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SMOOTHNESS OF LIPSCHITZ MINIMAL INTRINSIC GRAPHS IN HEISENBERG GROUPS $\mathbb{H}^n$, $n > 1$

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Abstract. We prove that Lipschitz intrinsic graphs in the Heisenberg groups $\mathbb{H}^n$, with $n > 1$, which are vanishing viscosity solutions of the minimal surface equation are smooth.

1. Introduction

The Heisenberg group is a Lie group with Lie algebra $\mathbb{R}^{2n+1}$ endowed with a stratification $V_1 \oplus V_2$, where V_1 has dimension $2n$, and $V_2 = [V_1, V_1]$ has dimension 1. Since we are interested in non-characteristic graphs, it is convenient that we use canonical coordinates of the second kind (the so called *polarized coordinates* [6]) and denote (s, x) the elements of the group, where $x = (x_1, ..., x_{2n})$. Accordingly we will choose a basis of the Horizontal tangent space V_1 as follows:

$$\begin{aligned} &X_s = \partial_s, X_i = \partial_i, \text{ for } i = 1, ..., n-1, \\ &X_i = \partial_i - x_{i-n+1}\partial_{2n}, \quad \text{for } i = n, ..., 2n-1. \end{aligned} \tag{1.1}$$

This set of vectors can be completed to be a basis of the tangent space by adding the vector

$$\partial_{2n} \in V_2.$$

The notion of intrinsic regular surface has been studied in [22], [13]. Such a surface is the graph of a function $u : \mathbb{R}^{2n} \to \mathbb{R}$, and can be represented as

$$M = \{(s, x) : \ s = u(x)\}.$$

Note that C^1 intrinsic graphs are always non-characteristic[1]. According to a version of the implicit function theorem ([22] and [13]), any level surface $\{f(s, x) = c\} \subset \mathbb{H}^n$ of functions $f : \mathbb{H}^n \to \mathbb{R}$ with continuous derivatives along the directions (1.1), can locally (near non-characteristic points) be expressed as an intrinsic graph of a function $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^{2n}$. Moreover the C^1_H smoothness of f implies that the function u is regular with respect to the projection on its domain of the vector fields in (1.1) (see [13] and [1]). Since X_s has null projection of the domain of u, the regularity of this function will be described in terms of the vector fields:

$$X_{i,u} = X_i \text{ for } i \leq 2n-2, X_{2n-1,u} = \partial_{2n-1} + u(x)\partial_{2n}, \tag{1.2}$$

In particular $X_{2n-1,u}$ is a non linear vector field, since it depends on u. Note that the vector fields $X_{1,u}, ..., X_{2n-1,u}$, satisfy Hörmander's finite rank condition in $\mathbb{R}^{2n}$.

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[1] i.e. $T_pM \neq span\{X_s, X_1, ..., X_{2n-1}\}(p)$, for all $p \in M$

Consequently they give rise to a control distance d_u, whose metric balls $B_u(x,r)$ have volume comparable to r^Q, with $Q = 2n+1$ the homogenous dimension of the space $(\mathbb{R}^{2n}, d_u)$.

The notion of mean curvature has been recently introduced as the first variation[2] of the area functional. Several first variation formula have been independently established in receent years, see for instance [18], [7], [8], [4], [32], [33], [27], [28], [35], [36]. For an introduction to the sub-Riemannian geometry of the Heisenberg group and a more detailed list of references see [6]. The prescribed mean curvature equation for intrinsic graphs (over $\Omega \subset \mathbb{R}^{2n}$) in the Heisenberg groups of dimension $n > 1$ has the following expression

$$Lu = \sum_{i=1}^{2n-1} X_{i,u}\left(\frac{X_{i,u}u}{\sqrt{1+|\nabla_u u|^2}}\right) = f, \text{ for } x \in \Omega \subset \mathbb{R}^{2n}. \tag{1.3}$$

where

$$\nabla_u = (X_{1,u}, \dots, X_{2n-1,u}).$$

If $u \in C^2(\Omega)$ is a solution of (1.3) for $f = 0$ then its graph is a critical point of the perimeter and consequently it is called a minimal intrinsic graph.

Properties of regular minimal surfaces have been investigated in [24], [31], [8], [7], [23], [19], [2] and [29].

Since minimal surfaces arise as critical points of the perimenter functional, the variational formulation naturally provides several notions of non regular solutions (see for instance [24], [31] and [7]). Indeed existence of BV minimizers of the perimeter is proved in [24], [31] using direct methods of the calculus of variations, More recently, existence of Lipschitz continuous vanishing viscosity solutions has been studied in [7]. Such solutions arise as the sub-Riemannian mean curvature equation is approximated by Riemannian problems which express the mean curvature in an approximating Riemannian metrics (see [31] and [7] for the relation between Riemannian and sub-Riemannian curvature). The Riemannian approximation of (1.3) is

$$L_\varepsilon u = \sum_{i=1}^{2n} X^\varepsilon_{i,u}\left(\frac{X^\varepsilon_{i,u}u}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}\right) = f, \text{ for } x \in \Omega \subset \mathbb{R}^{2n}. \tag{1.4}$$

where

$$X^\varepsilon_{i,u} = X_{i,u} \text{ for } i \le 2n-1, \quad X^\varepsilon_{2n,u} = \varepsilon\partial_{2n} \quad \text{and} \quad \nabla^\varepsilon_u = (X^\varepsilon_{1,u}, \dots, X^\varepsilon_{2n,u}). \tag{1.5}$$

Definition 1.1. *Letting C^1_E denote the standard Euclidean C^1 norm, we will say that an Euclidean Lipschitz continuous function u is a vanishing viscosity solution of* (1.3) *in an open set Ω, if there exists a sequence u_ε of smooth solutions of* (1.4) *in Ω such that for every compact set $K \subset \Omega$*

- *$||u_\varepsilon||_{C^1_E(K)} \le C$ for every ε;*
- *$u_\varepsilon \to u$ as $\varepsilon \to 0$ pointwise a.e. in Ω.*

As mentioned above, existence of this type of viscosity solutions in the case of $t-$graphs, i.e. graphs of the form $x_{2n} = g(s, x_1, ..., x_{2n-1})$, has been proved in [7, Theorem A and Theorem 4.5]. For such graph the corresponding PDE is more degenerate than (1.4) as characteristic points are allowed (indeed, much of

[2]For variations which do not move the characteristic set

the analysis in [7] and [8] is focused on the study of solutions near such points). In the same paper the authors prove that such solutions are minimizers of the perimeter and address questions of uniqueness and comparison theorems as well. The problem of regularity of minimal surfaces is still largely open. In this paper we address the issue of regularity away from characteristic points. Our goal is to prove the following

Theorem 1.2. *The Lipschitz continuous vanishing viscosity solutions of* (1.3) *with zero right-hand-side* $f = 0$ *are smooth functions.*

Invoking the implicit function theorem, we want to apply Theorem 1.2 to the study of the regularity away from the characteristic locus of the Lipschitz perimeter minimizers found in [7] for the case $\mathbb{H}^n$, $n > 1$. Here and in the following ∇_E denotes the Euclidean gradient in $\mathbb{R}^{2n}$.

Corollary 1.3. *Let* $O \subset \mathbb{R}^{2n}$ *be a strictly convex, smooth open set,* $\varphi \in C^{2,\alpha}(\bar{O})$ *and for each* $(s, x_1, ..., x_{2n-1}) \in O$ *denote by* $(s, x_1, ..., x_{2n-1})^* = (x_1, -s, x_3, -x_2, ...)$. *Consider the family*

$$\{g_\varepsilon(s, x_1, ..., x_{2n-1})\}_\varepsilon \quad \sup_O |g_\varepsilon| + \sup_O |\nabla_E g_\varepsilon| \le C \quad (\textit{uniformly in } \varepsilon),$$

of smooth solutions of the approximating minimal surface PDE

$$div\left(\frac{\nabla_E g_\varepsilon + (s, x_1, ..., x_{2n-1})^*}{\sqrt{\varepsilon^2 + |\nabla_E g_\varepsilon + (s, x_1, ..., x_{2n-1})^*|}}\right) = 0 \text{ in } O \quad \text{and } g_\varepsilon = \varphi \quad \text{in } \partial O$$

found in [7, Theorem 4.5]. *If for* $p_0 = (p_0^s, p_0^1, ..., p_0^{2n-1}) \in O$, $a > 0$ *and for every* $\varepsilon > 0$ *we have* $|\partial_s g_\varepsilon(p_0)| > a > 0$ *(or any other partial derivative is non-vanishing at* p_0 *uniformly in* ε*) then there is a sequence* $\varepsilon_k \to 0$ *such that the Lipschitz perimeter minimizer* $g = \lim_{\varepsilon_k \to 0} g_{\varepsilon_k}$ *is smooth in a neighborhood of the point* p_0.

Proof. The implicit function theorem implies that the level set of

$$x_{2n} - g_\varepsilon(s, x_1, ..., x_{2n-1})$$

can be written as smooth intrinsic graphs $s = u_\varepsilon(x)$ in a neighborhood Ω of $(p_0^1, ..., p_0^{2n-1}, g(p_0))$. The Lipschitz bounds on g_ε (proved in [7, Propositions 4.2-4]) yield uniform Lipschitz bounds on u_ε, thus allowing to apply Theorem 1.2 and conclude the proof. □

We remark that in the case $n = 1$ of the first Heisenberg group the regularity of vanishing viscosity minimal intrinsic graphs is quite different. In the forthcoming paper [5] we study this problem and prove a form of intrinsic regularity, with differentiability along the Legendrian foliation of the minimal graph.

Equation (1.3) is an uniformly elliptic approximation of a subelliptic equations. The defining vector fields have Lipschitz coefficients and satisfy a weak Hörmander condition, since together with their first order vector fields they span the space at every point. The main difficulty of the proof is to handle the vector field

$$X_{2n-1,u} = \partial_{2n-1} + u(x)\partial_{2n}$$

and the dependence on ε. A similar difficulty arises in problems of mathematical finance. For example in [16], [17], it was proved that the viscosity solutions of the following equation are C^∞

$$X_1^2 u + X_2 u = 0$$

where $X_1 = \partial_{xx}$,$X_2 = \partial_y u + u\partial_z u$, satisfy a weak Hörmander condition analogue to the one in the present paper. The techniques in [16], [17], provide the main inspiration for the proof of Theorem 1.2.

The regularity of solutions will be measured in terms of the natural norm of the intrinsic Hölder class $C^{1,\alpha}_u$, i.e. functions f such that $\nabla^\varepsilon_u f$ is Hölder continuous, with respect to the control distance d_u. The proof will be accomplished in two steps:

STEP 1 First prove that the Lipschitz continuous solutions are of class $C^{1,\alpha}_u$. Since the operator L_ε in (1.4) is represented in divergence form, then by differentiating the PDE and combining several horizontal and "vertical" energy estimates, it is possible to prove a Euclidean Cacciopoli-type inequality for the intrinsic gradient $\nabla^\varepsilon_u u$ of the solution. The Moser iteration technique will then lead to Hölder continuous estimates uniform in ε for the gradient. This step holds also for $n = 1$.

STEP 2 We prove the smoothness of the solution. In order to do so we first note that the operator L_ε can also be represented in a divergence form:

$$L_\varepsilon u = \sum_{i,j=1}^{2n} a^\varepsilon_{i,j}(\nabla^\varepsilon_u u) X^\varepsilon_{i,u} X^\varepsilon_{j,u} u, \tag{1.6}$$

where

$$a^\varepsilon_{ij} : R^{2n} \to R \quad a^\varepsilon_{ij}(p) = \delta_{ij} - \frac{p_i p_j}{1+|p|^2}.$$

For every fixed point x_0 we will approximate the assigned operator with a linear, uniformly subelliptic operator in divergence form L_{ε,x_0}, with C^∞ coefficients. The approximation is carried out through a ad-hoc *freezing* technique, where the function u in the coefficients of the vector field is substituted with polynomials, in a technique reminiscent of the work of Rothschild and Stein [34]. The novel difficulty arises from the non-smoothness of u, and has to be dealt with through a delicate bootstrap argument. The existence of a fundamental solution $\Gamma^\varepsilon_{x_0}$ for such operator as well as its estimates, uniform in ε, have previously been proved in the papers [3] and [12]. Eventually $\Gamma^\varepsilon_{x_0}$ will be used to define a parametrix for the fundamental solution of L_ε and to obtain estimates independent of ε, of the derivatives of any order of the solution.

2. Notations and known results

2.1. **Hölder classes.** In the sequel we will always keep fixed a function $\bar{u} \in C^\infty = C^\infty(\Omega)$, with $\Omega \subset \mathbb{R}^{2n}$ and consider the vector fields $X^\varepsilon_{i,\bar{u}}$ in (1.5), with coefficients depending on the fixed function $\bar{u}$. Let us define a new vector field

$$X^\varepsilon_{2n+1,\bar{u}} = \partial_{2n} = [X^\varepsilon_{1,\bar{u}}, X^\varepsilon_{n,\bar{u}}],$$

which act as a second order derivative, and call degree of σ_i the natural number $deg(\sigma_i) = 1$ for $\sigma_i \le 2n$, $deg(2n+1) = 2$. Correspondingly the degree of any multi-index $\sigma = (\sigma_1, \dots, \sigma_m)$, $\sigma_r \in \{1, \dots, 2n+1\}$, $1 \le r \le m \in \mathbb{N}$, will be:

$$deg(\sigma) = \sum_{i=1}^m deg(\sigma_i).$$

We will also denote the cardinality of $\sigma = (\sigma_1, \dots, \sigma_m)$ the number of its elements:

$$\#(\sigma) = m.$$

We define the intrinsic derivative

$$\nabla^{\varepsilon}_{\sigma,\bar u} = X^{\varepsilon}_{\sigma_1,\bar u}\cdots X^{\varepsilon}_{\sigma_m,\bar u}, \tag{2.1}$$

and $\nabla^{\varepsilon k}_{\bar u}$ the vector field with components $(\nabla^{\varepsilon}_{\sigma\bar u})_{deg(\sigma)=k}$.

Since the vector fields $X^{\varepsilon}_{1,\bar u},...,X^{\varepsilon}_{2n,\bar u}$, are the Riemannian completion of an Hörmander type set of vectors, they give rise to a control distance $d_{\varepsilon,\bar u}$. The corresponding metric balls are denoted $B_{\varepsilon,\bar u}(x,r)$. As $\varepsilon\to 0$ the metric space $(\Omega, d_{\varepsilon,\bar u})$ converge in the Gromov-Hausdorff sense to $(\Omega, d_{\bar u})$ (see [6]).

We next define the spaces of Hölder continuous functions related to the fixed function $\bar u$.

Definition 2.1. *Let $x_0\in\Omega$, $0<\alpha<1$, assume that $\bar u$ is a fixed Lipschitz continuous function, and that u is defined on Ω. We say that $u\in C^{\alpha}_{\bar u}(\Omega)$ if for every compact set K there exists a positive constant M such that for every $x, x_0\in K$ and $\varepsilon>0$*

$$|u(x)-u(x_0)|\le M d_{\varepsilon,\bar u}(x,x_0). \tag{2.2}$$

Iterating this definition, if $k\ge 1$, we say that $u\in C^{k,\alpha}_{\bar u}(\Omega)$, if $\nabla^{\varepsilon}_{\bar u}u\in C^{k-1,\alpha}_{\bar u}(\Omega)$.

2.2. **Taylor approximation.** The following result is well know for vector fields with C^∞ coefficients (see [30]) also holds for vector field is of the form $\partial_1+\bar u\partial_{2n}$, with $\bar u$ Lipschitz continuous with respect to the Euclidean distance. Let us first denote by $e_1,\dots,e_{2n}$ the canonical coordinates of a point x around x_0,

$$x = exp(\sum_{i=1}^{2n-1} e_i X^{\varepsilon}_{i,\bar u} + e_{2n}X^{\varepsilon}_{2n+1,\bar u})(x_0)$$

and, for a multi-index $\sigma=(\sigma_1,\dots,\sigma_{2n})$ we will denote $e_\sigma=(e_{\sigma_1},\dots,e_{\sigma_{2n}})$.

