Random Homogenization for the Stokes Flow through a Leaky Membrane

Razvan Florian Maris
Worcester Polytechnic Institute

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Random Homogenization of the Stokes Flow through a Leaky Membrane

by

Razvan Florian Maris

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APPROVED:

Professor Bogdan Vernescu, Advisor
Department of Mathematical Sciences
Worcester Polytechnic Institute

Professor Umberto Mosco
Department of Mathematical Sciences
Worcester Polytechnic Institute

Professor Ralph Showalter
Department of Mathematics
Oregon State University

Professor David Kinderlehrer
Department of Mathematical Sciences
Carnegie Mellon University

Professor Marcus Sarkis
Department of Mathematical Sciences
Worcester Polytechnic Institute

Professor Darko Volkov
Department of Mathematical Sciences
Worcester Polytechnic Institute
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Abstract

We study a random homogenization problem concerning the flow of a viscous fluid through a permeable membrane with a highly oscillatory geometry and nonlinear boundary condition on it. Along an interface we consider a periodic distribution of small permeable obstacles with a random geometry. Leak boundary conditions of threshold type are considered on the obstacle part of the membrane: the normal velocity of the fluid is zero until the jump of the normal component of the stress acting on it reaches a certain limit, and then the fluid may pass freely.

The problem is studied first in the deterministic case, and then in the random case, for which assumptions on the randomness of the solid obstacles are needed in order to obtain a limiting behaviour. The description of the obstacles is given in terms of a random set-valued variable defined on a probability space and a dynamical system acting on it. Effective boundary conditions for the fluid are derived, and these depend on the relative size of the obstacles. We establish two major cases, in one of them we obtain an effective permeability across the membrane and in the critical case a slip boundary condition of Navier type. If the dynamical system is assumed to be ergodic, the limiting behaviour of the fluid is deterministic.

The approach is based on the Mosco convergence, which also allows us to pass from the stationary case to the time dependent case via the convergence of the associated semigroups.
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Chapter 1

Introduction

The homogenization of transmission problems across perforated walls has been first studied, in the context of the Laplace equation with Neumann conditions on the wall, by Damlamian [1] (see also Attouch [2], Murat [3] and Picard [4] for different approaches). The corresponding spectral problem was studied by Onofrei and Vernescu [5]. Similar to the case of periodically distributed holes in the volume (as in Cioranescu and Murat [6, 7]), in these problems a critical size of the perforations was observed, for which in the limit the boundary conditions exhibited a different form.

The homogenization problem for flow of a viscous incompressible fluid through a perforated wall, with periodically distributed perforations, where the period and the size of the perforations were of the same order, was studied in a series of papers by Sanchez-Palencia [8, 9], using the asymptotic expansion method (see [10, 11] for other applications of this method), and then by Conca [12, 13, 14] using the oscillating test function method developed by Tartar in [15]. An important feature of the problem was that, in the limit, the velocity was normal to the wall and of constant magnitude. However in a neighborhood of the perforated wall the energy dissipation blew up, as stress concentrations were present.

Threshold slip and leak conditions were first introduced for viscous flow problems by Fujita [16, 17] who studied the existence and uniqueness for the Stokes problem with boundary conditions of the type

\[-(\sigma N)_r \in g \partial|u_r|,
\]

and respectively

\[-(\sigma N \cdot N) \in g \partial|u_N|,
\]  

(1.1)

where \(g \geq 0\) and for a convex function \(f\) we denote by \(\partial f\) the subdifferential of \(f\).
Here \( N \) denotes unit normal on the boundary, exterior to the fluid, and
\[
    u_N = u \cdot N, \quad u_\tau = u - (u \cdot N)N, \quad (1.2)
\]
\[
    \sigma_N = \sigma N \cdot N, \quad (\sigma N)_\tau = \sigma N - \sigma N N \quad (1.3)
\]
denote the normal and tangential projections of the velocity and of the normal stress, respectively. The threshold slip boundary condition can be written equivalently as
\[
    |(\sigma N)_\tau| < g \Rightarrow u_\tau = 0 \quad (1.4)
\]
\[
    |(\sigma N)_\tau| = g \Rightarrow (u_\tau = 0 \text{ or } u_\tau \neq 0) \text{ and } (u_\tau \neq 0 \Rightarrow (\sigma N)_\tau = -g \frac{u_\tau}{|u_\tau|}) \quad (1.5)
\]
and the threshold leak as:
\[
    |\sigma N \cdot N| \leq g \quad \text{and} \quad \left\{ \begin{array}{ll}
    |\sigma N \cdot N| < g \Rightarrow u_N = 0 \\
    |\sigma N \cdot N| = g \Rightarrow u_N = 0 \quad \text{or} \quad -\sigma N \cdot N = g \frac{u_N}{|u_N|} 
\end{array} \right. \quad (1.6)
\]
Regularity results of the solution to the Stokes problem with slip or leak boundary conditions was studied by Saito [18].

This thesis is structured as follows.

In Chapter 2 we formulate our problem and introduce the functional setting. We formulate first the continuous membrane problem in the stationary case with leak interface conditions on it and Dirichlet boundary condition on the boundary of the domain.

For a membrane immersed in a fluid, the leak condition becomes
\[
    -(|[\sigma N] \cdot N| \in g \partial|u_N|), \quad (1.7)
\]
or equivalently
\[
    |[\sigma N] \cdot N| \leq g \quad \text{and} \quad \left\{ \begin{array}{ll}
    |[\sigma N] \cdot N| < g \Rightarrow u_N = 0 \\
    |[\sigma N] \cdot N| = g \Rightarrow u_N = 0 \quad \text{or} \quad -[\sigma N] \cdot N = g \frac{u_N}{|u_N|} 
\end{array} \right. \quad (1.8)
\]
where \([h]\) denotes the jump of a field \( h \) across the membrane.

Here the geometry of the problem is described by a smooth interface \( \Sigma \) that separates a domain \( D \) into two subdomains \( D^+ \) and \( D^- \). Our problem becomes:
\[
    \begin{cases}
    -\Delta u + \nabla p = f & \text{in } D \setminus \Sigma, \\
    \nabla \cdot u = 0 & \text{in } D \setminus \Sigma, \\
    u = b & \text{on } \partial D, \\
    u_\tau = 0 & \text{on } \Sigma, \\
    -[\sigma N] \cdot N \in g \partial|u_N| & \text{a.e. on } \Sigma. \quad (1.9)
    \end{cases}
\]
The problem has a natural equivalent variational formulation for which we obtain existence and uniqueness results. Defining the sets:

\[ V = \left\{ u \in H^1(D)/\nabla \cdot u = 0 \text{ in } D \right\}, \]
\[ H^1_b(D) = \left\{ u \in H^1(D)/u = b \text{ on } \partial D \right\}, \]
\[ K = \left\{ u \in H^1_b(D)/u_{\tau} = 0 \text{ on } \Sigma \right\}, \]

we show the solution for the problem satisfies a variational inequality and u solves a minimization problem over a convex set:

\[
\min_{v \in K \cap V} \left\{ \frac{1}{2} a(v, v) + j(v) - \langle f, v \rangle_D \right\}
\]

where \(a(\cdot, \cdot)\) is a bilinear form, continuous and coercive and \(j\) is a convex functional. From here we derive the existence and uniqueness for the velocity. The solution for the pressure is not unique in the case when the flow decouples, as in this case on the interface the jump of the normal stress is below the threshold.

In Chapter 3 we study the effective interface conditions on a membrane with periodically distributed leaky parts. If the characteristic size of the period is denoted by \(\epsilon\) and the leaky parts are of characteristic size \(r_\epsilon\), then in deriving the limit behavior of the membrane, four different cases are distinguished: (i.) \(\lim_{\epsilon \to 0} \frac{r_\epsilon}{\epsilon} = \alpha > 0\), (ii.) \(\lim_{\epsilon \to 0} \frac{r_\epsilon}{\epsilon} = 0\) and \(\lim_{\epsilon \to 0} \frac{r_\epsilon^{n-2}}{\epsilon^{n-1}} = +\infty\), (iii.) \(\lim_{\epsilon \to 0} \frac{r_\epsilon^{n-2}}{\epsilon^{n-1}} = \beta \in (0, \infty)\) and (iv.) \(\lim_{\epsilon \to 0} \frac{r_\epsilon^{n-2}}{\epsilon^{n-1}} = 0\).
In the first case the limit problem consists of a leaky membrane for which an effective yield limit is derived; unlike in the permeable wall problem \cite{12, 13} the stress does not blow up along the membrane. In the third case a Navier-type slip condition \cite{19} is obtained, relating the jump of normal stress vector to the velocity on the membrane. In the intermediate case in the limit the tangential slip along the membrane cancels, whereas the normal velocity and stress are continuous. In the last case the membrane disappears in the limit.

The description of the geometry is as follows. \( D \) is a bounded domain in \( \mathbb{R}^n \), with Lipschitz boundary, that is separated by the hyperplane \( \{ x_n = 0 \} \) into two parts: \( D^+ = D \cap \{ x_n > 0 \} \) and \( D^- = D \cap \{ x_n < 0 \} \), both of them Lipschitz, and let \( \Sigma = D \cap \{ x_n = 0 \} \). We consider the unit cube in \( \mathbb{R}^{n-1} \), \( Y' = (-\frac{1}{2}, \frac{1}{2})^{n-1} \), and a fixed set \( \Gamma \subset \subset Y' \), the closure of a connected open set with Lipschitz boundary. Let \( N = \mathbb{Z}^{n-1} \) be the set of all points \( k \in \mathbb{Z}^{n-1} \) such that \( \epsilon(k + Y') \) is strictly included in \( \Sigma \) and denote by \( |N| \) their number. For any \( \epsilon > 0 \) and any \( k \in N \) we denote by \( \Gamma_k \) the set \( \epsilon k + r \Gamma \), where the sequence \( r \) satisfies \( 0 < r \leq \epsilon \), and represents the size of the obstacles. We define the following sets:

\[
\Gamma = \bigcup_{k \in N} \Gamma_k, \quad T = \Sigma \setminus \Gamma, \quad D = D \setminus \Gamma.
\]

\( \Gamma \) represents the union of the obstacles, distributed periodically in the cells \( \epsilon(k + Y') \) along \( \Sigma \) and \( T \) represents the holes. The fluid flows freely through the holes, but only above a certain stress through the obstacles. If \( u^\epsilon \) is the velocity of the fluid and \( p^\epsilon \) its pressure, the pair \( \{ u^\epsilon, p^\epsilon \} \) satisfies the Stokes equation in \( D \) and interface conditions

\[
\begin{aligned}
-\Delta u^\epsilon + \nabla p^\epsilon &= f & & \text{in } D, \\
\nabla \cdot u^\epsilon &= 0 & & \text{in } D, \\
\nabla \cdot u^\epsilon &= b & & \text{on } \partial D, \\
\n\nabla \cdot u^\epsilon &= 0 & & \text{on } \Gamma^\epsilon \text{ for all } i < n, \\
-\sigma^\epsilon e_n \cdot e_n &= g \partial |u^\epsilon| & & \text{on } \Gamma^\epsilon,
\end{aligned}
\]

when \( f \) is a distribution in \( \mathbf{H}^{-1}(D) \) and \( b \in \mathbf{H}^{1/2}(\partial D) \) satisfies some compatibility conditions. The i-th canonical vector of \( \mathbb{R}^n \) is denoted by \( e_i \) for \( 1 \leq i \leq n \) and \( g \) is a positive function from \( L^2(\Sigma) \), not necessarily constant, that represents the threshold for the appearance of the leak.

We define a convex functional \( F^\epsilon \) on \( \mathbf{H}^1(D) \) by:

\[
F^\epsilon(u) = \frac{1}{2} a(u, u) + j^\epsilon(u) - \langle f, u \rangle_D + I_{K^\epsilon \cap \nabla}(u)
\]
where $a$ is the bilinear form defined through $e(u) = \frac{1}{2}(\nabla u + \nabla^T u)$, the strain rate tensor

$$a(u, v) = 2(e(u), e(v))_D = 2\int_D \sum_{i,j=1}^n e_{ij}(u)e_{ij}(v)d\mathcal{L}^n \ 	ext{for} \ u, v \in \mathbf{H}^1(D),$$

$j^\epsilon$ is the convex functional defined on $\mathbf{H}^1(D)$

$$j^\epsilon(u) = \int_{\Gamma^\epsilon} g|u_n|d\mathcal{L}^{n-1},$$

and $K^\epsilon$ is the convex set

$$K^\epsilon = \{u \in \mathbf{H}^1_b(D)/u_i = 0 \ \text{on} \ \Gamma^\epsilon \ \forall i < n\}.$$

Also, $(\cdot, \cdot)_D$ is the duality between $\mathbf{H}^1_0(D)$ and $\mathbf{H}^{-1}(D)$ where $L^2(D)$ is the pivot space, $\mathbf{V}$ is the subspace of divergence free vector fields from $\mathbf{H}^1(D)$ and $I_S$ is the indicator function of the set $S$.

In the first case the limiting problem is

$$\begin{cases}
-\Delta u + \nabla p = f & \text{in } D \setminus \Sigma, \\
\nabla \cdot u = 0 & \text{in } D \setminus \Sigma, \\
u = b & \text{on } \partial D, \\
u_i = 0 & \text{on } \Sigma \ \text{for all } i < n, \\
[\sigma(u, p)e_n] \cdot e_n \in g\mathcal{L}^{n-1}(\Gamma) \cdot \partial|u_n| & \text{on } \Sigma.
\end{cases} \quad (1.11)$$
where $\mathcal{L}^d$ denotes the $d$–dimensional Lebesque measure.

This is a leaky membrane type of problem, with the threshold being the $L^2(\Sigma)$ function $g\mathcal{L}^{n-1}(\Gamma)$. This problem was studied in Chapter 2 and has the associated convex functional $F$:

$$F(u) = \frac{1}{2}a(u,u) + j(u) - \langle f, u \rangle_D + I_{K\cap V}$$

with

$$K = \{ u \in H^1_b(D)/u_r = 0 \text{ on } \Sigma \}$$

and

$$j(u) = \mathcal{L}^{n-1}(\Gamma) \int_{\Sigma} g|u_n|d\mathcal{L}^{n-1}.$$  

We show first the Mosco convergence ($M$–convergence) of $F^\epsilon$ to $F$ and then from here prove the convergence of $u^\epsilon$ to $u$ in the strong topology of $H^1(D)$ and the convergence of $p^\epsilon$ to $p$ in the strong topology of $L^2(D)/\mathbb{R}$, where $p$ is one of the solutions for the membrane problem.

In the third case the limiting problem is

\begin{equation}
\begin{cases}
-\Delta u + \nabla p = f & \text{in } D \setminus \Sigma, \\
\nabla \cdot u = 0 & \text{in } D \setminus \Sigma, \\
u = b & \text{on } \partial D, \\
-[\sigma(u,p)e_n] = \beta C u & \text{on } \Sigma,
\end{cases}
\end{equation}

where $C$ is a symmetric positive definite matrix. To define $C$ we need the solutions for the cell problems for $1 \leq i \leq n - 1$:

\begin{equation}
\begin{cases}
-\Delta \chi_i + \nabla \eta_i = 0 & \text{in } \mathbb{R}^n \setminus \Gamma, \\
\nabla \cdot \chi_i = 0 & \text{in } \mathbb{R}^n \setminus \Gamma, \\
\chi_i = e_i & \text{on } \Gamma, \\
\chi_i \to 0 & \text{at } \infty.
\end{cases}
\end{equation}

and $\chi_n = 0, \eta_n = 0$. We make use of these solutions to define

$$C_{ij} = \int_{\mathbb{R}^n} 2e(\chi_i)e(\chi_j)d\mathcal{L}^n, \text{ for } 1 \leq i \leq n.$$  

The law (1.12 d) obtained on $\Sigma$ is a Navier type of law, that essentially tells us that the tangential velocity of the fluid is proportional with the jump of the tangential component of the normal stress tensor. The functions $\{\chi_i, \eta_i\}$ that in general may be defined for $1 \leq i \leq n$ will play the same role in the case of the Stokes system.
as the capacitary potential for the set $\Gamma$ in the context of Laplace equation. In the case of vector field we will have $n$ of them due to the $n-$dimensional space for the boundary conditions and the matrix $\mathcal{C}$ comes naturally from these solutions. The problem 1.12 is simple to study, admits a unique solution, and the associated convex functional is:

$$F(u) = \frac{1}{2} a(u, u) - \langle f, u \rangle_D + \frac{\beta}{2} c(u, u) + I_{H_0^1(D) \cap \mathcal{V}}$$

where $c$ is the quadratic form, positive definite and continuous, defined through $\mathcal{C}$:

$$c(u, v) = \int_{\Sigma} C u \cdot v d\mathcal{L}^{n-1}.$$  

Also in this case, we show first $M-$convergence of $F^\epsilon$ to $F$ in $L^2(D)$, after we extend them with $+\infty$ outside $H^1(D)$, and from here we prove the convergence of $u^\epsilon$ to $u$ in the weak topology of $H^1(D)$ and the convergence of $p^\epsilon$ to $p$ in the weak topology of $L^2(D)/\mathbb{R}$.

In the other two cases, the limit can be derived from these two.

The study was done in the stationary case, and in all these cases the velocity of the fluid $u^\epsilon$ satisfies a minimization property, being the unique minimizer for a convex functional $F^\epsilon$. We showed $M-$convergence in some appropriate spaces for the sequence of functionals $F^\epsilon$ to the limit one $F$ and set up the corresponding system of PDEs. We then derived from the $M-$convergence the convergence of the solutions for these systems. These results may be found in [20].

In Chapter 4 we extended the problem to membranes with a random distribution of the leaky parts. The difficulty in choosing the right assumptions comes from the fact that on one hand we do not randomize the coefficients of PDEs but the geometry, and on the other hand from the fact that we need to capture all cases. The main assumption in homogenization of PDEs with random coefficients (see [21, 22] for the first study in the case of elliptic equations) is the stationarity. If $\Omega$ is a probability space, and $a : \mathbb{R}^n \times \Omega$, then the process $a$ is stationary if:

$$a(x_1, \omega), ... a(x_n, \omega) \overset{D}{=} a(x_1 + d, \omega), ... a(x_n + d, \omega)$$

for any $n \in \mathbb{Z}$, $x_i \in \mathbb{R}^n$ for every $1 \leq i \leq n$, $d \in \mathbb{R}^n$ and $\omega \in \Omega$. $\overset{D}{=}$ denotes equality in distribution or in law. Stationarity may also be defined through a group of measure preserving mappings $\tau_x : \Omega \rightarrow \Omega$ for every $x \in \mathbb{R}^n$ and:

$$a(x + y, \omega) = a(x, \tau_y \omega).$$
Another assumption is the ergodicity, which means that the only invariant sets under the mappings \( \tau_x \) are the trivial ones:

\[
\tau_x(A) = A \text{ for all } x \in \mathbb{R}^n \Rightarrow A \in \{\emptyset, \Omega\}.
\]

These two assumptions permit the extension of the periodic homogenization to ergodic theory, the averaging behaviour being the consequence of a law of large number type of result. It is worth mentioning that rigorous mathematical proofs become more challenging and the study of the rates of convergence becomes more complicated than in the periodic case and requires additional assumptions.

Many classical methods from the periodic homogenization were extended to the stochastic setting, among others, Tartar’s method (see [15, 23, 24]) by Papanicolaou and Varadhan in [22], \( G \)–convergence (see [25, 26]) by Zhikov, Kozlov, Oleinik and Ngoan in [27], \( \Gamma \)–convergence(see [28, 29]) by Dal Maso and Modica in [30, 31] and two-scale convergence(see [32, 33]) by Bourgeat, Mikelić and Wright in [34]. The elliptic problem in a perforated domain with a volume distribution of holes from [6, 7] was extended in the recent papers [35, 36] by Caffarelli and Mellet. The authors chose a discrete dynamical system to describe the randomness of the holes (obstacles) and the main assumption was the stationary ergodicity of the capacity of the holes, that appears as the extra term in the limit problem.

First we define the random geometry. The difference from the periodic case is the choice for every \( \epsilon > 0 \), \( \omega \in \Omega \) and \( k \in \mathbb{N}^\epsilon \) of a set in every cell \( \epsilon k + \epsilon Y' \), \( \Gamma^\epsilon_k(\omega) \subset \subset \epsilon(k + Y') \). We introduce the following sets:

\[
\Gamma^\epsilon(\omega) = \bigcup_{k \in \mathbb{N}^\epsilon} \Gamma^\epsilon_k(\omega), \quad T^\epsilon(\omega) = \Sigma \setminus \Gamma^\epsilon(\omega), \quad D^\epsilon(\omega) = D \setminus \Gamma^\epsilon(\omega)
\]

\( \Gamma^\epsilon(\omega) \) represents the membrane, still having a periodic distribution in the cells \( \epsilon(k + Y') \) along \( \Sigma \) but the sizes and shapes of the small obstacles from every cell is random. We keep the same type of interface conditions on \( \Gamma^\epsilon(\omega) \) and we study for every \( \omega \in \Omega \) the limiting behaviour when \( \epsilon \to 0 \) of the movement of an incompressible viscous fluid in \( D^\epsilon(\omega) \), where \( u^\epsilon(\omega) \) is the velocity of the fluid and \( p^\epsilon(\omega) \) the pressure. The pair \( \{u^\epsilon, p^\epsilon\} \) is the solution for

\[
\begin{align*}
-\Delta u^\epsilon(\omega) + \nabla p^\epsilon(\omega) &= f \quad \text{in } D^\epsilon(\omega), \\
\nabla \cdot u^\epsilon(\omega) &= 0 \quad \text{in } D^\epsilon(\omega), \\
u^\epsilon(\omega) &= b \quad \text{on } \partial D, \\
u^\epsilon_i(\omega) &= 0 \quad \text{on } \Gamma^\epsilon(\omega) \text{ for all } i < n, \\
-\sigma^\epsilon(\omega) e_n \cdot e_n &= g \partial|u^\epsilon_n(\omega)| \quad \text{on } \Gamma^\epsilon(\omega),
\end{align*}
\]

(1.14)
where $f, b \in H^{1/2}(\partial D)$ and $g$ are as in the periodic case. After we define the following random closed convex sets of $H^1(D)$

$$K^\epsilon(\omega) = \{ u \in H^1_b(D) / u_i = 0 \text{ on } \Gamma^\epsilon(\omega) \forall i < n \},$$

and the random convex functionals

$$j^\epsilon(\omega)(v) = \int_{\Gamma^\epsilon(\omega)} g|v_n|d\mathcal{L}^{n-1} \forall v \in H^1(D),$$

we introduce the random convex functional $F^\epsilon(\omega) : H^1(D) \to \mathbb{R} \cup \{\infty\}$,

$$F^\epsilon(\omega)(u) = \frac{1}{2}a(u,u) + j^\epsilon(\omega)(u) - \langle f,u \rangle_D + I_{K^\epsilon(\omega) \cap V}(u).$$

Next we formulate the assumptions, such that, in all the cases, we obtain for $F^\epsilon(\omega)(u)$ $M-$convergence almost surely to some convex functional $F$, and also $u^\epsilon(\omega)$, the unique minimizer of $F^\epsilon(\omega)$ to be Bochner measurable as a function defined on $\Omega$ with values in $H^1(D)$. Bochner measurability, which from Pettis’ theorem is equivalent with weak measurability due to the separability of $H^1(D)$ allows us to perform integration and eventually study rates of convergence.

Thus we assume that the sets $\Gamma^\epsilon_k(\omega)$ are of the form $\Gamma^\epsilon_k(\omega) = \epsilon k + r_{\epsilon}A(\tau_k \omega)$ where:

i) $A : \Omega \rightrightarrows Y'$ is a set valued mapping, measurable and compact valued. We also assume, in order to eliminate degenerate cases, that almost surely $\omega \in \Omega$, $A(\omega)$ contains a ball a a fixed radius $\delta > 0.$
ii) \( \tau \) is a \( n-1 \) dynamical system, measure-preserving and ergodic.

iii) \((r_\epsilon)_{\epsilon>0} \) is a sequence satisfying \( r_\epsilon \leq \epsilon \) for every positive \( \epsilon \).

One consequence of the assumptions is that \( F^\epsilon(\omega,u) \) is a convex normal integrand in the sense of Rockafellar ([37]) and then show the measurability of \( u^\epsilon \) (see also [38] for the use of the same idea, also in connection with \( M \)-convergence of convex functionals). Another consequence is that in each case, we obtain the stationary ergodicity of the set functionals that are critical, the measure of \( \Gamma_k^\epsilon \) in the first case, the vector capacity of \( \Gamma_k^\epsilon \) in the third case, so we are able to obtain the averaged behaviour.

In the first case the limiting problem is:

\[
\begin{align*}
-\Delta u + \nabla p &= f \quad \text{in } D \setminus \Sigma, \\
\nabla \cdot u &= 0 \quad \text{in } D \setminus \Sigma, \\
u &= b \quad \text{on } \partial D, \\
u_i &= 0 \quad \text{on } \Sigma \text{ for all } i<n, \\
-\left[\sigma(u,p)e_n\right] &\in g\int_\Omega \mathcal{L}^n(A(\omega))dP \cdot \partial|u_n| \quad \text{on } \Sigma,
\end{align*}
\]  

(1.15)

and thus extending the results from the periodic case.

In the third case the limiting problem is:

\[
\begin{align*}
-\Delta u + \nabla p &= f \quad \text{in } D \setminus \Sigma, \\
\nabla \cdot u &= 0 \quad \text{in } D \setminus \Sigma, \\
u &= b \quad \text{on } \partial D, \\
-\left[\sigma(u,p)e_n\right] &= C u \quad \text{on } \Sigma,
\end{align*}
\]  

(1.16)

where \( C \) is a symmetric positive definite matrix that can also be defined by:

\[
C = \int_\Omega C(A(\omega))dP
\]

where for a certain admissible set \( S \), the matrix \( C(S) \) has a similar definition with \( C \) defined in the periodic case for the fixed set \( \Gamma \). Although we did not defined it here because it was beyond our scope, \( C_{ij}(S) \) may be defined for any set \( S \), and it turns out that it is a Choquet capacity (see [39]).

