarily small. It follows that

$$\Big(\int (|\nabla_\varepsilon z|\phi^3)^6\Big)^{1/3} \le C_1$$

and consequently, by (5.6)

$$\int |\nabla_\varepsilon v|^4 \phi^6 \leq C_1$$

But this implies that $|\nabla_E(\nabla_\varepsilon u)| \leq |\nabla_\varepsilon^2 u| + |\nabla_\varepsilon v| + v^2 \in L^4_{loc}$. This implies, buy the standard Euclidean Sobolev Morrey inequality in $\mathbb{R}^2$ that

$$\nabla_\varepsilon u \in C_E^{1/2}.$$

By Theorem 3.1 (ii) it then follows that for every $r > 1$ there exists a constant $C > 0$ independent of ε such that

$$||\nabla_\varepsilon^2 u||_{W^{2,r}} \leq C_1.$$

□

In order to boostrap regularity we apply Theorem 4.5 to the non linear equation $L_\varepsilon u = 0$, and obtain immediately the following:

Lemma 5.2. *Let u be a smooth solution of $L_\varepsilon u = 0$, satisfying (2.1). Set $z = X_i u$, $i = 1, 2$ $v = \partial_2 u$. For every open set $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$, for every $p \geq 3$, and every integer $m \geq 2$ there exist a constant C which depend on p, m Ω_i and on M in (2.2), but is independent of ε such that the following estimates hold*

$$(5.9)\qquad ||z||^{p+1/2}_{W_\varepsilon^{m,p+1/2}(\Omega_1)} + ||v||^{p+1/2}_{W_\varepsilon^{m,p+1/2}(\Omega_1)} \leq$$
$$\leq C\Big(||v||^{4p+2}_{W_\varepsilon^{m-1,4p+2}(\Omega_2)} + ||z||^{4p+2}_{W_\varepsilon^{m-1,4p+2}(\Omega_2)} + ||v||^{2p/3}_{W_\varepsilon^{m,2p/3}(\Omega_2)} + ||z||^{2p/3}_{W_\varepsilon^{m,2p/3}(\Omega_2)} + 1\Big).$$

$$(5.10)\qquad ||z||^2_{W_\varepsilon^{m+1,2}(\Omega_1)} + ||v||^2_{W_\varepsilon^{m+1,2}(\Omega_1)} \leq$$
$$\leq C\Big(||v||^{14}_{W_\varepsilon^{m-1,14}(\Omega_2)} + ||z||^{14}_{W_\varepsilon^{m-1,14}(\Omega_2)} + ||v||^2_{W_\varepsilon^{m,2}(\Omega_2)} + ||z||^2_{W_\varepsilon^{m,2}(\Omega_2)} + 1\Big).$$

Proof. In view of Lemma 2.5 the function v solves an equation of the form

$$M_\varepsilon v = f_v,$$

with

$$(5.11)\qquad f_v = -\frac{a_{11}(\nabla_\varepsilon u)}{\sqrt{1 + |\nabla_\varepsilon u|^2}} v^3 - 3\frac{a_{1j}(\nabla_\varepsilon u)}{\sqrt{1 + |\nabla_\varepsilon u|^2}} v X_j v - X_i\Big(\frac{a_{i1}(\nabla_\varepsilon u)}{\sqrt{1 + |\nabla_\varepsilon u|^2}}\Big) v^2.$$

Analogously, the function $z = X_i u$ solves the equation

$$M_\varepsilon z = f_z,$$

of the form with

$$(5.12)\qquad f_z = -[X_k, X_i]\Big(\frac{X_i u}{\sqrt{1 + |\nabla_\varepsilon u|^2}}\Big) - X_i\Big(\frac{a_{ij}(\nabla_\varepsilon u)}{\sqrt{1 + |\nabla_\varepsilon u|^2}}[X_k, X_j]u\Big).$$

Hence

$$(5.13)\qquad |f_v| + |f_z| \leq C(1 + |\nabla_\varepsilon v| + \nabla_\varepsilon z|),$$

for some constant C depending only on M in (2.2).