We explicitly note that, since $X^{\varepsilon}_{2n,u}$ and $X^{\varepsilon}_{2n+1,u}$ are parallel, only one of them can appear in the definition of canonical coordinates, otherwise the values of e_i would not be uniquely determined. Due to this fact, for every multi-index $\sigma=(\sigma_1,\cdots\sigma_m)$, with components in $\{1,\cdots 2n\}$ we will denote $I(\sigma)=(\varrho_1,\cdots\varrho_m)$, where $\varrho_i=\sigma_i$ if $\sigma_i\ne 2n$, and $\varrho_i=2n+1$ if $\sigma_i=2n$.

Theorem 2.2. *Let $x_0\in\Omega$, $0<\alpha<1, k\in\mathbb{N}\cup\{0\}$ and assume that $u\in C^{k,\alpha}_{\bar u}(\Omega)$. Then we can define Taylor polynomial of order k the function*

$$P^k_{x_0}u(x) = \sum_{h=1}^{k}\sum_{\substack{deg(\sigma)=h,\\ \sigma_i\ne 2n+1}} \frac{1}{\#(\sigma)!} e_\sigma \nabla^{\varepsilon}_{I(\sigma),\bar u}u(x_0)$$

and we have

$$u(x) = P^k_{x_0}u(x) + O\left(d_{\varepsilon,\bar u}(x_0,x)^{k+\alpha}\right) \quad as\ \ x\to x_0. \tag{2.3}$$

We will also set $P^k_{x_0}u=0$ for any negative integer k.

Note that

$$P^1_{x_0}u(x) = \sum_{i=1}^{2n-1} e_i(x)X^{\varepsilon}_{i,\bar u}u(x_0) + e_{2n}(x)X^{\varepsilon}_{2n+1,\bar u}u(x_0).$$

From the explicit expression of the Taylor polynomials of order less than 4 it is possible to directly deduce the following result.

Remark 2.3. *If $u \in C^{k,\alpha}_{\bar u}(\Omega)$, $0 \le k \le 4$, K is a compact subset of Ω and σ is a multi-index, then there exists $C > 0$ such that*

$$\left|P^k_{x_0}u(\xi) - P^k_x u(\xi)\right| \le C d^\alpha_{\varepsilon,\bar u}(x_0,x) d^k_{\varepsilon,\bar u}(x_0,\xi), \tag{2.4}$$

for every $x, x_0, \xi \in K$, see [11, Lemma 3.6] *and* [16, Remarks 2.24 and 2.25].

2.3. **Derivatives and Frozen derivatives.** We will introduce here first order operators with polynomial coefficients which locally approximate the vector fields $X^\varepsilon_{i,\bar u}$. These new vector fields are defined in terms of the Taylor development of the coefficients of $X^\varepsilon_{i,\bar u}$. Precisely, for any fixed point x_0 we will call operator frozen at the point x_0

$$X^\varepsilon_{i,x_0} = X^\varepsilon_{i,\bar u} \text{ if } i \ne 2n-1, \quad X^\varepsilon_{2n-1,x_0} = \partial_{2n-1} + P^1_{x_0}\bar u(x)\partial_{2n}$$

and for every multi-index σ,

$$\nabla^\varepsilon_{\sigma,x_0} = X^\varepsilon_{\sigma_1,x_0}\cdots X^\varepsilon_{\sigma_m,x_0},$$

and $\nabla^{\varepsilon k}_{x_0}$ will be the vector field with components $(\nabla^\varepsilon_{\sigma,x_0})_{deg(\sigma)=k}$.

These frozen derivatives have been defined as approximation of the intrinsic derivatives, depending on $\bar u$. In order to clarify this point, we recall the following definition, given in [21] and [30]. If $\alpha \in R$ and $f(x,x_0) = O(d^\alpha_{\varepsilon,\bar u}(x,x_0))$ as $x \to x_0$, we will say that the differential operator $f(x,x_0)\nabla^\varepsilon_{\sigma,x_0}$ has degree $deg(\sigma) - \alpha$. We have

$$X^\varepsilon_{i,\bar u} = X^\varepsilon_{i,x_0} \quad \text{if } i \ne 2n-1$$
$$X^\varepsilon_{i,\bar u} = X^\varepsilon_{i,x_0} + (\bar u - P^1_{x_0}\bar u(x))\partial_{2n} \quad \text{if } i = 2n-1$$

Hence, if $\bar u$ is of class $C^{1,\alpha}_{\bar u}$, then $(\bar u - P^1_{x_0}\bar u(x))\partial_{2n}$ is a differential operator of degree $1-\alpha$, while $X^\varepsilon_{i,\bar u}$ and X^ε_{i,x_0} have degree 1. This means that the intrinsic derivative is expressed as the frozen derivative, plus a lower order term.

More generally the following approximation result holds:

Lemma 2.4. *If $\bar u \in C^{k-1,\alpha}_{\bar u}(\Omega)$, and σ is a multi-index such that $deg(\sigma) \le k$, then for every function $\varphi \in C^\infty_0(\Omega)$ the derivative $\nabla^\varepsilon_{\sigma,\bar u}\varphi$ can be represented as*

$$\nabla^\varepsilon_{\sigma,\bar u}\varphi = \nabla^\varepsilon_{\sigma,x_0}\varphi + \sum_{deg(\rho)-h\le deg(\sigma)} (\bar u - P^1_{x_0}\bar u)^h \sum_{\substack{deg(\mu_1)+\cdots+deg(\mu_k)\le k-1\\ deg(\mu_k)\ge 0}} C_{\rho,\mu_i,\sigma,h} \prod_{1\le deg(\mu_i)} \nabla^\varepsilon_{\mu,\bar u}(\bar u - P^1_{x_0}\bar u)\,\nabla^\varepsilon_{\rho,x_0}\varphi, \tag{2.5}$$

where $C_{\rho,\mu,\sigma,h}$ are suitable constants. In particular the operator $\nabla^\varepsilon_{\sigma,\bar u}\varphi(x)$ can be identified as a differential operator of degree $deg(\sigma)$ and represented in terms of frozen derivatives.

Proof. Since the function φ is of class $C^\infty_0(\Omega)$, its Lie derivatives can be simply computed as directional derivatives. By definition

$$X^\varepsilon_{i,\bar u}\varphi = X^\varepsilon_{i,x_0}\varphi + \delta_{i,2n-1}(\bar u - P^1_{x_0}\bar u)\,\partial_{2n}\varphi. \tag{2.6}$$

Hence the assertion is true if $deg(\sigma) = 1$. If the assertion is true for any σ such that $deg(\sigma) = k$, then we consider a multiindex σ such that $deg(\sigma) = k+1$. In this case

$$\sigma = (\sigma_1, \bar\sigma),$$

where

$$\begin{cases} deg(\bar\sigma) = k & \text{if } \sigma_1 \neq 2n+1 \\ deg(\bar\sigma) = k-1 & \text{if } k \geq 3 \text{ and } \sigma_1 = 2n+1 \end{cases}$$

We have

$$\nabla^\varepsilon_{\sigma,\bar u}\varphi(x) = X^\varepsilon_{\sigma_1,\bar u}\nabla^\varepsilon_{\bar\sigma,\bar u}\varphi(x) =$$

by inductive assumption

$$= X^\varepsilon_{\sigma_1,\bar u}\Big(\nabla^\varepsilon_{\bar\sigma,x_0}\varphi\Big) + \sum_{deg(\rho)-h\leq deg(\bar\sigma)} X^\varepsilon_{\sigma_1,\bar u}\Big((\bar u - P^1_{x_0}\bar u)^h \sum_{\substack{deg(\mu_1)+\cdots deg(\mu_k)\leq k-1 \\ deg(\mu_k)\geq 0}} C_{\rho,\mu_i,\bar\sigma,h} \prod_{1\leq deg(\mu_i)} \nabla^\varepsilon_{\mu,\bar u}(\bar u - P^1_{x_0}\bar u)\,\nabla^\varepsilon_{\rho,x_0}\varphi\Big),$$

(also using (2.6))

$$= X^\varepsilon_{\sigma_1,x_0}\nabla^\varepsilon_{\bar\sigma,x_0}\varphi + \delta_{\sigma_1 2n-1}(\bar u - P^1_{x_0}\bar u)\partial_{2n}\nabla^\varepsilon_{\bar\sigma,x_0}\varphi$$

$$+ \sum_{deg(\rho)-h\leq deg(\bar\sigma)} (\bar u - P^1_{x_0}\bar u)^{h-1} X^\varepsilon_{\sigma_1,\bar u}(\bar u - P^1_{x_0}\bar u) \sum_{\substack{deg(\mu_1)+\cdots deg(\mu_k)\leq k-1 \\ deg(\mu_k)\geq 0}} C_{\rho,\mu_i,\bar\sigma,h} \prod_{1\leq deg(\mu_i)} \nabla^\varepsilon_{\mu,\bar u}(\bar u - P^1_{x_0}\bar u)\,\nabla^\varepsilon_{\rho,x_0}\varphi +$$

$$+ \sum_{deg(\rho)-h\leq deg(\bar\sigma)} (\bar u - P^1_{x_0}\bar u)^h \sum_{\substack{deg(\mu_1)+\cdots+deg(\mu_k)\leq k-1 \\ deg(\mu_k)\geq 0}} C_{\rho,\mu_i,\bar\sigma,h} X^\varepsilon_{\sigma_1,\bar u}\Big(\prod_{1\leq deg(\mu_i)} \nabla^\varepsilon_{\mu,\bar u}(\bar u - P^1_{x_0}\bar u)\Big)\,\nabla^\varepsilon_{\rho,x_0}\varphi +$$

$$+ \sum_{deg(\rho)-h\leq deg(\bar\sigma)} (\bar u - P^1_{x_0}\bar u)^h \sum_{\substack{deg(\mu_1)+\cdots+deg(\mu_k)\leq k-1 \\ deg(\mu_k)\geq 0}} C_{\rho,\mu_i,\bar\sigma,h} \prod_{1\leq deg(\mu_i)} \nabla^\varepsilon_{\mu,\bar u}(\bar u - P^1_{x_0}\bar u)\,X^\varepsilon_{\sigma_1,x_0}\Big(\nabla^\varepsilon_{\rho,x_0}\varphi\Big) =$$

$$+\delta_{\sigma_1,2n-1} \sum_{deg(\rho)-h\leq deg(\bar\sigma)} (\bar u - P^1_{x_0}\bar u)^{h+1} \sum_{\substack{deg(\mu_1)+\cdots+deg(\mu_k)\leq k-1 \\ deg(\mu_k)\geq 0}} C_{\rho,\mu_i,\bar\sigma,h} \prod_{1\leq deg(\mu_i)} \nabla^\varepsilon_{\mu,\bar u}(\bar u - P^1_{x_0}\bar u)\,\partial_{2n}\nabla^\varepsilon_{\rho,x_0}\varphi.$$

Note that the first term satisfies

$$X^\varepsilon_{\sigma_1,x_0}\nabla^\varepsilon_{\bar\sigma,x_0}\varphi = \nabla^\varepsilon_{\sigma,x_0}\varphi.$$

The second term is $(\bar u - P^1_{x_0}\bar u)\partial_{2n}\nabla^\varepsilon_{\bar\sigma,x_0}\varphi$. It can be considered one of the term listed in the thesis, with $h = 1$, while $\partial_{2n}\nabla^\varepsilon_{\bar\sigma,x_0} = \nabla^\varepsilon_{\rho,x_0}$, for a suitable ϱ, of degree $deg(\varrho) = k+2$. Hence $deg(\varrho) - h = k+1$. Similarly, all the other terms are in the form, indicated in the thesis (in both case, $\sigma_1 \neq 2n+1$ or $k \geq 3$ and $\sigma_1 = 2n+1$). □

The vector fields X^ε_{i,x_0} satisfy an Hörmander type condition, hence they define a control distance $d_{\varepsilon,x_0}(x_0,\xi)$. The corresponding metric balls $B_{\varepsilon,x_0}(x,r)$ have volume comparable to r^{2n+1}, and we will call

$$Q = 2n+1 \tag{2.7}$$

the *homogeneous dimension* of the space $(\mathbb{R}^{2n}, d_{\varepsilon,x_0})$. Note that the homogeneous dimension is the same as the Hausdorff dimension of $(\mathbb{R}^{2n}, d_{\bar u})$, defined in the introduction. A simple modification of Proposition 2.4 in [16] yields the following relation between d_{ε,x_0} and the control distance $d_{\varepsilon,\bar u}$ associated to the vector fields $X^\varepsilon_{i,\bar u}$:

Proposition 2.5. *For every compact subset K of Ω, there exists a positive constant $C = C(K)$ such that for every $x, x_0, \in K$*
$C^{-1}d_{\varepsilon,x_0}(x_0,\xi) \leq d_{\varepsilon,\bar u}(x_0,\xi) \leq C d_{\varepsilon,x_0}(x_0,\xi)$,
$d_{\varepsilon,x_0}(x_0,x) \leq C(d_{\varepsilon,x_0}(x_0,\xi) + d_{\varepsilon,\xi}(\xi,x))$,
$d_{\varepsilon,\bar u}(x_0,x) \leq C(d_{\varepsilon,\bar u}(x_0,\xi) + d_{\varepsilon,\bar u}(\xi,x))$.

2.4. **Linearized and frozen operator.** In analogy with the definition of linear vector fields, in terms of a fixed function $\bar u$, we can also define a linearization $L_{\varepsilon,\bar u}$ of the operator L_ε, written in terms of the linearized vector fields $X^\varepsilon_{i,\bar u}$:

$$L_{\varepsilon,\bar u} u = \sum_{i,j=1}^{2n} a^\varepsilon_{i,j}(\nabla^\varepsilon_{\bar u}\bar u) X^\varepsilon_{i,\bar u} X^\varepsilon_{j,\bar u} u, \tag{2.8}$$

where $a^\varepsilon_{i,j}$ are defined in (1.6). Since the function $\bar u$ is fixed, the operator is a linear non divergence type operator, whose coefficients have the regularity of the function $\bar u$. In case $\bar u$ is not smooth, it is natural to approximate it with a frozen operator, defined in term of the vector fields X^ε_{i,x_0}:

$$L_{\varepsilon,x_0} u = \sum_{i,j=1}^{2n} a^\varepsilon_{i,j}(\nabla^\varepsilon_{\bar u}\bar u(x_0)) X^\varepsilon_{i,x_0} X^\varepsilon_{j,x_0} u, \tag{2.9}$$

where a^ε_{ij} are defined in (1.6). This is a divergence form uniformly subelliptic operator with C^∞ coefficients, which depends on ε. Hence it has a fundamental solution $\Gamma^\varepsilon_{x_0}$ (see [3]), and its dependence on ε which can be handled as in [12]. Since $\Gamma^\varepsilon_{x_0}$ depends on many variables, the notation

$$X^\varepsilon_{i,x_0}(x)\Gamma^\varepsilon_{x_0}(\,\cdot\,,\xi)$$

shall denote the X^ε_{i,x_0}-derivative of $\Gamma^\varepsilon_{x_0}(s,\xi)$ with respect to the variable s, evaluated at the point x.

Theorem 2.6. *([12] - Theorem 1.1) Let $x_0 \in \Omega$. For every compact set $K \subset \Omega$ and for every $p \in \mathbb{N}$ there exist two positive constants C, C_p independent of ε, such that*

$$|\nabla^\varepsilon_{\sigma,x_0}(x)\Gamma^\varepsilon_{x_0}(\cdot,\xi)| \le C_p \frac{d^{2-p}_{\varepsilon,x_0}(x,\xi)}{|B_{\varepsilon,x_0}(x, d_{\varepsilon,x_0}(x,\xi))|}, \quad deg(\sigma) = p \tag{2.10}$$

for every $x, \xi \in K$ with $x \ne \xi$, where $B_{\varepsilon,x_0}(x,r)$ denotes the ball with center x and radius r of the distance d_{ε,x_0}. If $p = 0$ we mean that no derivative are applied on $\Gamma^\varepsilon_{x_0}$.