In Chapter 5 we show the homogenization of the corresponding time dependent problems. The \( M \)-convergence introduced by Mosco in [40, 41] is used in the previous chapters to show the convergence of the solutions for variational inequalities. In this chapter we make use of further consequences of \( M \)-convergence, the convergence in the resolvent sense of the associated subdifferentials and the convergence of the associated semigroups, to obtain the convergence results for the non-stationary problem.
Chapter 2

Preliminaries

2.1 Permeable Membranes with Threshold Leak

Assume $D$ is a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary which is cut into two parts $D^+$ and $D^-$ by a $C^{1,1}$ surface in such a way that $D^+$ and $D^-$ are open domains with Lipschitz boundary and $D = D^+ \cup D^- \cup \Sigma$ where $\Sigma$ is the intersection of $D$ and the smooth surface. We will study first the stationary motion of a fluid in $D$ with prescribed boundary condition on $\partial D$ and leak, no-slip interface condition on $\Sigma$.

If $u$ is the velocity of the fluid and $p$ is the pressure, we denote by $\sigma = \sigma(u, p)$ the stress tensor defined by $\sigma(u, p) = -pI + 2e(u)$, where $e(u)$ is the strain rate tensor of the velocity field $u$, given by the symmetric gradient

$$e_{ij}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i) \text{ for } 0 \leq i, j \leq n,$$

and $N$ will represent the normal vector on $\Sigma$ pointing into $D^+$. The leak condition imposed on $\Sigma$ will be of the threshold type, meaning that there exists a positive function $g$ on $\Sigma$ such that

$$\begin{aligned}
[\sigma N] \cdot N \leq g, \\
[\sigma N] \cdot N < g \Rightarrow u_N = 0, \\
[\sigma N] \cdot N = g \Rightarrow u_N = 0 \text{ or } -[\sigma N] \cdot N = g \cdot \text{sgn}(u_N),
\end{aligned} \quad (2.1)$$

where $[\sigma N] \cdot N = \sigma^- N \cdot N - \sigma^+ N \cdot N$ represents the jump of the normal component of the stress across $\Sigma$ and $u_N = u \cdot N$ is the normal component of the velocity on $\Sigma$. The no-slip condition means that $u_r = 0$ on $\Sigma$, where $u_r = u - u_N N$ is the tangential component. The condition (2.1) for the appearance of the leak can also be written in
the equivalent form \(-[\sigma N] \cdot N \in g \partial |u_N|\). Therefore we consider the following system

\[
\begin{cases}
-\Delta u + \nabla p = f & \text{in } D \setminus \Sigma, \\
\nabla \cdot u = 0 & \text{in } D \setminus \Sigma, \\
u = b & \text{on } \partial D, \\
u_\tau = 0 & \text{on } \Sigma, \\
-[\sigma N] \cdot N \in g \partial |u_N| & \text{a.e. on } \Sigma,
\end{cases}
\]

(2.2)

where \(b\), \(f\) and \(g\) will belong to some spaces that will be specified later.

### 2.2 Weak Solutions

We will denote in the usual way the Lebesgue and Sobolev spaces of scalar functions and use bold letters for the similar spaces of \(n\)-dimensional vector fields. The inner product in \(L^2(D)\) with the \(n\)-dimensional Lebesgue measure \(L^n\) will be denoted by \((u, v)_D\), where

\[
(u, v)_D = \int_D \sum_{1 \leq i \leq n} u_i v_i dL^n,
\]

and the inner product in \(H^1(D)\) by \(((u, v))_D\)

\[
((u, v))_D = \int_D \sum_{1 \leq i \leq n} u_i v_i dL^n + \int_D \sum_{1 \leq i, j \leq n} \partial_j u_i \partial_j v_i dL^n.
\]

If \(u \in H^1(D)\), we can talk about the trace of \(u\) on \(\partial D\) as well as on \(\Sigma\) such that \(u|_{\partial D} \in L^2(\partial D)\) and \(u|_{\Sigma} \in L^2(\Sigma)\) so \(u|_{\partial D \cup \Sigma} \in L^2(\partial D \cup \Sigma)\) where the measure used on these spaces is \(n-1\) dimensional Hausdorff measure \(H^{n-1}\) (see [42], Ch. 2 for definition and properties for \(s\) dimensional Hausdorff measures).

The inner products on these \(L^2\) spaces will be denoted by \((\cdot, \cdot)_{\partial D}\), \((\cdot, \cdot)_{\Sigma}\) and \((\cdot, \cdot)_{\partial D \cup \Sigma}\) respectively. \(H^0_0(D)\) will be the closed subspace of \(H^1(D)\) with zero trace on \(\partial D\) and \(H^0_0(D^+ \cup D^-)\) the closed subspace of \(H^1(D)\) with 0 trace on \(\partial D \cup \Sigma\). The fractional Sobolev space \(H^{1/2}(\partial D \cup \Sigma)\) will be the dense subspace of \(L^2(\partial D \cup \Sigma)\) consisting of vector fields that are the traces on \(\partial D \cup \Sigma\) of vector fields from \(H^1(D)\). \(H^{1/2}(\partial D \cup \Sigma)\) is a Hilbert space embedded in \(L^2(\partial D \cup \Sigma)\) with the norm induced by \(H^1(D)/H^1_0(D^+ \cup D^-)\), i.e. the bijection

\[
u \in H^1(D)/H^1_0(D^+ \cup D^-) \rightarrow u|_{\partial D \cup \Sigma} \in H^{1/2}(\partial D \cup \Sigma)
\]

is an isometry. The dual space of \(H^{1/2}(\partial D \cup \Sigma)\) using \(L^2(\partial D \cup \Sigma)\) as the pivot space is \(H^{-1/2}(\partial D \cup \Sigma)\) and the duality will be denoted by \((\cdot, \cdot)_{\partial D \cup \Sigma}\).
In a similar way we may introduce $H^{1/2}(\partial D)$ and $H^{-1/2}(\partial D)$ with the duality $\langle \cdot, \cdot \rangle_{\partial D}$ and the isometry
\[ u \in H^1(D)/H^1_0(D) \rightarrow u|_{\partial D} \in H^{1/2}(\partial D). \]

In $H^{1/2}(\partial D \cup \Sigma)$ we have the closed subspace consisting of functions supported in $\Sigma$. We denote this space by $H^{1/2}(\Sigma)$ and the norm on this space will be the norm induced. The boundary of $\Sigma$ being negligible, this space will be injected in $L^2(\Sigma)$ and dense in $L^2(\Sigma)$ norm. It follows that the operator
\[ u \in H^1_0(D)/H^1_0(D^+ \cup D^-) \rightarrow u|_{\Sigma} \in H^{1/2}(\Sigma) \]
is also an isometry.

So $H^{1/2}(\partial D)$ is the algebraic complement of $H^{1/2}(\Sigma)$ in $H^{1/2}(\partial D \cup \Sigma)$ and the norm considered is stronger than the norm induced, being isometric with $H^{1/2}(\partial D \cup \Sigma)/H^{1/2}(\Sigma)$. The dual space of $H^{1/2}(\Sigma)$ using $L^2(\Sigma)$ as the pivot will be denoted by $H^{-1/2}(\Sigma)$ and the duality by $\langle \cdot, \cdot \rangle_{\Sigma}$.

Similar spaces may be defined for scalars and we will keep the same notations for inner products and dualities that will appear.

We are going to show that the problem (2.2) has a unique solution subject to $f \in H^{-1}(D)$, $b \in H^{1/2}(\partial D)$ and $g$ a positive function from $L^2(\Sigma)$, where the vector field $b$ has to satisfy the following compatibility conditions
\[ \int_{\partial D} b \cdot n \, d\sigma = 0 \quad (2.3) \]
and there exists $u_0 \in H^1(D)$ such that
\[
\begin{cases}
  u_0 = b \text{ on } \partial D, \\
  (u_0)_\tau = 0 \text{ on } \Sigma.
\end{cases} \quad (2.4)
\]

In order to define a weak solution we will introduce the following quadratic form on $H^1(D)$
\[ a(u,v) = 2(e(u),e(v))_D = 2 \int_D \sum_{i,j=1}^n e_{ij}(u)e_{ij}(v) \text{ for } u,v \in H^1(D). \quad (2.5) \]

Obviously $a(\cdot, \cdot)$ is continuous on $H^1(D)$ and as a consequence of Korn’s inequality (see [43],[44]) it is also coercive if restricted to $H^1_0(D)$. We will look for solutions $u \in H^1(D)$ satisfying the Stokes equation $-\Delta u + \nabla p = f$ in the weak sense in $D^+ \cup D^-$ and with $\nabla \cdot u = 0$. After some calculations we get
\[ \nabla \cdot \sigma = -\nabla p + \Delta u + \nabla(\nabla \cdot u) \text{ in } H^{-1}(D), \]
thus $\nabla \cdot \sigma = -f$ in $D^+ \cup D^-$. If $u \in H^1(D)$ with $\nabla \cdot u = 0$ and $\phi \in H^1_0(D^+ \cup D^-)$

$$-\langle f, \phi \rangle_D = \langle \nabla \cdot \sigma, \phi \rangle_D = (p, \nabla \cdot \phi)_D - a(u, \phi).$$

Define the following bounded linear operator on $H^1_0(D)$

$$\phi \to a(u, \phi) - (p, \nabla \cdot \phi)_D - \langle f, \phi \rangle_D$$

which is 0 on $H^1_0(D^+ \cup D^-)$. So there exists a distribution in $H^{-1/2}(\Sigma)$, denoted $[\sigma N]$, such that

$$\langle [\sigma N], \phi \rangle_\Sigma = a(u, \phi) - (p, \nabla \cdot \phi)_D - \langle f, \phi \rangle_D \quad \text{for every } \phi \in H^1_0(D). \quad (2.6)$$

Since $\Sigma$ is smooth, we are able to decompose $[\sigma N]$ into the normal and tangential component (see [45], Ch. 5), so $[\sigma N] \cdot N$ will be in $H^{-1/2}(\Sigma)$ such that for every $\phi \in H^{1/2}(\Sigma)$

$$\langle [\sigma N] \cdot N, \phi \rangle_\Sigma = \langle [\sigma N], \phi_N N \rangle_\Sigma. \quad (2.7)$$

In general $[\sigma N] \cdot N$ belongs to $H^{-1/2}(\Sigma)$, and the last condition from the definition of the solution will imply that $[\sigma N] \cdot N$ will be a function from $L^2(\Sigma)$ and will satisfy (2.1) a.e. on $\Sigma$.

Let us notice briefly that if $f \in L^2(D)$, then $\sigma^+ n^+$ belongs to $H^{-1/2}(\partial D^+)$ and $\sigma^- n^-$ belongs to $H^{-1/2}(\partial D^-)$ ([45], Th. 5.9). By $-\sigma^+ N$ and $\sigma^- N$ we will understand the restrictions of this bounded operators to $H^{1/2}(\Sigma)$. If $\phi \in H^1_0(D)$ we also have the following generalized Green’s formula

$$(f, \phi)_D = a(u, \phi) - (p, \nabla \cdot \phi)_D - \langle \sigma^- n^- + \sigma^+ n^+, \phi \rangle_\Sigma,$$

so in this case $[\sigma N] = \sigma^- N - \sigma^+ N$. In the general case when $f \in H^{-1}(D)$ only the jump is defined.

Let us define the following sets that we will use later for the definition of a weak solution for the problem (2.2) and for the existence and uniqueness results

$$V = \{ u \in H^1(D) / \nabla \cdot u = 0 \text{ in } D \},$$

$$H^1_0(D) = \{ u \in H^1(D) / u = b \text{ on } \partial D \};$$

$$K = \{ u \in H^1_0(D) / u_N = 0 \text{ on } \Sigma \}.$$

**Definition 1.** The pair $\{ u, p \}$ is a weak solution for problem (2.2) if:

i) $u \in K \cap V$ and $p \in L^2(D)$.

ii) $\{ u, p \}$ is a weak solution for the Stokes equation, i.e. for every $\phi \in H^1_0(D^+ \cup D^-)$

$$a(u, \phi) - (p, \nabla \cdot \phi)_D = \langle f, \phi \rangle_D.$$

iii) $-\sigma N \cdot N \in g \partial |u_N|$ a.e. on $\Sigma$. 

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2.3 Existence and Uniqueness Results

Theorem 1. \{u,p\} with \(u \in K \cap V\) and \(p \in L^2(D)\) is a weak solution for problem (2.2) if and only if

\[
a(u,v-u) - (p, \nabla \cdot (v-u))_D + j(v) - j(u) \geq \langle f, v-u \rangle_D \quad \forall v \in K,
\]

where

\[
j(v) = (g, |v_N|) = \int_\Sigma g|v_N|d\mathcal{H}^{n-1}.
\]

Moreover, \(u\) solves the following minimization problem:

\[
\min_{v \in K} \left\{ \frac{1}{2}a(v,v) + j(v) - \langle f, v \rangle_D \right\}.
\]

Proof. If \(\{u,p\}\) satisfies (2.8), then taking \(v = u \pm \phi\) with \(\phi \in H^1_0(D^+ \cup D^-)\) we get that \(\{u,p\}\) satisfies the weak Stokes equation. Also from the definition (2.6) of the normal stress

\[
a(u,v-u) - (p, \nabla \cdot (v-u))_D - \langle f, v-u \rangle_D = \langle [\sigma N], (v-u)_N N \rangle_\Sigma \quad \forall v \in K,
\]

so from (2.8) we get

\[
\langle [\sigma N], (v-u)_N N \rangle_\Sigma + \int_\Sigma (g|v_N| - g|u_N|)d\mathcal{H}^{n-1} \geq 0 \quad \forall v \in K.
\]

For every \(\phi \in K - u\) we obtain

\[
\langle -[\sigma N] \cdot N, \phi_N \rangle_\Sigma \leq \int_\Sigma (g|u_N + \phi_N| - g|u_N|)d\mathcal{H}^{n-1} \leq C||\phi_N||_{L^2(\Sigma)},
\]

The application \(\phi \in H^{1/2}(\Sigma) \mapsto (\phi_N, \phi - \phi_N N) \in H^{1/2}(\Sigma) \times H^{1/2}(\Sigma) \perp\) is surjective ([45], Theorem 5.6) so

\[
\langle -[\sigma N] \cdot N, \phi \rangle_\Sigma \leq C||\phi||_{L^2(\Sigma)} \quad \forall \phi \in H^{1/2}(\Sigma).
\]

\(H^{1/2}(\Sigma)\) being dense in \(L^2(\Sigma)\) will imply that \(-[\sigma N] N \in L^2(\Sigma)\) and from (2.12) we get easily that \(-[\sigma N] N \in g\partial |u_N|\) a.e. on \(\Sigma\).

Now let us show that a weak solution for the problem (2.2) satisfies equation (2.8). From (2.11) we only need to show that

\[
\int_\Sigma [\sigma N](v-u)_N Nd\mathcal{H}^{n-1} + \int_\Sigma (g|v_N| - g|u_N|)d\mathcal{H}^{n-1} \geq 0.
\]
But $-[\sigma N]N(v_N - u_N) \leq g|u_N + (v_N - u_N)| - g|u_N|$ follows from the definition of the subdifferential. We only have to prove that $u$ solves the minimization problem (2.10). From (2.8) it follows

$$a(u, v - u) + j(v) - j(u) \geq \langle f, v - u \rangle_D \quad \forall v \in K \cap V \Rightarrow$$

$$a(u, v - u) - \langle f, v \rangle_D + j(v) \geq - \langle f, u \rangle_D + j(u) \quad \forall v \in K \cap V.$$  

But $\frac{1}{2}a(v, v) - \langle f, v \rangle_D + j(v) \geq \frac{1}{2}a(u, u) + a(u, v - u) - \langle f, v \rangle_D + j(v) \geq \frac{1}{2}a(u, u) - \langle f, u \rangle_D + j(u) \quad \forall v \in K \cap V$, which proves the desired property. \qed

**Theorem 2.** There exists a solution $\{u, p\}$ for problem (2.2), with $u$ being unique and $p$ unique up to an additive constant if $u_N \neq 0$. When $u_N \equiv 0$ on $\Sigma$, $p$ exists up to two real constants with the difference belonging to a closed interval.

**Proof.** Because of the compatibility conditions (2.3) and (2.4) the convex set $K \cap V$ is nonempty. Indeed, it is enough to show that for any $h \in L^2(D)$ with mean zero, there exists $v \in H^1_0(D)$ with $v_\tau = 0$ on $\Sigma$ and $\nabla \cdot v = h$. So if $\phi' \in H^{1/2}(\Sigma)$ with $\int_{\Sigma} \phi' dH^{n-1} = \int_{D^+} h d\mathcal{L}^n$ and $\phi \in H^1_0(D)$ with $\phi = \phi' N$ on $\Sigma$, then the $L^2(D)$ function $h + \nabla \cdot \phi$ has mean zero in both $D^+$ and $D^-$ which are Lipschitz so we find $v' \in H^1_0(D^+ \cap D^-)$ that solves $\nabla v' = h + \nabla \cdot \phi$ in $D$. $v = v' - \phi$ is what we were looking for.

By making use of a classical result for variational inequalities [46] it is sufficient to observe that $j$ is convex and continuous and because $a$ is a bilinear, continuous form and also coercive the solution for the minimization problem (2.10) exists and is unique. For every $t \in (0, 1)$ and $\phi \in (K - u) \cap V$ we have

$$\frac{1}{2}a(u, u) - \langle f, u \rangle_D + j(u) \leq \frac{1}{2}a(u + t\phi, u + t\phi) - \langle f, u + t\phi \rangle_D + j(u + t\phi),$$

and after several calculations

$$a(u, \phi) + j(u + \phi) - j(u) \geq \langle f, \phi \rangle_D \quad \forall \phi \in (K - u) \cap V \Rightarrow$$

$$a(u, \phi) = \langle f, \phi \rangle_D \quad \forall \phi \in H^1_0(D^+ \cup D^-) \cap V.$$  

The existence of $p \in L^2(D)$, unique up to two additive constants, one for $D^+$ and one for $D^-$ such that

$$a(u, \phi) - (p, \nabla \cdot \phi)_D = \langle f, \phi \rangle_D \quad \forall \phi \in H^1_0(D^+ \cup D^-)$$

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follows as in [47]. Let $\phi'$ be a function from $H^{1/2}(\Sigma)$ such that $\int_{\Sigma} \phi' d\mathcal{H}^{n-1} = 0$ and let $\phi$ be a divergence free vector field in $H^1_0(D)$ with $\phi_r = 0$ and $\phi_N = \phi'$ on $\Sigma$. We obtain like in the previous proof

$$
\langle -[\sigma N], \phi_N \rangle_{\Sigma} \leq j(u + \phi) - j(u) \\
\langle -[\sigma N] \cdot N, \phi' \rangle_{\Sigma} \leq \int_{\Sigma} (g |u_N + \phi'| - g |u_N|) d\mathcal{H}^{n-1} \leq \int_{\Sigma} g |\phi'| d\mathcal{H}^{n-1},
$$

where the last inequality is true for every $\phi'$ in the kernel of the distribution from $H^{-1/2}(\Sigma)$ equal to 1. From Hahn Banach Theorem we obtain that $-[\sigma N] \cdot N$ is the sum between an $L^2$ function dominated by $g$ and a constant one. Adjusting one of the additive constants from the existence of $p$, we obtain the existence of $p$ in $L^2(D)$ up to an additive constant and keeping the same notation $|\sigma N| \cdot N \leq g$ pointwise.

Now we have to show the subdifferential inequality for the stress, eventually after modifying again $p$.

We have that for every $\phi' \in L^2(\Sigma)$ with $\int_{\Sigma} \phi' d\mathcal{H}^{n-1} = 0$

$$
\int_{\Sigma} -[\sigma N] \cdot N \phi' d\mathcal{H}^{n-1} \leq \int_{\Sigma} (g |u_N + \phi'| - g |u_N|) d\mathcal{H}^{n-1}. \quad (2.13)
$$

We notice that is if $\mathcal{H}^{n-1}(\{u_N = 0\}) > 0$, then (2.13) is true for every $\phi' \in L^2(\Sigma)$ so taking $\phi' = \pm u_N$ we obtain that

$$
\int_{\Sigma} -[\sigma N] \cdot N u_N d\mathcal{H}^{n-1} = \int_{\Sigma} g |u_N| d\mathcal{H}^{n-1},
$$

so $-[\sigma N] \cdot N \in g|\partial|u_N|$. Uniqueness for $p$ up to a real constant follows if $\mathcal{H}^{n-1}(\{u_N \neq 0\}) > 0$.

Also, if $\mathcal{H}^{n-1}(\{u_N > 0\}) > 0$ and $\mathcal{H}^{n-1}(\{u_N < 0\}) > 0$ there exists $t > 0$ such the function $u_N \chi_{\{u_N > 0\}} + tu_N \chi_{\{u_N < 0\}}$ has zero mean on $\Sigma$, where by $\chi_S$ we denote the characteristic function of a set $S$

$$
\chi_S(x) = \begin{cases} 
1 & x \in S, \\
0 & x \notin S.
\end{cases}
$$

We may assume $t \in (0,1]$ and using $\phi' = \pm (u_N \chi_{\{u_N > 0\}} + tu_N \chi_{\{u_N < 0\}})$ in (2.13) we obtain

$$
\int_{\{u_N > 0\}} (-[\sigma N] \cdot N - g) u_N d\mathcal{H}^{n-1} + t \int_{\{u_N < 0\}} ([\sigma N] \cdot N + g) u_N d\mathcal{H}^{n-1} = 0,
$$

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so in this case also $-\{\sigma N\} \cdot N \in g\partial|u_N|$ and $p$ is unique up to a constant. If $u_N > 0$ almost everywhere on $\Sigma$, let $\phi' \in L^\infty(\Sigma)$ with zero mean value supported in $\{u_N > \epsilon\}$ for $\epsilon > 0$. Using $\pm t\phi'$ in (2.13) with small $t > 0$ we obtain

$$\int_{\{u_N > \epsilon\}} -\{\sigma N\} \cdot N \phi' d\mathcal{H}^{n-1} = \int_{\{u_N > \epsilon\}} g\phi' d\mathcal{H}^{n-1}$$

for every $\phi' \in L^\infty(\{u_N > \epsilon\})$ with zero mean value. This means that $-\{\sigma N\} \cdot N - g$ is constant on $\{u_N > \epsilon\}$ for every $\epsilon > 0$, so $-[\sigma N] \cdot N + c = g$ on $\Sigma$. We adjust again one of the constants from the pressure and obtain that $-\{\sigma N\} \cdot N \in \partial g|u_N|$ also in this case and uniqueness for $p$ up to a constant.

In the case $u_N \equiv 0$ on $\Sigma$ we already have $| - [\sigma N]| \leq g$ so $-\{\sigma N\} \cdot N \in \partial g|u_N|$ which means that $p$ exists. Uniqueness up to a real constant we obtain if and only if $\text{ess inf}(g + [\sigma N] \cdot N) = \text{ess inf}(g - [\sigma N] \cdot N) = 0$. In general $p$ will be unique up to two constants, one taking values in $\mathbb{R}$ and one in the close interval $[\text{ess sup}([\sigma N] \cdot N - g), \text{ess inf}(g + [\sigma N] \cdot N)]$. □
Chapter 3

Membranes with Periodic Distribution of Leaky Obstacles

3.1 Formulation of the Problem

In the following, $D$ will be a bounded domain in $\mathbb{R}^n$, with Lipschitz boundary, and for simplicity we will assume it to be separated by the hyperplane $\{x_n = 0\}$ into two parts $D^+ = D \cap \{x_n > 0\}$ and $D^- = D \cap \{x_n < 0\}$ such that $D^\pm$ have also Lipschitz boundary. Define $\Sigma = D \cap \{x_n = 0\}$. Any point $x \in \mathbb{R}^n$ will have the coordinates $x = (x', x_n)$ where $x' \in \mathbb{R}^{n-1}$.

Let $Y' = (-\frac{1}{2}, \frac{1}{2})^{n-1}$ be the unit cube in $\mathbb{R}^{n-1}$ and $\Gamma \subset \subset Y'$, the closure of a connected open set with Lipschitz boundary. For every $\epsilon > 0$, let $N^\epsilon$ be the set of all points $k \in \mathbb{Z}^{n-1}$ such that $\epsilon(k + Y')$ is strictly included in $\Sigma$ and denote by $|N^\epsilon|$ the total number of them. For any $\epsilon > 0$ and any $k \in N^\epsilon$ we denote by $\Gamma_k^\epsilon$ the set $\epsilon k + r_\epsilon \Gamma$, where the sequence $r_\epsilon$ satisfies $0 < r_\epsilon \leq \epsilon$, and we will define the following sets

$$
\Gamma^\epsilon = \bigcup_{k \in N^\epsilon} \Gamma_k^\epsilon,
T^\epsilon = \Sigma \setminus \Gamma^\epsilon,
D^\epsilon = D \setminus \Gamma^\epsilon.
$$

$\Gamma^\epsilon$ will represent the membrane, distributed periodically in the cells $\epsilon(k + Y')$ along $\Sigma$ and $T^\epsilon$ will represent the holes. The fluid will pass freely through the holes, but only above a certain stress through the membrane. We will study for every case the
limiting behaviour when $\epsilon \to 0$ of the movement of an incompressible viscous fluid in $D^\epsilon$ with leak interface condition on $\Gamma^\epsilon$.

If $u^\epsilon$ is the velocity of the fluid and $p^\epsilon$ its pressure the pair $\{u^\epsilon, p^\epsilon\}$ will satisfy the Stokes equation in $D^\epsilon$ with the following interface condition

$$
\begin{align*}
-\Delta u^\epsilon + \nabla p^\epsilon &= f & \text{in } D^\epsilon, \\
\nabla \cdot u^\epsilon &= 0 & \text{in } D^\epsilon, \\
\nabla \cdot u^\epsilon &= b & \text{on } \partial D, \\
\n\nabla \cdot u^\epsilon &= 0 & \text{on } \Gamma^\epsilon \text{ for all } i < n, \\
\left[-[\sigma^\epsilon e_n] \cdot e_n\right] = g & \text{on } \Gamma^\epsilon,
\end{align*}
$$

(3.1)

where $f$ is a distribution in $H^{-1}(D)$ and $b \in H^{1/2}(\partial D)$ satisfying the same compatibility conditions (2.3) and (2.4). The i-th canonical vector of $\mathbb{R}^n$ is represented by $e_i$ for $1 \leq i \leq n$. Also $g$ is a positive function from $L^2(\Sigma)$, not necessarily constant, that will represent the threshold for the appearance of the leak.