Applying Theorem 4.5 to z and to v, yields:

$$||z||^{p+1/2}_{W_\varepsilon^{m,p+1/2}(\Omega_1)} + ||v||^{p+1/2}_{W_\varepsilon^{m,p+1/2}(\Omega_1)} + \sum_{|I|=m} |||\nabla_\varepsilon^I z|^{(p-1)/2}||^2_{W_\varepsilon^{1,2}(\Omega_1)} + \sum_{|I|=m} |||\nabla_\varepsilon^I v|^{(p-1)/2}||^2_{W_\varepsilon^{1,2}(\Omega_1)} \leq$$
$$\leq C\Big(||f_z||^{(2p+1)/5}_{W_\varepsilon^{m-1,(2p+1)/5}(\Omega_2)} + ||f_v||^{(2p+1)/5}_{W_\varepsilon^{m-1,(2p+1)/5}(\Omega_2)} +$$

$$+||v||^{4p+2}_{W^{m-1,4p+2}_\varepsilon(\Omega_2)} + ||u||^{4p+2}_{W^{m,4p+2}_\varepsilon(\Omega_2)} + ||u||^{2p/3}_{W^{m+1,2p/3}_\varepsilon(\Omega_2)} +$$

$$+||z||^{4p+2}_{W^{m-1,4p+2}_\varepsilon(\Omega_2)} + ||z||^{2p/3}_{W^{m,2p/3}_\varepsilon(\Omega_2)} + ||v||^{4p+2}_{W^{m-1,4p+2}_\varepsilon(\Omega_2)} + ||v||^{2p/3}_{W^{m,2p/3}_\varepsilon(\Omega_2)} + 1\Big) \leq$$

(substituting (5.13) in the latter)

$$\leq C\Big(||\nabla_\varepsilon z||^{(2p+1)/5}_{W^{m-1,(2p+1)/5}_\varepsilon(\Omega_2)} + ||\nabla_\varepsilon v||^{(2p+1)/5}_{W^{m-1,(2p+1)/5}_\varepsilon(\Omega_2)} +$$

$$+||z||^{4p+2}_{W^{m-1,4p+2}_\varepsilon(\Omega_2)} + ||z||^{2p/3}_{W^{m,2p/3}_\varepsilon(\Omega_2)} + ||v||^{4p+2}_{W^{m-1,4p+2}_\varepsilon(\Omega_2)} + ||v||^{2p/3}_{W^{m,2p/3}_\varepsilon(\Omega_2)} + 1\Big) \leq$$

(using Hölder inequality)

$$\leq C\Big(||z||^{4p+2}_{W^{m-1,4p+2}_\varepsilon(\Omega_2)} + ||z||^{2p/3}_{W^{m,2p/3}_\varepsilon(\Omega_2)} + ||v||^{4p+2}_{W^{m-1,4p+2}_\varepsilon(\Omega_2)} + ||v||^{2p/3}_{W^{m,2p/3}_\varepsilon(\Omega_2)} + 1\Big).$$

We have proved (5.9) and substituting $p = 3$ yields (5.10)

□

The main result of this section is the following a priory regularity estimates for solutions of the approximating non linear equation:

Theorem 5.3. *Let u be a smooth solution of*

$$L_\varepsilon u = 0,$$

in $\Omega \subset \mathbb{R}^2$, satisfying (2.1). For every open set $\Omega_1 \subset\subset \Omega$, for every $p \geq 3$, and every integer $m \geq 2$ there exist a constant C which depends on p, m Ω_1 and on M in (2.2), but is independent of ε such that the following estimates holds

$$||u||_{W^{m,p}_\varepsilon(\Omega_1)} + ||\partial_2 u||_{W^{m,p}_\varepsilon(\Omega_1)} \leq C. \tag{5.14}$$

Proof. The proof follows from the estimate

$$||X_1 u||_{W^{m-1,p}_\varepsilon(\Omega_1)} + ||\partial_2 u||_{W^{m-1,p}_\varepsilon(\Omega_1)} + ||X_1 u||_{W^{m,2}_\varepsilon(\Omega_1)} + ||\partial_2 u||_{W^{m,2}_\varepsilon(\Omega_1)} \leq C, \tag{5.15}$$

which we prove by induction.