Remark 2.7. *With the same notation as in preceding theorem, from Lemma 2.4, and inequality (2.10) it follows that*

$$|\nabla^\varepsilon_{\sigma,\bar u}(x)\Gamma^\varepsilon_{x_0}(\cdot,\xi)| \le C_p \frac{d^{2-p}_{\varepsilon,x_0}(x,\xi)}{|B_{\varepsilon,x_0}(x, d_{\varepsilon,x_0}(x,\xi))|}, \quad deg(\sigma) = p \tag{2.11}$$

for every $x, \xi \in K$ with $x \ne \xi$.

Hence, using Proposition 2.20 and 2.21 in ([16]) we have:

Proposition 2.8. *Let $k \in \mathbb{N}$, $2 \le k \le 6$. Let $\bar u \in C^{k-1,\alpha}_{\bar u}(\Omega)$ and K be a compact subset of Ω. There is a positive constant C independent of ε, such that*

$$\begin{aligned} &\left|\left(\nabla^\varepsilon_{\sigma,\bar u}(x) - \nabla^\varepsilon_{\sigma,\bar u}(x_0)\right)\Gamma^\varepsilon_{x_0}(\cdot,\xi)\right| \\ &\le C\left(d_{\varepsilon,x_0}(x_0,x)d_{\varepsilon,x_0}(x_0,\xi)^{-Q-deg(\sigma)+1} + d_{\varepsilon,x_0}(x_0,x)^\alpha d_{\varepsilon,x_0}(x_0,\xi)^{-Q-deg(\sigma)+2}\right), \end{aligned} \tag{2.12}$$

and

$$\begin{aligned}
&\left|\nabla^\varepsilon_{\sigma,\bar{u}}(x)\Gamma^\varepsilon_x(\cdot,\xi) - \nabla^\varepsilon_{\sigma,\bar{u}}(x)(x_0)\Gamma^\varepsilon_{x_0}(\cdot,\xi)\right| \\
&\quad \le C\left(d_{\varepsilon,x_0}(x_0,x)d_{\varepsilon,x_0}(x_0,\xi)^{-Q-deg(\sigma)+1} + d_{\varepsilon,x_0}(x_0,x)^\alpha d_{\varepsilon,x_0}(x_0,\xi)^{-Q-deg(\sigma)+2}\right), \qquad (2.13)
\end{aligned}$$

for every multi-index σ, $deg(\sigma) = k$, and for every $x, x_0 \in K$ and ξ such that $d_{\varepsilon,x_0}(x_0,\xi) \ge M d_{\varepsilon,x_0}(x_0,x)$, for suitable $M > 0$. The constant Q is the homogeneous dimension of the space, defined in (2.7).

Estimates of this type for the fundamental solution are the key elements used in Proposition 3.9 in [16] to prove the following result:

Proposition 2.9. *Let $k \in \mathbb{N}$, $2 \le k \le 4$. Assume that u is a function of class $C^{k-1}_{\bar{u}}(\Omega)$ and that there are open sets $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset \Omega$ such that for every $x \in \Omega_1$ the function u admits the following representation*

$$\begin{aligned}
u(x) = &\int_\Omega \Gamma^\varepsilon_{x_0}(x,\xi)N_1(\xi,x_0)d\xi + \int_\Omega \Gamma^\varepsilon_{x_0}(x,\xi)N_{2,k}(\xi,x_0)d\xi \\
&+ \sum_{i=1}^{2n}\int_\Omega X^\varepsilon_{i,x_0}\Gamma^\varepsilon_{x_0}(x,\xi)N_{2,ki}(\xi,x_0)d\xi + \int_\Omega \Gamma^\varepsilon_{x_0}(x,\xi)N_{3,k}(\xi,x_0)d\xi. \qquad (2.14)
\end{aligned}$$

Also assume that for every x_0 fixed $\in \Omega_1$, the kernels $N_i(\cdot,x_0)$ are supported in $\overline{\Omega_3}$, as functions of their first variable and there exists a constant C_1 such that the kernels satisfy the following conditions:

(i) if x_0 is fixed $\in \Omega_1$ the $N_1(\cdot,x_0)$ is supported in $\overline{\Omega_3 - \Omega_2}$

$$|N_1(\xi,x_0) - N_1(\xi,x)| \le C_1\, d^\alpha_{\varepsilon,x_0}(x_0,x); \qquad (2.15)$$

(ii) $N_{2,k}(\cdot,x_0)$ and $N_{2,ki}(\cdot,x_0)$ are smooth functions and all derivatives are uniformly Hölder continuous in the variable x_0, satisfying condition (2.15) with the same constant C_1 as N_1;

(iii) for every $\xi \in \Omega_3$ and $x, x_0 \in \Omega_1$

$$|N_{3,k}(\xi,x_0)| \le C_1 d^{k-2+\alpha}_{\varepsilon,x_0}(x_0,\xi), \qquad (2.16)$$

and

$$|N_{3,k}(\xi,x_0) - N_{3,k}(\xi,x)| \le C_1 d^\alpha_{\varepsilon,x_0}(x_0,x)d^{k-2}_{\varepsilon,x_0}(x_0,\xi). \qquad (2.17)$$

Then $u \in C^k_{\bar{u}}$ and for every σ such that $deg(\sigma) = k$

$$\begin{aligned}\nabla^\varepsilon_{\sigma,\bar u}(u\varphi)(x_0) = & \int_\Omega \nabla^\varepsilon_{\sigma,\bar u}(x_0)\Gamma^\varepsilon_{x_0}(\cdot,\xi)N_1(\xi,x_0)d\xi \\ & + \int_\Omega \Gamma^\varepsilon_{x_0}(\xi,0)\nabla^\varepsilon_{\sigma,\bar u}(x_0)\big(N_{2,k}(x_0\circ\xi^{-1},x_0)\big)d\xi \\ & + \sum_{i=1}^{2n}\int_\Omega X^\varepsilon_{i,x_0}(x_0)\Gamma^\varepsilon_{x_0}(\xi,0)\nabla^\varepsilon_{\sigma,\bar u}(x_0)\big(N_{2,ki}(x_0\circ\xi^{-1},x_0)\big)d\xi \\ & + \int_\Omega \nabla^\varepsilon_{\sigma,\bar u}(x_0)\Gamma^\varepsilon_{x_0}(x,\xi)N_{3,k}(\xi,x_0)d\xi.\end{aligned} \tag{2.18}$$

Besides, for any $\alpha' < \alpha$, there exists a constant C only dependent on C_1 and on C_p in (2.10) such that

$$||u||_{C^{k,\alpha'}_{\bar u}} \le C.$$

Remark 2.10. *The derivatives $\nabla^\varepsilon_{\sigma,\bar u}(x_0)$ in (2.18) can be computed by Lemma 2.4 in term of the frozen derivatives $\nabla^\varepsilon_{\sigma',x_0}(x_0)$. In particular the frozen derivatives $\nabla^\varepsilon_{\sigma,x_0}(x_0)(h(\cdot\circ\xi^{-1}))$ can be calculated by formula in Proposition 2.23 in* [16].

Remark 2.11. *It is not difficult to prove that the same result is still true, if u has a more general representation*

$$\begin{aligned} u(x) = & \int_\Omega \Gamma^\varepsilon_{x_0}(x,\xi)N_1(\xi,x_0)d\xi + \int_\Omega \Gamma^\varepsilon_{x_0}(x,\xi)N_{2,k}(\xi,x_0)d\xi \\ & + \sum_{i=1}^{2n}\int_\Omega X^\varepsilon_{i,x_0}\Gamma^\varepsilon_{x_0}(x,\xi)N_{2,ki}(\xi,x_0)d\xi + \int_\Omega \Gamma^\varepsilon_{x_0}(x,\xi)N_{3,k}(\xi,x_0)d\xi \\ & + \sum_{i=1}^{2n}\int_\Omega X^\varepsilon_{i,x_0}\Gamma^\varepsilon_{x_0}(x,\xi)N_{4,ki}(\xi,x_0)d\xi, \end{aligned} \tag{2.19}$$

where the kernels $N_{4,ki}(\xi,x_0)$ satisfy assumptions similar to $N_{3,k}$:

for every $\xi\in\Omega_3$ and $x,x_0\in\Omega_1$

$$|N_{4,ki}(\xi,x_0)| \le C_1 d^{k-1+\alpha}_{\varepsilon x_0}(x_0,\xi), \tag{2.20}$$

and

$$|N_{4,ki}(\xi,x_0) - N_{4,ki}(\xi,x)| \le C_1 d^\alpha_{\varepsilon,x_0}(x_0,x) d^{k-1}_{\varepsilon,x_0}(x_0,\xi). \tag{2.21}$$

3. From *Lip* to $C^{1,\alpha}_{\bar u}$.

Let us now start the first step in the proof of the regularity result. Using in full strength the nonlinearity of the operator L_ε, we prove here some Cacciopoli-type inequalities for the intrinsic gradient of u, and for the derivative $\partial_{2n}u$. The main novelty of the proof is that putting together two intrinsic subelliptic Cacciopoli inequalities we will end up with an Euclidean Cacciopoli inequality. In this way we can obtain the Hölder-regularity of the gradient via a standard Moser procedure.

We first observe that

$$\partial_{2n} X^\varepsilon_{i,u} u = -(X^\varepsilon_{i,u})^* \partial_{2n} u,$$

where $(X^\varepsilon_{i,u})^*$ is the L^2 adjoint of the differential operator $X^\varepsilon_{i,u}$ and

(3.1)
$$(X^\varepsilon_{i,u})^* = -X^\varepsilon_{i,u}, \quad \text{if} \quad i = 1, ..., 2n-2, 2n, \text{ and } \quad (X^\varepsilon_{2n-1,u})^* = -X^\varepsilon_{2n-1,u} - \partial_{2n} u.$$

We now prove that if u is a smooth solution of $L_\varepsilon u = 0$ in $\Omega \subset \mathbb{R}^{2n}$ then its derivatives $\partial_{2n} u$ and $X^\varepsilon_{k,u} u$ are solution of a similar mean curvature type equation with different right hand side:

Lemma 3.1. *If u is a smooth solution of $L_\varepsilon u = 0$ then $\omega = \partial_{2n} u + 2||u||_{Lip}$ is a solution of the equation*

$$\sum_{i,j} (X^\varepsilon_{i,u})^* \Big(\frac{a_{ij}(\nabla^\varepsilon_u u)}{\sqrt{1 + |\nabla^\varepsilon_u u|^2}} (X^\varepsilon_{j,u})^* \omega \Big) = 0, \tag{3.2}$$

where a_{ij} are defined in (1.6).

Proof. Differentiating the equation $L_\varepsilon u = 0$ with respect to ∂_{2n} we obtain

$$\partial_{2n} \Big(X^\varepsilon_{i,u} \Big(\frac{X^\varepsilon_{i,u} u}{\sqrt{1 + |\nabla^\varepsilon_u u|^2}} \Big) \Big) = 0$$

Using the previous remark

$$(X^\varepsilon_{i,u})^* \Big(\partial_{2n} \Big(\frac{X^\varepsilon_{i,u} u}{\sqrt{1 + |\nabla^\varepsilon_u u|^2}} \Big) \Big) = 0$$

Note that

$$\partial_{2n} \Big(\frac{X^\varepsilon_{i,u} u}{\sqrt{1 + |\nabla^\varepsilon_u u|^2}} \Big) = \frac{\partial_{2n} X^\varepsilon_{i,u} u}{\sqrt{1 + |\nabla^\varepsilon_u u|^2}} - \frac{X^\varepsilon_{i,u} u \, X^\varepsilon_{j,u} u \, \partial_{2n} X^\varepsilon_{j,u} u}{(1 + |\nabla^\varepsilon_u u|^2)^{3/2}}$$
$$= -\frac{(X^\varepsilon_{i,u})^* \partial_{2n} u}{\sqrt{1 + |\nabla^\varepsilon_u u|^2}} + \frac{X^\varepsilon_{i,u} u \, X^\varepsilon_{j,u} u \, (X^\varepsilon_{j,u})^* \partial_{2n} u}{(1 + |\nabla^\varepsilon_u u|^2)^{3/2}}.$$

The result follows immediately. □

Differentiating the equation $L_\varepsilon u = 0$ with respect to $X^\varepsilon_{k,u}$ we obtain

Lemma 3.2. *If u is a smooth solution of $L_\varepsilon u = 0$ then $z = X^\varepsilon_{k,u} u + 2||u||_{Lip}$ with $k \leq 2n - 1$ is a solution of the equation*

$$\sum_{i,j} X^\varepsilon_{i,u} \Big(\frac{a_{ij}(\nabla^\varepsilon_u u)}{\sqrt{1 + |\nabla^\varepsilon_u u|^2}} X^\varepsilon_{j,u} z \Big) =$$

$$= -\sum_i [X^\varepsilon_{k,u}, X^\varepsilon_{i,u}] \Big(\frac{X^\varepsilon_{i,u} u}{\sqrt{1 + |\nabla^\varepsilon_u u|^2}} \Big) - \sum_{i,j} X^\varepsilon_{i,u} \Big(\frac{a_{ij}(\nabla^\varepsilon_u u)}{\sqrt{1 + |\nabla^\varepsilon_u u|^2}} [X^\varepsilon_{k,u}, X^\varepsilon_{j,u}] u \Big), \tag{3.3}$$

where a_{ij} are defined in (1.6).

Proof. Differentiating the equation $L_\varepsilon u = 0$ with respect to $X^\varepsilon_{k,u}$ we obtain

$$X^\varepsilon_{k,u}\Big(X^\varepsilon_{i,u}\Big(\frac{X^\varepsilon_{i,u}u}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}\Big)\Big) = 0$$

$$[X^\varepsilon_{k,u}, X^\varepsilon_{i,u}]\Big(\frac{X_{i,u}u}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}\Big) + X^\varepsilon_{i,u}\Big(X^\varepsilon_{k,u}\Big(\frac{X^\varepsilon_{i,u}u}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}\Big)\Big) = 0$$

Note that

$$X^\varepsilon_{k,u}\Big(\frac{X^\varepsilon_{i,u}u}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}\Big) = \frac{X^\varepsilon_{k,u}X^\varepsilon_{i,u}u}{\sqrt{1+|\nabla^\varepsilon_u u|^2}} - \frac{X^\varepsilon_{i,u}u\, X^\varepsilon_{j,u}u\, X^\varepsilon_{k,u}X^\varepsilon_{j,u}u}{(1+|\nabla^\varepsilon_u u|^2)^{3/2}} =$$

$$= \frac{a_{ij}(\nabla^\varepsilon_u u)}{(1+|\nabla^\varepsilon_u u|^2)^{1/2}}([X^\varepsilon_{k,u}X^\varepsilon_{j,u}]u + X^\varepsilon_{j,u}z)$$

concluding the proof. □

Remark 3.3. *It is useful to compute explicitly the commutators that appear in the previous result.*

If $k \le n-1$ and $i = k+n-1$ or $i \le n-1$ and $k = i+n-1$, then

$$[X^\varepsilon_{k,u}, X^\varepsilon_{i,u}] = sign(k-i)\partial_{2n};$$

If $i = 2n-1$ and $k < 2n-1$, then

$$[X^\varepsilon_{k,u}, X^\varepsilon_{i,u}] = X^\varepsilon_{k,u}u\,\partial_{2n};$$

If $k = 2n-1$ and $i \neq 2n-1$ then

$$[X^\varepsilon_{k,u}, X^\varepsilon_{i,u}] = -X^\varepsilon_{i,u}u\partial_{2n}$$

If $k = 2n$ and $i = 2n-1$ then

$$[X^\varepsilon_{k,u}, X^\varepsilon_{i,u}] = X^\varepsilon_{k,u}u\,\partial_{2n}.$$

All other commutators vanish. As a consequence, if u is a smooth solution of $L_\varepsilon u = 0$ then $|[X^\varepsilon_{k,u}, X^\varepsilon_{i,u}]|$ is always bounded by $1 + ||u||^2_{Lip}$.