We will define shortly the spaces that will appear in the problem: $H^1_0(D^\epsilon)$ is the subspace of $H^1(D)$ with 0 trace on $\partial D$ as well as on $\Gamma^\epsilon$. $H^{1/2}(\Gamma^\epsilon)$ is the subspace of $L^2(\Gamma^\epsilon)$ that are traces on $\Gamma^\epsilon$ of vector fields in $H^1(D)$ or equivalently, because in our case the distance from $\Gamma^\epsilon$ to $\partial D$ is strictly positive, traces of vector fields in $H^1_0(D)$. The norm will be the one induced by $H^1_0(D)/H^1_0(\epsilon^\epsilon)$). The dual of $H^{1/2}(\Gamma^\epsilon)$ is denoted by $H^{-1/2}(\Gamma^\epsilon)$ and the duality by $\langle \cdot, \cdot \rangle_{\Gamma^\epsilon}$.

**Remark 1.** We remark that $H^{1/2}(\Gamma^\epsilon)$ is a subspace of $H^{1/2}(\Sigma)$ but with a stronger norm, namely for every $\phi \in H^{1/2}(\Gamma^\epsilon)$ there exists $\hat{\phi} \in H^{1/2}(\Sigma)$ such that $\hat{\phi}|_{\Gamma^\epsilon} = \phi$ and

$$
||\phi||_{H^{1/2}(\Gamma^\epsilon)} = \inf \{||\hat{\phi}||_{H^{1/2}(\Sigma)} / \hat{\phi}|_{\Gamma^\epsilon} = \phi\},
$$

which will imply that $H^{-1/2}(\Gamma^\epsilon)$ is the subspace of $H^{-1/2}(\Sigma)$ supported in $\Gamma^\epsilon$ with the norm induced.

By $[\sigma^\epsilon e_n] \cdot e_n$ we will understand the distribution in $H^{-1/2}(\Gamma^\epsilon)$ defined similarly as in (2.6) and (2.7) so it will satisfy

$$
\langle [\sigma^\epsilon e_n] \cdot e_n, \phi \rangle_{\Gamma^\epsilon} = \langle [\sigma^\epsilon e_n], \phi e_n \rangle_{\Gamma^\epsilon} \quad \forall \phi \in H^{1/2}(\Gamma^\epsilon),
$$

(3.2)

where

$$
\langle [\sigma^\epsilon e_n], \phi \rangle_{\Gamma^\epsilon} = a(u^\epsilon, \phi) - (p^\epsilon, \nabla \phi)_D - \langle f, \phi \rangle_D \quad \forall \phi \in H^1_0(D).
$$

(3.3)

By the previous remark, we may consider $[\sigma^\epsilon e_n]$ as a distribution from $H^{-1/2}(\Sigma)$ that is zero outside $\Gamma^\epsilon$ so we can rewrite (3.2) and (3.3) as

$$
\langle [\sigma^\epsilon e_n] \cdot e_n, \phi \rangle_{\Sigma} = \langle [\sigma^\epsilon e_n], \phi e_n \rangle_{\Sigma} \quad \forall \phi \in H^{1/2}(\Sigma)
$$

(3.4)
\[
\langle [\sigma^\epsilon e_n], \phi \rangle_\Sigma = a(u^\epsilon, \phi) - (p^\epsilon, \nabla \cdot \phi)_D - \langle f, \phi \rangle_D \quad \forall \phi \in H^1_0(D).
\]

We will also define the following closed convex sets of \( H^1(D) \)

\[
K^\epsilon = \left\{ u \in H^1_b(D) \middle| u_i = 0 \text{ on } \Gamma \text{ } \forall i < n \right\}.
\]

We are hoping that the solution of the system (3.1) will converge when \( \epsilon \to 0 \) to the solution of a system that we want to determine. The relative size of the holes (given here through \( r_\epsilon \)) should also be an important factor so we will expect different limit problems. As in the previous section, we have the following characterization of a weak solution with a similar proof as Theorem 1:

**Theorem 3.** \( \{ u^\epsilon, p^\epsilon \} \) with \( u^\epsilon \in K^\epsilon \cap V \) and \( p^\epsilon \in L^2(D) \) is a weak solution for problem (3.1) if and only if

\[
a(u^\epsilon, v - u^\epsilon) - (p^\epsilon, \nabla \cdot (v - u^\epsilon))_D + j^\epsilon(v) - j^\epsilon(u^\epsilon) \geq \langle f, v - u^\epsilon \rangle_D \quad \forall v \in K^\epsilon, \tag{3.4}
\]

where \( j^\epsilon(v) = \int_{\Gamma_\epsilon} g|v_n| d\mathcal{L}^{n-1} \quad \forall v \in H^1(D) \).

Also, \( u^\epsilon \) may be characterized as the minimizer for the following problem

\[
\min_{v \in K^\epsilon \cap V} \left\{ \frac{1}{2} a(v, v) + j^\epsilon(v) - \langle f, v \rangle_D \right\}. \tag{3.5}
\]

There is a corresponding theorem for existence of a solution, where we will have uniqueness also for \( p^\epsilon \) because the stress has to be continuous across \( \Sigma \setminus \Gamma^\epsilon \) (see Remark 1):

**Theorem 4.** There exists a solution \( \{ u^\epsilon, p^\epsilon \} \) for problem (3.1), with \( u^\epsilon \) being unique and \( p^\epsilon \) unique up to an additive constant.

**Proof.** The existence and uniqueness for \( u^\epsilon \) is similar as in Theorem 2. We only need to show existence and uniqueness for \( p^\epsilon \).

Let \( D^+, D^- \) and \( D \setminus \Gamma^\epsilon \times [-\delta, \delta] \) for \( \delta \) small enough represent an open covering with Lipschitz domains of \( D^\epsilon \). From the existence of \( u^\epsilon \) we derive that \( \langle f + \Delta u^\epsilon, \phi \rangle_D = 0 \) for every divergence free vector field \( \phi \) compactly supported in any of the elements of the covering. So there exist \( p_1 \in L^2(D^+) \), \( p_2 \in L^2(D^-) \), \( p_3 \in L^2(D \setminus \Gamma^\epsilon \times [-\delta, \delta]) \), unique up to additive constants such that \( f + \Delta u^\epsilon = \nabla p_1 \) in \( D^+ \), \( f + \Delta u^\epsilon = \nabla p_2 \) in \( D^- \) and \( f + \Delta u^\epsilon = \nabla p_3 \) in \( D \setminus \Gamma^\epsilon \times [-\delta, \delta] \).

It follows that there exist unique constants \( c_1 \) and \( c_2 \) such that \( p_1 + c_1 = p_3 \) on
Let $\Gamma^\epsilon \times [-\delta, \delta]$ and $p_2 + c_2 = p_3$ on $D^- \setminus \Gamma^\epsilon \times [-\delta, \delta]$. Let $p^\epsilon$ be equal with $p_3$ in $D \setminus \Gamma^\epsilon \times [-\delta, \delta]$, with $p_1 + c_1$ in $D^+$ and with $p_2 + c_2$ in $D^-$. Obviously $p^\epsilon$ belongs to $L^2(D^\epsilon)$ and $\{u^\epsilon, p^\epsilon\}$ satisfies the weak Stokes equation in all three open sets, i.e. $\langle f + \Delta u^\epsilon - \nabla p^\epsilon, \phi \rangle_D = 0$ for every vector field $\phi$ compactly supported in one of the sets $D^+, D^-, D \setminus \Gamma^\epsilon \times [-\delta, \delta]$. Using the partition of unity we obtain that $\{u^\epsilon, p^\epsilon\}$ satisfies the weak Stokes equation in $D^\epsilon$. We get as a consequence also the uniqueness of $p^\epsilon$ up to an additive constant.

It remains to prove that pointwise $- [\sigma^\epsilon e_n] \cdot e_n \in g\partial|u_n|$ a.e. on $\Sigma$. But from (3.5) $u^\epsilon$ will satisfy

$$a(u^\epsilon, \phi) + j^\epsilon(u^\epsilon + \phi) - j^\epsilon(u^\epsilon) \geq \langle f, \phi \rangle$$

for every $\phi \in V \cap H^1_0(D)$ with $\phi_r \equiv 0$ on $\Sigma$, where $j$ is the convex function defined on $H^1(D)$

$$j^\epsilon(v) = \int_{\Sigma} g\chi_{\Gamma} |v_n| d\mathcal{L}^{n-1}.$$
in the Mosco sense to $F$ and be able to characterize the minimizer for $F$. As already stated, the limit functional will depend on the rate of convergence to 0 of $r_{\epsilon}$, so we will divide the problem in four cases: $r_{\epsilon} \approx \epsilon$, $r_{\epsilon} \approx \epsilon^{\frac{1}{n-2}}$ and the intermediate ones.
3.2 Homogenization Results

3.2.1 Case I: Concentrated Obstacles

We assume that \( \lim_{\epsilon \to \epsilon^*} \frac{r_\epsilon}{\epsilon} = \alpha > 0 \). Let us introduce now the unique solution to the problem

\[
\begin{aligned}
-\Delta u + \nabla p &= f \\
\nabla \cdot u &= 0 \\
u &= b \\
u_i &= 0 \\
-\left[\sigma(u, p) e_n\right] \cdot e_n &\in g\mathcal{L}^{n-1}(\Gamma) \cdot \partial|u_n| \quad \text{on } \Sigma.
\end{aligned}
\]  

(3.6)

This is a problem of type (2.2), with the threshold being the \( L^2(\Sigma) \) function \( g\mathcal{L}^{n-1}(\Gamma) \).

According to Theorem 2, there exists a solution \( \{u, p\} \) with \( u \in K \cap V \), \( p \in L^2(D)/\mathbb{R} \), with \( u \) being unique and if \( u|\Sigma \neq 0 \) \( p \) is also unique. If the unique solution \( u \) satisfies \( u|\Sigma \equiv 0 \), and if \( p \in L^2(D)/\mathbb{R} \) is a solution, then any function of the type

\[
p_c(x) = \begin{cases} 
p(x) & x \in D^+, \\
p(x) + c & x \in D^-. \end{cases}
\]

is still a solution if and only if \( c \in [\text{ess sup}(\sigma N) \cdot N - g), \text{ess inf}(g + [\sigma N] \cdot N)] \). The energy functional associated to the problem (3.6) will be \( F : H^1(D) \to \mathbb{R} \cup \{\infty\} \)

\[
F(u) = \frac{1}{2} a(u, u) + j(u) - \langle f, u \rangle_D + I_{K \cap V},
\]

where

\[
K = \{ u \in H^1_b(D)/u_\tau = 0 \text{ on } \Sigma \}
\]

and

\[
j(u) = \mathcal{L}^{n-1}(\Gamma) \int_{\Sigma} g|u_n|d\mathcal{L}^{n-1}.
\]

We will prove that this problem is the limit problem for (3.1), namely we will show prove \( M \)-convergence of the sequence of functionals \( F^\epsilon \) to \( F \), and as a consequence we will obtain that \( u^\epsilon \) will converge strongly to \( u \) in \( H^1(D) \) and \( p^\epsilon \) will converge strongly in \( L^2(D)/\mathbb{R} \) to \( p \), where \( \{u, p\} \) is a solution for the problem (3.6).

Theorem 5. The sequence of functionals \( F^\epsilon \) \( M \)-converges to \( F \) in \( H^1(D) \).
Proof. (i) $M - \limsup_{\epsilon \to 0} F^\epsilon(u) \leq F(u)$. For the first part of the proof we need to show that for every $u$ satisfying $F(u) < +\infty$ there exists a sequence $u^\epsilon$ strongly convergent to $u$ in $H^1(D)$ such that $F^\epsilon(u^\epsilon)$ converges to $F(u)$. We will take the constant sequence $u^\epsilon = u$ and because $K \cap V \subset K^\epsilon \cap V$ we only need to show that $j^\epsilon(u)$ converges to $j(u)$. This should follow from the weak* convergence of $\chi_{\Gamma^\epsilon}$ to $L^{n-1}(\Gamma)$ in $L^\infty(\Sigma)$, and because the functions $\chi_{\Gamma^\epsilon}$ are uniformly bounded in $L^\infty$ it is sufficient to prove that

$$\lim_{\epsilon \to 0} \int_U \chi_{\Gamma^\epsilon} dL^{n-1} = L^{n-1}(\Gamma) \int_U dL^{n-1}$$

for every $U \subset \Sigma$ open with Lipschitz boundary. But

$$\lim_{\epsilon \to 0} \int_U \chi_{\Gamma^\epsilon} dL^{n-1} = \lim_{\epsilon \to 0} \sum_{(\epsilon k + Y') \subset U} L^{n-1}(\Gamma^\epsilon_k) + \lim_{\epsilon \to 0} \sum_{(\epsilon k + Y') \cap \partial U \neq \emptyset} L^{n-1}(\Gamma^\epsilon_k \cap U)$$

$$= L^{n-1}(\Gamma) \lim_{\epsilon \to 0} \sum_{(\epsilon k + Y') \subset U} \epsilon^{n-1} + O(\epsilon)$$

$$= L^{n-1}(\Gamma) L^{n-1}(U).$$

(ii) $M - \liminf_{\epsilon \to 0} F^\epsilon(u) \geq F(u)$. For the second part we need to show that for every sequence $u^\epsilon$ convergent to $u$ in the weak topology of $H^1(D)$ we have that

$$\liminf_{\epsilon \to 0} F^\epsilon(u^\epsilon) \geq F(u)$$

The bilinear functional $a$ is convex and continuous in the strong topology so

$$a(u, u) \leq \liminf_{\epsilon \to 0} a(u^\epsilon, u^\epsilon).$$

From the weak* convergence of $\chi_{\Gamma^\epsilon}$ to $L^{n-1}(\Gamma)$ in $L^\infty(\Sigma)$ and the strong convergence of $u^\epsilon_i$ to $u_i$ in $L^2(\Sigma)$ for all $1 \leq i \leq n$ ([49])

$$\lim_{\epsilon \to 0} j^\epsilon(u^\epsilon) = \lim_{\epsilon \to 0} \int_{\Sigma} g|u^\epsilon_n|^2 \chi_{\Gamma^\epsilon} dL^{n-1} = L^{n-1}(\Gamma) \int_{\Sigma} g|u_n|^2 dL^{n-1} = j(u),$$

and for all $i < n$

$$0 = \lim_{\epsilon \to 0} \int_{\Sigma} |u^\epsilon_i|^2 \chi_{\Gamma^\epsilon} dL^{n-1} = L^{n-1}(\Gamma) \int_{\Sigma} |u_i|^2 dL^{n-1}.$$
It follows that $u_i \equiv 0$ on $\Sigma$. The weak convergence in $H^1(D)$ of $u^\epsilon$ to $u$ implies that $\nabla \cdot u = 0$ in $D$ so $u \in K \cap V$ and
\[
\liminf_{\epsilon \to 0} F^\epsilon(u^\epsilon) \geq F(u).
\]

Now we can prove as a consequence the convergence of the solutions.

**Theorem 6.** $u^\epsilon$ converges strongly to $u$ in $H^1(D)$ and $p^\epsilon$ converges strongly in $L^2(D)/\mathbb{R}$ to $p$, where $\{u, p\}$ is a solution for the problem (3.6).

**Proof.** From the $M-$convergence of functionals $F^\epsilon$ to $F$ and equicoercivity of $F^\epsilon$ we deduce the convergence of energies as well as the convergence of minimizers ([50] Theorem 7.8 and Corollary 7.17), so $u^\epsilon$ converges weakly in $H^1(D)$ to $u$ and $F^\epsilon(u^\epsilon)$ converges to $F(u)$. Part (i) of $M-$convergence implies that $u^\epsilon = u_1^\epsilon + u_2^\epsilon$ where $u_1^\epsilon \in H^1_0(D) \cap V$ converges weakly to zero in $H^1_0(D)$, $u_2^\epsilon \in K^\epsilon \cap V$ $u_1^\epsilon$ converges strongly to $u$ in $H^1(D)$ and $F^\epsilon(u^\epsilon) - F^\epsilon(u_2^\epsilon)$ converges to zero. After elementary calculation we obtain that $a(u_1^\epsilon, u_1^\epsilon)$ converges to 0 so $u_1^\epsilon$ will converge strongly to zero ([51]) which implies the strong convergence of $u^\epsilon$ to $u$.

Because $u^\epsilon - u \in H^1_0(D)$, $||\nabla u^\epsilon - \nabla u||_{L^2(D)} = ||\Delta u^\epsilon - \Delta u||_{H^{-1}(D)}$ and converges to 0 when $\epsilon \to 0$. Therefore, if $p$ is a solution for the limit problem, $\nabla (p^\epsilon - p)|_{D^\pm}$ converges strongly in $H^{-1}(D^\pm)$ to 0. The function $q \mapsto ||\nabla q||_{H^{-1}(D^\pm)}$ being an equivalent norm on $L^2(D^\pm)/\mathbb{R}$ ([47]) will imply that $p^\epsilon|_{D^\pm}$ will converge to $p|_{D^\pm}$ strongly in $L^2(D^\pm)/\mathbb{R}$. Hence, we can assume that there is a constant $c$ such that $p^\epsilon$ converges strongly to $p_c$ in $L^2(D)/\mathbb{R}$, where
\[
p_c(x) = \begin{cases} 
p(x) & x \in D^+,
\quad \quad \quad \quad p(x) + c & x \in D^-.
\end{cases}
\]

From the variational characterization of the solution (3.4), we have that for every $\phi \in H^1_0(D)$ with $\phi_\tau = 0$ on $\Sigma$
\[
a(u^\epsilon, \phi) - (p^\epsilon, \nabla \cdot \phi)_D + j^\epsilon(u^\epsilon + \phi) - j^\epsilon(u^\epsilon) \geq \langle f, \phi \rangle_D,
\]
and after taking the limit in $\epsilon$ we obtain that $\{u, p_c\}$ is a solution for problem (3.6).
3.2.2 Case II: Dilute Obstacles

We assume that \( \lim_{\epsilon \to 0} \frac{r_\epsilon}{\epsilon} = 0 \) and \( \lim_{\epsilon \to 0} \frac{r_{n-2}}{\epsilon^{n-1}} = +\infty \). We will prove that in this case the limit problem will be the following

\[
\begin{cases}
-\Delta u + \nabla p = f & \text{in } D \setminus \Sigma, \\
\nabla \cdot u = 0 & \text{in } D \setminus \Sigma, \\
u = b & \text{on } \partial D, \\
u_i = 0 & \text{on } \Sigma \text{ for all } i < n, \\
-\left[\sigma(u,p)e_n\right] \cdot e_n = 0 & \text{on } \Sigma.
\end{cases}
\] (3.7)

According to Theorem 2, the solution \( \{u, p\} \) to this problem with \( u \in K \cap V, \ p \in L^2(D)/\mathbb{R} \) exists, is unique and \( u \) minimizes the functional \( F : L^2(D) \to \mathbb{R} \cup \{\infty\} \)

\[
F(u) = \frac{1}{2} a(u, u) + I_{K \cap V} - \langle f, u \rangle_D.
\]

**Theorem 7.** The sequence of functionals \( F^\epsilon \) \( M \)-converges to \( F \) in \( H^1(D) \).

**Proof.** Most part of the proof works using similar argument as in Theorem (5). The only difference appears in the second part when we have to show that \( u_i = 0 \) on \( \Sigma \) for all \( i < n \). In this case we cannot use the same argument because \( \chi_{\Gamma^\epsilon} \) converges to 0.

Let \( \tilde{D} \) be a bounded Lipschitz domain in \( \mathbb{R}^n \) such that \( D \cup \Sigma \times (-1, 1) \subset \subset \tilde{D} \) and let \( P : H^1(D) \to H^1_0(\tilde{D}) \) be an extension operator. As a consequence of the weak convergence of \( u^\epsilon \) to \( u \) in \( H^1(D) \) we have that \( \|\nabla Pu^\epsilon\|_{(L^2(D))^n} \leq C \) for every \( 0 \leq i \leq n \) for some constant \( C \).

Let us define the capacity of a set \( A \) in \( \mathbb{R}^n \) as in ([42])

\[
\text{cap}(A) = \inf_{v \in K_A} \int_{\mathbb{R}^n} |\nabla v|^2 dx,
\]

where \( K_A = \{ v \in L^{2n/n-2}(\mathbb{R}^n), \ \nabla v \in L^2(\mathbb{R}^n), \ v \geq 0, \ A \subset \{ v \geq 1 \}^3 \} \) and if the set \( A \) is bounded with smooth boundary the infimum is obtained and the function that realizes the minimum for \( A \) is called the capacitary potential of the set \( A \) and is the solution to the following boundary value problem

\[
\begin{cases}
-\Delta w = 0 & \text{in } \mathbb{R}^n \setminus A, \\
w = 1 & \text{on } A, \\
w \to 0 & \text{at } \infty.
\end{cases}
\] (3.8)
Let $Y = Y' \times (-1, 1)$. If $w \in H^1(Y)$, such that $\text{Cap}(\overline{Y} \cap \{w = 0\}) > 0$, we will get from Poincare inequality ([52], Corollary 4.5.2.)

$$
\int_Y w^2 d\mathcal{L}^n \leq \frac{C}{\text{Cap}(\overline{Y} \cap \{w = 0\})} \int_Y |\nabla w|^2 d\mathcal{L}^n,
$$

where $C$ is a constant depending only on $Y$ and $n$. As a consequence we get

$$
\int_{Y'} w^2 d\mathcal{L}^{n-1} \leq C \left( \frac{1}{\text{Cap}(\overline{Y} \cap \{w = 0\})} + 1 \right) \int_Y |\nabla w|^2 d\mathcal{L}^n,
$$

(3.9)

for $C = C(Y, n)$.

For any $k \in \mathbb{N}$ and $1 \leq i \leq n-1$, the function $Pu_i(\epsilon(y+k))$ is defined on $Y$ and is zero on the set $\overline{\mathcal{E}} \setminus \epsilon \Gamma$. Applying (3.9), and using the fact that $\text{Cap}(\lambda A) = \lambda^{n-2} \text{Cap}(A)$ for any $\lambda \in \mathbb{R}$ and $A \subset \mathbb{R}^n$ ([42], Section 4.7, Theorem 2) we get

$$
\int_{Y'} (Pu_i(\epsilon(y+k)))^2 d\mathcal{L}^{n-1} \leq C \left( \frac{\epsilon^{n-2}}{\text{Cap}(\Gamma) r_{\epsilon}^{n-2}} + 1 \right) \int_Y |\nabla Pu_i|_2^2(\epsilon(y+k)) dy.
$$

After a change of variable

$$
\epsilon^{1-n} \int_{\epsilon(Y'+k)} (Pu_i(\epsilon(x)))^2 d\mathcal{L}^{n-1} \leq C \epsilon^{2-n} \left( \frac{\epsilon^{n-2}}{\text{Cap}(\Gamma) r_{\epsilon}^{n-2}} + 1 \right) \int_{\epsilon(Y+k)} |\nabla Pu_i|_2^2(x) d\mathcal{L}^n,
$$

and then we obtain

$$
\int_{\epsilon(Y'+k)} (Pu_i(\epsilon(x)))^2 d\mathcal{L}^{n-1} \leq C \left( \frac{\epsilon^{n-1}}{\text{Cap}(\Gamma) r_{\epsilon}^{n-2}} + \epsilon \right) \int_{\epsilon(Y+k)} |\nabla Pu_i|_2^2(x) d\mathcal{L}^n.
$$

Summing over all $k \in \mathbb{N}$ and using the fact that $\epsilon(Y+k) \subset \mathcal{D}$

$$
\int_\Sigma (Pu_i(\epsilon(x)))^2 d\mathcal{L}^{n-1} \leq C \left( \frac{\epsilon^{n-1}}{\text{Cap}(\Gamma) r_{\epsilon}^{n-2}} + \epsilon \right) \int_\mathcal{U} |\nabla Pu_i|_2^2(x) d\mathcal{L}^n,
$$

which will imply that $\lim_{\epsilon \to 0} \int_\Sigma (u_i(\epsilon))^2 d\mathcal{L}^{n-1} = 0$. As a consequence we get that $u_i = 0$ on $\Sigma$ for all $i < n$. \hfill \Box

As a result we have the following theorem that gives the convergence of solutions with a similar proof as Theorem 6.

**Theorem 8.** $u^\epsilon$ converges strongly to $u$ in $H^1(D)$ and $p^\epsilon$ converges strongly in $L^2(D)/\mathbb{R}$ to $p$, where $\{u, p\}$ is the unique solution for the problem (3.7).
3.2.3 Case III: Critical Case

We assume that \( r_n^{-2} \to 0 \) as \( n \to \infty \). We will show that in this case the limit problem will be the following

\[
\begin{align*}
-\Delta u + \nabla p &= f & \text{in } D \setminus \Sigma, \\
\nabla \cdot u &= 0 & \text{in } D \setminus \Sigma, \\
u &= b & \text{on } \partial D,
\end{align*}
\]

where \( \Sigma \) is a symmetric positive definite matrix that we will define later.

We will define next a weak solution for this equation, show that the solution exists and derive the variational formulation for it.

**Definition 2.** The pair \( \{u,p\} \) is a weak solution for problem (3.10) if:

i) \( u \in H^1_0(D) \cap \mathcal{V} \) and \( p \in L^2(D) \)

ii) \( a(u,\phi) - (p, \nabla \cdot \phi)_D = \langle f, \phi \rangle_D \quad \forall \phi \in H^1_0(D^+ \cup D^-) \)

iii) \( -[\sigma(u,p)e_n] = \beta Cu \) a.e. on \( \Sigma \).