First step: $m = 2$

By Lemma 5.1 we already know that there exists a constant such that for every p

$$||u||^p_{W^{2,p}_\varepsilon(\Omega_1)} \leq C_1. \tag{5.16}$$

We need to show that $v \in W^{1,p}_{\varepsilon loc}$ for every p, and that $X_1 u, v \in W^{2,2}_{\varepsilon loc}$.

Note that we can not yet invoke Lemma 5.2, since it only apply to higher order derivatives.

Recall that v is a solution of $M_\varepsilon v = f_v$, where f_v is defined in (5.11) By Theorem 4.3 there exist a constant C such that

$$\int |\nabla_\varepsilon v|^{p+1/2}\phi^{2p} + \int |\nabla_\varepsilon\big(|\nabla_\varepsilon v|^{(p-1)/2}\big)|^2\phi^{2p} \leq$$

$$\leq C\Big(\int |v|^{4p+2}\phi^{2p} + \int \big(\phi^2 + |\nabla_\varepsilon\phi|^2\big)^p + \int |\nabla_\varepsilon(\partial_2 u)|^p\phi^{2p} +$$

$$+\int \big(|f_v|^{(2p+1)/7}\phi^{2p} + |f_v|^{(2p+1)/5}(|\nabla_\varepsilon\phi| + \phi)^{(2p+1)/5}\phi^{(8p-1)/5}\big) + \int |\nabla^2_\varepsilon u|^{2p}\phi^{2p}\Big) \leq$$

(using (5.13), (5.1) and (5.16))

$$\leq C\Big(1 + \int |\nabla_\varepsilon v|^{(2p+1)/5}(|\nabla_\varepsilon\phi| + \phi)^{(2p+1)/5}\phi^{(8p-1)/5} + \int |\nabla_\varepsilon(\partial_2 u)|^p\phi^{2p}\Big) \leq$$

$$\leq C\Big(1+\int |\nabla_\varepsilon v|^p(|\nabla_\varepsilon \phi|+\phi)^{(2p+1)/5}\phi^{(8p-1)/5}\Big).$$

From Lemma 5.1 the right hand side is bounded for $p=4$. An obvious boostrap argument yields $\nabla_\varepsilon v \in L^p_{loc}$ for every p. Moreover, choosing $p=3$, we also infer

$$\nabla^2_\varepsilon v \in L^2_{loc}.$$

To conclude the first iteration step we observe that the function $z = X_1 u$ solves $M_\varepsilon z = f_z$, where f_z is defined in (5.12). Theorem 4.3 and estimate (5.13) yield that there exists a constant C such that

$$\int |\nabla_\varepsilon z|^{p+1/2}\phi^{2p} + \int |\nabla_\varepsilon\big(|\nabla_\varepsilon z|^{(p-1)/2}\big)|^2\phi^{2p} \leq C.$$

Choosing $p=3$ we obtain $X_1 u \in W^{2,2}_{\varepsilon loc}$

Main iteration step: $m>2$

Assume (5.15) holds for for a fixed value of m.

Let Ω_2 be as in Lemma 5.2. In view of that result we infer

$$||z||^{p+1/2}_{W^{m,p+1/2}_\varepsilon(\Omega_1)} + ||v||^{p+1/2}_{W^{m,p+1/2}_\varepsilon(\Omega_1)} \leq \tag{5.17}$$

$$\leq C\Big(||v||^{4p+2}_{W^{m-1,4p+2}_\varepsilon(\Omega_2)} + ||z||^{4p+2}_{W^{m-1,4p+2}_\varepsilon(\Omega_2)} + ||v||^{2p/3}_{W^{m,2p/3}_\varepsilon(\Omega_2)} + ||z||^{2p/3}_{W^{m,2p/3}_\varepsilon(\Omega_2)} + 1\Big) \leq$$

by induction assumption

$$\leq C\Big(||v||^{2p/3}_{W^{m,2p/3}_\varepsilon(\Omega_2)} + ||z||^{2p/3}_{W^{m,2p/3}_\varepsilon(\Omega_2)} + 1\Big).$$

The same boostrap argument used above implies $v, z \in W^{m,p}_{\varepsilon loc}$ for every p.