Proposition 3.4. *(First Cacciopoli type inequality for $X^\varepsilon_{k,u}u$) If u is a smooth solution of $L_\varepsilon u = 0$ in $\Omega \subset \mathbb{R}^{2n}$, and $z = X^\varepsilon_{k,u}u + 2||u||_{Lip}$ with $k \le 2n$ then for every $p \neq 2$ there exists a constant C, only dependent on the bounds on the spatial gradient and on p such that for every $\varphi \in C^\infty_0$*

$$\int |\partial_{2n}z|^2 z^{p-2}\varphi^2 \le C\left(\int |\nabla^\varepsilon_u z|^2 z^{p-2}\varphi^2 + \int z^p(\varphi^2 + |\nabla^\varepsilon_u\varphi|^2)\right).$$

The constant C is bounded if p is bounded away from 2. If $p = 2$ the inequality holds in the form

$$\int |\partial_{2n}z|^2\varphi^2 \le C\int z^p(\varphi^2 + |\nabla^\varepsilon_u\varphi|^2).$$

Proof. Calling $\omega = \partial_{2n}u + 2||u||_{Lip}$, we have

$$\int |\nabla^\varepsilon_u \omega|^2 z^{p-2}\varphi^2 \leq C \int \frac{a_{ij}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}} X^\varepsilon_{i,u}\omega X^\varepsilon_{j,u}\omega z^{p-2}\varphi^2 =$$

$$= -C\int \frac{a_{ij}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}(X^\varepsilon_{i,u})^*\omega\, X^\varepsilon_{j,u}\omega\, z^{p-2}\,\varphi^2 - C\int \frac{a_{2n-1\,j}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}} X^\varepsilon_{j,u}\omega\,\omega\partial_{2n}u\, z^{p-2}\varphi^2 =$$

(integrating by parts $X^\varepsilon_{j,u}$ in the first integral)

$$= C\int (X^\varepsilon_{j,u})^*\Big(\frac{a_{ij}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}(X^\varepsilon_{i,u})^*\omega\Big)\omega z^{p-2}\varphi^2 + C\int \frac{a_{2n-1\,j}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}(X^\varepsilon_{i,u})^*\omega\,\omega\partial_{2n}u\, z^{p-2}\varphi^2$$

$$+C(p-2)\int \frac{a_{ij}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}(X^\varepsilon_{i,u})^*\omega\,\omega X^\varepsilon_{j,u}z\, z^{p-3}\varphi^2 + 2C\int \frac{a_{ij}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}(X^\varepsilon_{i,u})^*\omega\,\omega z^{p-2}\varphi X^\varepsilon_{j,u}\varphi$$

$$-C\int \frac{a_{2n-1\,j}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}} X^\varepsilon_{j,u}\omega\,\omega\partial_{2n}u z^{p-2}\varphi^2.$$

The first integral vanishes by Lemma 3.1. In the other integrals we can use the fact that

$$\Big|\frac{a_{ij}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}\Big| \leq 1 \quad \text{and } |(X^\varepsilon_{i,u})^*\omega| \leq (1+||u||_{Lip})|\nabla^\varepsilon_u\omega|$$

where $||u||_{Lip}$ is bounded uniformly in ε by assumption. Then

$$\int |\nabla^\varepsilon_u\omega|^2 z^{p-2}\varphi^2 \leq C\Big(\int |\nabla^\varepsilon_u\omega| z^{p-2}(\varphi^2 + |\varphi\nabla^\varepsilon_u\varphi|) + (p-2)\int |\nabla^\varepsilon_u\omega||\nabla^\varepsilon_u z| z^{p-3}\varphi^2\Big)$$

(by Hölder inequality and the fact that z is bounded away from 0)

$$\leq \delta\int |\nabla^\varepsilon_u\omega|^2 z^{p-2}\varphi^2 + C(\delta)\int |\nabla^\varepsilon_u z|^2 z^{p-2}\varphi^2 + C(\delta)\int z^p(\varphi^2 + |\nabla^\varepsilon_u\varphi|^2).$$

For δ sufficiently small this implies that

$$\int |\nabla^\varepsilon_u\omega|^2 z^{p-2}\varphi^2 \leq C\int |\nabla^\varepsilon_u z|^2 z^{p-2}\varphi^2 + C\int z^p(\varphi^2 + |\nabla^\varepsilon_u\varphi|^2) \tag{3.4}$$

The constant C above is bounded as long as p is away from 2. The special case $p=2$ follows along similar computations.

Note that, if $k \neq 2n-1$, we have

$$|\partial_{2n}z| = |\partial_{2n}X^\varepsilon_{k,u}u| = |X^\varepsilon_{k,u}\partial_{2n}u| \leq |\nabla^\varepsilon_u\omega|$$

If $k = 2n-1$, we have

$$|\partial_{2n}z| = |\partial_{2n}X^\varepsilon_{2n-1,u}u| = |X^\varepsilon_{2n-1,u}\partial_{2n}u| + |\partial_{2n}u|^2 \leq |\nabla^\varepsilon_u\omega| + |\partial_{2n}u|^2$$

Hence, using again the boundness of $|\partial_{2n}u|$ and the fact that z is bounded from below, together with inequality (3.4) we conclude the proof. □

Proposition 3.5. *(Intrinsic Cacciopoli type inequality for $X^\varepsilon_{k,u}u$) If u is a smooth solution of $L_\varepsilon u = 0$ in $\Omega \subset \mathbb{R}^{2n}$ and $z = X^\varepsilon_{k,u}u + 2||u||_{Lip}$, with $k \leq 2n$, then for every $p \neq 1$ there exists a constant $C > 0$, only dependent on the bounds on the spatial gradient and on p such that for every $\varphi \in C^\infty_0$*

$$\int |\nabla^\varepsilon_u z|^2 z^{p-2}\varphi^2 \leq C\int z^p(\varphi^2 + |\nabla^\varepsilon_u\varphi|^2 + |\varphi\partial_{2n}\varphi|).$$

The constant C is bounded if p is bounded away from the values 1 and 2.

Proof. Multiplying the equation (3.3) by $z^{p-1}\varphi^2$ and integrating we obtain

$$\int X^\varepsilon_{i,u}\Big(\frac{a_{ij}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}X^\varepsilon_{j,u}z\Big)z^{p-1}\varphi^2 =$$

$$= -\int [X^\varepsilon_{k,u}, X^\varepsilon_{i,u}]\Big(\frac{X^\varepsilon_{i,u}u}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}\Big)z^{p-1}\varphi^2 - \int X^\varepsilon_{i,u}\Big(\frac{a_{ij}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}[X^\varepsilon_{k,u}, X^\varepsilon_{j,u}]u\Big)z^{p-1}\varphi^2.$$

We denote by I_1 and I_2 the integrals in the right hand side. Let us consider the left hand side

$$\int X^\varepsilon_{i,u}\Big(\frac{a_{ij}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}X^\varepsilon_{j,u}z\Big)z^{p-1}\varphi^2 =$$

(since $(X^\varepsilon_{i,u})^* = -X^\varepsilon_{i,u} - \delta_{i,2n-1}\partial_{2n}u$)

$$= -(p-1)\int \frac{a_{ij}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}X^\varepsilon_{j,u}z\, X^\varepsilon_{i,u}z\, z^{p-2}\,\varphi^2 - 2\int \frac{a_{ij}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}X^\varepsilon_{j,u}z\, z^{p-1}\,\varphi\, X^\varepsilon_{i,u}\varphi -$$

$$-\int \frac{a_{2n-1j}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}X^\varepsilon_{j,u}z\,\partial_{2n}u\ z^{p-2}\,\varphi^2.$$

Using the uniform ellipticity of a_{ij} and the boundeness of a_{ij} and $\partial_{2n}u$, we obtain

$$\int |\nabla^\varepsilon_u z|^2 z^{p-2}\varphi^2 \le C\Big(2\int |\nabla^\varepsilon_u z|\, z^{p-1}\,\varphi\,|\nabla^\varepsilon_u\varphi| + \int |\nabla^\varepsilon_u z|\, z^{p-2}\,\varphi^2\Big) + |I_1| + |I_2|.$$

From here, using an Hölder inequality and the boundeness of z from below one has

$$\int |\nabla^\varepsilon_u z|^2 z^{p-2}\varphi^2 \le C\int z^{p-2}\,(\varphi^2 + |\nabla^\varepsilon_u\varphi|^2) + |I_1| + |I_2|. \tag{3.5}$$

Next we estimate separately the terms I_1 and I_2. We begin with the latter and observe that integrating by parts the expression of I_2, we have

$$I_2 = (p-1)\int \frac{a_{ij}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}[X^\varepsilon_{k,u}, X^\varepsilon_{j,u}]u\, X^\varepsilon_{i,u}z\, z^{p-2}\,\varphi^2$$

$$+2\int \frac{a_{ij}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}[X^\varepsilon_{k,u}, X^\varepsilon_{j,u}]u\, z^{p-1}\,\varphi\, X_{i,u}\varphi$$

$$+\int \frac{a_{2n-1\,j}(\nabla^\varepsilon_u u)}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}[X^\varepsilon_{k,u}, X^\varepsilon_{j,u}]u\,\partial_{2n}u\, z^{p-1}\,\varphi^2$$

(using the fact that both a_{ij} and the brakets, computed in Remark 3.3, are bounded)

$$\le C\int |\nabla^\varepsilon_u z| z^{p-2}\varphi^2 + C\int z^{p-1}\varphi|\nabla^\varepsilon_u\varphi| + C\int z^{p-1}\varphi^2 \le$$

(since z is bounded from below)

$$\le \delta\int |\nabla^\varepsilon_u z|^2 z^{p-2}\varphi^2 + C(\delta)\int z^p(\varphi^2 + |\nabla^\varepsilon_u\varphi|^2),$$

for every $\delta > 0$.

In order to estimate I_1 we first consider separately the case $k \ne 2n-1$. If $k \le n-1$ then by Remark 3.3

$$I_1 = -sign(k-n)(1-\delta_{k,2n})\int \partial_{2n}\Big(\frac{X^\varepsilon_{k+n-1,u}u}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}\Big)z^{p-1}\,\varphi^2 - \int X^\varepsilon_{k,u}u\partial_{2n}\Big(\frac{X^\varepsilon_{2n-1,u}u}{\sqrt{1+|\nabla^\varepsilon_u u|^2}}\Big)z^{p-1}\varphi^2 \le$$

(integrating by parts, using the boundness of $\nabla^\varepsilon_u u$, and the fact that $\partial_{2n} X^\varepsilon_{k,u} u = \partial_{2n} z$)

$$\leq C \int |\partial_{2n} z| z^{p-2} \varphi^2 + C \int z^{p-1} |\varphi \partial_{2n} \varphi| + C \int |\partial_{2n} z| z^{p-1} \varphi^2 \leq$$

(using Hölder inequality and the fact that z is bounded away from 0)

$$\leq \delta \int |\partial_{2n} z|^2 z^{p-2} \varphi^2 + C(\delta) \int z^p (|\varphi \partial_{2n} \varphi| + \varphi^2) \leq$$

(by Proposition 3.4)

$$\leq \delta \int |\nabla^\varepsilon_u z|^2 z^{p-2} \varphi^2 + C(\delta) \int z^p (|\varphi \partial_{2n} \varphi| + \varphi^2 + |\nabla^\varepsilon_u \varphi|^2).$$

This estimate can be proved with a similar argument in the case $n \leq k \leq 2n-2$ and $k = 2n$. Hence if $k \neq 2n-1$ the conclusion follows by choosing δ sufficiently small.

If $k = 2n-1$, then Remark 3.3 yields

$$I_1 = -\int X^\varepsilon_{i,u} u \partial_{2n} \Big(\frac{X^\varepsilon_{i,u} u}{\sqrt{1 + |\nabla^\varepsilon_u u|^2}} \Big) z^{p-1} \varphi^2 =$$

(directly computing the derivative with respect to ∂_{2n})

$$= -\int (X^\varepsilon_{i,u} u)^2 \partial_{2n} \Big(\frac{1}{\sqrt{1 + |\nabla^\varepsilon_u u|^2}} \Big) z^{p-1} \varphi^2 - \int X^\varepsilon_{i,u} u \frac{\partial_{2n} X^\varepsilon_{i,u} u}{\sqrt{1 + |\nabla^\varepsilon_u u|^2}} z^{p-1} \varphi^2$$

(since $\sum_i (X^\varepsilon_{i,u} u)^2 = |\nabla^\varepsilon_u u|^2$)

$$= -\int |\nabla^\varepsilon_u u|^2 \partial_{2n} \Big(\frac{1}{\sqrt{1 + |\nabla^\varepsilon_u u|^2}} \Big) z^{p-1} \varphi^2 - \frac{1}{2} \int \frac{\partial_{2n} |\nabla^\varepsilon_u u|^2}{\sqrt{1 + |\nabla^\varepsilon_u u|^2}} z^{p-1} \varphi^2.$$

If we set $F(s) = \frac{1}{\sqrt{1+s}}$, then one easily computes

$$\partial_{2n} F(|\nabla^\varepsilon_u u|^2) = \Big(|\nabla^\varepsilon_u u|^2 \partial_{2n} \Big(\frac{1}{\sqrt{1 + |\nabla^\varepsilon_u u|^2}} \Big) + \frac{1}{2} \frac{\partial_{2n} |\nabla^\varepsilon_u u|^2}{\sqrt{1 + |\nabla^\varepsilon_u u|^2}} \Big)$$

so that the previous integral becomes:

$$I_1 = -\int \partial_{2n} F(|\nabla^\varepsilon_u u|^2) z^{p-1} \varphi^2 =$$

(integrating by parts)

$$= (p-1) \int F(|\nabla^\varepsilon_u u|^2) \partial_{2n} z z^{p-2} \varphi^2 + 2 \int F(|\nabla^\varepsilon_u u|^2) z^{p-1} \varphi \partial_{2n} \varphi \leq$$

(using the fact that F is bounded)

$$\leq C \int |\partial_{2n} z| z^{p-2} \varphi^2 + C \int z^{p-1} |\varphi \partial_{2n} \varphi| \leq$$

(by Proposition 3.4 and an Hölder inequality)

$$\leq \delta \int |\nabla^\varepsilon_u z|^2 z^{p-2} \varphi^2 + C(\delta) \int z^p (|\varphi \partial_{2n} \varphi| + \varphi^2 + |\nabla^\varepsilon_u \varphi|^2),$$

thus concluding the proof. □

Next, we note that, from Propositions 3.4 and 3.5 one can derive a Euclidean Caccioppoli type inequality for $z = X^\varepsilon_{k,u}u + 2||u||_{Lip}$, with $k \le 2n$. Here and in the following ∇_E denotes the Euclidean gradient in $\mathbb{R}^{2n}$.

Proposition 3.6. *(Euclidean Cacciopoli inequality) If u is a Lipschitz continuous solution of $L_\varepsilon u = 0$ in $\Omega \subset \mathbb{R}^{2n}$, and $z = X^\varepsilon_{k,u}u + 2||u||_{Lip}$, with $k \le 2n$, then for every $p \ne 1$ there exists a constant C, only dependent on the bounds on the spatial gradient and on p such that for every $\varphi \in C^\infty_0$*

$$\int |\nabla_E z|^2 z^{p-2}\varphi^2 \le C \int z^p(\varphi^2 + |\nabla_E \varphi|^2). \tag{3.6}$$

The constant C is bounded if p is bounded away from the values 1 and 2.