**Theorem 9.** \( \{u,p\} \) with \( u \in H^1_0(D) \cap \mathcal{V} \) and \( p \in L^2(D) \) is a weak solution for (3.10) if and only if

\[
a(u,\phi) - (p, \nabla \cdot \phi)_D + \beta c(u,\phi) = \langle f, \phi \rangle_D \quad \forall \phi \in H^1_0(D). \tag{3.11}
\]

where \( c \) is the bilinear form, positive definite and continuous, defined on the space \( H^1(D) \) by

\[
c(u,v) = \int_\Sigma Cu \cdot v d\mathcal{L}^{n-1}.
\]

Moreover, \( u \) solves the following minimization problem

\[
\min_{v \in H^1_0(D) \cap \mathcal{V}} \left\{ \frac{1}{2} a(v,v) + \frac{\beta}{2} c(v,v) - \langle f, v \rangle_D \right\}. \tag{3.12}
\]

**Proof.** If \( \{u,p\} \) satisfies (3.11), then taking \( \phi \in H^1_0(D^+ \cup D^-) \) we get that \( \{u,p\} \) satisfies condition ii). Also

\[
a(u,\phi) - (p, \nabla \cdot \phi)_D - \langle f, \phi \rangle_D = \langle [\sigma(u,p)e_n], \phi \rangle_\Sigma \quad \forall \phi \in H^1_0(D),
\]

so we get

\[
\beta \int_\Sigma Cu \cdot \phi d\mathcal{L}^{n-1} = - \langle [\sigma(u,p)e_n], \phi \rangle_\Sigma \quad \forall \phi \in H^1_0(D),
\]

and from here we get condition iii). The opposite direction is obvious.

To show that \( u \) solves the minimization problem (3.12) we use standard arguments. \( \square \)
**Theorem 10.** There exists a unique solution \( \{u,p\} \), \( u \in H^1_0(D) \cap V \) and \( p \in L^2(D) \cap \mathbb{R} \) for problem (3.10).

**Proof.** Consider the functional
\[
F(u) = \frac{1}{2} a(u,u) - \langle f,u \rangle_D + \frac{\beta}{2} c(u,u) + I_{H^1_0(D) \cap V},
\]
and let \( u \) be the minimizer, which exists based on the coercivity of the bilinear form \( a \) and the positive definiteness of \( c \). From the minimization property we obtain that
\[
a(u,\phi) - \langle f,\phi \rangle_D + \beta c(u,\phi) = 0 \quad \forall \phi \in H^1_0(D) \cap V.
\]
\( \phi \mapsto a(u,\phi) - \langle f,\phi \rangle_D + \beta c(u,\phi) \) is a bounded linear function on \( H^1_0(D) \) which is zero if \( \phi \) is divergence free, so there exists \( p \in L^2(D) \) unique up to a constant such that
\[
a(u,\phi) - \langle f,\phi \rangle_D + b(u,\phi) = (p, \nabla \cdot \phi)_D \quad \text{for every } \phi \in H^1_0(D).
\]

In order to define the matrix \( \mathcal{C} \) we will introduce the solutions \( \{\chi_i, \eta_i\}_{1 \leq i \leq n} \) for the following Stokes problems in free space and fixed boundary values on the set \( \Gamma \)
\[
\begin{cases}
-\Delta \chi_i + \nabla \eta_i = 0 & \text{in } \mathbb{R}^n \setminus \Gamma, \\
\nabla \cdot \chi_i = 0 & \text{in } \mathbb{R}^n \setminus \Gamma, \\
\chi_i = e_i & \text{on } \Gamma, \\
\chi_i \to 0 & \text{at } \infty,
\end{cases}
\]
where \( e_i \) is the \( i \)-th canonical vector in \( \mathbb{R}^n \).

The matrix \( \mathcal{C} \) will be defined by
\[
\mathcal{C}_{ij} = \int_{\mathbb{R}^n} 2 e(\chi_i) e(\chi_j) d\mathcal{L}^n, \quad \text{for } 1 \leq i \leq n - 1.
\]

To prove the desired \( M \)-convergence result we will not be able to use the functions \( \chi_i \) and \( \eta_i \) in the approximating sequences, because the domain \( D \) is bounded, so we need to introduce the functions \( \chi_i^R \) and \( \eta_i^R \) that are the solutions for a similar Stokes system in \( B_R \), the ball of radius \( R \) centered at zero, for \( R \) big enough such that \( \Gamma \subset B_R \)
\[
\begin{cases}
-\Delta \chi_i^R + \nabla \eta_i^R = 0 & \text{in } B_R \setminus \Gamma, \\
\nabla \cdot \chi_i^R = 0 & \text{in } B_R \setminus \Gamma, \\
\chi_i^R = e_i & \text{on } \Gamma, \\
\chi_i^R \to 0 & \text{on } \partial B_R.
\end{cases}
\]

For every such \( R \) let the matrix \( \mathcal{C}^R \) be defined by
\[
\mathcal{C}_{ij}^R = \int_{B_R} e(\chi_i^R) e(\chi_j^R) d\mathcal{L}^n, \quad \text{for } 1 \leq i \leq n - 1.
\]
Let us notice now the similarity between the equation (3.13) and the equation (3.8) that defines the capacitary potential. So the functions \( \{ \chi_i, \eta_i \}_{1 \leq i \leq n} \) will play the same role as the capacitary potential but in the context of divergence free vector fields. We will have \( n \) of them due to the \( n \)-dimensional space for the boundary conditions.

We will study shortly the systems (3.14) and (3.13) and then derive some estimates for their solutions that we will need later. We will start with the system (3.14) because the domain is bounded.

We look for solutions \( \chi_i^R \in H_0^1(B_R) \) with \( \chi_i^R = e_i \) on \( \Gamma \), \( \nabla \cdot \chi_i^R = 0 \) and \( \eta_i^R \in L^2(B_R) \) that satisfy the following weak formulation for every divergence free vector field \( v \) in \( H_0^1(B_R \setminus \Gamma) \)

\[
a_{B_R}(\chi_i^R, v) - \langle \eta_i^R, \nabla \cdot v \rangle_{B_R} = 0,
\]

where \( \langle \cdot, \cdot \rangle_{B_R} \) denotes the \( L^2(B_R) \) inner product by \( a_{B_R}(\cdot, \cdot) \) the bilinear form defined on \( H_0^1(B_R) \) by \( a_{B_R}(u, v) = 2(e(u), e(v))_{B_R} \). It follows that \( \chi_i^R \) minimizes the quadratic form \( a_{B_R} \) on the convex set

\[
K_i^R = \{ v \in H_0^1(B_R) / v = e_i \text{ on } \Gamma, \nabla \cdot v = 0 \}.
\]

We only need to show that the set \( K_i^R \) is not empty, and the existence and uniqueness of \( \chi_i^R \) will follow based on Korn’s inequality and Poincare inequality.

Let \( R_0 \) be a fixed radius such that \( Y' \subset B_{R_0} \).

**Lemma 1.** There exists a constant \( C \) such that \( ||q||_{L^2(B_{2R_0})} \leq C||\nabla q||_{H^{-1}(B_{R_0} \setminus Y')} \) for every \( q \in L^2(B_{R_0}) \). As a consequence, there exists a constant \( C \), independent of \( R \), such that if \( \phi \) belongs to \( L^2(B_{R_0}) \) and has mean zero, there exists a vector field \( u \in H_0^1(B_{R_0} \setminus Y') \) with \( \nabla \cdot u = \phi \) and \( ||\nabla u||_{L^2(B_{R_0})} \leq C||\phi||_{L^2(B_{R_0})} \) for every \( R \geq R_0 \).

**Proof.** The first assertion of the lemma is equivalent with the fact that the operator \( q \in L^2(B_{R_0}) \mapsto \nabla q \in H^{-1}(B_{R_0} \setminus Y') \) has closed range ([47, 53]). So let \( q_n \) be a sequence in \( L^2(B_{R_0}) \) such that \( \nabla q_n \) is convergent to some \( f \) in \( H^{-1}(B_{R_0} \setminus Y') \). Then if \( \phi \in C_0^\infty(B_{R_0} \setminus Y') \) and is divergence free we have \( \langle f, \phi \rangle_{B_{R_0}} = 0 \). It follows that \( f = \nabla q \) for some \( q \in L^2(B_{R_0}) \) (see **Remark 2**) which proves the first part.

As a consequence of the Closed Range Theorem ([51], Ch. II, Th. II.18.) the range of the adjoint operator \( u \in H_0^1(B_{R_0} \setminus Y') \mapsto \nabla \cdot u \in L^2(B_{R_0}) \) is the closed subspace of mean zero functions, and the last estimate is a consequence of the Open Mapping Theorem and a change of variable.

The non emptiness of \( K_i^R \) follows now as a simple consequence of **Lemma 1**. We take \( v \in H_0^1(B_R) \) with \( v = e_i \) on \( \Gamma \). Because \( \nabla \cdot v \) has mean zero over \( B_R \) we can
take then \( u \in H_0^1(B_R \setminus Y') \) with \( \nabla \cdot u = \nabla \cdot v \). \( v - u \) will belong to \( K_i^R \). Finally, Remark 2 gives also the existence of \( \eta_i^R \).

Lemma 2. There exists a constant \( C \), independent of \( R \) and \( i \), such that

\[
a_{B_R}(\chi_i^R, \chi_i^R) \leq C, \quad ||\eta_i^R||_{L^2(B_R)/\mathbb{R}} \leq C.
\]

Proof. The first estimate is an immediate consequence of the variational characterization of \( \chi_i^R \) which implies that \( a(\chi_i^R, \chi_i^R) \) is decreasing as a function of \( R \). For the second one, the function \( \eta_i^R - \int_{B_R} \eta_i^R d\mathcal{L}^n \) has mean zero over \( B_R \), so according to the previous lemma there exists a vector field \( v \in H_0^1(B_R \setminus Y') \) with \( \nabla \cdot v = \eta_i^R - \int_{B_R} \eta_i^R d\mathcal{L}^n \) and \( ||\nabla v||_{L^2(B_R)^n} \leq C ||\eta_i^R - \int_{B_R} \eta_i^R d\mathcal{L}^n||_{L^2(B_R)} \), where \( C \) is also independent of \( R \) and \( i \). But

\[
a_{B_R}(\chi_i^R, v) = (\eta_i^R, \nabla \cdot v)_{B_R},
\]

which implies after elementary calculations that

\[
||\eta_i^R - \int_{B_R} \eta_i^R d\mathcal{L}^n||_{L^2(B_R)}^2 \leq C a_{B_R}(\chi_i^R, \chi_i^R)^{1/2}||\eta_i^R - \int_{B_R} \eta_i^R d\mathcal{L}^n||_{L^2(B_R)}^2,
\]

which shows the second estimate.

Now we will study the similar system in free space (3.13). Following ([47], Th. 2.3) we consider the Hilbert space \( X \), the closure of the divergence free vector fields from \( C_0^\infty(\mathbb{R}^n) \) with the inner product given by \( a_{\mathbb{R}^n}(u, v) = 2(e(u), e(v))_{\mathbb{R}^n} \). Sobolev inequality implies that \( X \) is a subspace of \( \{ v \in L^{2n/n-2}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)^n, \nabla \cdot v = 0 \text{ in } \mathbb{R}^n \} \).

The vector field \( \chi_i \) will be chosen from the space \( X \), as the minimizer of the bilinear form \( a_{\mathbb{R}^n}(\cdot, \cdot) \) over the convex set \( K_i = \{ v \in X,v = e_i \text{ on } \Gamma \} \) which is not empty because \( K_i^R \subset K_i \) for every \( R \). This will imply that \( a_{\mathbb{R}^n}(\chi_i, v) = 0 \) for every divergence free vector field from \( C_0^\infty(\mathbb{R}^n \setminus \Gamma) \). Using ([47], Remark 1.4) we obtain the existence of \( \eta_i \in L^2_{loc}(\mathbb{R}^n) \) such that

\[
a_{\mathbb{R}^n}(\chi_i, v) - (\eta_i, \nabla \cdot v)_{\mathbb{R}^n} = 0
\]

for every \( v \in C_0^\infty(\mathbb{R}^n \setminus \Gamma) \).

A similar proof as in Lemma 2 shows that for any \( R \geq R_0 \)

\[
||\eta_i - \int_{B_R} \eta_i d\mathcal{L}^n||_{L^2(B_R)} \leq C.
\]
For any $R' > R$

$$||\eta_i - \int_{B_{R'}} \eta_i d\mathcal{L}^n||_{L^2(B_{R'})} \leq C,$$

so

$$||\int_{B_{R'}} \eta_i(x) d\mathcal{L}^n - \int_{B_R} \eta_i(x) d\mathcal{L}^n||_{L^2(B_{R'})} \leq 2C.$$  

This will imply that the sequence $R \mapsto \int_{B_{R'}} \eta_i(x) d\mathcal{L}^n$ will be convergent to a limit $l$ that will satisfy $||\eta_i - l||_{L^2(B_{R'})} \leq C$, and because $\eta_i$ is unique up to a constant we can take $l = 0$ and obtain $||\eta_i||_{L^2(\mathbb{R}^n)} \leq C$. This gives the existence of $\eta_i \in L^2(\mathbb{R}^n)$ such that

$$a_{\mathbb{R}^n}(x, \chi_i, v) - (\eta_i, \nabla \cdot v)_{\mathbb{R}^n} = 0$$

for every $v \in C_0^\infty(\mathbb{R}^n \setminus \Gamma)$, and now we can pass to the limit and obtain that this is true for every $v \in X$ such that $v = 0$ on $\Gamma$.

**Lemma 3.** If we consider $\chi_i^R$ and $\eta_i^R$ extended by 0 outside $B_R$ and impose the condition $\int_{B_R} \eta_i^R = 0$, then $\chi_i^R$ converges to $\chi_i$ in $X$ and $\eta_i^R$ converges to $\eta_i$ in $L^2(\mathbb{R}^n)$ as $R \to +\infty$.

**Proof.** According to **Lemma 2** $\chi_i^R$ will converge weakly in $X$ and $\eta_i^R$ weakly in $L^2(\mathbb{R}^n)$, and the limit will satisfy 3.13 so by uniqueness it has to be $\{\chi_i, \eta_i\}$. Let $\{v^R, q^R\}$ solve the system for $R > R_0$

\[
\begin{cases}
-\Delta v^R + \nabla q^R = -\Delta \chi_i + \Delta \chi_i^{R_0} & \text{in } B_R \setminus \Gamma, \\
\nabla \cdot v^R = 0 & \text{in } B_R \setminus \Gamma, \\
v^R = 0 & \text{on } \partial B_R \cup \Gamma.
\end{cases}
\]

We obtain from the variational formulation that $a_{\mathbb{R}^n}(v^R, v^R) = a_{\mathbb{R}^n}(v^R, \chi_i - \chi_i^{R_0})$ which implies that $a_{\mathbb{R}^n}(v^R, v^R) \leq a_{\mathbb{R}^n}(\chi_i - \chi_i^{R_0}, \chi_i - \chi_i^{R_0})$. Also, as in **Lemma 2** we obtain an uniform bound in $L^2(\mathbb{R}^n)$ for $q^R$ if we chose them to have mean zero in $B_R$. If $v$ is the weak limit in $X$ for $v^R$ and $q$ the weak limit for $q^R$ in $L^2(\mathbb{R}^n)$, then $\{q, v\}$ solves the system

\[
\begin{cases}
-\Delta v + \nabla q = -\Delta \chi_i + \Delta \chi_i^{R_0} & \text{in } \mathbb{R}^n \setminus \Gamma, \\
\nabla \cdot v = 0 & \text{in } \mathbb{R}^n \setminus \Gamma, \\
v = 0 & \text{on } \Gamma,
\end{cases}
\]

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which implies that \( v = \chi - \chi^R\) and \( q = 0 \). But from \( a_{\mathbb{R}^n}(v^R, v^R) \leq a_{\mathbb{R}^n}(\chi - \chi^R, \chi - \chi^R) \) we obtain that \( q^R \) converges strongly to \( q = \chi - \chi^R \) in \( X \). So \( q^R + \chi^R \) converges strongly to \( \chi \). The minimizing properties for \( \chi_i \) and \( \chi^R_i \) gives

\[
a_{\mathbb{R}^n}(\chi_i, \chi_i) \leq a_{\mathbb{R}^n}(\chi^R_i, \chi^R_i) \leq a_{\mathbb{R}^n}(q^R + \chi^R_i, q^R + \chi^R_i),
\]

so \( a_{\mathbb{R}^n}(\chi^R_i, \chi^R_i) \) will converge to \( a_{\mathbb{R}^n}(\chi_i, \chi_i) \), which together with the weak convergence implies that \( \chi^R_i \) converges strongly to \( \chi_i \) in \( X \).

Let \( u^R \in X \), such that \( u^R \in H^1_0(B^R \setminus \Gamma) \), \( \nabla \cdot u^R = \eta \), \( \int_{B^R} \eta_i dL^n = \eta_i^R \) and \( ||\nabla u^R||_{L^2(\mathbb{R}^n)} \leq C||\nabla \cdot u^R||_{L^2(\mathbb{R}^n)} \) as given in Lemma 1. From \( a_{\mathbb{R}^n}(\chi_i - \chi^R_i, u^R) = (\eta_i - \eta_i^R, u^R)_{\mathbb{R}^n} \) we obtain that \( ||u^R||_{L^2(\mathbb{R}^n)} \to 0 \). Jensen’s inequality implies that \( \int_{B^R} \eta_i dL^n \to 0 \) so \( \eta_i^R \) will converge strongly to \( \eta_i \) in \( L^2(\mathbb{R}^n) \).

Now we can prove the main theorem. We consider the same functionals \( F^\epsilon \) and \( F \) defined on \( L^2(D) \), extended with \( +\infty \) outside \( H^1(D) \).

**Theorem 11.** The sequence of functionals \( F^\epsilon \) \( M \)-converges to \( F \) in \( L^2(D) \).

**Proof.** (i) \( M - \limsup_{\epsilon \to 0} F^\epsilon(u) \leq F(u) \).

This means that for every \( u \in H^1_0(D) \cap V \) there exists a sequence \( u^\epsilon \in K^\epsilon \cap V \) converging to \( u \) strongly in \( L^2(D) \) such that

\[
\lim_{\epsilon \to 0} F^\epsilon(u^\epsilon) \leq F(u).
\]

Let \( b_0 \) be a function in \( H^1_0(D) \cap V \) such that \( (b_0)_i = 0 \) for every \( 1 \leq i \leq n - 1 \) and assume first that \( u \) is chosen such that \( u - b_0 \) is smooth.

For every \( k \in N^\epsilon \), let \( B^\epsilon_k \) be the ball centered at \( \epsilon k \) with radius \( \epsilon \). For every such ball \( B^\epsilon_k \) and every \( 1 \leq i \leq n - 1 \), consider the solutions to the following problems

\[
\begin{cases}
-\Delta \chi^\epsilon_{k,i} + \nabla \eta^\epsilon_{k,i} = 0 & \text{in } B^\epsilon_k \setminus \Gamma^\epsilon_k, \\
\nabla \cdot \chi^\epsilon_{k,i} = 0 & \text{in } B^\epsilon_k \setminus \Gamma^\epsilon_k, \\
\chi^\epsilon_{k,i} = e_i & \text{on } \Gamma^\epsilon_k, \\
\chi^\epsilon_{k,i} = 0 & \text{on } \partial B^\epsilon_k. 
\end{cases}
\tag{3.15}
\]

After a change of variable we get that \( \chi^\epsilon_{k,i}(r, x + \epsilon k) \) and \( r \eta^\epsilon_{k,i}(r, x + \epsilon k) \) are solutions for a problem of type (3.14), namely \( \chi^{\epsilon/r}_i \) and \( \eta^{\epsilon/r}_i \).
Let us prove that the sequence \( \bar{u} = \sum_{i=1}^{n-1} \sum_{k \in \mathbb{N}} \chi_{k,i} u_i(\epsilon k) \) converges strongly to 0 in \( L^2(D) \). First we will prove that the sequence is bounded in \( H^1(D) \). After a change of variables
\[
\|\nabla \chi_{k,i}^\epsilon\|_{L^2(D)^n} = r_{\epsilon}^{n-2} \|\nabla \chi_i^{\epsilon/r_{\epsilon}}\|_{L^2(B_{r_{\epsilon}/r_{\epsilon}})} \leq C\epsilon^{n-1}.
\]
Since the first \( n-1 \) components of \( u \) are smooth on \( \Sigma \), and the number of terms is of the order of \( \epsilon^{-n} \) it follows that \( \bar{u} \) is bounded in \( H^1(D) \). The measure of the support of this sequence being convergent to 0, we get strong convergence to 0 in \( \mathbb{W}^{1,1}(D) \) so the sequence will be strongly convergent to 0 in \( L^2(D) \), in fact weakly convergent to 0 in \( H^1(D) \).

The sequence \( u - \bar{u} \) satisfies all the properties except that the values \( u_i - \bar{u}_i \) on \( \Gamma^\epsilon \) are not quite 0, but very small. We need to modify it in the following.

If \( v \) is a smooth function defined on \( Y' \), as a consequence of Lemma 1, there exists a function \( \tilde{v} \in H^1_0(B_{R_0}) \) with divergence zero in \( B_{R_0} \), equal to \( v \) on \( Y' \), and satisfying
\[
||\tilde{v}||_{H^1_0(B_{R_0})} \leq C||v||_{H^{1/2}(Y')}.
\]
Let \( [v]_{H^{1/2}(Y')} \) be the \( H^{1/2}(Y') \) seminorm of \( v \)
\[
[v]_{H^{1/2}(Y')} = \left( \int_{Y' \times Y'} \frac{|v(x') - v(y')|^2}{|x' - y'|^n} d\mathcal{L}^{n-1}(x')d\mathcal{L}^{n-1}(y') \right)^{1/2}.
\]

Then
\[
[v]_{H^{1/2}(Y')}^2 \leq C||v||_{L^\infty(Y')}||\nabla v||_{L^\infty(Y')}^{3/2} \int_0^{R_0} t^{-1/2} dt
\]
which implies that
\[
||\tilde{v}||_{H^1_0(B_{R_0})}^2 \leq C||v||_{L^\infty(Y')}||v||_{\mathbb{W}^{1,\infty}(Y')}^{3/2}.
\]

Let \( B_{R_0^\epsilon r_{\epsilon}} \) be the ball with the center at \( \epsilon k \) and radius \( R_0 r_{\epsilon} \). After rescaling, we obtain that for any smooth function \( v \) defined on \( \epsilon k + r_\epsilon Y' \) there exists a function \( \tilde{v} \in H^1_0(B_{R_0^\epsilon r_{\epsilon}}) \) such that
\[
\begin{align*}
\nabla \cdot \tilde{v} &= 0, & \text{in } B_{R_0^\epsilon r_{\epsilon}}, \\
\tilde{v} &= v, & \text{on } r_\epsilon Y',
\end{align*}
\]
where the constant \( C \) is independent of \( \epsilon \) and \( k \).

Let \( \tilde{v}_k \) be the function satisfying (3.16) when we take \( v \) the function defined compo-
nentwise \( v_i = u_i - u_i(\epsilon k) \) for \( 1 \leq i \leq n - 1 \) and \( v_n = 0 \). Then \( \tilde{v}_k^\epsilon \) will satisfy

\[
\begin{align*}
\nabla \cdot \tilde{v}_k^\epsilon &= 0 & \text{in } B_{2r}^{2r}, \\
(\tilde{v}_k^\epsilon)_i &= u_i - u_i(\epsilon k) & \text{on } r_Y' \text{ for } i < n - 1, \\
(\tilde{v}_k^\epsilon)_n &= 0 & \text{on } r_Y', \\
||\nabla \tilde{v}_k^\epsilon||^2_{L^2(B_{2r}^{2r})} &\leq C r_n^{-2} \max_{1 \leq i \leq n - 1} ||u_i - u_i(\epsilon k)||^{1/2}_{L^\infty(r_Y)},
\end{align*}
\]

where the constant \( C \) depends on \( u \) but not on \( k \) or \( \epsilon \).

The function \( \tilde{u}^\epsilon = \sum_{k \in \mathbb{N}^\epsilon} \tilde{v}_k^\epsilon \) is divergence free and from the uniform continuity of \( u_i \) for \( 1 \leq i \leq n - 1 \) on \( \Sigma \) will converge to 0 strongly in \( H^1(D) \). Now we can define the sequence \( u^\epsilon \) that we need

\[
u^\epsilon = u - \bar{u}^\epsilon - \tilde{u}^\epsilon.
\]

Based on what we proved \( u^\epsilon \) is in \( K^\epsilon \cap V \) and converges to \( u \) strongly in \( L^2(D) \). Also

\[
\lim_{\epsilon \to 0} F^\epsilon(u^\epsilon) = \lim_{\epsilon \to 0} \frac{1}{2} a(u^\epsilon, u^\epsilon) - \langle f, u^\epsilon \rangle_D + j^\epsilon(u^\epsilon) = \frac{1}{2} a(u, u) - \langle f, u \rangle_D + I_1
\]

where \( I_1 = \frac{1}{2} \lim_{\epsilon \to 0} a\left( \sum_{i=1}^{n-1} \sum_{k \in \mathbb{N}^\epsilon} \chi_i^k u_i(\epsilon k), \sum_{i=1}^{n-1} \sum_{k \in \mathbb{N}^\epsilon} \chi_i^k u_i(\epsilon k) \right) \).

Making a change of variables and using the fact that \( \chi_i^k \) and \( u_i^\epsilon \) have disjoint supports for \( k \neq l \), we obtain

\[
I_1 = \frac{1}{2} \lim_{\epsilon \to 0} \sum_{k \in \mathbb{N}^\epsilon} \sum_{i,j=1}^{n-1} a(\chi_i^{\epsilon r}, \chi_j^{\epsilon r}) r_n^{-2} u_i(\epsilon k) u_j(\epsilon k) \\
= \frac{1}{2} \lim_{\epsilon \to 0} \sum_{k \in \mathbb{N}^\epsilon} \sum_{i,j=1}^{n-1} C_{ij} u_i(\epsilon k) u_j(\epsilon k) r_n^{-2}
\]

From Lemma 3 and because \( r_n^{-2} \) is of the same order with \( \epsilon^{n-1} \) we get that

\[
I_1 = \beta \lim_{\epsilon \to 0} \sum_{i,j=1}^{n-1} \sum_{k \in \mathbb{N}^\epsilon} C_{ij} u_i(\epsilon k) u_j(\epsilon k) \epsilon^{n-1},
\]

which will imply

\[
I_1 = \beta \sum_{i,j=1}^{n-1} C_{ij} \int_{\Sigma} u_i u_j.
\]
This shows that
\[
\lim_{\epsilon \to 0} F^\epsilon(u^\epsilon) = \frac{1}{2} a(u, u) - \langle f, u \rangle_D + \frac{\beta}{2} c(u, u) = F(u),
\]
and we have proved so far that \( M - \limsup_{\epsilon \to 0} F^\epsilon(u) \leq F(u) \) for every smooth \( u \) in \( H^1_b(D) \cap V \). We may choose a countable dense set for the strong topology of \( H^1_b(D) \cap V \) such that the property holds for every \( u \) from this subset. We will complete the proof using a diagonalization argument ([2], Corollary 1.18) to show that the property holds for every \( u \in H^1_b(D) \cap V \).