Invoking (5.10)

$$||z||^2_{W^{m+1,2}_\varepsilon(\Omega_1)} + ||v||^2_{W^{m+1,2}_\varepsilon(\Omega_1)} \leq \tag{5.18}$$

$$\leq C\Big(||v||^{14}_{W^{m-1,14}_\varepsilon(\Omega_2)} + ||z||^{14}_{W^{m-1,14}_\varepsilon(\Omega_2)} + ||v||^2_{W^{m,2}_\varepsilon(\Omega_2)} + ||z||^2_{W^{m,2}_\varepsilon(\Omega_2)} + 1\Big) \leq C,$$

concluding the proof. □

6. Estimates for the viscosity solution

In this section we turn our attention to the proof of regularity for vanishing viscosity solutions u of equation (1.2). The regularity is expressed in terms of the intrinsic Sobolev spaces $W^{k,p}_0(\Omega)$ and rests on the *a priori* estimates proved in the previous section in the limit $\varepsilon \to 0$.

Let u be a vanishing viscosity solution, and (u_j) denote its approximating sequence, as defined in Definition 1.1. For each ε_j and function u_j we set $X_{1,j} = \partial_1 + u_j\partial_{x_2}$, $X_{2,j} = \varepsilon_j\partial_{x_2}$ the corresponding vector fields, and let ∇_{ε_j} and $W^{k,p}_{\varepsilon_j}(\Omega)$ denote the natural gradient and Sobolev spaces. We also let u , $X_1 = \partial_1 + u\partial_2$, and $\nabla_0 = (X_1, 0)$ denote the coefficients and vector fields associated to the limit equation and the limit solution u, while $W^{k,p}_0(\Omega)$ will be the associated Sobolev space. Note that ∇_E and $W^{k,p}_E(\Omega)$ are the usual gradient and Sobolev space.

Theorem 6.1. *Let $u \in Lip(\Omega)$ be a vanishing viscosity solution of* (1.2)*, and set $v_j = \partial_2 u_j$. For every ball $B(R) \subset\subset \Omega$ and $p>1$ there exists a constant $C>0$ such that*

$$||\nabla_{\varepsilon_j} u_j||_{W^{1,p}_E(B(R))} + ||v_j||_{L^\infty(B(R))} + ||v_j||_{W^{1,2}_{\varepsilon_j}(B(R))} \leq C \tag{6.1}$$

and

$$X_{1,j}u_j \to Xu, \quad X_{2,j}u_j \to 0 \tag{6.2}$$

as $j \to +\infty$ *weakly in* $W^{1,2}_{E,loc}(\Omega)$*. Moreover equation (1.2) can be represented as*

$$X^2 u = 0$$

and is satisfied weakly in the Sobolev sense, and hence, pointwise a.e. in Ω*, i.e.*

$$\int_\Omega X u X^* \phi = 0 \text{ for all } \phi \in C_0^\infty(\Omega).$$

Proof. The uniform bound on $||v_j||_{L^\infty(B(R))}$ follows from the definition of vanishing viscosity solution. The bound on $||v_j||_{W^{1,2}_{\varepsilon_j}(B(R))}$ is a consequence of (5.14). To prove the remaining estimate observe that for any function w: $\partial_2 X_{1,j} w = X_{1,j}\partial_2 w + \partial_2 u_j \partial_2 w$. Substituting $w = u_j$ and in view of (5.14) we see that there exists positive constants C_1, C_2 depending only on the uniform bound on $||v_j||_{L^\infty(B(R))}$ such that for any $p \geq 1$,