Proof. Observe that there exists $C > 0$ depending only on $||u||_{Lip}$ and Ω such that for all points in Ω,

$$|\nabla_E z|^2 \le C\Big(\sum_{k<2n} |X^\varepsilon_{k,u}z|^2 + |\partial_{2n}z|^2\Big).$$

Hence using Propositions 3.4 and 3.5 and observing that

$$|\nabla^\varepsilon_u \varphi|^2 + |\partial_{2n}\varphi|^2 \le C|\nabla_E\varphi|^2$$

we obtain (3.6). □

From Proposition 3.6, using the classical Moser procedure in the Euclidean setting, we can immediately deduce the following regularity result:

Proposition 3.7. *Let u be a solution of $L_\varepsilon u = 0$ in $\Omega \subset \mathbb{R}^{2n}$ and set $z = X^\varepsilon_{k,u}u + 2||u||_{Lip}$, with $k \le 2n$. For every compact set $K \subset\subset \Omega$ then there exist a real number α and a constant C, only dependent on the bounds on the spatial gradient and on the choice of the compact set such that*

$$||z||_{C^\alpha_u(K)} \le C.$$

In particular we have the estimate

$$\sum_{i=1}^{2n+1}\sum_{j=1}^{2n} ||X^\varepsilon_{i,u}X^\varepsilon_{j,u}u||_{L^2(K)} + ||u||_{C^{1,\alpha}_u(K)} \le C.$$

Proof. For $p \ne 1, 2$ and z as in the statement of the proposition define the function

$$w = \begin{cases} z^{\frac{p}{2}} & \text{if } p \ne 0; \\ \ln z & \text{if } p = 0. \end{cases}$$

If $p \ne 0$ then the Caccioppoli inequality (3.6) and the Euclidean Sobolev embedding Theorem yield

$$\Big(\int |\varphi w|^{2\theta}\Big)^{\frac{1}{\theta}} \le Cp^2 \int w^2|\nabla_E\varphi|^2, \tag{3.7}$$

for some $\theta > 1$. Let $0 < r_1 < r_2$ be sufficiently small so that the Euclidean ball B_{r_2} is contained in Ω. With an appropriate choice of test function (3.7) implies

$$(3.8)\qquad \Big(\int_{B_{r_1}} |z|^{\theta p}\Big)^{\frac{1}{p\theta}} \le \Big(\frac{Cp^2}{r_2 - r_1}\Big)^{\frac{2}{p}} \Big(\int_{B_{r_2}} z^p\Big)^{\frac{1}{p}} \qquad \text{if } p > 0$$

$$(3.9)\qquad \Big(\int_{B_{r_1}} |z|^{\theta p}\Big)^{\frac{1}{p\theta}} \Big(\frac{Cp^2}{r_2 - r_1}\Big)^{\frac{2}{|p|}} \ge \Big(\int_{B_{r_2}} z^p\Big)^{\frac{1}{p}}. \qquad \text{if } p < 0.$$

If $p = 0$, (3.6) implies

$$\int |\varphi \nabla_E w|^2 \le C \int |\nabla_E \varphi|^2.$$

Let $r > 0$ sufficiently small so that the Euclidean ball $B_r \subset \Omega$. A standard choice of test function and Hölder inequality yield

$$\int_{B_r} |\nabla_E w| \le C R^{2n+1}.$$

Recalling that $\Omega \subset \mathbb{R}^{2n}$ and using Poincare' inequality we obtain $w \in BMO(\Omega)$.

At this point, using (3.8), the John-Nirenberg Lemma and following the standard Moser iteration process (see for instance [25, Chapter 8]) we obtain the Hölder regularity of z. □

4. From $C^{1,\alpha}_{\bar u}$ to $C^\infty_{\bar u}$.

In this section we will conclude the proof of the regularity result. The section is organized in 3 steps. We fix a function $\bar u$, and study solutions of the linearized equation

$$L_{\varepsilon,\bar u} u = \sum_{i,j=1}^{2n} a^\varepsilon_{i,j}(\nabla^\varepsilon_{\bar u} \bar u) X^\varepsilon_{i,\bar u} X^\varepsilon_{j,\bar u} u, = 0$$

defined in (2.8), and represented in non-divergence form. The solutions u well be represented in terms of the fundamental solution $\Gamma^\varepsilon_{x_0}$ of the approximating operator L_{ε,x_0}, defined in (2.9):

$$L_{\varepsilon,x_0} u = \sum_{i,j=1}^{2n} a^\varepsilon_{i,j}(\nabla^\varepsilon_{\bar u} \bar u(x_0)) X^\varepsilon_{i,x_0} X^\varepsilon_{j,x_0} u,$$

where a^ε_{ij} are defined in (1.6). Since estimates of $\Gamma^\varepsilon_{x_0}$ uniform in ε are well known (and have been recalled in section 2), from these representation formulas we will deduce a priori estimates for the solution u, in terms of the fixed solution $\bar u$. Choosing $\bar u = u$ we will obtain a priori estimates of the solutions of the non linear equation $L_\varepsilon u = 0$. Finally, letting ε go to 0, we will conclude the proof of the estimates of the vanishing viscosity solutions of $Lu = 0$.

4.1. **Representation formulas.**

Lemma 4.1. *The difference between the operator $L_{\varepsilon,\bar u}$ and its frozen operator can be expressed as follows:*

$$
\begin{aligned}
(L_{\varepsilon,x_0}-L_{\varepsilon,\bar u})u(\xi) = &\sum_{ij=1}^{2n}\Big(a^{\varepsilon}_{ij}(\nabla^{\varepsilon}_{\bar u}\bar u(x_0)) - a_{ij}(\nabla^{\varepsilon}_{\bar u}\bar u(\xi))\Big)X^{\varepsilon}_{i,\bar u}X^{\varepsilon}_{j,\bar u}u(\xi)\\
&-\sum_{j=1}^{2n} a^{\varepsilon}_{2n-1j}(\nabla^{\varepsilon}_{\bar u}\bar u(x_0))(\bar u(\xi)-P^1_{x_0}\bar u(\xi))\partial_{2n}X^{\varepsilon}_{j,\bar u}u(\xi)\\
&-\sum_{j=1}^{2n} a^{\varepsilon}_{2n-1j}(\nabla^{\varepsilon}_{\bar u}\bar u(x_0))X^{\varepsilon}_{j,x_0}\Big((\bar u(\xi)-P^1_{x_0}\bar u(\xi))\partial_{2n}\Big)u(\xi).
\end{aligned}
\tag{4.1}
$$

Proof. Observe that

$$
(L_{\varepsilon,x_0}-L_{\varepsilon,\bar u})u(\xi) = \sum_{ij=1}^{2n}\Big(a^{\varepsilon}_{ij}(\nabla^{\varepsilon}_{\bar u}\bar u(x_0)) - a^{\varepsilon}_{ij}(\nabla^{\varepsilon}_{\bar u}\bar u(\xi))\Big)X^{\varepsilon}_{i,\bar u}X^{\varepsilon}_{j,\bar u}u(\xi)-
$$

$$
-\sum_{ij=1}^{2n} a^{\varepsilon}_{ij}(\nabla^{\varepsilon}_{\bar u}\bar u(x_0))\Big(X^{\varepsilon}_{i,\bar u}X^{\varepsilon}_{j,\bar u} - X^{\varepsilon}_{i,x_0}X^{\varepsilon}_{j,x_0}\Big)u(\xi) =
$$

$$
= \sum_{ij=1}^{2n}\Big(a^{\varepsilon}_{ij}(\nabla^{\varepsilon}_{\bar u}\bar u(x_0)) - a^{\varepsilon}_{ij}(\nabla^{\varepsilon}_{\bar u}\bar u(\xi))\Big)X^{\varepsilon}_{i,\bar u}X^{\varepsilon}_{j,\bar u}u(\xi)-
$$

$$
-\sum_{ij=1}^{2n} a^{\varepsilon}_{ij}(\nabla^{\varepsilon}_{\bar u}\bar u(x_0))\Big((X^{\varepsilon}_{i,\bar u}-X^{\varepsilon}_{i,x_0})X^{\varepsilon}_{j,\bar u} + X^{\varepsilon}_{i,x_0}(X^{\varepsilon}_{j,\bar u}-X^{\varepsilon}_{j,x_0})\Big)u(\xi)
$$

$$
= \sum_{ij=1}^{2n}\Big(a^{\varepsilon}_{ij}(\nabla^{\varepsilon}_{\bar u}\bar u(x_0)) - a^{\varepsilon}_{ij}(\nabla^{\varepsilon}_{\bar u}\bar u(\xi))\Big)X^{\varepsilon}_{i,\bar u}X^{\varepsilon}_{j,\bar u}u(\xi)-
$$

$$
-\sum_{ij=1}^{2n} a^{\varepsilon}_{ij}(\nabla^{\varepsilon}_{\bar u}\bar u(x_0))\Big(\delta_{i,2n-1}(\bar u(\xi)-P^1_{x_0}\bar u(\xi))\partial_{2n}X^{\varepsilon}_{j,\bar u}+X^{\varepsilon}_{i,x_0}(\delta_{j,2n-1}(\bar u(\xi)-P^1_{x_0}\bar u(\xi))\partial_{2n})\Big)u(\xi)
$$

where δ is the Kroeneker function. From this the thesis immediately follows □

Let us first represent the solutions of the equation $L_{\varepsilon,\bar u}u = 0$ it in terms of the fundamental solution $\Gamma^{\varepsilon}_{x_0}$ of the operator L_{ε,x_0} defined in (2.9).

Proposition 4.2. *Let us assume that $\bar u$ is a fixed function of class $C^{\infty}(\Omega)$, and that u is a classical solution of $L_{\varepsilon,\bar u}u = g \in C^{\infty}(\Omega)$, Then for any $\varphi \in C^{\infty}_0(\Omega)$ the function $u\varphi$ can be represented as*

$$
\begin{aligned}
u\varphi(x) = &\int_{\Omega}\Gamma^{\varepsilon}_{x_0}(x,\xi)N_1(\xi,x_0)d\xi + \int_{\Omega}\Gamma^{\varepsilon}_{x_0}(x,\xi)L_{\varepsilon,\bar u}u(\xi)\,\varphi(\xi)\,d\xi+\\
&+\sum_{ij=1}^{2n}\int_{\Omega}\Gamma^{\varepsilon}_{x_0}(x,\xi)\Big(b^{\varepsilon}_{ij}(\nabla^{\varepsilon}_{\bar u}\bar u(x_0)) - b^{\varepsilon}_{ij}(\nabla^{\varepsilon}_{\bar u}\bar u(\xi))\Big)X^{\varepsilon}_{i,\bar u}X^{\varepsilon}_{j,\bar u}u(\xi)\varphi(\xi)d\xi\\
&+\sum_{ijs=1}^{2n}\int_{\Omega}X^{\varepsilon}_{s,x_0}\Gamma^{\varepsilon}_{x_0}(x,\xi)(\bar u(\xi)-P^1_{x_0}\bar u(\xi))h_{sij}(x_0)X^{\varepsilon}_{i,\bar u}X^{\varepsilon}_{j,\bar u}u(\xi)\varphi(\xi)d\xi.
\end{aligned}
\tag{4.2}
$$

The expressions of N_1, b_{ij} and h_{sij} are the following:

$$
\begin{aligned}
N_1(\xi, x_0) = {} & u(\xi) L_{\varepsilon,x_0}\varphi(\xi) + \\
& + \sum_{ij=1}^{2n} a^\varepsilon_{ij}(\nabla^\varepsilon_{\bar u}\bar u(x_0))\Big(X^\varepsilon_{i,\bar u}u(\xi)X^\varepsilon_{j,x_0}\varphi(\xi) + X^\varepsilon_{j,\bar u}u(\xi)X^\varepsilon_{i,x_0}\varphi(\xi)\Big) \\
& - \sum_{i=1}^{2n} a^\varepsilon_{i\,2n-1}(\nabla^\varepsilon_{\bar u}\bar u(x_0))(\bar u(\xi) - P^1_{x_0}\bar u(\xi))\partial_{2n}u(\xi)\, X^\varepsilon_{i,x_0}\varphi(\xi) \\
& + \sum_{i=1}^{2n} a^\varepsilon_{i\,2n-1}(\nabla^\varepsilon_{\bar u}\bar u(x_0))(\bar u(\xi) - P^1_{x_0}\bar u(\xi))X^\varepsilon_{n,\bar u}X^\varepsilon_{i,\bar u}u(\xi)X^\varepsilon_{1\bar u}\varphi(\xi) \\
& - \sum_{i=1}^{2n} a^\varepsilon_{i\,2n-1}(\nabla^\varepsilon_{\bar u}\bar u(x_0))(\bar u(\xi) - P^1_{x_0}\bar u(\xi))X^\varepsilon_{1,\bar u}X^\varepsilon_{i,\bar u}u(\xi)X^\varepsilon_{n,\bar u}\varphi(\xi).
\end{aligned}
\tag{4.3}
$$

$$b_{ij} : \mathbb{R}^{2n} \to \mathbb{R},$$

$$b_{i,j}(p) = -\delta_{ik}a_{kj}(p) + a_{2n-1j}(\nabla^\varepsilon_{\bar u}\bar u(x_0))p_1\delta_{in} - p_n\delta_{i1}a_{2n-1j}(\nabla^\varepsilon_{\bar u}\bar u(x_0)) \tag{4.4}$$

Finally h_{sij} are real numbers, only dependent on x_0, defined as

$$h_{sij}(x_0) = -a^\varepsilon_{2n-1,s}(\nabla^\varepsilon_{\bar u}\bar u(x_0))(\delta_{i1}\delta_{jn} - \delta_{in}\delta_{j1}) + a^\varepsilon_{2n-1,j}(\nabla^\varepsilon_{\bar u}\bar u(x_0))(\delta_{s1}\delta_{in} - \delta_{sn}\delta_{i1}). \tag{4.5}$$

Proof. By definition of fundamental solution, we have

$$
\begin{aligned}
u\varphi(x) &= \int_\Omega \Gamma^\varepsilon_{x_0}(x,\xi)L_{\varepsilon,x_0}(u\varphi)(\xi)d\xi \\
&= \int_\Omega \Gamma^\varepsilon_{x_0}(x,\xi)\left(u\, L_{\varepsilon,x_0}\varphi + \sum_{ij=1}^{2n} a^\varepsilon_{ij}(\nabla^\varepsilon_{\bar u}\bar u(x_0))\Big(X^\varepsilon_{i,x_0}uX^\varepsilon_{j,x_0}\varphi + X^\varepsilon_{j,x_0}uX^\varepsilon_{i,x_0}\varphi\Big)\right)d\xi \\
&+ \int_\Omega \Gamma^\varepsilon_{x_0}(x,\xi)L_{\varepsilon,\bar u}u(\xi)\,\varphi(\xi)d\xi + \int_\Omega \Gamma^\varepsilon_{x_0}(x,\xi)(L_{\varepsilon,x_0} - L_{\varepsilon,\bar u})u(\xi)\,\varphi(\xi)d\xi.
\end{aligned}
\tag{4.6}
$$

We can use the expression of $L_{\varepsilon,x_0} - L_{\varepsilon,\bar u}$ computed in (4.1). Let us consider the second term in the right hand side, multiplied by the fundamental solution. Since $\partial_{2n} = [X^\varepsilon_{1,x_0}, X^\varepsilon_{n,x_0}]$, it becomes:

$$\sum_{j=1}^{2n} a^\varepsilon_{2n-1j}(\nabla^\varepsilon_{\bar u}\bar u(x_0))\int_\Omega \Gamma^\varepsilon_{x_0}(x,\xi)(\bar u - P^1_{x_0}\bar u)\partial_{2n}X^\varepsilon_{j,\bar u}u\varphi d\xi \tag{4.7}$$

$$= \sum_{j=1}^{2n} a^\varepsilon_{2n-1j}(\nabla^\varepsilon_{\bar u}\bar u(x_0))\int_\Omega \Gamma^\varepsilon_{x_0}(x,\xi)(\bar u - P^1_{x_0}\bar u)[X^\varepsilon_{1,x_0}, X^\varepsilon_{n,x_0}]\big(X^\varepsilon_{j,\bar u}u\big)\varphi d\xi =$$