(ii) \( M - \liminf_{\epsilon \to 0} F^\epsilon(u) \geq F(u) \).

Which means that for every sequence \( u^\epsilon \in L^2(D) \) converging weakly to \( u \)
\[
\liminf_{\epsilon \to 0} F^\epsilon(u^\epsilon) \geq F(u).
\]

We may assume that \( \liminf_{\epsilon \to 0} F^\epsilon(u^\epsilon) < \infty \) so \( u^\epsilon \in K^\epsilon \cap V \) and based on uniform coercivity of \( F^\epsilon \) up to a subsequence \( u^\epsilon \) will be weakly convergent to \( u \) in \( H^1(D) \). Let us consider first \( v \) such that \( v - b_0 \) is smooth and the corresponding sequence from the previous part \( v^\epsilon \) weakly convergent to \( v \) in \( H^1(D) \) such that \( F^\epsilon(v^\epsilon) \to F(v) \).

Then by a subdifferential type inequality
\[
F^\epsilon(u^\epsilon) = \frac{1}{2} a(u^\epsilon, u^\epsilon) - \langle f, u^\epsilon \rangle_D + j^\epsilon(u^\epsilon) \geq F^\epsilon(v^\epsilon) + a(u^\epsilon - v^\epsilon, v^\epsilon) - \langle f, u^\epsilon - v^\epsilon \rangle_D + j^\epsilon(u^\epsilon) - j^\epsilon(v^\epsilon).
\]

From the weak convergences and the first part
\[
\liminf_{\epsilon \to 0} F^\epsilon(u^\epsilon) \geq F(v) - \langle f, u - v \rangle_D + \liminf_{\epsilon \to 0} a(u^\epsilon - v^\epsilon, v^\epsilon).
\]

So it is enough to study
\[
\liminf_{\epsilon \to 0} a(u^\epsilon - v^\epsilon, v^\epsilon).
\]

We remember that \( v^\epsilon = v - \overline{v}^\epsilon - \tilde{v}^\epsilon \), where \( \overline{v}^\epsilon \) converges weakly to 0 in \( H^1(D) \) and \( \tilde{v}^\epsilon \) converges strongly. Then
\[
\liminf_{\epsilon \to 0} a(u^\epsilon - v^\epsilon, v^\epsilon) = a(u - v, v) - \limsup_{\epsilon \to 0} a(u^\epsilon - v^\epsilon, v^\epsilon).
\]

We will estimate the following limit
\[
I_2 = \limsup_{\epsilon \to 0} |a(u^\epsilon - v^\epsilon, \overline{v}^\epsilon)|,
\]

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where
\[ a(u^\epsilon - v^\epsilon, v^\epsilon) = \sum_{i=1}^{n-1} \sum_{k \in \mathcal{N}_e} v_i(\epsilon k) \int_{B_k^\epsilon} 2e(u^\epsilon - v^\epsilon)e(\chi_{k,i}) d\mathcal{L}^n. \]

Let us assume for the time being that we are able to show that
\[ I_2 \leq C \|v\|_{H^1(D)} \cdot \|v - u\|_{H^1(D)}. \]

We will obtain that for every \( v \) in a dense subset of \( H^1_b(D) \cap V \)
\[ \liminf_{\epsilon \to 0} F^\epsilon(u^\epsilon) \geq F(v) - \langle f, u - v \rangle_D + a(u - v, v) - C \|v\|_{H^1(D)} \cdot \|v - u\|_{H^1(D)}. \]

Now we let \( v \) converge to \( u \) strongly, use the continuity of \( F \) with respect to the strong topology and get the desired inequality.

In order to show the estimate for \( I_2 \) we notice that this can be restated in the following form
\[ \limsup_{\epsilon \to 0} \sum_{k \in \mathcal{N}_e} \int_{B_k^\epsilon} e(w^\epsilon)e(\chi_{k,i}) d\mathcal{L}^n \leq C \|w\|_{H^1(D)}, \]
for any \( 0 \leq i \leq n - 1 \) and for any sequence \( w^\epsilon \in H^1_b(D) \cap V \) such that \( w_j^\epsilon \equiv 0 \) on \( \Gamma^\epsilon \) for all \( 0 \leq j \leq n - 1 \) and weakly convergent to \( w \). This will be shown in the next lemma.

**Lemma 4.** Let \( w^\epsilon \) a sequence from \( w \in H^1_b(D) \cap V \) such that \( w_j^\epsilon \equiv 0 \) on \( \Gamma^\epsilon \) for all \( 0 \leq j \leq n - 1 \) which is weakly convergent in \( H^1_0(D) \) to some \( w^\prime \). If \( 0 \leq i \leq n - 1 \) and \( \chi_{k,i}^\epsilon \) is defined like in 3.15, then there is a constant \( C \) independent of \( w \) and the sequence \( w^\epsilon \) such that
\[ \limsup_{\epsilon \to 0} \sum_{k \in \mathcal{N}_e} \int_{B_k^\epsilon} e(w^\epsilon)e(\chi_{k,i}) d\mathcal{L}^n \leq C \|w\|_{H^1_0(D)}. \]

**Proof.** Let us notice first that after integrating by parts \( \int_{B_k^\epsilon} e(w^\epsilon)e(\chi_{k,i}) d\mathcal{L}^n \) depends only on the values of \( w^\epsilon \) on the boundaries of \( B_k^\epsilon \) because of the symmetry of \( \chi_{k,i}^\epsilon \) and the fact that the tangential components of \( w^\epsilon \) are zero on the sets \( \Gamma_k^\epsilon \).

Let \( v^\epsilon \) be from \( H^1(B_{e/r^\epsilon}) \) defined by
\[ v^\epsilon(y) = \frac{1}{|N^\epsilon|} \sum_{k \in N^\epsilon} w^\epsilon(r^\epsilon y + \epsilon k). \]
Then \( v^i \) is zero on \( \Gamma \) for all \( 1 \leq i \leq n - 1 \) and
\[
\int_{B_{\epsilon/r_{\epsilon}}} |\nabla v|^2 d\mathcal{L}^n = \frac{1}{|N^\epsilon| r_{\epsilon}^{n-2}} \sum_{k \in N^\epsilon} \int_{B_k^\epsilon} |\nabla w^i|^2 d\mathcal{L}^n \leq C.
\]
The limit we need to compute becomes
\[
\limsup_{\epsilon \to 0} \frac{1}{|N^\epsilon| r_{\epsilon}^{n-2}} \int_{B_{\epsilon/r_{\epsilon}}} e(v^i) e(\chi_i^{\epsilon/r_{\epsilon}}) d\mathcal{L}^n = C \limsup_{\epsilon \to 0} \int_{B_{\epsilon/r_{\epsilon}}} e(v^i) e(\chi_i^{\epsilon/r_{\epsilon}}) d\mathcal{L}^n.
\]
We notice that the limit depends only on the values of \( v^i \) on the boundary of \( B_{\epsilon/r_{\epsilon}} \) and the limit becomes easy to estimate if \( v^i \) is constant on \( \partial B_{\epsilon/r_{\epsilon}} \) and we know how to estimate the constants. So first we will try to modify \( v^i \) making it constant on \( \partial B_{\epsilon/r_{\epsilon}} \) such that the limit does not change too much and then we will try to estimate those constants.

Given a divergence free vector field \( v \) in \( H^1(B_2 \setminus B_1) \), we want to find \( \tilde{v} \) in \( H^1(B_2 \setminus B_1) \), constant on \( \partial B_2 \) and divergence free, such that \( ||\nabla v - \nabla \tilde{v}||_{L^2(B_2 \setminus B_1)^n} \) is minimum. Regularity theorem ([47], Proposition 2.2) allows us to choose \( v - \tilde{v} \) with boundary values 0 on \( \partial B_1 \) and \( v - a \) on \( \partial B_2 \) where \( a \) is a constant such that \( ||v - \tilde{v}||_{H^1(B_2 \setminus B_1)} \leq C ||v - a||_{H^{1/2}(\partial B_2)} \leq C ||v - a||_{H^1(B_2 \setminus B_1)} \). So we can choose \( \tilde{v} \) such that \( ||\nabla v - \nabla \tilde{v}||_{L^2(B_2 \setminus B_1)^n} \leq C ||\nabla v||_{L^2(B_2 \setminus B_1)^n} \) if \( a \) is \( \int_{B_2 \setminus B_1} v d\mathcal{L}^n \). Obviously the result holds true for \( B_{2R} \setminus B_R \) with the same constant.

Let \( \tilde{v}^i \) be the modification of \( v^i \) in \( B_{\epsilon/r_{\epsilon}} \setminus B_{\epsilon/2r_{\epsilon}} \), so \( \tilde{v}^i \) is equal to \( v^i \) on \( B_{\epsilon/2r_{\epsilon}} \) and \( ||\nabla \tilde{v}^i||_{L^2(B_{\epsilon/r_{\epsilon}})^n} \leq C \). Then
\[
\limsup_{\epsilon \to 0} \int_{B_{\epsilon/r_{\epsilon}}} e(v^i) e(\chi_i^{\epsilon/r_{\epsilon}}) d\mathcal{L}^n \leq \limsup_{\epsilon \to 0} \int_{B_{\epsilon/r_{\epsilon}}} e(\tilde{v}^i) e(\chi_i^{\epsilon/r_{\epsilon}}) d\mathcal{L}^n + A,
\]
where
\[
A = \limsup_{\epsilon \to 0} \int_{B_{\epsilon/r_{\epsilon}}} e(v^i - \tilde{v}^i) e(\chi_i^{\epsilon/r_{\epsilon}}) d\mathcal{L}^n.
\]
But applying Cauchy’s inequality and then Lemma 3
\[
A^2 \leq \limsup_{\epsilon \to 0} \int_{B_{\epsilon/r_{\epsilon}}} e(\tilde{v}^i - \tilde{v}^i) e(v^i - \tilde{v}^i) d\mathcal{L}^n \int_{cB_{\epsilon/2r_{\epsilon}}} e(\chi_i^{\epsilon/r_{\epsilon}}) e(\chi_i^{\epsilon/r_{\epsilon}}) d\mathcal{L}^n
\]
\[
\leq C \limsup_{\epsilon \to 0} \int_{cB_{\epsilon/2r_{\epsilon}}} e(\chi_i^{\epsilon/r_{\epsilon}}) e(\chi_i^{\epsilon/r_{\epsilon}}) d\mathcal{L}^n
\]
\[
\leq C \limsup_{\epsilon \to 0} \int_{cB_{\epsilon/2r_{\epsilon}}} e(\chi_i) e(\chi_i) d\mathcal{L}^n = 0.
\]
Now we will study $\limsup_{\epsilon \to 0} \int_{B_{\epsilon/\epsilon}} e(\tilde{v}^\epsilon) e(\chi_{i/\epsilon}^\epsilon) d\mathcal{L}^n$. The integral $\int_{B_{\epsilon/\epsilon}} e(\tilde{v}^\epsilon) e(\chi_{i/\epsilon}^\epsilon) d\mathcal{L}^n$ depends only on the values of $\tilde{v}^\epsilon$ on $\partial B_{\epsilon/\epsilon}$, so it will be equal with

$$-\sum_{j=1}^n a_j^{\epsilon} \int_{B_{\epsilon/\epsilon}} e(\chi_j^{\epsilon/\epsilon}) e(\chi_i^{\epsilon/\epsilon}) d\mathcal{L}^n,$$

where $a_j^{\epsilon} = \int_{B_{\epsilon/\epsilon} \setminus B_{\epsilon/2\epsilon}} w_j^{\epsilon} d\mathcal{L}^n$. So from Lemma 2

$$\limsup_{\epsilon \to 0} \int_{B_{\epsilon/\epsilon}} e(\tilde{v}^\epsilon) e(\chi_{i/\epsilon}^\epsilon) d\mathcal{L}^n \leq C \max_j \limsup_{\epsilon \to 0} |a_j^{\epsilon}|.$$

Let $A^\epsilon = \bigcup_{k \in \mathbb{N}} B_{k} \setminus B_{k/2}$. If $\phi$ is a continuous function defined in $D$ it is easy to compute the limit

$$\lim_{\epsilon \to 0} \int_{A^\epsilon} \phi d\mathcal{L}^n = \int_{\Sigma} \phi d\mathcal{L}^n,$$

which implies that the sequence $\frac{1}{\mathcal{L}^n(A^\epsilon)} \chi_{A^\epsilon}$ converges weakly in $L^2(D)$ to $\frac{1}{\mathcal{L}^n(\Sigma)} \chi_{\Sigma}$. This implies that $a_j^{\epsilon}$ converges to $\int_{\Sigma} w_j$. We obtain from here that

$$\limsup_{\epsilon \to 0} \int_{B_{\epsilon/\epsilon}} e(\tilde{v}^\epsilon) e(\chi_{i/\epsilon}^\epsilon) d\mathcal{L}^n \leq C||w||_{L^2(\Sigma)},$$

which proves the lemma. $\square$

As a consequence we have the following result:

**Theorem 12.** $u^\epsilon$ converges weakly to $u$ in $H^1(D)$ and $p^\epsilon$ converges weakly tp $p$ in $L^2(D)/\mathbb{R}$, where $\{u, p\}$ is the solution for the problem (3.10).

**Proof.** Weak convergence for $u^\epsilon$ to $u$ follows from ([50] and Corollary 7.17) after we notice that we proved also $\Gamma -$convergence for the functionals $F^\epsilon$ in the weak topology of $H^1(D)$. From here it follows immediately the weak convergence in $H^{-1}(D)$ for $\nabla p^\epsilon$ which proves the theorem. $\square$
3.2.4 Case IV: Vanishing Obstacles

We assume that $\lim_{\varepsilon \to 0} \frac{\varepsilon^{n-2}}{\varepsilon^{n-1}} = 0$. In this case the limit problem will be trivial

$$
\begin{cases}
-\Delta u + \nabla p = f & \text{in } D, \\
\nabla \cdot u = 0 & \text{in } D, \\
u = b & \text{on } \partial D.
\end{cases} 
(3.19)
$$

We obtain easily that the sequence of functionals $F^\varepsilon$ $M$-converges to $F$ in $L^2(D)$ where

$$
F(u) = \frac{1}{2} a(u, u) - \langle f, u \rangle_D + I_{H_0^1(D) \cap \mathcal{V}},
$$

and we have also weak convergences for $u^\varepsilon$ and $p^\varepsilon$ as in Theorem 12. To show the lim sup inequality we use comparison arguments and Theorem 11 that gives us that $M - \limsup_{\varepsilon \to 0} F^\varepsilon(u) \leq F(u) + \beta \langle u, u \rangle$ for every positive number $\beta$. The lim inf inequality follows from the fact that $F^\varepsilon(u) \geq F(u)$ for every $u$. 

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Chapter 4

Membranes with Random Distribution of Leaky Obstacles

4.1 Preliminaries and Assumptions

As in the previous chapter, $D$ is a bounded domain in $\mathbb{R}^n$, with Lipschitz boundary, separated by the hyperplane $\{x_n = 0\}$ into two $D^+ = D \cap \{x_n > 0\}$ and $D^- = D \cap \{x_n < 0\}$ both of them with Lipschitz boundary, and $\Sigma = D \cap \{x_n = 0\}$. $Y' = (-\frac{1}{2}, \frac{1}{2})^{n-1}$ will be the unit cube in $\mathbb{R}^{n-1}$ and let $(\Omega, \mathcal{F}, P)$ be a complete probability space. For every $\epsilon > 0$, $N^\epsilon$ is the set of all points $k \in \mathbb{Z}^{n-1}$ such that $\epsilon(k + Y')$ is strictly included in $\Sigma$ and $|N^\epsilon|$ the total number of them. Now, for every $\epsilon > 0$, $\omega \in \Omega$, $k \in N^\epsilon$ we consider a compact set of $\mathbb{R}^{n-1}$, $\Gamma^\epsilon_k(\omega) \subset \subset \epsilon(k + Y')$ such that $\Gamma^\epsilon_k(\omega)$ is the closure of a connected open set with Lipschitz boundary.

We will define the following sets:

$$\Gamma^\epsilon(\omega) = \bigcup_{k \in N^\epsilon} \Gamma^\epsilon_k(\omega),$$

$$T^\epsilon(\omega) = \Sigma \setminus \Gamma^\epsilon(\omega),$$

$$D^\epsilon(\omega) = D \setminus \Gamma^\epsilon(\omega).$$

$\Gamma^\epsilon(\omega)$ will represent the membrane, still having a periodic distribution in the cells $\epsilon(k + Y')$ along $\Sigma$ but the sizes and shapes of the small obstacles from every cell will be random. We will keep the same type of interface condition on $\Gamma^\epsilon(\omega)$ and we will study for every $\omega \in \Omega$ the limiting behaviour when $\epsilon \to 0$ of the movement of an incompressible viscous fluid in $D^\epsilon(\omega)$. 

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For any given \( \omega \), we denote by \( u'(\omega) \) the velocity of the fluid and by \( p'(\omega) \) its pressure, so the pair \( \{u'(\omega), p'(\omega)\} \) will satisfy the Stokes equation in \( D'(\omega) \) with the following boundary conditions

\[
\begin{cases}
-\Delta u'(\omega) + \nabla p'(\omega) = f & \text{in } D'(\omega), \\
\nabla \cdot u'(\omega) = 0 & \text{in } D'(\omega), \\
u'(\omega) = b & \text{on } \partial D, \\
u_i'(\omega) = 0 & \text{on } \Gamma^i(\omega) \text{ for all } i < n, \\
-\left[\sigma'(\omega)e_n\right] \cdot e_n \in g\partial|u_n'(\omega)| & \text{on } \Gamma^e(\omega),
\end{cases}
\]

(4.1)

where \( f \) is a distribution in \( H^{-1}(D) \) and \( b \in H^{1/2}(\partial D) \) satisfying the same compatibility conditions (2.3) and (2.4). \( e_i \) represents the \( i \)-th canonical vector of \( \mathbb{R}^n \) for \( 1 \leq i \leq n \) and \( g \) is a positive function from \( L^2(\Sigma) \) that represents the threshold for the appearance of the leak.

The difference from the previous chapter is that the domain depends on a random parameter \( \omega \), and for each \( \omega \) the obstacles in each cell are not copies of each other. What we are hoping is that under suitable assumptions, the solutions of the system (4.1) will converge when \( \epsilon \to 0 \), for almost all \( \omega \in \Omega \), to a solution of a deterministic system that we want to determine. The relative size of the holes (given in the previous chapter through \( r_\epsilon \)) should also be an important factor so we will expect also different limit problems.

We define the following random closed convex sets of \( H^1(D) \)

\[
K'(\omega) = \{ u \in H^1_b(D) / u_i = 0 \text{ on } \Gamma^i(\omega) \forall i < n \},
\]

and the random convex functionals

\[
j'(\omega)(v) = \int_{\Gamma^e(\omega)} g|v_n|d\mathcal{L}^{n-1} \forall v \in H^1(D).
\]

As in the previous chapter, we have the following characterization of a weak solution and the theorem of existence and uniqueness for \( \{u'(\omega), p'(\omega)\} \). The proofs are similar with the ones from the previous chapter so we will only state them because we will refer to them later:

**Theorem 13.** \( \{u'(\omega), p'(\omega)\} \) with \( u'(\omega) \in K'(\omega) \cap V \) and \( p'(\omega) \in L^2(D) \) is a weak solution for problem (4.1) if and only if for every \( v \in K'(\omega) \) we have

\[
a(u'(\omega), v-u'(\omega)) - (p'(\omega), \nabla \cdot (v-u'(\omega)))_D + j'(\omega)(v) - j'(\omega)(u'(\omega)) \geq \langle f, v - u'(\omega) \rangle_D.
\]

(4.2)
Also, \( u^*(\omega) \) may be characterized as the minimizer for the following problem:

\[
\min_{v \in \mathbf{K}^* \cap \mathbf{V}} \left\{ \frac{1}{2} a(v, v) + j^*(\omega)(v) - \langle f, v \rangle_D \right\}.
\]

(4.3)

**Theorem 14.** For every \( \omega \in \Omega \) there exists a solution \( \{u^*(\omega), p^*(\omega)\} \) for problem (4.1), with \( u^*(\omega) \) being unique and \( p^*(\omega) \) unique up to an additive constant.

We need several assumptions upon the sets \( \Gamma_k^\epsilon(\omega) \) that will assure the existence of some kind of average and for this we need several preliminaries about stochastic geometry and measurable set valued functions:

**Definition 3.** On a probability space \( (\Omega, \mathcal{F}, P) \), a \( d \)-dimensional dynamical system \( \tau \) is a family of mappings \( (\tau_k)_{k \in \mathbb{Z}^d} \) that satisfy the following properties:

i) Group property: \( \tau_0 \) is the identity and

\[
\tau_k \circ \tau_l = \tau_{k+l} \quad \text{for all } k, l \in \mathbb{Z}^d.
\]

ii) Invariance: The mappings \( (\tau_k)_{k \in \mathbb{Z}^d} \) are measurable and measure preserving, meaning that

\[
P(\tau_k^{-1}B) = P(B) \quad \text{for all } k, l \in \mathbb{Z}^d \text{ and } B \in \mathcal{F}.
\]

If the system satisfies the additional property

iii) Ergodicity:

\[
\tau_k B = B \quad \forall k \in \mathbb{Z}^d \Rightarrow B \in \{\emptyset, \Omega\}
\]

then the system will be called ergodic.

To any measurable function \( f \) in \( (\Omega, \mathcal{F}, P) \) we will associate for every \( k \in \mathbb{Z}^d \) the function \( \tau_k f = f \circ \tau_k \) which from the invariance property has the same distribution as \( f \). Therefore, the dynamical system will induce a \( d \)-parameter group of isometries on \( L^p(\Omega) \) for \( 1 \leq p \leq \infty \).

**Definition 4.** A set \( B \in \mathcal{F} \) is called \( \tau \)-invariant if \( P(\tau_k B \Delta B) = 0 \) for every \( k \in \mathbb{Z}^d \), where \( \Delta \) denotes symmetric difference. A measurable function \( f \) is called \( \tau \)-invariant if for every \( k \in \mathbb{Z}^d \) \( \tau_k f = f \) almost surely \( \omega \in \Omega \).

It can be easily shown that the \( \tau \)-invariant sets form a sub \( \sigma \)-algebra of \( \mathcal{F} \), denoted by \( \mathcal{I} \), and a function is \( \tau \)-invariant if and only if is measurable with respect to \( \mathcal{I} \). Then, the ergodicity property is equivalent to any of the following:

\[
B \text{ } \tau \text{-invariant } \Rightarrow P(B) \in \{0, 1\},
\]

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Given $\tau$ a $d$-dimensional dynamical system we will associate to any function $f$ from $L^1(\Omega)$ an additive process on finite subsets $F$ of $\mathbb{Z}^d$ with values in $L^1(\Omega)$

$$S(F, f)(\omega) = \sum_{k \in F} \tau_k f(\omega),$$

and for every $F$ we will denote by $A(F, f)$ the following average of $f$ over $F$

$$A(F, f)(\omega) = \frac{1}{|F|} S(F, f)(\omega),$$

where by $|F|$ we denote the cardinal of the set $F$. We recall the following results from Ergodic Theory ([54], Ch. 6 or [55] Ch. 8):

Let $1 \leq p < \infty$ and $f$ from $L^p(\Omega)$. For $u, v \in \mathbb{Z}^d$ we say that $u \leq v$ if $u_i \leq v_i$ for every $i \in \{1, 2, \ldots, d\}$ and $u < v$ if $u_i < v_i$ for every $i \in \{1, 2, \ldots, d\}$. Assume $u < v$ and let $nQ = [nu, nv] = \{k \in \mathbb{Z}^d / nu \leq k \leq nv\}$ for every $n$ positive integer.

**Theorem 15. (Mean Ergodic Theorem)** The sequence $A(nQ, f)$ will converge strongly in $L^p(\Omega)$ to $E(f|I)$, the conditional expectation of $f$ with respect to the $\sigma$-algebra $I$. As a consequence, if $\tau$ is ergodic, then $E(f|I) = \int_{\Omega} f dP$ almost surely $\omega \in \Omega$.

**Theorem 16. (Pointwise Ergodic Theorem)** The sequence $A(nQ, f)$ will converge pointwise to $E(f|I)$ almost surely $\omega \in \Omega$. If $\tau$ is ergodic, then $A(nQ, f)$ converges pointwise to $\int_{\Omega} f dP$ almost surely $\omega \in \Omega$.

We will use these results in a more convenient way for our problem in the following way: Assume $f \in L^\infty(\Omega)$. For any positive $\epsilon$, by $\epsilon \mathbb{Z}^d$ we mean the set of points in $\mathbb{R}^d$ of the form $\epsilon k$ where $k \in \mathbb{Z}^d$.

**Theorem 17.** There exists a set of full probability on which the sequence $A(\epsilon \mathbb{Z}^d \cap U, f)$ will converge when $\epsilon$ goes to 0 pointwise to $E(f|I)$, for every bounded open set $U$ in $\mathbb{R}^d$ with Lipschitz boundary. We have also strong convergence in $L^p(\Omega)$ for $1 \leq p < \infty$.