$$\begin{aligned} &||\partial_1 \nabla_{\varepsilon_j} u_j||_{L^p(B(R)} + ||\partial_2 \nabla_{\varepsilon_j} u_j||_{L^p(B(R)} \\ &\quad \leq ||X_{1,j} \nabla_{\varepsilon_j} u_j||_{L^p(B(R)} + ||(1+|u_j|)\partial_2 \nabla_{\varepsilon_j} u_j||_{L^p(B(R)} \\ &\qquad \leq ||u_j||_{W^{2,p}_\varepsilon(B(R))} + C_1 ||v_j||_{W^{1,p}_\varepsilon(B(R))} + C_2 \leq C, \end{aligned} \tag{6.3}$$

for a new constant $C > 0$ independent of j. The weak regularity of u and the weak Sobolev convergence follow in a standard fashion.

Next we address the PDE: Since for every j the approximating solution u_j is of class C^∞ then we can use the non divergence form of the equation

$$\sum_{h,k=1}^{2} a_{hk}(\nabla_j u_j) X_{h,j} X_{k,j} u_j = 0.$$

Here

$$a_{h,k}(\nabla_j u_j) \to a_{h,k}(\nabla_0 u) = \delta_{h1}\delta_{k1} \text{ in } L^p,$$

while

$$X_{1,j} u_j \to X u, \quad X_{2,j} u_j \to 0 \tag{6.4}$$

as $j \to +\infty$ weakly in $W^{1,2}_{loc}(\Omega)$. Hence letting j go to ∞ in the non divergence form equation we conclude

$$X^2 u = 0$$

in the Sobolev sense. □

An analogous result holds for higher order derivatives:

Proposition 6.2. *For every* $k \in N$ *for every* $p > 1$ *and for every multiindex* I *of length* k*, the sequence* $(\nabla^I_{\varepsilon_j} u_j)$ *is bounded in* $W^{1,p}_{E,loc}(\Omega)$*. Moreover*

$$X^k_{1,j} u_j \to X^k u, \text{ and } X^k_{2,j} u_j \to 0$$

weakly in $W^{1,p}_E(\Omega)$ *as* $j \to \infty$*. We will express this convergence in the notation*

$$\nabla^I_{\varepsilon_j} u_j \to D^I_0 u \text{ as } j \to +\infty, \text{ weakly in } W^{1,p}_E(\Omega).$$

Proof. Arguing as in the previous result and using (5.14), and (6.1) we deduce that for every ball $B(R) \subset\subset \Omega$ there exist $C_1, C_2, \widetilde{C}$ independent of j, such that

$$||\partial_2 D^I_{\varepsilon_j} u_j||_{L^p(B(R))} \le C_1 \sum_{|J|\le|I|} ||D^J_{\varepsilon_j} v_j||_{L^p(B(R))} + C_2 \le \widetilde{C},$$

$$||\partial_x D^I_{\varepsilon_j} u_j||_{L^p(B(R))} \le ||X_{1,j} D^I_{\varepsilon_j} u_j||_{L^p(B(R))} + ||u_j \partial_2 D^I_{\varepsilon_j} u_j||_{L^p(B(R))} \le \widetilde{C},$$

so that $(\nabla^I_{\varepsilon_j} u_j)$ is bounded in $W^{1,p}_{E,\mathrm{loc}}(\Omega)$ for every $p > 1$, and every multi-index I. Let us prove that the weak limit is $D^I_0 u$. If $|I| = 1$ the assertion is true by (6.2). If I is a multi-index such that $|I| = k$, we can assume by simplicity that $I = (1, I')$, where $|I'| = k - 1$. We can also assume by inductive hypothesis that

$$\partial_2 u_j \to \partial_2 u \;\; as \;\; j \to \infty \;\; weakly \;\; in \;\; L^p(\Omega)$$

$$u_j \to u \;\; as \;\; j \to \infty \;\; in \;\; L^p_{loc}(\Omega)$$

$$\nabla^{I'}_{\varepsilon_j} u_j \to D^{I'}_0 u \;\; as \;\; j \to \infty \;\; in \;\; L^p_{loc}(\Omega).$$