(integrating by part and using the fact that $X^\varepsilon_{1,x_0} = X^\varepsilon_{1,\bar u}$ and $X^\varepsilon_{n,x_0} = X^\varepsilon_{n,\bar u}$.)

$$= -\sum_{j=1}^{2n} a^\varepsilon_{2n-1j}(\nabla^\varepsilon_{\bar u}\bar u(x_0))\int_\Omega X^\varepsilon_{1x_0}\Gamma^\varepsilon_{x_0}(x,\xi)(\bar u(\xi) - P^1_{x_0}\bar u(\xi))X^\varepsilon_{n,\bar u}X^\varepsilon_{j,\bar u}u\varphi\, d\xi$$

$$-\sum_{j=1}^{2n} a^\varepsilon_{2n-1j}(\nabla^\varepsilon_{\bar u}\bar u(x_0))\int_\Omega \Gamma^\varepsilon_{x_0}(x,\xi)X^\varepsilon_{1,\bar u}(\bar u(\xi)-P^1_{x_0}\bar u(\xi))X^\varepsilon_{n,\bar u}X^\varepsilon_{j,\bar u}u\varphi\, d\xi$$

$$-\sum_{j=1}^{2n} a^\varepsilon_{2n-1j}(\nabla^\varepsilon_{\bar u}\bar u(x_0))\int_\Omega \Gamma^\varepsilon_{x_0}(x,\xi)(\bar u(\xi)-P^1_{x_0}\bar u(\xi))X^\varepsilon_{n\bar u}X^\varepsilon_{j,\bar u}uX^\varepsilon_{1\bar u}\varphi\, d\xi$$

$$+\sum_{j=1}^{2n} a^\varepsilon_{2n-1j}(\nabla^\varepsilon_{\bar u}\bar u(x_0))\int_\Omega X^\varepsilon_{n,x_0}\Gamma^\varepsilon_{x_0}(x,\xi)(\bar u(\xi)-P^1_{x_0}\bar u(\xi))X^\varepsilon_{1,\bar u}X^\varepsilon_{j,\bar u}u\varphi\, d\xi$$

$$+\sum_{j=1}^{2n} a^\varepsilon_{2n-1j}(\nabla^\varepsilon_{\bar u}\bar u(x_0))\int_\Omega \Gamma^\varepsilon_{x_0}(x,\xi)X^\varepsilon_{n\bar u}(\bar u(\xi)-P^1_{x_0}\bar u(\xi))X^\varepsilon_{1,\bar u}X^\varepsilon_{j,\bar u}u\varphi\, d\xi$$

$$+\sum_{j=1}^{2n} a^\varepsilon_{2n-1j}(\nabla^\varepsilon_{\bar u}\bar u(x_0))\int_\Omega \Gamma^\varepsilon_{x_0}(x,\xi)(\bar u(\xi)-P^1_{x_0}\bar u(\xi))X^\varepsilon_{1,\bar u}X^\varepsilon_{j,\bar u}uX^\varepsilon_{n,\bar u}\varphi\, d\xi.$$

The third term in (4.1) becomes

$$\sum_{i=1}^{2n} a^\varepsilon_{2n-1j}(\nabla^\varepsilon_{\bar u}\bar u(x_0))\int_\Omega \Gamma^\varepsilon_{x_0}(x,\xi)X^\varepsilon_{j,x_0}\Big((\bar u(\xi)-P^1_{x_0}\bar u(\xi))\partial_{2n}\Big)u(\xi)\varphi(\xi)d\xi \tag{4.8}$$

(integrating by part)

$$\begin{aligned}&=-\sum_{j=1}^{2n} a^\varepsilon_{2n-1j}(\nabla^\varepsilon_{\bar u}\bar u(x_0))\int_\Omega X^\varepsilon_{j,x_0}\Gamma^\varepsilon_{x_0}(x,\xi)\,(\bar u(\xi)-P^1_{x_0}\bar u(\xi))\partial_{2n}u(\xi)\,\varphi(\xi)d\xi\\ &\quad-\sum_{j=1}^{2n} a^\varepsilon_{2n-1j}(\nabla^\varepsilon_{\bar u}\bar u(x_0))\int_\Omega \Gamma^\varepsilon_{x_0}(x,\xi)\,(\bar u(\xi)-P^1_{x_0}\bar u(\xi))\partial_{2n}u(\xi)\,X^\varepsilon_{j,x_0}\varphi(\xi)d\xi\end{aligned} \tag{4.9}$$

Inserting (4.7) and (4.8) in (4.1) and using the expression of b_{ij} and $h_{sij}(x_0)$, we obtain

$$\begin{aligned}&\int_\Omega \Gamma^\varepsilon_{x_0}(x,\xi)(L_{\varepsilon,x_0}-L_{\varepsilon,\bar u})u(\xi)\,\varphi(\xi)d\xi\\ &\quad=\sum_{ij=1}^{2n}\int_\Omega \Gamma^\varepsilon_{x_0}(x,\xi)\Big(a^\varepsilon_{ij}(\nabla^\varepsilon_{\bar u}\bar u(x_0))-a^\varepsilon_{ij}(\nabla^\varepsilon_{\bar u}\bar u(\xi))\Big)X^\varepsilon_{i,\bar u}X^\varepsilon_{j,\bar u}u(\xi)\varphi(\xi)d\xi\\ &\quad+\sum_{ij=1}^{2n}\int_\Omega \Gamma^\varepsilon_{x_0}(x,\xi)\Big(b^\varepsilon_{ij}(\nabla^\varepsilon_{\bar u}\bar u(x_0))-b^\varepsilon_{ij}(\nabla^\varepsilon_{\bar u}\bar u(\xi))\Big)X^\varepsilon_{i,\bar u}X^\varepsilon_{j,\bar u}u(\xi)\varphi(\xi)d\xi\\ &\quad+\sum_{ijs=1}^{2n}\int_\Omega X^\varepsilon_{s,x_0}\Gamma^\varepsilon_{x_0}(x,\xi)(\bar u(\xi)-P^1_{x_0}\bar u(\xi))h_{sij}(x_0)X^\varepsilon_{i,\bar u}X^\varepsilon_{j,\bar u}u(\xi)\varphi(\xi)d\xi\end{aligned} \tag{4.10}$$

$$
\begin{aligned}
&-\sum_{j=1}^{2n} a^{\varepsilon}_{2n-1j}(\nabla^{\varepsilon}_{\bar u}\bar u(x_0))\int_{\Omega}\Gamma^{\varepsilon}_{x_0}(x,\xi)(\bar u(\xi)-P^1_{x_0}\bar u(\xi))X^{\varepsilon}_{n\bar u}X^{\varepsilon}_{j,\bar u}uX^{\varepsilon}_{1\bar u}\varphi\, d\xi\\
&+\sum_{j=1}^{2n} a^{\varepsilon}_{2n-1j}(\nabla^{\varepsilon}_{\bar u}\bar u(x_0))\int_{\Omega}\Gamma^{\varepsilon}_{x_0}(x,\xi)(\bar u(\xi)-P^1_{x_0}\bar u(\xi))X^{\varepsilon}_{1,\bar u}X^{\varepsilon}_{j,\bar u}uX^{\varepsilon}_{n,\bar u}\varphi\, d\xi\\
&-\sum_{j=1}^{2n} a^{\varepsilon}_{2n-1j}(\nabla^{\varepsilon}_{\bar u}\bar u(x_0))\int_{\Omega}\Gamma^{\varepsilon}_{x_0}(x,\xi)\,(\bar u(\xi)-P^1_{x_0}\bar u(\xi))\partial_{2n}u(\xi)\,X^{\varepsilon}_{j,x_0}\varphi(\xi)d\xi
\end{aligned}
\tag{4.11}
$$

From this expression, equation (4.6), and the expression of N_1 in (4.3) we obtain the asserted representation formula. □

The following representation formula will be used to estimate higher order derivatives of the solutions.

Proposition 4.3. *Let us assume that $\bar u$ is a fixed function of class $C^\infty(\Omega)$, and assume that u is a classical solution of $L_{\varepsilon,\bar u}u = g \in C^\infty(\Omega)$. Then for any $\varphi \in C^\infty_0(\Omega)$ the function $u\varphi$ can be represented as*

$$
\begin{aligned}
u\varphi(x) = &\int_{\Omega}\Gamma^{\varepsilon}_{x_0}(x,\xi)N_1(\xi,x_0)d\xi + \int_{\Omega}\Gamma^{\varepsilon}_{x_0}(x,\xi)N_{2,k}(\xi,x_0)\varphi(\xi)d\xi\\
&+\sum_{s=1}^{2n}\int_{\Omega}X^{\varepsilon}_{s,x_0}\Gamma^{\varepsilon}_{x_0}(x,\xi)N_{2,ks}(\xi,x_0)d\xi + \int_{\Omega}\Gamma^{\varepsilon}_{x_0}(x,\xi)N_{3,k}(\xi,x_0)d\xi\\
&+\sum_{i=1}^{2n}\int_{\Omega}X^{\varepsilon}_{i,x_0}\Gamma^{\varepsilon}_{x_0}(x,\xi)N_{4,ki}(\xi,x_0)d\xi,
\end{aligned}
\tag{4.12}
$$

where $N_1(\xi,x_0)$ is defined in (4.3). If b_{ij} is the function defined in (4.4), we call $b_{ij\bar u} = b_{ij}(\nabla^{\varepsilon}_{\bar u}\bar u)$, and the other kernel are expressed:

$$
N_2(\xi,x_0) = P^{k-2}_{x_0}g(\xi) + \sum_{ij=1}^{2n}\Big(P^{k-2}_{x_0}b_{ij\bar u}(\xi) - b_{ij\bar u}(x_0)\Big)P^{k-3}_{x_0}(X^{\varepsilon}_{i,\bar u}X^{\varepsilon}_{j,\bar u}u)(\xi)
$$

$$
N_{2,ks}(\xi,x_0) = \sum_{ij=1}^{2n}\Big(P^{k-1}_{x_0}\bar u(\xi) - P^1_{x_0}\bar u(\xi)\Big)h_{sij}(x_0)P^{k-3}_{x_0}(X^{\varepsilon}_{i,\bar u}X^{\varepsilon}_{j,\bar u}u)(\xi)
$$

$$
\begin{aligned}
N_{3,k}(\xi,x_0) = &\Big(g(\xi) - P^{k-2}_{x_0}g(\xi)\Big)+\\
&+\Big(b_{ij\bar u}(\xi) - b_{ij\bar u}(x_0)\Big)\Big(X^{\varepsilon}_{i,\bar u}X^{\varepsilon}_{j,\bar u}u(\xi) - P^{k-3}_{x_0}(X^{\varepsilon}_{i,\bar u}X^{\varepsilon}_{j,\bar u}u)(\xi)\Big)+\\
&+\Big(b_{ij\bar u}(\xi) - P^{k-2}_{x_0}b_{ij\bar u}(\xi)\Big)P^{k-3}_{x_0}(X^{\varepsilon}_{i,\bar u}X^{\varepsilon}_{j,\bar u}u)(\xi)
\end{aligned}
\tag{4.13}
$$

(4.14)

$$
\begin{aligned}
N_{4,ks}(\xi,x_0) = &\Big(\bar u(\xi) - P^{k-1}_{x_0}\bar u(\xi)\Big)h_{sij}(x_0)P^{k-3}_{x_0}(X^{\varepsilon}_{i,\bar u}X^{\varepsilon}_{j,\bar u}u)(\xi)+\\
&+\Big(\bar u(\xi) - P^1_{x_0}\bar u(\xi)\Big)h_{sij}(x_0)\Big(X^{\varepsilon}_{i,\bar u}X^{\varepsilon}_{j,\bar u}u(\xi) - P^{k-3}_{x_0}(X^{\varepsilon}_{i,\bar u}X^{\varepsilon}_{j,\bar u}u)(\xi)\Big).
\end{aligned}
$$

Proof. We represent $u\varphi$ as in formula (4.2), and we study each term separately. Let us start with the second term in (4.2):

$$g(\xi) = L_{\varepsilon,\bar u}u(\xi) = P^{k-2}_{x_0}g(\xi) + (g(\xi) - P^{k-2}_{x_0}g(\xi)) \tag{4.15}$$

The kernel in the third term of (4.2) will developed as follows:

$$\begin{aligned}\Big(b_{ij\bar u}(\xi) - b_{ij\bar u}(x_0)\Big)&X^\varepsilon_{i,\bar u}X^\varepsilon_{j,\bar u}u(\xi) = \\ &= \Big(P^{k-2}_{x_0}b_{ij\bar u}(\xi) - b_{ij\bar u}(x_0)\Big)P^{k-3}_{x_0}(X^\varepsilon_{i,\bar u}X^\varepsilon_{j,\bar u}u)(\xi) + \\ &+ \Big(b_{ij\bar u}(\xi) - b_{ij\bar u}(x_0)\Big)\Big(X^\varepsilon_{i,\bar u}X^\varepsilon_{j,\bar u}u(\xi) - P^{k-3}_{x_0}(X^\varepsilon_{i,\bar u}X^\varepsilon_{j,\bar u}u)(\xi)\Big) \\ &+ \Big(b_{ij\bar u}(\xi) - P^{k-2}_{x_0}b_{ij\bar u}(\xi)\Big)P^{k-3}_{x_0}(X^\varepsilon_{i,\bar u}X^\varepsilon_{j,\bar u}u)(\xi).\end{aligned} \tag{4.16}$$

The first terms in (4.15) and (4.16) define $N_{2,k}$, the sum of the other terms defines $N_{3,k}$.