**Proof.** An elementary computation gives that for any two finite sets $F_1$ and $F_2$, we
have
\[
A(F_2, f)(\omega) - A(F_1, f)(\omega) \leq \frac{S(F_1, f)(\omega)}{|F_2|} + \frac{S(F_1 \Delta F_2, |f|)(\omega)}{|F_2|} - \frac{S(F_1, f)(\omega)}{|F_1|}
\]
\[
\leq \frac{|F_1 \Delta F_2|}{|F_2|} A(F_1 \Delta F_2, |f|)(\omega) + \frac{|F_1 \Delta F_2|}{|F_2|} A(F_1, f)(\omega)
\]
\[
\leq \frac{|F_1 \Delta F_2|}{|F_2|} (A(F_1 \Delta F_2, |f|)(\omega) + A(F_1, |f|)(\omega)),
\]
so we obtain
\[
|A(F_2, f)(\omega) - A(F_1, f)(\omega)| \leq \frac{|F_1 \Delta F_2|}{|F_2|} (A(F_1 \Delta F_2, |f|)(\omega) + A(F_1, |f|)(\omega)). \quad (4.4)
\]
We apply this estimate for the sets \(\frac{1}{\epsilon} Q \cap \mathbb{Z}^d\) and \([\frac{1}{\epsilon}] Q \cap \mathbb{Z}^d\), where \(Q\) is a cube with integer coordinates ([\(a\)] denotes the integer part of \(a\)).

\[
\limsup_{\epsilon \to 0} |A(\frac{1}{\epsilon} Q \cap \mathbb{Z}^d, f)(\omega) - A([\frac{1}{\epsilon}] Q, f)(\omega)| \leq C \limsup_{\epsilon \to 0} \frac{[\frac{1}{\epsilon} + 1]^d - [\frac{1}{\epsilon}]^d}{\frac{1}{\epsilon}^d} = 0.
\]

So there exists a set of full probability \(\Omega'\) such that the limit holds for any cube with rational coordinates and as a consequence for any disjoint finite union of such cubes. For any \(S\) a nonempty set in \(\mathbb{R}^d\) the distance function to \(S\), \(d_S : \mathbb{R}^d \to [0, +\infty)\) is defined by

\[
d_S(x) = \inf_{s \in S} ||x - s||,
\]
and let \(\tilde{d}_S : \mathbb{R}^d \to (-\infty, +\infty)\) be defined by

\[
\tilde{d}_S(x) = d_S(x) - d_{\mathbb{R}^d \setminus S}(x).
\]
If \(U\) is a bounded open set with Lipschitz boundary we notice that for every small positive \(\epsilon\) there exists a constant \(C\) such that \(L^d(|\tilde{d}_U| < \epsilon) \leq C \epsilon H^{d-1}(\partial U)\) and \(L^d(\tilde{d}_U < -C \epsilon) < \epsilon \mathbb{Z}^d \cap U|\epsilon^d < L^d(\tilde{d}_U < C \epsilon)\). Let \(V_n\) be a sequence of sets that can be written as finite unions of cubes with rational coefficients and such that \(\{\tilde{d}_U < -\frac{1}{n}\} \subset V_n \subset U\). Then using (4.4)

\[
\limsup_{\epsilon \to 0} |A(U \cap \epsilon \mathbb{Z}^d, f)(\omega) - A(V_n \cap \epsilon \mathbb{Z}^d, f)(\omega)| \leq \frac{C \epsilon H^{d-1}(\partial U)}{n L^d(U)},
\]
so because the limit holds on \(\Omega'\) for all \(V_n\) and taking the limit in \(n\)

\[
\limsup_{\epsilon \to 0} |A(U \cap \epsilon \mathbb{Z}^d, f)(\omega) - E(f)(\omega)| = 0 \quad \forall \omega \in \Omega'.
\]
Strong convergence in $L^p(\Omega)$ for $1 \leq p < \infty$ follows from the Dominated Convergence Theorem.

In theorem (17) we showed the convergence of the averages $A(\epsilon \mathbb{Z}^d \cap U, f)$ to $E(f|\mathcal{I})(\omega)$ or essentially equivalent of $\sum_{\epsilon \in U} \epsilon^d f(\tau_k \omega)$ to $E(f|\mathcal{I})(\omega)\mathcal{L}^d(U)$ on a set of full probability, for every $U$ bounded, open with Lipschitz boundary. The following theorem represents an improvement, in the sense that we can add different weights when we compute the averages or we can use a different measure instead of $\mathcal{L}^d$. This theorem will be needed in proving the convergence of the main result.

**Theorem 18.** Assume $f \in L^\infty(\Omega)$.

i) There exists a set of full probability such that the sequence

$$\sum_{\epsilon \in U} \epsilon^d f(\tau_k \omega)u(\epsilon k)$$

converges pointwise to

$$E(f|\mathcal{I})(\omega) \int_U u(x)d\mathcal{L}^d$$

for any $U$ open, bounded with Lipschitz boundary and $u \in C(\overline{U})$, the space of continuous functions on the closure of $U$.

ii) There exists a set of full probability such that the sequence

$$\sum_{\epsilon \in U} f(\tau_k \omega)\int_{\epsilon + [0,1]^d} u(x)d\mathcal{L}^d$$

converges pointwise to

$$E(f|\mathcal{I})(\omega) \int_U u(x)d\mathcal{L}^d$$

for any $U$ open, bounded with Lipschitz boundary and $u \in L^1(U)$.

**Proof.** i) Take $\Omega'$ from the previous theorem and $\omega \in \Omega'$. For any $V$ open, bounded with Lipschitz boundary and $u \in C(\overline{V})$ we obviously have

$$\limsup_{\epsilon \to 0} \left| \sum_{\epsilon \in V} \epsilon^d f(\tau_k \omega)u(\epsilon k) - E(f|\mathcal{I})(\omega) \int_V u(x)d\mathcal{L}^d \right| \leq \|f\|_{L^\infty} (\sup_V u - \inf_V u)\mathcal{L}^d(V).$$

Let $U$ and $u$ be as in the assumptions of the theorem and let us partition $U$ into a finite number of sets with $(V_i)_{i \in F}$ with Lipschitz boundary. We get

$$\limsup_{\epsilon \to 0} \sum_{\epsilon \in U} \epsilon^d f(\tau_k \omega)u(\epsilon k) - E(f|\mathcal{I})(\omega) \int_U u(x)d\mathcal{L}^d \leq \|f\|_{L^\infty} \sum_{i \in F} (\sup_{V_i} u - \inf_{V_i})\mathcal{L}^d(V_i).$$
If we use a partition with the diameters of the elements converging uniformly to 0, based of the uniform continuity of \( u \) we obtain that
\[
\lim_{\epsilon \to 0} \sum_{\epsilon k \in U} \epsilon^d f(\tau_k \omega) u(\epsilon k) = E(f|\mathcal{I})(\omega) \int_U u(x) d\mathcal{L}^d \quad \forall \omega \in \Omega'.
\]

ii) The proof works in the same way as in part i) for \( u \in C(U) \) using instead of the weights \( u(\epsilon k) \),
\[
\int_{\epsilon(k+[0,1]^d)} u(x) d\mathcal{L}^d
\]
and then by approximation extends to \( L^1(U) \).

We will give now the definition for the measurability assumption we will use:

**Definition 5.** A set valued function \( A \) defined on a probability space \( (\Omega, \mathcal{F}, P) \), taking values into a metric space \( X \) is called measurable if the inverse image of any open set is measurable, where the inverse image of a set \( C \) is defined by \( A^{-1}(C) = \{ \omega \in \Omega: A(\omega) \cap C \neq \emptyset \} \).

\( A \) is called closed (compact) valued if \( A(\omega) \) is closed (compact) for every \( \omega \in \Omega \). If the space \( X \) is a Banach space, then \( A \) is called weakly measurable if for every continuous linear functional \( x' \in X' \) the set valued function \( x' \circ A \) defined on \( (\Omega, \mathcal{F}, P) \) with values in \( \mathbb{R} \) is measurable.

It is well known that in general the measurability property is different if we require inverse images of open sets to be measurable or inverse images of closed sets, unlike in the case of a random variable ([56]). In our case, because \( X \) will be separable, locally compact, \( (\Omega, \mathcal{F}, P) \) is complete and \( A \) is compact valued these properties will be equivalent.

Assumptions:
The sets \( \Gamma_k^\epsilon(\omega) \) are of the form \( \Gamma_k^\epsilon(\omega) = \epsilon k + r_\epsilon A(\tau_k \omega) \) where:

i) \( A: \Omega \to Y' \) is a set valued mapping, measurable and compact valued. Also almost surely \( \omega \in \Omega \), \( A(\omega) \) is the closure of a connected open set with Lipschitz boundary that contains a ball of fixed radius \( \delta > 0 \).

ii) \( \tau \) is a \( n-1 \) discrete dynamical system, measure-preserving and ergodic.

iii) \( (r_\epsilon)_{\epsilon > 0} \) is a sequence satisfying \( r_\epsilon \leq \epsilon \) for every positive \( \epsilon \).

This means that the walls are distributed in every periodic cell \( \epsilon k + \epsilon Y' \), within a distance \( r_\epsilon \) from the center, they have random shape and measure, but satisfy certain measurability conditions with respect to the \( \omega \) (that will ensure for example that the measure \( \omega \mapsto \mathcal{L}^{n-1}(\Gamma_k^\epsilon(\omega)) \) is a random variable, as well as the capacity). We obtain easily the periodic case if we eliminate the dependence on \( \omega \). So the randomness applies here only to the geometry of the walls and their positions inside the cells, but not to their distribution in space which is still assumed to be periodic. This will be
important in finding a limit problem in third case when $r_{\epsilon}^{n-2} \approx \epsilon^{n-1}$, where relevant will be suitable defined vector valued set functionals.

The fact that under our assumptions the measure of $A(\omega)$ is a random variable is well known, but we give a simple proof that will be also used later to prove measurability for other functions that will appear:

**Lemma 5.** The function $(x, \omega) \mapsto d(x, A(\omega))$, defined on $Y' \times \Omega$ is measurable with respect to the product sigma algebra $\mathcal{B}(Y') \times \mathcal{F}$. As a consequence, the function $\omega \mapsto L^{n-1}(A(\omega))$ is in $L^\infty(\Omega)$.

**Proof.** Let $(y_i)_{i \in \mathbb{N}}$ be a countable dense subset of $Y'$. For any $\epsilon > 0$ consider the countable family of sets $(D_j)_{j \in \mathbb{N}}$ that can be written as a finite union of balls of radius $\epsilon$ and centered in some $y_i$.

Let $A_i$ be the set of $y \in Y'$ such that $i$ is the first element for which $d(y, y_i) < \epsilon$ and $B_j$ the set of $\omega \in \Omega$ such that $j$ is the first element for which $A(\omega) \subseteq D_j$. Because of the compactness of $A(\omega)$ and the measurability of $A$, $(A_i \times B_j)_{i,j \in \mathbb{N}}$ is a measurable partition of $Y' \times \Omega$.

The function that takes the value $d(x, B_j)$ on $(A_i \times B_j)$ is clearly measurable and is a $2\epsilon$ approximation of $(x, \omega) \mapsto d(x, A(\omega))$, hence $(x, \omega) \mapsto d(x, A(\omega))$ is measurable.

From here we get that the set \{$(x, \omega) / x \in A(\omega)$\} is measurable as the level set of a measurable function, so $\omega \mapsto L^{n-1}(A(\omega))$ is measurable and being bounded by 1, is in $L^p(\Omega)$ for every $1 \leq p \leq \infty$.

**Lemma 6.** The function $\omega \mapsto \text{Cap}(A(\omega))$ is measurable.

**Proof.** All we have to show is that the level sets \{$(x, \omega) / \text{Cap}(A(\omega)) < t$\} are measurable for every $t \in \mathbb{R}$. The space $Y'$ being separable, there exists a countable family of sets $(D_i)_{i \geq 0}$, such that if $K$ is a compact set and $D$ is an open set with $K \subseteq D$, there exists an element of the family $D_j$ such that $K \subseteq D_j \subseteq D$.

Then, \{$(x, \omega) / \text{Cap}(A(\omega)) < t$\} = \{$(x, \omega) / \exists D_i$ with $\text{Cap}(D_i) < t$ and $A(\omega) \subseteq D_i$\} which is a measurable set by the measurability of $A$. \qed

Under these assumptions, we will show that the solution {$u'(\omega), p'(\omega)$} will converge almost surely $\omega \in \Omega$ to a solution of a deterministic system (as a consequence of the ergodicity assumption), the topology of convergence and the limit depending on the rate of convergence of $r_{\epsilon}$ to 0.

We will use $M-$ convergence so we will define the random convex functionals that will appear in the problem. For any $\omega \in \Omega$ let $F'(\omega) : H^1(D) \rightarrow \mathbb{R} \cup \{\infty\}$,

$$F'(\omega)(u) = \frac{1}{2}a(u,u) + j'(\omega)(u) - \langle f, u \rangle_D + I_{K(\omega) \cap V}(u)$$
From (4.2), \( u^\epsilon(\omega) \) is the minimizer for the functional \( F^\epsilon(\omega) \) so we have to find another functional \( F \) such that the sequence \( F^\epsilon(\omega) \) will converge in the Mosco sense to \( F \) almost surely \( \omega \in \Omega \) and be able to characterize the minimizer for \( F \). As in the previous chapter, the limit functional will depend on the rate of convergence to 0 of \( r_\epsilon \), so we will have four cases also.

If we consider the functionals \( F^\epsilon \) defined on \( \Omega \times H^1(D) \), \( F^\epsilon(\omega,u) = F^\epsilon(\omega)(u) \), we know that \( F^\epsilon \) is convex and lower semicontinuous with respect to \( u \) and we expect from the assumptions we made to be measurable with respect to \( \omega \). We will show that in fact \( F^\epsilon \) is jointly measurable.

**Theorem 19.** Let \( \mathcal{B}(H^1(D)) \) be the \( \sigma \)–algebra generated by the open sets in \( H^1(D) \). Then \( F^\epsilon : \Omega \times H^1(D) \to \mathbb{R} \cup \{+\infty\} \) is measurable with respect to the product \( \sigma \)–algebra \( \mathcal{F} \times \mathcal{B}(H^1(D)) \).

**Proof.** Obviously \( \omega \sim \Gamma^\epsilon(\omega) \) is measurable as a finite union of measurable mappings. Define the mapping 

\[
\omega \sim G^\epsilon(\omega) = \{ u \in H^1_b(D) \mid u_i = 0 \text{ on a neighbourhood of } \Gamma^\epsilon(\omega) \forall i < n \},
\]

and we get that for any \( u \in H^1(D) \), the set \( \{ \omega \in \Omega \mid u \in G^\epsilon(\omega) \} \) is measurable. Separability of \( H^1(D) \) implies that for any \( t > 0 \) the set \( \{ \omega \in H^1(D) \mid \omega/d(u,G^\epsilon(\omega)) < t \} \) is measurable, so \( G^\epsilon \) is measurable and by definition, the closure of \( G^\epsilon \) is also measurable. Now we notice that \( K^\epsilon(\omega) = G^\epsilon(\omega) \). \( V \) being closed implies that \( \omega \sim K^\epsilon(\omega) \cap V \) is a measurable mapping. By Lemma 5 we obtain that \( I_{K^\epsilon(\omega) \cap V}(u) \) is jointly measurable. Now we only have to take care of \( j^\epsilon(\omega)(u) \).

But again from Lemma 5 we obtain that \( \chi_{\Gamma^\epsilon(\omega)}(x) \) is jointly measurable and then by Fubini Theorem \( \omega \to \int \Sigma g|u_n|\chi_{\Gamma^\epsilon(\omega)}d\mathcal{L}^{n-1} \) is measurable for any fixed \( u \). It follows that \( (\omega,u) \to \int \Sigma g|u_n|\chi_{\Gamma^\epsilon(\omega)}d\mathcal{L}^{n-1} \) is a Carathéodory function, being obviously continuous with respect to \( u \), and because \( H^1(D) \) is a separable metric space will be jointly measurable. \( \square \)

**Remark 3.** We showed that \( F^\epsilon(\omega,u) \) is a convex normal integrand in the sense of Rockafeller, as defined in ([37]), which is equivalent with the measurability of the closed convex valued mapping \( \omega \sim epiF^\epsilon(\omega,\cdot) \) ([57]). It follows also that \( (F^\epsilon)^*(\omega,u) \) is a convex normal integrand, where \( (F^\epsilon)^*(\omega,\cdot) \) is the conjugate of \( F^\epsilon(\omega,\cdot) \).

As a consequence, it follows then easily that the set

\[
\{(\omega,u,v) \in \Omega \times H^1(D) \times H^1(D) \mid F^\epsilon(\omega,u) + (F^\epsilon)^*(\omega,v) = \langle u,v \rangle_{H^1(D)}\}
\]
is measurable, so \{ (\omega, u) \in \Omega \times H^1(D) / 0 \notin \partial F(\omega, u) \} is measurable which, following Pettis’ theorem, implies that \( u^\epsilon : \Omega \to H^1(D) \) is Bochner measurable. This will allow us to integrate \( u^\epsilon(\omega) \) over measurable sets from \( \Omega \).
4.2 Homogenization Results

4.2.1 Case I: Concentrated Obstacles

Assume that \( \lim_{\epsilon \to 0} \frac{r_\epsilon}{\epsilon} = \alpha > 0 \). We will introduce the unique solution to the problem

\[
\begin{cases}
-\Delta u + \nabla p = f & \text{in } D \setminus \Sigma, \\
\nabla \cdot u = 0 & \text{in } D \setminus \Sigma, \\
u = b & \text{on } \partial D, \\
u_i = 0 & \text{on } \Sigma \text{ for all } i < n, \\
-[\sigma(u,p)e_n] \cdot e_n \in g \int_\Omega \mathcal{L}^{n-1}(A(\omega))dP \cdot \partial |u_n| & \text{on } \Sigma.
\end{cases}
\]

(4.5)

This is a problem of type (2.2) with the threshold given by the positive function \( g \int_\Omega \mathcal{L}^{n-1}(A(\omega))dP \). **Theorem 2** gives us existence of a solution \( \{u, p\} \) with \( u \in K \cap V \), \( p \in L^2(D)/\mathbb{R} \). We have also uniqueness for \( u \), and for \( p \) only if \( u \) is not identically 0 on \( \Sigma \).

The energy functional associated to the problem (4.5) will be \( F : H^1(D) \to \mathbb{R} \cup \{\infty\} \)

\[
F(u) = \frac{1}{2} a(u,u) + j(u) - \langle f, u \rangle_D + I_{K\cap V},
\]

where

\[
K = \{u \in H^1_b(D)/u_r = 0 \text{ on } \Sigma\}
\]

and

\[
j(u) = \int_\Omega \mathcal{L}^{n-1}(A(\omega))dP \int_\Sigma g|u_n|d\mathcal{L}^{n-1}.
\]

We will prove that this problem is the limit problem for \( \{u^\epsilon(\omega), p^\epsilon(\omega)\} \) in the following sense:

**Theorem 20.** Almost surely \( \omega \in \Omega \), \( u^\epsilon(\omega) \) converges strongly to \( u \) in \( H^1(D) \) and \( p^\epsilon(\omega) \) converges strongly in \( L^2(D)/\mathbb{R} \) to \( p \), where \( \{u, p\} \) is a solution for the problem (4.5).

We will prove \( M \)-convergence of the sequence of functionals \( F^\epsilon(\omega) \) to \( F \), almost surely \( \omega \in \Omega \), which will imply as a consequence the convergence of the minimizers based on the equicoercivity of the functionals \( F^\epsilon(\omega) \).

**Theorem 21.** The sequence of functionals \( F^\epsilon(\omega) \) \( M \)-converges to \( F \) almost surely \( \omega \in \Omega \).
Proof. (i) $M - \limsup_{\epsilon \to 0} F^\epsilon(\omega)(u) \leq F(u)$ a.s. $\omega \in \Omega$.

For the first part of the proof we need to show that a.s. $\omega \in \Omega$ and for every $u$ satisfying $F(u) < +\infty$ there exists a sequence $u^\epsilon$ strongly convergent to $u$ in $H^1(D)$ such that $F^\epsilon(\omega)(u^\epsilon)$ converges to $F(u)$. We will take a constant sequence $u^\epsilon = u$ and because $K \cap V \subset K'(\omega) \cap V$ we only need to show that $j^\epsilon(\omega)(u)$ converges to $j(u)$ almost surely. This will follow from the weak* convergence of $\chi_{\Gamma^\epsilon(\omega)}$ to $\int_\Omega L_{n-1}(A(\omega)) dP$ in $L^\infty(\Sigma)$, and because the functions $\chi_{\Gamma^\epsilon(\omega)}$ are uniformly bounded in $L^\infty(\Omega)$ it is sufficient to prove that almost surely $\omega \in \Omega$

$$\lim_{\epsilon \to 0} \int_\Sigma \chi_{\Gamma^\epsilon(\omega)}(u) dL^{n-1} = \int_\Omega L^{n-1}(A(\omega)) dP \int_\Sigma udL^{n-1}$$

for every $v \in C_0(\Sigma)$. But from Lemma 5 and Theorem 18

$$\lim_{\epsilon \to 0} \sum_{\epsilon k \in \Sigma} \epsilon^{-n} L^{n-1}(A(\tau_k \omega)) v(\epsilon k) = \int_\Omega L^{n-1}(A(\omega)) dP \int_\Sigma vdL^{n-1}$$

on a set of full probability and the rest follows based on the continuity of $v$ and compactness of its support.

(ii) $M - \liminf_{\epsilon \to 0} F^\epsilon(\omega)(u) \geq F(u)$ a.s. $\omega \in \Omega$.

For the second part we will show that almost surely $\omega \in \Omega$ and for every sequence $u^\epsilon$ convergent to $u$ in the weak topology of $H^1(D)$ we have that

$$\liminf_{\epsilon \to 0} F^\epsilon(\omega)(u^\epsilon) \geq F(u)$$

The bilinear functional $a$ is convex and continuous in the strong topology so

$$a(u, u) \leq \liminf_{\epsilon \to 0} a(u^\epsilon, u^\epsilon).$$

The sequence $u^\epsilon_i$ will converge strongly to $u_i$ in $L^2(\Sigma)$ for all $1 \leq i \leq n$ ([49]) so if $\omega$ is such that $\chi_{\Gamma^\epsilon(\omega)}$ converges weakly* in $L^\infty(\Sigma)$ to $\int_\Omega L^{n-1}(A(\omega)) dP$, then

$$\lim_{\epsilon \to 0} j^\epsilon(\omega)(u^\epsilon) = \lim_{\epsilon \to 0} \int_\Sigma g|u^\epsilon| \chi_{\Gamma^\epsilon(\omega)} dL^{n-1} = \int_\Omega L^{n-1}(A(\omega)) dP \int_\Sigma g|u| dL^{n-1} = j(u),$$

and for all $i < n$

$$0 = \lim_{\epsilon \to 0} \int_\Sigma |u^\epsilon_i| \chi_{\Gamma^\epsilon(\omega)} dL^{n-1} = \int_\Omega L^{n-1}(A(\omega)) dP \int_\Sigma |u_i| dL^{n-1}.$$
It follows that $u_i \equiv 0$ on $\Sigma$ because $\mathcal{L}^{n-1}(A(\omega))$ is bounded from below by a constant. We obviously have that $\nabla \cdot u = 0$ in $D$ so $u \in K \cap V$ and

$$\liminf_{\epsilon \to 0} F^\epsilon(\omega)(u^\epsilon) \geq F(u).$$

Theorem 20 follows now as a consequence, similarly as in the periodic case, showing first the weak convergence of $u^\epsilon(\omega)$ almost surely and then also the convergence of the norms.
4.2.2 Case II: Dilute Obstacles

Assume that \( \lim_{\epsilon \to 0} \frac{r}{\epsilon} = 0 \) and \( \lim_{\epsilon \to 0} \frac{r^{n-2}}{\epsilon^{n-1}} = +\infty \). The limit problem will be in this case the same as in the periodic case

\[
\begin{aligned}
- \Delta u + \nabla p &= f \quad \text{in } D \setminus \Sigma, \\
\nabla \cdot u &= 0 \quad \text{in } D \setminus \Sigma, \\
u &= b \quad \text{on } \partial D, \\
u_i &= 0 \quad \text{on } \Sigma \quad \text{for all } i < n, \\
- [\sigma(u, p) e_n] \cdot e_n &= 0 \quad \text{on } \Sigma.
\end{aligned}
\] (4.6)

According to Chapter 2 we introduce the functional \( F : L^2(D) \to \mathbb{R} \cup \{ \infty \} \)

\[
F(u) = \frac{1}{2} a(u, u) + I_{K \cap V} - \langle f, u \rangle_D.
\]

**Theorem 22.** The sequence of functionals \( F^\epsilon(\omega) \) \( \epsilon \)-converges to \( F \) almost surely \( \omega \in \Omega \).

**Proof.** The \( \lim \sup \) inequality follows from Theorem 21 and comparison arguments. We obtain that for any positive number \( t \)

\[
M - \limsup_{\epsilon \to 0} F^\epsilon(u) \leq F(u) + \int_{\Omega} L^{n-1}(A(\omega)) dP \int_{\Sigma} t g |u_n| d\mathcal{L}^{n-1}
\]
on a set of full measure and then we make \( t \to 0 \).

The \( \lim \inf \) inequality can be reduced to

\[
M - \liminf_{\epsilon \to 0} I_{K^\epsilon(\omega) \cap V} \geq I_{K \cap V}
\]
almost surely or equivalently

\[
M - \limsup_{\epsilon \to 0} K^\epsilon(\omega) \cap V \subseteq K \cap V \text{ a.s. } \omega \in \Omega.
\]

Fix \( \omega \) such that \( A(\tau_k \omega) \) contains a ball of radius \( \delta \) for each \( k \in \mathbb{Z}^{n-1} \). From our assumptions this set has full measure. Let \( u^\epsilon \) converge weakly to \( u \) in \( H^1(D) \). For such a sequence, it follows as in Theorem 7 that

\[
\int_{\Sigma} (u^\epsilon_i)^2 d\mathcal{L}^{n-1} \leq C \left( \frac{\epsilon^{n-1}}{C_{\delta} \epsilon^{n-2}} + \epsilon \right)
\]
where \( C_{\delta} \) represents the capacity of the \( n-1 \) dimensional ball of radius \( \delta \) in \( \mathbb{R}^n \) and \( C \) is a constant that depends on the sequence \( u^\epsilon \). So \( \lim_{\epsilon \to 0} \int_{\Sigma} (u^\epsilon_i)^2(x') d\mathcal{L}^{n-1} = 0 \) which implies that \( u_i = 0 \) on \( \Sigma \). \( V \) being closed with respect to the weak topology of \( H^1(D) \) the claim follows. \( \square \)
The convergence of solutions follows from here. We just state the theorem, the proof being similar with the periodic case:

**Theorem 23.** Almost surely \( \omega \in \Omega \), \( u^\omega \) converges strongly to \( u \) in \( H^1(D) \) and \( p^\omega \) converges strongly in \( L^2(D)/\mathbb{R} \) to \( p \), where \( \{u, p\} \) is the unique solution of the problem (4.6).
4.2.3 Case III: Critical Case

Assume that \( \lim_{\epsilon \to 0} \frac{r_{n-2}}{\epsilon^{n-1}} = \beta \in (0, \infty) \). In this case the limit problem will be the following

\[
\begin{cases}
-\Delta u + \nabla p = f & \text{in } D \setminus \Sigma, \\
\nabla \cdot u = 0 & \text{in } D \setminus \Sigma, \\
u = b & \text{on } \partial D, \\
-\left[\sigma(u, p)\epsilon_n\right] = C u & \text{on } \Sigma,
\end{cases}
\]

where \( C \) is a symmetric positive definite matrix that we will define next.