Then integrating by parts

$$\lim_{j\to\infty} \int \nabla^I_{\varepsilon_j} u_j \phi = - \lim_{j\to\infty} \int \nabla^{I'}_{\varepsilon_j} u_j X_j \phi - \int \partial_2 u_j \nabla^{I'}_{\varepsilon_j} u_j \phi =$$

$$= - \int D^{I'}_0 u X\phi - \int \partial_2 u D^{I'}_0 u \phi,$$

and this ensures the weak convergence of $(D^I_{\varepsilon_j} u_j)$ to $D^I_0 u$. □

Remark 6.3. *In view of the Ascoli-Arzelá theorem and the Morrey-Sobolev embedding one has convergence $X^k_{1,j} u_j \to X^k u$ in the C^α norm on compact subsets of Ω for all $\alpha \in (0, 1)$.*

We can now prove the main regularity properties of the limit function u:

Proposition 6.4. *For every k, and for every $p > 1$ the function $z = X^k u$ belongs to $W^{1,p}_{E,loc}(\Omega)$ and it is an a.e. solution of*

$$X^2 z = 0 \quad in \;\; \Omega. \tag{6.5}$$

In particular

$$X^k u \in C^\alpha_{loc}(\Omega) \tag{6.6}$$

for every $0 < \alpha < 1$.

Proof. Since u is a vanishing viscosity solution of $X^2 u = 0$ in Ω, then Proposition 6.2 implies $X^2 u \in W^{1,p}_{E,\mathrm{loc}}(\Omega)$ for all $p \ge 1$. As $X^2 u = 0$ a.e. in Ω, then a simple iteration shows that all the derivatives $X^2 X^k u$ vanish a.e. in Ω. The Hölder regularity (6.6) follows from the classical Morrey-Sobolev embedding theorem. □

We can now give a new pointwise definition of derivative in the direction of vector fields X_1 and X_2.

Definition 6.5. *Let X be a Lipschitz vector field on Ω and let $\xi_0 \in \Omega$ and γ be a solution to problem $\gamma' = X(\gamma)$, $\gamma(0) = \xi_0$.*

We say that a function $f \in C^\alpha_{loc}(\Omega)$, with $\alpha \in]0, 1[$, has Lie-derivative in the direction of the vector field X in ξ_0 if there exists

$$\frac{d}{dh}(f \circ \gamma)_{|h=0},$$

and we will denote its value by $Xf(\xi_0)$.

If the weak derivative of a function f is sufficiently regular, then the two notions of derivatives coincide. For the proof of the following result see [13, Remark 5.6].

Proposition 6.6. *If $f \in C^{\alpha}_{loc}(\Omega)$ for some $\alpha \in]0,1[$ and its weak derivatives $Xf \in C^{\alpha}_{loc}(\Omega), \partial_2 f \in L^p_{loc}(\Omega)$ with $p > 1/\alpha$, then for all $\xi \in \Omega$ the Lie-derivatives $Xf(\xi)$ exist and coincide with the weak ones.*

We are now ready to prove the result concerning the foliation

Proof of Corollary 1.6 The equation $\gamma' = X(\gamma)$ has an unique solution, of the form

$$\gamma(x) = (x, y(x)),$$

where $y'(x) = u(x, y(x))$. In view of the regularity of u and of the previous proposition then $y''(x) = Xu(x, y(x))$, and $y'''(x) = X^2 u(x, y(x)) = 0$. This shows that γ is a polynomial of order 2 and concludes the proof. □

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Department of Mathematical Sciences, University of Arkansas, Fayetteville, AR 72701
E-mail address: `lcapogna@uark.edu`

Dipartimento di Matematica, Piazza Porta S. Donato 5, 40126 Bologna, Italy
E-mail address: `citti@dm.unibo.it`

Dipartimento di Matematica, Piazza Porta S. Donato 5, 40126 Bologna, Italy
E-mail address: `manfredi@dm.unibo.it`