The kernel in the last term of (4.2) can be represented as

$$\begin{aligned}\Big(\bar u(\xi) - P^1_{x_0}\bar u(\xi)\Big)&h_{kij}(x_0)X^\varepsilon_{i,\bar u}X^\varepsilon_{j,\bar u}u(\xi) = \\ &= \Big(\bar u(\xi) - P^1_{x_0}\bar u(\xi)\Big)h_{sij}(x_0)\Big(X^\varepsilon_{i,\bar u}X^\varepsilon_{j,\bar u}u(\xi) - P^{k-3}_{x_0}(X^\varepsilon_{i,\bar u}X^\varepsilon_{j,\bar u}u)(\xi)\Big) \\ &+ \Big(\bar u(\xi) - P^{k-1}_{x_0}\bar u(\xi)\Big)h_{sij}(x_0)P^{k-3}_{x_0}(X^\varepsilon_{i,\bar u}X^\varepsilon_{j,\bar u}u)(\xi) + \\ &+ \Big(P^{k-1}_{x_0}\bar u(\xi) - P^1_{x_0}\bar u(\xi)\Big)h_{sij}(x_0)P^{k-3}_{x_0}(X^\varepsilon_{i,\bar u}X^\varepsilon_{j,\bar u}u)(\xi).\end{aligned} \tag{4.17}$$

The first two terms of this expression define $N_{4,ks}$, the third defines $N_{2,ks}$. □

4.2. A priori estimates of the solution of the linear operator.

Proposition 4.4. *Assume that u and $\bar u$ are of class $C^\infty_{\bar u}(\Omega)$ and that $L_{\varepsilon,\bar u}u = g \in C^\infty_{\bar u}(\Omega)$. Also assume that there exists a compact set $K \subset \Omega$ and constant C_0 such that*

$$||\bar u||_{C^{1,\alpha}_{\bar u}(K)} + ||g||_{C^{1,\alpha}_{\bar u}(K)} + \sum_{deg(\sigma)\le 2} ||\nabla^\varepsilon_{\sigma,\bar u}u||_{L^\infty(K)} \le C_0.$$

Then, for every compact set $K_1 \subset\subset K$, for every $\alpha' < \alpha$ there exists a constant $C > 0$ only dependent on C_0, α and the compact sets, such that

$$\sum_{deg(\sigma)=2} ||\nabla^\varepsilon_{\sigma,\bar u}u||_{C^{\alpha'}_{\bar u}(K_1)} \le C. \tag{4.18}$$

Moreover, for every choice of compact sets K_2, K_3 such that $K_1 \subset\subset K_2 \subset\subset K_3 \subset\subset K$ for every function $\varphi \in C^\infty_0(int(K))$ such that $\varphi \equiv 1$ in K_2, for every multi-index

σ of length 2, we have the following representation

(4.19)

$$\nabla^{\varepsilon}_{\sigma,\bar u}(u\varphi)(x_0) = \int_\Omega \nabla^{\varepsilon}_{\sigma,\bar u}(x_0)\Gamma^{\varepsilon}_{x_0}(\cdot,\xi)N_1(\xi,x_0)d\xi$$

$$+ \int_\Omega \Gamma^{\varepsilon}_{x_0}(\xi,0)\nabla^{\varepsilon}_{\sigma,\bar u}(x_0)P^1_{x_0}g(\xi)\,\varphi(x_0\circ\xi^{-1})\,d\xi$$

$$+ \int_\Omega \nabla^{\varepsilon}_{\sigma,\bar u}(x_0)\Gamma^{\varepsilon}_{x_0}(\cdot,\xi)\Big(g(\xi) - P^1_{x_0}g(\xi)\Big)\varphi(\xi)d\xi$$

$$+ \sum_{ij=1}^{2n}\int_\Omega \nabla^{\varepsilon}_{\sigma,\bar u}(x_0)\Gamma^{\varepsilon}_{x_0}(\cdot,\xi)\Big(b^{\varepsilon}_{ij}(\nabla^{\varepsilon}_{\bar u}\bar u(x_0)) - b^{\varepsilon}_{ij}(\nabla^{\varepsilon}_{\bar u}\bar u(\xi))\Big)X^{\varepsilon}_{i,\bar u}X^{\varepsilon}_{j,\bar u}u(\xi)\varphi(\xi)d\xi$$

$$+ \sum_{ijs=1}^{2n}\int_\Omega \nabla^{\varepsilon}_{\sigma,\bar u}X^{\varepsilon}_{i,x_0}(x_0)\Gamma^{\varepsilon}_{x_0}(\cdot,\xi)\Big(\bar u(\xi) - P^1_{x_0}\bar u(\xi)\Big)h_{sij}(x_0)X^{\varepsilon}_{i,\bar u}X^{\varepsilon}_{j,\bar u}u(\xi)\varphi(\xi)d\xi.$$

where N_1 is defined in Proposition 4.2.

Proof. By Proposition 4.2, we know that the function $u\varphi$ admits a representation in terms of the fundamental solution of the frozen operator, and suitable kernels. Let us verify that these kernels satisfy the assumptions of Lemma 2.9 with $k = 2$.

From the expression of N_1 in (4.3), we see that N_1 is a sum of derivatives of the function φ. Since φ is constantly equal to 1 on the set K_2, then N_1 vanishes on the same set. Hence the point (i) of Lemma 2.9 regarding the support of N_1 is satisfied. On the other side N_1 depends on x_0 only through the first derivatives of $\bar u$, while it depends on ξ through the derivatives up to second order of the function u. Hence it is Hölder continuous in x_0 locally uniformly in ξ. Hence there exists a constant C_1 only depending on C_0 such that

$$|N_1(\xi,x_0) - N_1(\xi,x)| \le C_1 d^{\alpha}_{\varepsilon,x_0}(x_0,x),$$

for every $x, x_0 \in K_1$ and $\xi \in K_3$ and this conclude the proof of assumption (2.15).

The second term in (4.2) is the convolution of the fundamental solution with the function $g(\xi) = L_{\varepsilon,\bar u}u(\xi) = P^1_{x_0}g(\xi) + \Big(g(\xi) - P^1_{x_0}g(\xi)\Big)$. The function $P^1_{x_0}g(\xi)\varphi(\xi)$ will play the role of the kernel N_2 in Lemma 2.9. Since g is of class $C^{1,\alpha}_{\bar u}(K)$, its first order Taylor polynomial is Hölder continuous in x_0 locally uniformly in ξ and there exists a constant C_1 only depending on C_0 such that

$$|P^1_{x_0}g(\xi)\varphi(\xi) - P^1_{x}g(\xi)\varphi(\xi)| \le C_1 d^{\alpha}_{\varepsilon,x_0}(x_0,x),$$

for every $x, x_0 \in K_1$ and $\xi \in K_3$. And this conclude the proof of assumption (ii).

The function $(g - P^1_{x_0}g)(\xi)$ satisfies the assumptions (2.16) and (2.17) of the kernel $N_{3,k}$, in the same lemma with $k = 2$. Indeed, from the definition (2.2) of Taylor polynomial we deduce that

$$|(g - P^1_{x_0}g)(\xi)| \le C_1 d^{1+\alpha}_{\varepsilon,x_0}(x_0,\xi),$$

if x, x_0 are fixed in K_1 and $\xi \in K_3$. Similarly, from (2.4) we deduce that

$$\big|P^1_{x_0}g(\xi) - P^1_{x}g(\xi)\big| \le C_1 d_{\varepsilon,x_0}(x_0,x)^{\alpha} d_{\varepsilon,x_0}(x_0,\xi),$$

again with a constant C_1, depending on the $C^{1,\alpha}_{\bar u}$ norm of g. This ensures that (2.17) is satisfied.

In the same way, using the regularity properties of $\bar u$, we deduce that the function

$$\Big(b^\varepsilon_{ij}(\nabla^\varepsilon_{\bar u}\bar u(x_0)) - b^\varepsilon_{ij}(\nabla^\varepsilon_{\bar u}\bar u(\xi))\Big)X^\varepsilon_{i,\bar u}X^\varepsilon_{j,\bar u}u(\xi)$$

satisfies the assumption of the kernel $N_{3,k}$ in Lemma 2.9.

Finally, from the property (2.4) of the Taylor polynomials, we deduce that the function

$$(\bar u(\xi) - P^1_{x_0}\bar u(\xi))h_{sij}(x_0)X^\varepsilon_{i,\bar u}X^\varepsilon_{j,\bar u}u(\xi)\varphi(\xi)$$

satisfies the assumptions of the kernels $N_{4,ks}$ in Remark 2.11. □

In order to obtain an a-priori estimates of the second derivatives of the solution u in terms of its L^p norms we need to improve slightly the previous result.

Lemma 4.5. *Assume that u and $\bar u$ are C^∞ functions, and that u is a classical solution of $L_{\varepsilon,\bar u}u = g \in C^\infty_{\bar u}(\Omega)$. Also assume that there exist a compact $K \subset \Omega$, and real numbers $p > 1$, $\alpha < 1$ and $C_0 > 0$ such that*

$$||\bar u||_{C^{1,\alpha}_{\bar u}(K)} + ||g||_{C^{1,\alpha}_{\bar u}(K)}+$$

$$+||u||_{C^{1,\alpha}_{\bar u}(K)} + \sum_{deg(\sigma)=2}||\nabla^\varepsilon_{\sigma,\bar u}u||_{L^p(K)} \le C_0. \tag{4.20}$$

- *If $Q - p\,\alpha > 0$, then, for every compact set $K_1 \subset\subset K$ there exists a constant $C > 0$ only depending on C_0 such that*

$$\sum_{deg(\sigma)=2}||\nabla^\varepsilon_{\sigma,\bar u}u||_{L^r(K_1)} \le C, \tag{4.21}$$

  where $r = \frac{Qp}{Q-p\,\alpha}$, and Q is the homogeneous dimension of the space, defined in (2.7).
- *If $Q - p\,\alpha < 0$, then for every compact set $K_1 \subset\subset K$ there exists a constant $C > 0$ only depending on C_0 such that*

$$\sum_{deg(\sigma)=2}||\nabla^\varepsilon_{\sigma,\bar u}u||_{L^\infty(K_1)} \le C. \tag{4.22}$$

Proof. The proof follows from the representation of the derivatives of $u\varphi$ provided in Proposition 4.4.

Let us consider the first integral in (4.19) We first note that, by the expression (4.3) of N_1, there exist constants C_3 and C_4 only dependent on C_0 such that

$$|N_1(\xi,x_0)| \le C_3 + C_4\sum_{ij}|X_{i\bar u}X_{i\bar u}u|$$

for any $\xi, x_0 \in K$. On the other hand N_1 is a sum of derivatives of the function φ, constant in K_2, the support of the kernel N_1 is a subset of $K_3 - K_2$. This implies that if $x \in K_1$ and $\xi \in supp(N_1)$, then $d_{\varepsilon,\bar u}(x,\xi) \ge d_{\varepsilon,\bar u}(K_1, K_3 - K_2)$ and the function $\nabla^\varepsilon_{\sigma,\bar u}(x_0)\Gamma^\varepsilon_{x_0}(x,\xi)$, is bounded uniformly in ε, by condition (2.11). Then, for every $x_0 \in K_1$

$$\Big|\int \nabla^\varepsilon_{\sigma,\bar u}\Gamma^\varepsilon(x,\xi)N_1(\xi,x_0)d\xi\Big| \le \int_{K_3}|N_1(\xi,x_0)|d\xi \le C\Big(1 + ||X_{i\bar u}X_{i\bar u}u||_{L^1(K)}\Big) \le C_1,$$

where C only depends on C_0.

The second term in (4.19) is the convolution of the fundamental solution with a regular function,

$$\nabla^{\varepsilon}_{\sigma,\bar u}(x_0)\big(P^1_{x_0}g(\xi)\,\varphi(x_0\circ\xi^{-1})\big)$$

whose L^∞ norm only depends on $||g||_{C^{1,\alpha}_{\bar u}}$ on the support K of the function φ.

In the third term of (4.19), using the property (2.2) of the Taylor polynomial, and the estimate (2.10) of the fundamental solution, we obtain

$$\int_\Omega \Big|\nabla^{\varepsilon}_{\sigma,\bar u}(x_0)\Gamma^{\varepsilon}_{x_0}(\cdot,\xi)\Big(g(\xi)-P^1_{x_0}g(\xi)\Big)\Big|\varphi(\xi)d\xi \le C\int_{K_3} d^{\alpha-Q}_{\varepsilon,x_0}(x_0,\xi)d\xi \le C_1$$

for a suitable constant C_1 depending on $||g||_{C^{1,\alpha}_{\bar u}}$, and on the compact set K. Consequently, these terms belong to L^∞_{loc}.

Using again (2.2) and (2.10) the last two terms in representation formula (4.19) can be estimated by

$$\int_\Omega d^{-Q+\alpha}_{\varepsilon,x_0}(x_0,\xi)|X^{\varepsilon}_{i,\bar u}X^{\varepsilon}_{j,\bar u}u(\xi)|d\xi. \tag{4.23}$$

Thanks to Theorem 2.6, we can then apply the standard theory of singular integrals and deduce that if $|X^{\varepsilon}_{i,\bar u}X^{\varepsilon}_{j,\bar u}u(\xi)|\in L^p(K)$, with $Q-p\alpha>0$ then (4.21) follows.

In order to prove (4.22) it suffices to apply Hölder inequality to (4.23) and use the fact that $Q-p\alpha<0$. This immediately leads to the desired L^∞ bounds on the second derivatives of the solution. □

Proposition 4.6. *Let us assume that $\bar u$ is of class $C^\infty(\Omega)$ and that u is a classical solution of $L_{\varepsilon,\bar u}u = g\in C^\infty_{\bar u}(\Omega)$. Let us also assume that there exists a compact set $K\subset\Omega$ and constant C_0 such that assumption (4.20) is satisfied, and such that*

$$||u||_{C^{k-1,\alpha}_{\bar u}(K)} + ||\bar u||_{C^{k-1,\alpha}_{\bar u}(K)} + ||g||_{C^{k-2,\alpha}_{\bar u}(K)} \le C_0.$$

Then, for any $2\le k\le 4$, for every compact set $K_1\subset\subset K$ there exists a constant $C>0$ depending only on the choice of the compact sets, C_0, k and α such that

$$||u||_{C^{k,\alpha}_{\bar u}(K_1)} \le C.$$

Proof. A direct computation shows that the kernel in representation formula in Proposition 4.3 satisfy assumptions of Proposition 2.9. □

4.3. **A priori estimates of the solution of the nonlinear operator.** We start with the follow iteration result:

Lemma 4.7. *Assume that z is a smooth function satisfying*

$$\sum_{ij} a_{ij}(\nabla^{\varepsilon}_{\bar u}\bar u)X^{\varepsilon}_{i\bar u}X^{\varepsilon}_{j\bar u}z + f_0 = 0 \ \text{in}\ \Omega \tag{4.24}$$

then the function $v_h = X^{\varepsilon}_{h,\bar u}z$ satisfies the equation

$$\sum_{ij} a_{ij}(\nabla^{\varepsilon}_{\bar u}\bar u)X^{\varepsilon}_{i\bar u}X^{\varepsilon}_{j\bar u}v_h + f_h = 0,$$

on the same set Ω, where f_h depends on $\nabla^{\varepsilon 2}_{\bar u}z$, $\partial_{2n}\nabla^{\varepsilon}_{\bar u}z$, $X^{\varepsilon}_{h,\bar u}f_0$, $\nabla^{\varepsilon 2}_{\bar u}\bar u$.

Proof. Differentiating the equation (4.24) with respect to $X^\varepsilon_{h,\bar u}$ we obtain

$$0 = X^\varepsilon_{h,\bar u}\Big(\sum_{ij} a_{ij}(\nabla^\varepsilon_{\bar u}\bar u)X^\varepsilon_{i\bar u}X^\varepsilon_{j\bar u}z\Big) + X^\varepsilon_{h,\bar u}f_0 =$$

$$= \sum_{ijk}\partial_{p_k}a_{ij}X^\varepsilon_{h,\bar u}X^\varepsilon_{k\bar u}\bar u X^\varepsilon_{i\bar u}X^\varepsilon_{j\bar u}z + \sum_{ij}a_{ij}(\nabla^\varepsilon_{\bar u}\bar u)[X^\varepsilon_{h,\bar u}, X^\varepsilon_{i\bar u}]X^\varepsilon_{j\bar u}z +$$

$$+ \sum_{ij}a_{ij}(\nabla^\varepsilon_{\bar u}\bar u)X^\varepsilon_{i\bar u}[X^\varepsilon_{h,\bar u}, X^\varepsilon_{j\bar u}]z + \sum_{ij}a_{ij}(\nabla^\varepsilon_{\bar u}\bar u)X^\varepsilon_{i\bar u}X^\varepsilon_{j\bar u}(X^\varepsilon_{h,\bar u}z) + X_{h,\bar u}f_0.$$

Then

$$\sum_{ij}a_{ij}(\nabla^\varepsilon_{\bar u}u)X^\varepsilon_{i\bar u}X^\varepsilon_{j\bar u}v_h + f_h = 0,$$

where

$$f_h = \sum_{ij}^{2n}\partial_{p_k}a_{ij}X^\varepsilon_{h,\bar u}X^\varepsilon_{k\bar u}\bar u\ X^\varepsilon_{i\bar u}X^\varepsilon_{j\bar u}z +$$

$$\sum_{ij}a_{ij}(\nabla^\varepsilon_{\bar u}u)\Big(X^\varepsilon_{h,\bar u}b_i - X^\varepsilon_{i,\bar u}b_h\Big)\partial_{2n}X^\varepsilon_{j\bar u}z +$$

$$+ \sum_{ij}a_{ij}(\nabla^\varepsilon_{\bar u}u)X^\varepsilon_{i\bar u}\Big(X^\varepsilon_{h,\bar u}b_j - X^\varepsilon_{j,\bar u}b_h\Big)\partial_{2n}z +$$

$$+ \sum_{ij}a_{ij}(\nabla^\varepsilon_{\bar u}u)\Big(X^\varepsilon_{h,\bar u}b_j - X^\varepsilon_{j,\bar u}b_h\Big)\partial_{2n}X^\varepsilon_{i\bar u}z$$

$$+ \sum_{ij}a_{ij}(\nabla^\varepsilon_{\bar u}u)\Big(X^\varepsilon_{h,\bar u}b_j - X^\varepsilon_{j,\bar u}b_h\Big)\partial_{2n}b_i\partial_{2n}z + X^\varepsilon_{h,\bar u}f_0.$$

Here the function f_h clearly depends on $\nabla^{\varepsilon 2}_{\bar u}z$, $\partial_{2n}\nabla^\varepsilon_{\bar u}z$, $X^\varepsilon_{h,\bar u}f_0$, $\nabla^{\varepsilon 2}_{\bar u}\bar u$. □

Lemma 4.8. *Assume that z is a smooth function satisfying*

$$\sum_{ij}a_{ij}(\nabla^\varepsilon_{\bar u}\bar u)X^\varepsilon_{i\bar u}X^\varepsilon_{j\bar u}z + f_0 = 0 \ \text{in}\ \Omega \tag{4.25}$$

then the function $v = \partial_{2n}z$, satisfies

$$\sum_{ij}a_{ij}(\nabla^\varepsilon_{\bar u}u)X^\varepsilon_{i\bar u}X^\varepsilon_{j\bar u}v + f = 0,$$

where f depends on $\nabla^{\varepsilon 2}_{\bar u}z$, $\partial_{2n}\nabla^\varepsilon_{\bar u}z$, $\partial_{2n}f_0$.