A weak solution for this problem is already defined in Definition 2 from the periodic case and we have also proved there the existence and uniqueness. We remind that the solution \( u \) is the minimizer for

\[
\min_{v \in H^1_b(D) \cap V} \left\{ \frac{1}{2} a(v, v) + \frac{\beta}{2} c(v, v) - \langle f, v \rangle_D \right\},
\]

where \( c \) is the bilinear form, positive definite and continuous, defined on the space \( H^1(D) \) by

\[
c(u, v) = \int_{\Sigma} C u \cdot v d\mathcal{L}^{n-1}.
\]

In order to define the matrix \( C \) we will introduce the solutions \( \{\chi^R_v(S), \eta^R_v(S)\} \) for the following Stokes problems in \( B_R \), the ball of radius \( R \) and centered in 0. This will be done here only for a class \( A \) of admissible sets \( S \) which are the closure of an open set from \( \mathbb{R}^{n-1} \) included in \( Y' \) with Lipschitz boundary, for \( R > R_0 \), where \( R_0 \) is such that \( Y' \subset \subset B_{R_0} \) and for any vector \( v \) from \( \mathbb{R}^n \).

\[
\begin{cases}
-\Delta \chi^R_v(S) + \nabla \eta^R_v(S) = 0 & \text{in } B_R \setminus S \\
\nabla \cdot \chi^R_v(S) = 0 & \text{in } B_R \setminus S \\
\chi^R_v(S) = v & \text{on } S \\
\chi^R_v(S) = 0 & \text{on } \partial B_R
\end{cases}
\]

(4.8)

Also, \( \{\chi_v(S), \eta_v(S)\} \) will represent the solutions for a similar system in free space

\[
\begin{cases}
-\Delta \chi_v(S) + \nabla \eta_v(S) = 0 & \text{in } \mathbb{R}^n \setminus S, \\
\nabla \cdot \chi_v(S) = 0 & \text{in } \mathbb{R}^n \setminus S \\
\chi_v(S) = v & \text{on } S, \\
\chi_v(S) \to 0 & \text{at } \infty.
\end{cases}
\]

(4.9)

Similar problems were studied in the previous chapter but for a fixed set \( \Gamma \) (see 3.14, 3.13), where also we defined the matrices \( (C_{ij})_{1 \leq i,j \leq n-1} \) and \( (C^R_{ij})_{1 \leq i,j \leq n-1} \) associated
to those systems. We intend in the following to define in a similar manner $C_v(S)$, for a fixed vector $v$ and for any admissible set $S$. Even if we restrict the definition to the class $A$, this can be easily extended to any subset of $\mathbb{R}^n$ and we get for any fixed vector $v$ a set functional which will be in fact a Choquet capacity (see [39] for the definition).

We will remind from the previous chapter the properties of $\chi_v(S)$ and $\chi^R_v(S)$ that follow from the definition:

$\chi^R_v(S)$ minimizes the bilinear form $a_{B_R}$ over the closed convex set

$$K^R_v(S) = \{ w \in H^1_0(B_R)/w = v \text{ on } S, \nabla \cdot w = 0 \text{ in } B_R \},$$

where $a_{B_R}(u, v) = 2(e(u), e(v))_{B_R}$ for every $u, v \in H^1(B_R)$.

The Hilbert space $X$ was defined as the closure of the divergence free vector fields from $C_0^\infty(\mathbb{R}^n)$ with the inner product given by $a_{\mathbb{R}^n}(u, v) = 2(e(u), e(v))_{\mathbb{R}^n}$ and $\chi_v(S)$ minimizes the bilinear form $a_{\mathbb{R}^n}$ over the closed convex set

$$K_v(S) = \{ w \in X/w = v \text{ on } S, \nabla \cdot w = 0 \text{ in } \mathbb{R}^n \}.$$

To any such admissible set $S \subset Y^\prime$, $v \in \mathbb{R}^n$ we will associate the following set functional that will play the role of the capacity

$$C_v(S) = a_{\mathbb{R}^n}(\chi_v(S), \chi_v(S))$$

and the relative ones to $B_R$ with $R > R_0$

$$C^R_v(S) = a_{B_R}(\chi^R_v(S), \chi^R_v(S)).$$

From the variational characterizations for $\chi^R_v(S)$ and $\chi_v(S)$ we have the following obvious inequalities:

$$S_1 \subset S_2 \Rightarrow C^R_v(S_1) \leq C^R_v(S_2), \quad C_v(S_1) \leq C_v(S_2), \quad (4.10)$$

$$R_1 \leq R_2 \Rightarrow C_v(S) \leq C^R_v(S) \leq C^R_v(S). \quad (4.11)$$

It can be shown easily that $C^R_v(S)$ will converge to $C_v(S)$ using the same arguments as in the previous chapter, but we want to show that the convergence in uniform for all $S$ admissible and $v$ in bounded sets so we will try a different approach. The idea is that the behaviour at infinity of the fundamental solution for the Stokes system is similar to the one of the fundamental solution for the Laplace equation and we expect that $\chi_v(S)$ to follow the same behaviour. We will show next that the uniform convergence will be a consequence of a pointwise uniform decrease to 0 of $\nabla \chi_v(S)$.
which follows from the fact that \( \{ \chi_v(S), \eta_v(S) \} \) satisfy Stokes systems in \( \mathbb{R}^n \) with 
\[-\Delta \chi_v(S) + \nabla \eta_v(S) \]
being distributions from \( H^{-1}(\mathbb{R}^n) \) which are compactly supported in \( B_{R_0} \) and have uniformly bounded norms with respect to \( S \in \mathcal{A} \) and \( v \) in bounded sets.

**Lemma 7.** \( \chi_v(S) \) and \( \eta_v(S) \) satisfy the following pointwise estimates for \( |x| > R_0 \):

\[
\begin{align*}
|D^\alpha \chi_v(S)(x)| &\leq \frac{C|v|}{|x|^{n-2+|\alpha|}} & |\alpha| &\leq 2, \\
|D^\alpha \eta_v(S)(x)| &\leq \frac{C|v|}{|x|^{n-1+|\alpha|}} & |\alpha| &\leq 1,
\end{align*}
\]

(4.12)

where the constant \( C \) is independent of \( v \) and \( S \).

**Proof.** \( v \to ||\nabla \chi_v(S)||_{L^2(\mathbb{R}^n)} \) is a norm on \( \mathbb{R}^n \) for every \( S \). This together with (4.10) gives the existence of a constant \( C \) such that \( ||\nabla \chi_v(S)||_{L^2(\mathbb{R}^n)} \leq C|v| \) for every \( S \in \mathcal{A} \).

The idea is to transform the system 4.9 into a system in the free space and then make use of the regularity theorem and the fundamental solution of the Stokes system to estimate \( \chi_v(S) \) pointwise.

Let \( \phi \in C_0^\infty(\mathbb{R}^n) \), \( \phi = 0 \) on a neighbourhood of \( Y' \), \( \phi = 1 \) outside \( B_{R_0} \). Then, after elementary calculations, \( \phi \chi_v(S) \) and \( \phi \eta_v(S) \) will be the solution for the following Stokes system in \( \mathbb{R}^n \)

\[
\begin{align*}
-\Delta (\phi \chi_v(S)) + \nabla (\phi \eta_v(S)) &= f \quad \text{in } \mathbb{R}^n, \\
\nabla \cdot (\phi \chi_v(S)) &= g \quad \text{in } \mathbb{R}^n,
\end{align*}
\]

(4.13)

where \( f \in L^2(\mathbb{R}^n) \), \( g \in H^1(\mathbb{R}^n) \) are compactly supported in \( B_{R_0} \) and are given by

\[
\begin{align*}
f &= -\Delta \phi \cdot \chi_v(S) + 2\nabla \phi \cdot \nabla \chi_v(S) + \nabla \phi \cdot \eta_v(S) \\
g &= (\nabla \cdot \phi) \cdot \chi_v(S).
\end{align*}
\]

So there exists a constant \( C \) such that

\[
||f||_{L^2(\mathbb{R}^n)} \leq C \left( ||\chi_v(S)||_{H^1(B_{R_0})} + ||\eta_v(S)||_{L^2(B_{R_0})} \right)
\]

and

\[
||g||_{L^2(\mathbb{R}^n)} \leq C ||\chi_v(S)||_{L^2(B_{R_0})}.
\]

From Sobolev inequality \( ||\chi_v(S)||_{L^{2n/(n-2)}(\mathbb{R}^n)} \leq C|v| \) so we get the following estimates:

\[
||g||_{L^1(B_{R_0})} \leq C|v|
\]
\[ \|f\|_{L^1(B_{R_0})} \leq C |v| + C \|\eta_v(S)\|_{L^1(B_{R_0})}. \]

For any fixed \( R > R_0 \) the function \( x \mapsto \eta_v(S)(x) - \int_{B_R} \eta_v(S)(x)d\mathcal{L}^n \) has mean zero over \( B_R \), so there exists a function \( \phi_R \) from \( H^1_0(B_R \setminus Y') \) such that

\[ \nabla \cdot \phi_R(x) = \eta_v(S)(x) - \int_{B_R} \eta_v(S)(x)d\mathcal{L}^n, \]

and

\[ \|\phi_R\|_{H^1_0(B_R \setminus Y')} \leq C \|\eta_v(S)(x) - \int_{B_R} \eta_v(S)(x)d\mathcal{L}^n\|_{L^2(B_R)}. \]

Using this in the first equation of (4.9)

\[ \int_{B_R} \nabla \chi_v(s) \nabla \phi_R d\mathcal{L}^n - \int_{B_R} \left( \eta_v(S) - \int_{B_R} \eta_v(S)(x)d\mathcal{L}^n \right)^2 d\mathcal{L}^n = 0, \]

which will imply that

\[ \|\eta_v(S) - \int_{B_R} \eta_v(S)(x)d\mathcal{L}^n\|_{L^2(B_R)} < C |v|. \]

The function \( \eta_v(S) \) is in \( L^2(\mathbb{R}^n) \), so Jensen inequality implies that \( \int_{B_R} \eta_v(S)(x)d\mathcal{L}^n \) converges to zero when \( R \) goes to infinity.

From here we obtain that \( \|\eta_v(S)\|_{L^2(\mathbb{R}^n)} < C |v| \). We showed so far that

\[ \|g\|_{L^1(B_{R_0})}, \|f\|_{L^1(B_{R_0})} \leq C |v| \]

The solutions to the problem (4.13) are zero on a neighbourhood of \( Y' \) and regularity theorem tells us that \( \chi_v(S) \) is smooth away from \( S \). Now we can find easily \( \chi_v(S) \) and \( \eta_v(S) \) by the use of Fourier transform

\[ \phi \chi_v(S)(x) = \frac{1}{2(n-2)n\alpha(n)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}}dy + \frac{1}{2n\alpha(n)} \int_{\mathbb{R}^n} \frac{(x-y) \cdot f(y)}{|x-y|^n}(x-y)dy \]

and

\[ \phi \eta_v(S)(x) = g(x) + \frac{1}{n\alpha(n)} \int_{\mathbb{R}^n} \frac{(x-y) \cdot f(y)}{|x-y|^n}dy, \]

where \( \alpha(n) \) represents the volume of the \( n \)-dimensional ball of radius 1. Now, the estimates follow because \( f \) and \( g \) have compact support in \( B_{R_0} \) and \( \phi = 1 \) outside \( B_{R_0} \).

\( \square \)
As a consequence of the estimates (4.12) we have the following lemma:

**Lemma 8.** \( a_{BR}(\chi^R_v(S), \chi^R_v(S)) \) converges to \( a_{R}(\chi_v(S), \chi_v(S)) \) uniformly for \( S \subset Y' \) and \( v \) in bounded sets. Also, there exists a constant \( C \) such that for every \( R > R_0 \) and \( S \)

\[
||\sigma(\chi^R_v(S), \chi^R_v(S))||_{L^2(\partial B_R)}^2 \leq \frac{C|v|^2 R}{R^n}. \tag{4.14}
\]

**Proof.** We will omit in this proof to write the dependence on \( S \) understanding that the estimates are uniform with respect to \( S \). Also \( C \) will represent a generic constant, independent of \( R, S \) and \( v \).

First let us notice that (4.10) and (4.11) and the linearity with respect to \( v \) of \( \chi^R_v \) gives us a constant \( C \) such that

\[
a_{BR}(\chi^R_v(S), \chi^R_v(S)) \leq C|v|^2 \quad \text{for every} \quad R \quad \text{and} \quad v.
\]

We are interested only in the behaviour at infinity so we will take again a smooth cutoff function \( \phi \), equal to 0 on a neighbourhood of \( Y \) and equal to 1 outside \( B_{R_0} \). If we choose \( \eta^R_v \) to have mean zero over \( B_R \) we can easily show like in Lemma 2, as a consequence of Lemma 1 that

\[
||\eta^R_v||_{L^2(B_R)} \leq C|v| \quad \text{for every} \quad R \quad \text{and} \quad v.
\]

The pair \( \{\phi\chi^R_v, \phi\eta^R_v\} \) will satisfy a Stokes system

\[
\begin{cases}
-\Delta(\phi\chi^R_v) + \nabla(\phi\eta^R_v) = f^R_v & \text{in} \ B_R, \\
\nabla \cdot (\phi\chi^R_v) = g^R_v & \text{in} \ B_R, \\
\phi\chi^R_v = 0 & \text{on} \ \partial B_R,
\end{cases}
\tag{4.15}
\]

where \( f^R_v \) and \( g^R_v \) are compactly supported in \( B_{R_0} \) and have the estimates

\[
||f^R_v||_{L^2(B_R)} \leq C|v|,
\]

\[
||g^R_v||_{H^1_0(B_R)} \leq C|v|.
\]

The previous lemma tells us that the similar solutions in the free space, \( \tilde{\chi}_v \) and \( \tilde{\eta}_v \), will satisfy certain pointwise estimates at infinity:

\[
\begin{cases}
|D^\alpha \tilde{\chi}_v(x)| \leq \frac{C|v|}{|x|^{n-2+|\alpha|}} & |\alpha| \leq 2, \\
|D^\alpha \tilde{\eta}_v(x)| \leq \frac{C|v|}{|x|^{n-1+|\alpha|}} & |\alpha| \leq 1.
\end{cases}
\tag{4.16}
\]

After elementary calculations, the estimates (4.16) will imply that

\[
||D^\alpha \tilde{\chi}_v||_{L^2(R_{2R}\setminus B_R)}^2 \leq \frac{C|v|^2 R^{4-2|\alpha|}}{R^n} \quad |\alpha| \leq 2
\]
\[ ||D^\alpha \tilde{\eta}_v||_{L^2(R_{2R} \setminus B_R)} \leq \frac{C|v|^2 R^{2-2|\alpha|}}{R^n} \quad |\alpha| \leq 1 \]

for \( R > R_0 \) and \( v \in \mathbb{R}^n \).

We obtain first the following estimate for the stress

\[ ||\sigma(\chi_v^R(S), \chi_v^R(S))||_{L^2(\partial B_R)}^2 \leq \frac{C|v|^2 R}{R^n}. \quad (4.17) \]

After a change of variable we obtain

\[ ||\tilde{\chi}_v \left( \frac{R}{R_0} \right) \|_{H^2(B_{2R_0} \setminus B_{R_0})} \leq \frac{C|v| R^2}{R^n} \]

and

\[ ||\tilde{\eta}_v \left( \frac{R}{R_0} \right) \|_{H^1(B_{2R_0} \setminus B_{R_0})} \leq \frac{C|v| R}{R^n}. \]

Let \( u_v(x) = \phi \chi_v^R \left( \frac{R}{R_0} x \right) - \tilde{\chi}_v \left( \frac{R}{R_0} x \right) \) and \( p_v(x) = \frac{R}{R_0} \phi \eta_v^R \left( \frac{R}{R_0} x \right) - \frac{R}{R_0} \tilde{\eta}_v \left( \frac{R}{R_0} x \right) \); then \( \{u_v, p_v\} \) is the solution for

\[
\begin{cases}
-\Delta u_v + \nabla p_v = 0 & \text{in } B_{R_0}, \\
\nabla \cdot u_v = 0 & \text{in } B_{R_0}, \\
u_v = -\tilde{\chi}_v \left( \frac{R}{R_0} x \right) & \text{on } \partial B_{R_0}.
\end{cases}
\]

From the regularity theorem ([47], Proposition 2.2) we know that there exists a constant \( C \) such that

\[ ||p_v||_{H^1(B_{R_0})} + ||u_v||_{H^2(B_{R_0})} \leq C \|\tilde{\chi}_v\|_{H^{3/2}(\partial B_{R_0})}, \]

which implies that

\[ ||p_v||_{H^1(B_{R_0})} + ||u_v||_{H^2(B_{R_0})} \leq \frac{C|v| R^2}{R^n}. \]

We derive from here two estimates. The first one is

\[ ||\sigma(u_v, p_v)||_{L^2(\partial B_{R_0})} \leq \frac{C|v|^2 R^2}{R^n}, \]

which gives

\[ ||\sigma(\chi_v^R, \eta_v^R) - \sigma(\tilde{\chi}_v, \tilde{\eta}_v)||_{L^2(\partial B_R)}^2 \leq \frac{C|v|^2 R}{R^n}. \]
that together with 4.17 gives 4.14.

The second one is

\[ \|u_v\|_{L^2(\partial B_{R_0})}^n \leq \frac{C|v|^2 R^2}{R^n}. \]

After applying back a change of variables gives that

\[ \|\chi_v^R - \bar{\chi}_v\|_{L^2(\partial B_R)}^2 \leq \frac{C|v|^2 R^3}{R^n} \]

which after 4.16 implies

\[ \|\chi_v^R\|_{L^2(\partial B_R)}^2 \leq \frac{C|v|^2 R^3}{R^n}. \]

Now all we have to do is to use the variational formulations for \( \chi_v^R \) and \( \chi_v \)

\[ a_{BR}(\chi_v^R - \chi_v, \chi_v^R - \chi_v) = \int_{\partial B_R} \sigma(\chi_v^R - \chi_v, \eta_v^R - \eta_v)(\chi_v^R - \chi_v)d\mathcal{H}^{n-1} \]

\[ \leq \frac{C|v|^2 R^2}{R^n}, \]

which shows that \( \chi_v^R \) converges in \( X \) to \( \chi_v \) and proves the lemma. \( \square \)

Now we can define the matrix \( C \). For \( 1 \leq i, j \leq n, R > 0, \omega \in \Omega: \)

\[ C_{ij}(\omega) = a_{BR}(\chi_v^R(A(\omega)), \chi_v^R(A(\omega))) \]

if \( 1 \leq i, j \leq n - 1 \) and \( C_{ij}(\omega) = 0 \) otherwise.

\[ C_{ij}(\omega) = a_{R^\omega}(\chi_{e_i}(A(\omega)), \chi_{e_j}(A(\omega))) \]

if \( 1 \leq i, j \leq n - 1 \) and \( C_{ij}(\omega) = 0 \) otherwise.

**Lemma 9.** Let \( R_0 < R \leq \infty \). Then:

(i) \( \omega \rightarrow C^R_{ij}(\omega) \) is measurable.

(ii) \( \lim_{R \rightarrow \infty} C^R_{ij}(\omega) = C_{ij}(\omega) \) uniformly in \( \omega \).

The matrix \( C \) will be defined by

\[ C_{ij} = \int_{\Omega} C_{ij}(\omega)dP = \lim_{R \rightarrow \infty} C^R_{ij} = \int_{\Omega} C^R_{ij}(\omega)dP. \]

**Proof.** i) We want to use the same idea as in the proof for the measurability of capacity **Lemma 6**, so all we need to show is that for fixed \( v \) and \( R \) the set function \( S \rightarrow a_{BR}(\chi_v^R(S)), \chi_v^R(S)) \), which we defined only for certain type of sets by using the solution of a Stokes system, is continuous on decreasing sequences.

So let \( S \) be an admissible set and \( \chi_v^R(S) \) the solution of the Stokes system (4.8).

Assume \( (S_n)_{n \geq 1} \) is a sequence of admissible sets satisfying \( S \subset S_{n+1} \subset S_n \subset Y' \) for every \( n \geq 1 \) and \( S = \bigcap_{n \geq 1} S_n \). We want to prove that \( a_{BR}(\chi_v^R(S_n)), \chi_v^R(S_n)) \)

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converges to \( a_{B_R}(\chi^R_v(S)), \chi^R_v(S) \).

Let \( S_\epsilon = \{ x' \in Y' \mid d(x', S) \leq \epsilon \} \). The interior of \( S_\epsilon \) is \( \{ x' \in Y'/d(x', S) < \epsilon \} \) which is connected because \( S \) is connected and the boundary is \( \{ x' \in Y'/d(x', S) = \epsilon \} \) which is the level set of a Lipschitz function. Also, for \( \epsilon \) small enough \( S_\epsilon \subset Y' \). Let \( \tilde{S}_\epsilon = S_\epsilon \times [-\epsilon, \epsilon] \) which is a subset of \( B_R \) for \( \epsilon \) small enough. Let us denote for every \( t \in [-\epsilon, \epsilon] \) by \( S_\epsilon(t) = \{ x \in \tilde{S}_\epsilon:x_n = t \} \).

Let \( \phi \) be a smooth function on \( \tilde{S}_\epsilon \). We have

\[
\phi(x', t'e_n) = \phi(x', 0) + \int_0^t \frac{\partial \phi}{\partial x_n}(x', se_n)ds \Rightarrow
\]

\[
\phi^2(x', t'e_n) \leq 2\phi^2(x', 0) + 2 \int_0^t 1ds \int_0^t \left( \frac{\partial \phi}{\partial x_n}(x', se_n) \right)^2 dx' \int_{S_\epsilon} \phi \leq ||\phi||_{L^2(\tilde{S}_\epsilon)} \sqrt{2} \int_{S_\epsilon} |\nabla \phi|^2 dx \int_{S_\epsilon} \phi \int_{S_\epsilon} \phi \leq ||\phi||_{L^2(\tilde{S}_\epsilon)} \sqrt{2} \int_{S_\epsilon} |\nabla \phi|^2 dx \int_{S_\epsilon} \phi \]

integrating over \( S_\epsilon \)

\[
\int_{S_\epsilon} \phi^2(x)d\mathcal{L}^{n-1} \leq 2 \int_{S_\epsilon} \phi^2(x)d\mathcal{L}^{n-1} + 2t \int_{\tilde{S}_\epsilon} |\nabla \phi|^2 dx \int_{S_\epsilon} \phi \]

and integrating with respect to \( t \in [-\epsilon, \epsilon] \)

\[
||\phi||_{L^2(\tilde{S}_\epsilon)} \leq 4\epsilon||\phi||_{L^2(S_\epsilon)} + \epsilon^2 ||\nabla \phi||_{L^2(\tilde{S}_\epsilon)}. \tag{4.18}
\]

Passing to the limit we obtain that the inequality holds for every \( H^1(\tilde{S}_\epsilon) \) function, so

\[
||v - \chi^R_v(S)||_{L^2(S_\epsilon)} \leq 4\epsilon||v - \chi^R_v(S)||_{L^2(S_\epsilon)} + \epsilon^2 ||\nabla \chi^R_v(S)||_{L^2(\tilde{S}_\epsilon)}. \tag{4.18}
\]

Let \( \theta_\epsilon \) be a smooth function taking values in \([0,1]\), equal to 0 on \( \tilde{S}_\epsilon \), equal to 1 on the complement of \( \tilde{S}_2 \) and with the gradient \(|\nabla \theta_\epsilon(x)| \leq 1/\epsilon \) pointwise. Let \( u_\epsilon = v - \theta_\epsilon(v - \chi^R_v(S)) \). Then \( u_\epsilon = \chi^R_v(S) \) outside \( \tilde{S}_2 \), and \( u_\epsilon = v \) in \( \tilde{S}_\epsilon \). Also

\[
||u_\epsilon - \chi^R_v(S)||_{H^1(S_2\epsilon)} \leq ||v - \chi^R_v(S)||_{H^1(S_\epsilon)} + \frac{1}{\epsilon^2} ||v - \chi^R_v(S)||_{L^2(S_\epsilon)}
\]

so using inequality (4.18)

\[
||u_\epsilon - \chi^R_v(S)||_{H^1(S_2\epsilon)} \leq 5||v - \chi^R_v(S)||_{H^1(S_\epsilon)} + 4\epsilon^2 ||v - \chi^R_v(S)||_{L^2(S_\epsilon)}.
\]

We will apply to the function of \(|v - \chi^R_v(S)|^2 \) the coarea formula using the Lipschitz function \( d(\cdot, S) \)

\[
\int_{S_2\epsilon} |v - \chi^R_v(S)|^2(x) \cdot |\nabla d(x, S)|d\mathcal{L}^n = \int_{-2\epsilon}^{2\epsilon} \int_{\partial S_t} |v - \chi^R_v(S)|^2(x) d\mathcal{L}^{n-1}(x)dt.
\]

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The function $d(\cdot, S)$ is differentiable almost everywhere, and its gradient is equal to the exterior normal to $S_t$ if $x \in \partial S_t$, so we obtain

$$||v - \chi_v^R(S)||_{L^2(S_{2\epsilon})} = \int_{-2\epsilon}^{2\epsilon} \int_{\partial S_t} |v - \chi_v^R(S)|^2(x) d\mathcal{L}^{n-1}(x) dt,$$

and

$$||u_\epsilon - \chi_u^R(S)||_{H^1(S_{2\epsilon})} \leq 5||v - \chi_v^R(S)||_{H^1(S_{2\epsilon})} + \frac{4}{\epsilon} \int_{-2\epsilon}^{2\epsilon} \int_{\partial S_t} |v - \chi_v^R(S)|^2(x) d\mathcal{L}^{n-1}(x) dt.$$

So $u_\epsilon$ converges to $\chi_v^R(S)$ in $H^1(B_R)$ because outside $\tilde{S}_{2\epsilon}$ they are equal. The function $\tilde{u}_\epsilon$ with the properties

$$\begin{cases}
\nabla \cdot \tilde{u}_\epsilon = 0 & \text{in } B_R, \\
\tilde{u}_\epsilon = u_\epsilon & \text{on } Y' \cup \partial B_R, \\
||\tilde{u}_\epsilon - u_\epsilon||_{H^1_0(B_R \setminus Y')} \leq C||u_\epsilon - \chi_v^R(S)||_{H^1(B_R)},
\end{cases}$$

will also converge to $\chi_v^R(S)$ in $H^1(B_R)$, will be equal with 0 on $\partial B_R$, with $v$ on $S_\epsilon$ and will be divergence free. This will imply

$$\lim_{\epsilon \to 0} a_{B_R}(\chi_v^R(S_\epsilon), \chi_v^R(S_\epsilon)) = a_{B_R}(\chi_v^R(S)), \chi_v^R(S)),$$

and because of the compactness

$$\lim_{n \to \infty} a_{B_R}(\chi_v^R(S_n), \chi_v^R(S_n)) = a_{B_R}(\chi_v^R(S)), \chi_v^R(S)).$$

The proof continues like in Lemma 6.

ii) This is a consequence of the Lemma 8, where $v$ is taken to be for $1 \leq i, j \leq n - 1$ of the form $e_i$, $e_i$ and $e_i + e_j$ and using the bilinearity of $a$ and the fact that $A$ is $Y'$ valued.