Proof. Differentiating the equation (4.25) with respect to ∂_{2n} we obtain

$$0 = \partial_{2n}\Big(\sum_{ij}a_{ij}(\nabla^\varepsilon_{\bar u}\bar u)X^\varepsilon_{i\bar u}X^\varepsilon_{j\bar u}z + f_0\Big) =$$

$$= \sum_{ijk}\partial_{p_k}a_{ij}\partial_{2n}X^\varepsilon_{k\bar u}\bar u X^\varepsilon_{i\bar u}X^\varepsilon_{j\bar u}z + \sum_{ij}a_{ij}(\nabla^\varepsilon_{\bar u}\bar u)[\partial_{2n}, X^\varepsilon_{i\bar u}]X^\varepsilon_{j\bar u}z +$$

$$+ \sum_{ij}a_{ij}(\nabla^\varepsilon_{\bar u}\bar u)X^\varepsilon_{i\bar u}[\partial_{2n}, X^\varepsilon_{j\bar u}]z + \sum_{ij}a_{ij}(\nabla^\varepsilon_{\bar u}\bar u)X^\varepsilon_{i\bar u}X^\varepsilon_{j\bar u}(\partial_{2n}z) + \partial_{2n}f_0.$$

The latter can be rewritten as

$$\sum_{ij}a_{ij}(\nabla^\varepsilon_{\bar u}u)X^\varepsilon_{i\bar u}X^\varepsilon_{j\bar u}v + f = 0,$$

where the function

$$f = \sum_{ijk} \partial_{p_k} a_{ij} \partial_{2n} X^{\varepsilon}_{k\bar{u}} \bar{u} X^{\varepsilon}_{i\bar{u}} X^{\varepsilon}_{j\bar{u}} z + \sum_{ij} a_{ij}(\nabla^{\varepsilon}_{\bar{u}} \bar{u}) \delta_{i2n-1} \partial_{2n} \bar{u} \partial_{2n} X^{\varepsilon}_{j\bar{u}} z +$$

$$+ \sum_{ij} a_{ij} \delta_{j2n-1} (\nabla^{\varepsilon}_{\bar{u}} \bar{u}) X^{\varepsilon}_{i\bar{u}} (\partial_{2n} \bar{u} \partial_{2n} z) + \sum_{ij} a_{ij}(\nabla^{\varepsilon}_{\bar{u}} u) X^{\varepsilon}_{i\bar{u}} X^{\varepsilon}_{j\bar{u}} v + \partial_{2n} f_0$$

depends on $\nabla^{\varepsilon}_{\bar{u}} v$, $\nabla^{\varepsilon 2}_{\bar{u}} z$, $\partial_{2n} \nabla^{\varepsilon}_{\bar{u}} z$, $\nabla^{\varepsilon 2}_{\bar{u}} \bar{u}$. □

In order to study of the nonlinear equation we apply the previous lemma with $u = \bar{u}$,

Lemma 4.9. *Let σ be a multi-index with all components smaller than $2n$. Then the function $v_\sigma = \nabla^{\varepsilon}_{\sigma,u} u$ satisfies*

$$\sum_{ij} a_{ij}(\nabla^{\varepsilon}_{u} u) X^{\varepsilon}_{iu} X^{\varepsilon}_{ju} v_\sigma + f_\sigma = 0,$$

where f_σ depends on $\nabla^{\varepsilon(k+1)}_{u} u$, $\partial_{2n} \nabla^{\varepsilon k}_{u} u$ with $k = deg(\sigma)$.

Proof. By Lemma 4.7 the assertion is true for the derivatives of order one. Assume that it is true for $deg(\sigma) = k$. Then, let us consider a multi-index σ of degree $k+1$. By definition $\sigma = (\sigma_1, \widetilde{\sigma})$, with $deg(\widetilde{\sigma}) = k$. Then by inductive assumption the function $z = \nabla^{\varepsilon}_{\widetilde{\sigma},u} u$ satisfies

$$\sum_{ij} a_{ij}(\nabla^{\varepsilon}_{u} u) X^{\varepsilon}_{iu} X^{\varepsilon}_{ju} z + f_0 = 0,$$

where f_0 depends on $\nabla^{\varepsilon k+1}_{u} u$, $\partial_{2n} \nabla^{\varepsilon k}_{u} u$. Applying Lemma 4.7 we deduce that the function $v_\sigma = X^{e}_{\sigma_1 u} z$ is a solution of

$$\sum_{ij} a_{ij}(\nabla^{\varepsilon}_{u} u) X^{\varepsilon}_{i\bar{u}} X^{\varepsilon}_{ju} v_\sigma + f_{\sigma_1} = 0,$$

where f_{σ_1} depends on $\nabla^{\varepsilon 2}_{u} z$, $\partial_{2n} \nabla^{\varepsilon}_{\bar{u}} z$ and $X_{\sigma_1,u} f_0$. Since f_0 depends on $\nabla^{\varepsilon k+1}_{u} u$, $\partial_{2n} \nabla^{\varepsilon k}_{u} u$, then $X_{\sigma_1,u} f_0$ depends on $X^{\varepsilon}_{\sigma_1,u} \nabla^{\varepsilon k+1}_{u} u$ and

$$X^{\varepsilon}_{\sigma_1,u} \partial_{2n} \nabla^{\varepsilon k}_{u} u = \delta_{\sigma_1 2n-1} \partial_{2n} u \partial_{2n} \nabla^{\varepsilon k}_{u} u + \partial_{2n} X^{\varepsilon}_{\sigma_1,u} \nabla^{\varepsilon k}_{u} u.$$

□

Theorem 4.10. *Let u be a smooth classical solution of the nonlinear equation $L_\varepsilon u = 0$. Let us fix a compact set $K \subset\subset \Omega$ and assume that there exist constants $\alpha < 1$, $p > 1$ and $C_0 > 0$ such that*

$$||u||_{C^{1,\alpha}_u(K)} + \sum_{deg(\sigma)=2} ||\nabla^{\varepsilon}_{\sigma,u} u||_{L^p(K)} \le C_0. \tag{4.26}$$

Then, for every $\beta < 1$, for every compact set $K_1 \subset\subset K$ there exists a constant $\widetilde{C}_\beta$ such that

$$||u||_{C^{3,\beta}_u(K_1)} + ||\partial_{2n} u||_{C^{2,\beta}_u(K_1)} \le \widetilde{C}_\beta.$$

Proof. We first prove that for every $r > 1$ for every compact set K_2 such that $K_1 \subset\subset K_2 \subset\subset K$ there exists a constant C_r, only depending on C_0 and on the choice of the compact sets, such that for every multi-index σ of degree 2, we have

$$||\nabla^{\varepsilon}_{\sigma,u} u||_{L^r(K_2)} \le C_r. \tag{4.27}$$

Indeed, for every compact set K_3 such that $K_2 \subset\subset K_3 \subset\subset K$ we can apply the first assertion of Lemma 4.5, with $u = \bar{u}$ and we obtain

$$||\nabla^{\varepsilon}_{\sigma,u} u||_{L^{r_1}(K_3)} \le C_{r_1}, \tag{4.28}$$

where $r_1 = \frac{n2}{n-2\,\alpha} > 2$. If $n - r_1\alpha > 0$ we can apply again Lemma 4.5 on a new compact set compact set K_4 such that $K_2 \subset\subset K_4 \subset\subset K_3$ and we have

$$||\nabla^{\varepsilon}_{\sigma,\bar{u}} u||_{L^{r_2}(K_4)} \le C_{r_2}, \tag{4.29}$$

with

$$r_2 = \frac{nr_1}{n - r_1\,\alpha} = \frac{2n}{n - 4\,\alpha} > r_1.$$

For every fixed number r, after a finite number of iterations of this same argument, we can prove the estimate (4.27).

Consequently by (4.22) we have $||\nabla^{\varepsilon}_{\sigma,\bar{u}} u||_{L^{\infty}_{loc}} \le C$ then for every compact set K_5 such that $K_1 \subset\subset K_2$ and for every $\beta < 1$ there exists a constant $\widetilde{C}_\beta$ such that

$$||\nabla^{\varepsilon}_{\bar{u}} u||_{C^{\beta}_{\bar{u}}(K_5)} \le \widetilde{C}_\beta.$$

As a consequence of Proposition 4.4 we deduce that for every $\beta < 1$ for every compact set K_6 such that $K_1 \subset\subset K_6 \subset\subset K_5$ there exists a constant $\widetilde{C}_\beta$ such that

$$||u||_{C^{2,\beta}_{\bar{u}}(K_6)} \le \widetilde{C}_\beta.$$

By Proposition 4.6 we deduce that for every $\beta < 1$ and for every compact set K_7 such that $K_1 \subset\subset K_7 \subset\subset K_6$ there exists a constant $\widetilde{C}_\beta$ such that

$$||u||_{C^{3,\beta}_{\bar{u}}(K_7)} \le \widetilde{C}_\beta.$$

Applying again the same proposition with $k = 4$, we deduce that

$$||u||_{C^{4,\beta}_{\bar{u}}(K_1)} \le \widetilde{C}_\beta,$$

which implies in particular that there exists a constant $\widetilde{C}_\beta$ such that

$$||u||_{C^{3,\beta}_{\bar{u}}(K_1)} + ||\partial_{2n} u||_{C^{2,\beta}_{\bar{u}}(K_1)} \le \widetilde{C}_\beta.$$

□

Theorem 4.11. *Let u be a smooth classical solution of $L_\varepsilon u = 0$. Let us also assume that assumption (4.26) is satisfied. Then, for any compact set $K_1 \subset\subset K$, for every $k \in N$ and $\alpha < 1$, there exists a constant $C > 0$ depending only on C_0, k, α and K_1 such that*

$$||u||_{C^{k,\alpha}_u(K_1)} \le C.$$

Proof. For every $k \in N$, for every σ with $deg(\sigma) = k$, and components in $\{1, \dots, 2n\}$ we prove by induction that

$$\nabla^{\varepsilon}_{\sigma,u} u \in C^{3,\alpha}_u(\Omega), \quad \partial_{2n} \nabla^{\varepsilon}_{\sigma,u} u \in C^{2,\alpha}_u(\Omega),$$

and that for every compact set K_1 such that $K_1 \subset\subset K$ there exists a constant $C > 0$ depending only on C_0, k, α such that

$$||\nabla^\varepsilon_{\sigma,u} u||_{C^{3,\alpha}_u(K_1)} + ||\partial_{2n} \nabla^\varepsilon_{\sigma,u} u||_{C^{2,\alpha}_u(K_1)} \leq C.$$

The thesis is true for $deg(\sigma) = 0$ by Theorem 4.10.

We assume by induction that it is true for $deg(\sigma) = k$. Call $z = \nabla^\varepsilon_{\sigma,u} u$, then by Lemma 4.9 the function z satisfies $L_{\varepsilon,u} z = f_\sigma$ in Ω with $f_\sigma = f_\sigma(\nabla^{\varepsilon(k+1)}_u u, \partial_{2n} \nabla^{\varepsilon k}_u u) \in C^{2,\alpha}_u(\Omega)$, by inductive assumption. By Lemma 4.8, the function $v = \partial_{2n} z$ satisfies $L_{\varepsilon,u} v = f$, in Ω, where the function $f = f(\nabla^{\varepsilon 2}_u z, \partial_{2n} \nabla^\varepsilon_u z) \in C^{1,\alpha}_u(\Omega)$.

It follows by Proposition 4.6 that $v = \partial_{2n} z \in C^{3,\alpha}_u(\Omega)$, and if K_2 is a compact set such that $K_1 \subset\subset K_2 \subset\subset K$ there exists a constant C depending only on C_0, k, α such that

$$||v||_{C^{3,\alpha}_u(K_2)} \leq C_1 ||f||_{C^{1,\alpha}_u(K_2)} = C_2.$$

This argument, applied to any multi-index σ with $deg(\sigma) = k$, implies that $\partial_{2n} \nabla^{\varepsilon k}_u u \in C^{3,\alpha}_u$. Consequently $\partial_{2n} \nabla^{\varepsilon k+1}_u u \in C^{2,\alpha}_u$, and

$$||\partial_{2n} \nabla^{\varepsilon k}_u u||_{C^{2,\alpha}_u(K_2)} \leq C_1,$$

for an other constant C_1, only dependent on C_0, k, α.

Moreover by Lemma 4.7 the function $v_h = X^\varepsilon_{hu} z$ satisfies $Lv_h = f_h$, with $f_h = f_h(\nabla^{\varepsilon 2}_u z, \partial_{2n} \nabla^\varepsilon_u z) \in C^{1,\alpha}_u(\Omega)$. Again Proposition 4.6 implies that $X^\varepsilon_{hu} \nabla^\varepsilon_{\sigma,u} u \in C^{3,\alpha}(\Omega)$, and that

$$||\nabla^{\varepsilon(k+1)}_u u||_{C^{3,\alpha}_u(K_1)} \leq C_2,$$

for a constant C_2 dependent on C_0, k, α. This concludes the proof. □

Proof of Theorem 1.2

Proof. Let $(u_\varepsilon)_\varepsilon$ be a smooth approximating sequence of u such that

$$L_\varepsilon u_\varepsilon = 0 \quad \text{in } \Omega.$$

Let K_1 be an arbitrary compact set in Ω. Then there exist compact sets K and K_2 such that

$$K_1 \subset\subset K_2 \subset\subset K.$$

By assumption there exists a positive constant C_0 such that

$$||\nabla_E u_\varepsilon||_{L^\infty(K)} \leq C_0,$$

for every ε. By proposition 3.7 there exists a constant C_1 such that

$$||\nabla^\varepsilon_{u_\varepsilon} u_\varepsilon||_{C^{1,\alpha}_{u_\varepsilon}(K_2)} + ||\partial_{2n} u_\varepsilon||_{L^\infty(K_2)} + ||\nabla^{\varepsilon 2}_{u_\varepsilon} u_\varepsilon||_{L^2(K_2)} \leq C_1.$$

Then by Theorem 4.11 for every k there is C_k such that

$$||u_\varepsilon||_{C^{2k,\alpha}_{u_\varepsilon}(K_1)} \leq C_k.$$

In particular,

$$||u_\varepsilon||_{C^k_E(K_1)} \leq C_k.$$

Since all the constants are independent of ε, letting ε go to 0 we obtain estimates of u in C^k_E for every k. Consequently $u \in C^\infty_E$. □

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