Now we can prove the main theorem. The $M-$ convergence obtained in this case will be in $L^2(D)$ so we will consider the functionals extended with $+\infty$ outside $H^1(D)$.

**Theorem 24.** The sequence of functionals $F^\epsilon(\omega)$ $M-$converges to $F$ in $L^2(D)$ on a set of full probability.
Proof. (i) \( M - \limsup_{\epsilon \to 0} F'(\omega)(u) \leq F(u) \) a.s. \( \omega \in \Omega \).

Which means that for almost every \( \omega \in \Omega \) and for every \( u \in H^1_b(D) \cap V \) we can find a sequence \( u^\epsilon(\omega) \in K^\epsilon(\omega) \cap V \) converging to \( u \) strongly in \( L^2(D) \) such that

\[
\lim_{\epsilon \to 0} F'(\omega)(u^\epsilon(\omega)) \leq F(u).
\]

Let \( b_0 \) be a function in \( H^1_b(D) \cap V \) such that \((b_0)_i = 0\) for every \( 1 \leq i \leq n - 1 \) and assume first that \( u \) is chosen such that \( u - b_0 \) is smooth.

For every \( k \in N^c \), let \( B_k^\epsilon \) be the ball centered at \( \epsilon k \) with radius \( \epsilon \). For every such a ball \( B_k^\epsilon \), every \( 1 \leq i \leq n - 1 \) and every \( \omega \in \Omega \) consider the solutions to the following problems

\[
\begin{cases}
-\Delta u_{k,i}^\epsilon(\omega) + \nabla p_{k,i}^\epsilon(\omega) = 0 & \text{in } B_k^\epsilon \setminus \Gamma_k^\epsilon(\omega), \\
\nabla \cdot u_{k,i}^\epsilon(\omega) = 0 & \text{in } B_k^\epsilon \setminus \Gamma_k^\epsilon(\omega), \\
u_{k,i}^\epsilon(\omega) = e_i & \text{on } \Gamma_k^\epsilon(\omega), \\
u_{k,i}^\epsilon(\omega) = 0 & \text{on } \partial B_k^\epsilon.
\end{cases}
\tag{4.20}
\]

After a change of variable we get that \( u_{k,i}^\epsilon(\omega)(r_\epsilon x + \epsilon k) \) and \( r_\epsilon p_{k,i}^\epsilon(\omega)(r_\epsilon x + \epsilon k) \) are solutions for a problem of type (4.8), namely \( \chi_i^{\epsilon/r_\epsilon}(\tau_k \omega) \) and \( \eta_i^{\epsilon/r_\epsilon}(\tau_k \omega) \), so using estimates (4.14) we have the following estimate for the stress tensor if \( \epsilon \) is small enough

\[
||\sigma(u_{k,i}^\epsilon(\omega), p_{k,i}^\epsilon(\omega))||_{L^2(\partial B_k^\epsilon)^n} \leq C\epsilon^{n-1}.
\tag{4.21}
\]

The sequence \( \overline{w}(\omega) = \sum_{i=1}^{n-1} \sum_{k \in N^c} u_{k,i}^\epsilon(\omega)u_i(\epsilon k) \) converges strongly to 0 in \( L^2(D) \).

We obtain after a change of variable and some calculations

\[
||\nabla \overline{w}(\omega)||_{L^2(D)^n} \leq C \sum_{i=1}^{n-1} \sum_{k \in N^c} r_\epsilon^{n-2}||\nabla \chi_i^{\epsilon/r_\epsilon}(\tau_k \omega)||_{L^2(B_{r_\epsilon/r_\epsilon})^n} \leq C|N^c|r_\epsilon^{n-2} \leq C,
\]

and

\[
||\overline{w}(\omega)||_{L^2(D)} \leq C \sum_{i=1}^{n-1} \sum_{k \in N^c} r_\epsilon^n||\chi_i^{\epsilon/r_\epsilon}(\tau_k \omega)||_{L^2(B_{r_\epsilon/r_\epsilon})}
\leq C \sum_{i=1}^{n-1} \sum_{k \in N^c} r_\epsilon^n||\chi_i^{\epsilon/r_\epsilon}(\tau_k \omega)||_{L^{2n/(n-2)}(B_{r_\epsilon/r_\epsilon})}(\epsilon/r_\epsilon)^2
\leq Cr_\epsilon^n|N^c|(\epsilon/r_\epsilon)^2 \leq C\epsilon^2,
\]

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so \( \overline{u} (\omega) \) will converge in fact weakly to 0 in \( \mathbf{H}^1 (D) \).

Let us consider \( v_k^\varepsilon = u - u_n - \sum_{i=1}^{n-1} \epsilon_i u_i (\epsilon k) \) defined in \( B_k^\varepsilon \). Uniform continuity of \( u_i \) for \( 1 \leq i \leq n - 1 \) gives that

\[
||v_k^\varepsilon||_{\mathbf{H}^{1/2} (\epsilon k + r^\varepsilon Y')} = r^{n-1}_\varepsilon o(\varepsilon).
\]

As a consequence of Lemma 1 and rescaling properties we can always take \( \tilde{v}_k^\varepsilon \) such that

\[
\begin{cases}
\nabla \cdot \tilde{v}_k^\varepsilon = 0 & \text{in } B_k^r, \\
v_k^\varepsilon = \tilde{v}_k^\varepsilon & \text{on } \epsilon k + r \varepsilon Y', \\
||\nabla \tilde{v}_k^\varepsilon||_{L^2 (B_k^r)^n}^2 \leq C r \epsilon ||v_k^\varepsilon||_{\mathbf{H}^{1/2} (\epsilon k + r \varepsilon Y')} ,
\end{cases}
\]

where the constant \( C \) is independent of \( \varepsilon \) and \( k \).

The function \( \tilde{u}^\varepsilon = \sum_{k \in \mathbb{N}^\varepsilon} \tilde{v}_k^\varepsilon \) is divergence free and \( ||\nabla \tilde{u}^\varepsilon||_{L^2 (\varepsilon k)^n}^2 \) converges to 0. Now we can define the sequence \( u^\varepsilon (\omega) \) that we need

\[
u^\varepsilon (\omega) = u - \overline{u} (\omega) - \tilde{u}^\varepsilon.
\]

Based on what we proved \( u^\varepsilon (\omega) \) is in \( \mathbf{K}^\varepsilon (\omega) \cap \mathbf{V} \) and converges to \( u \) strongly in \( L^2 (D) \). Also

\[
\lim_{\varepsilon \to 0} F^\varepsilon (\omega) (u^\varepsilon (\omega)) = \frac{1}{2} a (u, u) - \langle f, u \rangle_D + I_1,
\]

where \( I_1 = \frac{1}{2} \lim_{\varepsilon \to 0} a \left( \sum_{k \in \mathbb{N}^\varepsilon} \sum_{i=1}^{n-1} u^\varepsilon_{k,i}(\omega) u_i (\epsilon k), \sum_{i=1}^{n-1} \sum_{k \in \mathbb{N}^\varepsilon} u^\varepsilon_{k,i}(\omega) u_i (\epsilon k) \right) \).

Making a change of variables and using the fact that \( u^\varepsilon_{k,i}(\omega) \) and \( u^\varepsilon_{k,j}(\omega) \) have disjoint supports for \( k \neq l \), we obtain

\[
I_1 = \frac{1}{2} \lim_{\varepsilon \to 0} \sum_{k \in \mathbb{N}^\varepsilon} \sum_{i,j=1}^{n-1} a (\chi_{i/r^\varepsilon}^\varepsilon (\tau_k \omega), \chi_{j/r^\varepsilon}^\varepsilon (\tau_k \omega) ) r^{n-2}\varepsilon u_i (\epsilon k) u_j (\epsilon k)
\]

\[
= \frac{1}{2} \lim_{\varepsilon \to 0} \sum_{k \in \mathbb{N}^\varepsilon} \sum_{i,j=1}^{n-1} C_{ij}^\varepsilon (\tau_k \omega) u_i (\epsilon k) u_j (\epsilon k) r^{n-2}\varepsilon,
\]

From Lemma 9 and because \( \frac{r^{n-2}\varepsilon}{\varepsilon^{n-1}} \) converges to \( \beta \) we get that

\[
I_1 = \beta \lim_{\varepsilon \to 0} \sum_{i,j=1}^{n-1} \sum_{k \in \mathbb{N}^\varepsilon} C_{ij} (\tau_k \omega) u_i (\epsilon k) u_j (\epsilon k) \varepsilon^{n-1},
\]

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which will imply by Lemma 18

\[ I_1 = \frac{1}{2} \sum_{i,j=1}^{n-1} C_{ij} \int_{\Sigma} u_i u_j \quad \text{a.s. } \omega \in \Omega. \]

So for \( u \)

\[ \lim_{\epsilon \to 0} F^\epsilon(\omega)(u^\epsilon(\omega)) = \frac{1}{2} a(u, u) - \langle f, u \rangle_D + \frac{1}{2} \sum_{i,j=1}^{n-1} C_{ij} \int_{\Omega} u_i u_j = F(u) \quad \text{a.s. } \omega \in \Omega. \]

We proved so far that a.s. \( \omega \in \Omega, M - \limsup_{\epsilon \to 0} F^\epsilon(u) \leq F(u) \) for every \( u \) in \( H^1_b(D) \cap V \) such that \( u - b_0 \) is smooth. We may choose a countable dense set for the strong topology of \( H^1_b(D) \cap V \) so for every \( \omega \) in a set of full measure the property holds for this subset. We will complete the proof using a diagonalization argument ([2], Corollary 1.18) to show that the property holds for every \( u \in H^1_b(D) \cap V \) and \( \omega \) in a set of full measure in \( \Omega \).

(ii) \( M - \liminf_{\epsilon \to 0} F^\epsilon(\omega)(u) \geq F(u) \) a.s. \( \omega \in \Omega \).

Which means that almost surely \( \omega \in \Omega \), for every sequence \( u^\epsilon \in H^1_b(D) \cap V \) converging weakly to \( u \) in \( L^2(D) \)

\[ \liminf_{\epsilon \to 0} F^\epsilon(\omega)(u^\epsilon) \geq F(u). \]

We may assume that \( \liminf_{\epsilon \to 0} F^\epsilon(\omega)(u^\epsilon) < \infty \) so \( u^\epsilon \in K(\omega) \cap V \) and based on uniform coercivity of \( F^\epsilon(\omega) \) up to a subsequence \( u^\epsilon \) will be weakly convergent to \( u \) in \( H^1(D) \). Let \( \omega \) be from the set of full measure obtained in the first part and consider first \( v \) such that \( v - b_0 \) is smooth and the corresponding sequence \( v^\epsilon(\omega) \) weakly convergent to \( v \) in \( H^1(D) \) such that \( F^\epsilon(\omega)(v^\epsilon(\omega)) \to F(v) \). Then by a subdifferential type inequality

\[ F^\epsilon(\omega)(u^\epsilon) = \frac{1}{2} a(u^\epsilon, u^\epsilon) - \langle f, u^\epsilon \rangle_D + j^\epsilon(\omega)(u^\epsilon) \geq F^\epsilon(\omega)(v^\epsilon(\omega)) + a(u^\epsilon - v^\epsilon(\omega), v^\epsilon(\omega)) - \langle f, u^\epsilon - v^\epsilon(\omega) \rangle_D + j^\epsilon(\omega)(u^\epsilon) - j^\epsilon(\omega)(v^\epsilon(\omega)). \]

From what was proved in the first part we derive

\[ \liminf_{\epsilon \to 0} F^\epsilon(\omega)(u^\epsilon) \geq F(v) - \langle f, u - v \rangle_D + \liminf_{\epsilon \to 0} a(u^\epsilon - v^\epsilon(\omega), v^\epsilon(\omega)). \]

So it is enough to study the limit

\[ \liminf_{\epsilon \to 0} a(u^\epsilon - v^\epsilon(\omega), v^\epsilon(\omega)). \]
Because \( v^\epsilon(\omega) = v - \overline{v}^\epsilon(\omega) - \tilde{v}^\epsilon \), where \( \overline{v}(\omega) \) converges weakly to 0 in \( H^1(D) \) and \( \tilde{v}^\epsilon \) converges strongly we get

\[
\liminf_{\epsilon \to 0} a(u^\epsilon - v^\epsilon(\omega), v^\epsilon(\omega)) = a(u - v, v) - \limsup_{\epsilon \to 0} a(u^\epsilon - v^\epsilon(\omega), v^\epsilon(\omega)).
\]

We will estimate now the limit

\[
I_2 = \limsup_{\epsilon \to 0} |a(u^\epsilon - v^\epsilon(\omega), \overline{v}^\epsilon(\omega))|,
\]

where

\[
a(u^\epsilon - v^\epsilon(\omega), \overline{v}^\epsilon(\omega)) = \sum_{i=1}^{n-1} \sum_{k \in N^\epsilon} v_i(\epsilon k) \int_{B_k^\epsilon} 2e(u^\epsilon - v^\epsilon(\omega))e(v_{k,i}^\epsilon(\omega))dL^n.
\]

Integrating by parts we obtain

\[
\int_{B_k^\epsilon} 2e(u^\epsilon - v^\epsilon(\omega))e(v_{k,i}^\epsilon(\omega))dL^n = \int_{\partial B_k^\epsilon} \sigma(v_{k,i}^\epsilon(\omega), \eta_{k,i}^\epsilon(\omega))N \cdot (u^\epsilon - v^\epsilon(\omega))dH^{n-1},
\]

and then using the estimate (4.21) we get

\[
|\int_{B_k^\epsilon} 2e(u^\epsilon - v^\epsilon(\omega))e(v_{k,i}^\epsilon(\omega))dL^n| \leq C \left( \epsilon^{n-1} \int_{\partial B_k^\epsilon} |u^\epsilon - v^\epsilon(\omega)|^2dH^{n-1} \right)^{1/2}.
\]

Now we apply Cauchy’s inequality and obtain

\[
|a(u^\epsilon - v^\epsilon(\omega), \overline{v}^\epsilon(\omega))|^2 \leq \sum_{i=1}^{n-1} \left( \sum_{k \in N^\epsilon} v_i(\epsilon k)^2 \right) \left( \sum_{k \in N^\epsilon} \int_{B_k^\epsilon} 2e(u^\epsilon - v^\epsilon(\omega))e(v_{k,i}^\epsilon(\omega))dL^n \right)^2.
\]

The functions \( v_i \) being smooth for all \( 1 \leq i \leq n - 1 \), then

\[
\lim_{\epsilon \to 0} \epsilon^{n-1} v_i^\epsilon(\epsilon k) = \int_{\Sigma} v_i^2 dL^{n-1},
\]

so we get

\[
|a(u^\epsilon - v^\epsilon(\omega), \overline{v}^\epsilon(\omega))|^2 \leq C \int_{\Sigma} v_i^2 dH^{n-1} \cdot \sum_{k \in N^\epsilon} \int_{\partial B_k^\epsilon} |u^\epsilon - v^\epsilon(\omega)|^2dH^{n-1}
\]

\[
\leq C \int_{\Sigma} v_i^2 dH^{n-1} \cdot \sum_{k \in N^\epsilon} \left( \int_{B_k^\epsilon \cap \Sigma} |u^\epsilon - v^\epsilon(\omega)|^2dH^{n-1} + \epsilon \int_{B_k^\epsilon} |\nabla(u^\epsilon - v^\epsilon(\omega))|^2dL^n \right).
\]

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So we derive finally that

$$\limsup_{\epsilon \to 0} |a(u^\epsilon - v^\epsilon(\omega), v^\epsilon(\omega))|^2 \leq C \int_\Sigma v^2 d\mathcal{L}^{n-1} \cdot \int_\Sigma |v - u|^2 d\mathcal{L}^{n-1},$$

which implies that

$$I_2 \leq C \|v\|_{L^2(\Sigma)} \|v - u\|_{L^2(\Sigma)}.$$

So $$\liminf_{\epsilon \to 0} F^\epsilon(\omega)(u^\epsilon(\omega)) \geq F(v) - \langle f, u - v \rangle_D + a(u - v, v) - C \|v\|_{H^1(D)} \|v - u\|_{H^1(D)}.$$

Now letting $$v$$ to converge to $$u$$ strongly in $$H^1(D)$$ and using the continuity of $$F$$ with respect to this topology we get the desired result.

As a consequence of this we have also the theorem for convergence of solutions:

**Theorem 25.** Almost surely $$\omega \in \Omega$$, $$u^\epsilon(\omega)$$ converges weakly to $$u$$ in $$H^1(D)$$ and $$p^\epsilon(\omega)$$ converges weakly in $$L^2(D)/\mathbb{R}$$ to $$p$$, where $$\{u, p\}$$ is the unique solution of the problem (4.7).
4.2.4 Case IV: Vanishing Obstacles

Assume that \( \lim_{\epsilon \to 0} \frac{r^n - 2}{\epsilon^{n-1}} = 0 \). In the last case we get similarly as in the periodic case as a limit a Stokes system in the domain \( D \) with Dirichlet boundary conditions

\[
\begin{cases}
-\Delta u + \nabla p = f & \text{in } D, \\
\nabla \cdot u = 0 & \text{in } D, \\
u = b & \text{on } \partial D.
\end{cases}
\tag{4.24}
\]

The convex functional in this case is

\[
F(u) = \frac{1}{2} a(u, u) - \langle f, u \rangle_D + I_{H^1_0(D) \cap V},
\]

and based on what we proved in the previous case and using comparison arguments we obtain for almost all \( \omega \in \Omega \), \( M \)-convergence of \( F^\epsilon \) to \( F \) and from here we obtain also also weak convergences for \( u^\epsilon \) and \( p^\epsilon \).
Chapter 5

Time Dependent Case

5.1 Convergence of Evolutionary Problems

In this part we will explain shortly how the convergence results obtained in the stationary case can be extended to the time dependent case. We will recall first shortly some standard definitions and results concerning maximal monotone operators on real Hilbert spaces, the case of the subdifferential of a convex function and then the connection between $M-$convergence and convergence of solutions for a class of evolution problems.

If $H$ is a Hilbert space with the inner product $(\cdot, \cdot)_H$, an operator $A$ on $H$ (which will be identified through its graph with a subset of $H \times H$) will be called monotone if $(x_1 - x_2, y_1 - y_2)_H \geq 0$ for all $(x_1, y_1), (x_2, y_2) \in A$. This is equivalent in a Hilbert space with $A$ being accretive, i.e. $||x_1 - x_2||_H \leq ||x_1 + \lambda y_1 - x_2 - \lambda y_2||_H$ for all $(x_1, y_1), (x_2, y_2) \in A$ and any $\lambda > 0$ or equivalently $(I + \lambda A)^{-1}$ is a contraction on the range of $A$, $R(A)$ for all $\lambda > 0$.

A monotone operator $A$ will be called m-monotone if in addition $R(I + A) = H$ or equivalently $R(I + \lambda A) = H$ for all $\lambda > 0$ ([58], Ch. IV, Lemma 1.3). In the case of a Hilbert space this is equivalent with $A$ being maximal monotone, i.e. it does not have a proper monotone extension ([59], Ch. II, Th. 3.1). If $A$ is a monotone operator on $H$ we denote by $(J^A_\lambda)_{\lambda>0}$ the corresponding resolvents of $A$, defined by $J^A_\lambda = (I + \lambda A)^{-1}$ for all $\lambda > 0$. If $A$ is m-monotone $(J_\lambda)_{\lambda>0}$ is a family of contractions on $H$, $D(A)$ is convex, and the following property holds ([58], Ch. IV, Prop. 1.7):

$$\lim_{\lambda \to 0} J_\lambda(x) = \text{Proj}_{D(A)}(x).$$

For a m-monotone operator $A$ on $H$ we denote by $(A_\lambda)_{\lambda>0}$ the corresponding Yosida approximations of $A$, defined by $A_\lambda = \lambda^{-1}(I - J^A_\lambda)$ for all $\lambda > 0$ and by $A^0$ the
minimal section of $A$, i.e. $A^0(x)$ is the unique element of minimal norm from the closed convex set $A(x)$. For any $\lambda > 0$, $(A_\lambda)$ is a m-monotone operator, Lipschitz with constant $\lambda^{-1}$. $||A_\lambda(x)||_H$ is increasing for every $x \in H$ when $\lambda \to 0$ and bounded if and only if $x \in D(A)$ ([58], Ch. IV, Th. 1.1). Also for every $x \in D(A)$

$$\lim_{\lambda \to 0} A_\lambda(x) = A^0(x).$$

Let $\phi : H \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous, convex, proper function. The subdifferential of $\phi$ at $x \in D(\phi)$ is denoted by $\partial \phi(x)$ and represents the set of all subgradients at $x$, i.e.

$$\partial \phi(x) = \{y \in H/(y, x' - x)_H \leq \phi(x') - \phi(x) \forall x' \in H\}.$$  

The subdifferential of $\phi$, denoted $\partial \phi$ is the operator on $H$ defined by

$$(x, y) \in \partial \phi \iff y \in \partial \phi(x)$$

which is a maximal monotone operator ([60]).

For any $\lambda > 0$ the Yosida approximations of $\phi$, denoted $\phi_\lambda$ are defined as the inf-convolutions between $\phi$ and $(2\lambda)^{-1}|| \cdot ||_H^2$

$$\phi_\lambda(x) = \inf_{x' \in H} \left\{ \phi(x') + \frac{||x - x'||_H^2}{2\lambda} \right\}.$$  

For any $\lambda > 0$, $\phi_\lambda$ is convex, Fréchet differentiable with the derivative $\phi'_\lambda(x)$ equal to $(\partial \phi)_\lambda(x)$ and

$$\phi_\lambda(x) = \phi \left( J_{\lambda}^{\partial \phi}(x) \right) + \frac{||x - J_{\lambda}^{\partial \phi}(x)||_H^2}{2\lambda}.$$  

Also for every $x \in H$, $\phi_\lambda(x)$ is increasing and converges to $\phi(x)$ when $\lambda \to 0$ ([58], Ch. IV, Prop. 1.8).

Given a maximal monotone operator $A$ on the Hilbert space $H$ the theory of nonlinear semigroups ([61]) gives us the associated semigroup of contractions $\{S^A(t), \ t \geq 0\}$, defined on the closed convex set $D(A)$ such that for every $x \in D(A)$, $S^A(t)(x)$ is the unique solution for the evolution equation

$$\left\{ \begin{array}{ll}
\frac{du}{dt} + A(u(t)) & \geq 0, \ 0 < t, \\
u(0) & = x.
\end{array} \right.$$  

The generator of $\{S^A(t), \ t \geq 0\}$ is $-A^0$ and the correspondence in between maximal monotone operators and nonlinear semigroups of contractions on closed convex sets of $H$ is one to one. The correspondence is also continuous in the following sense:
Theorem 26. (Brezis) Let \((A_n)_{n \in \mathbb{N}}\), \(A\) be a sequence of maximal monotone operators on \(H\). The following are equivalent:

i) For every \(x \in D(A)\), there exists a sequence \(x_n\) strongly convergent to \(x\) in \(H\) such that \(A_n^0(x_n)\) converges strongly to \(A^0(x)\) in \(H\).

ii) For every \(x \in D(A)\) and every \(\lambda > 0\), \(J_{A_n}^\lambda(x)\) converges strongly to \(J_{A_\lambda}^\lambda(x)\) in \(H\).

iii) For every \(x \in D(A)\), there exists a sequence \(x_n\) strongly convergent to \(x\) in \(H\), such that \(x_n \in D(A_n)\) for every \(n \in \mathbb{N}\), and \(S_{A_n}(t)(x_n)\) converges strongly to \(S_{A}(t)(x)\) in \(H\).

If \(A = \partial \phi\) for a convex, proper, lower semicontinuous function \(\phi : H \to \mathbb{R}\), then a smoothing effect on initial conditions takes place, i.e. for every \(x \in D(A)\) and almost every \(t > 0\), \(S_{\partial \phi}(t)(x) \in D(A)\). In this case we also have the following result:

Theorem 27. (Brezis) Let \(f\) be given in \(L^2(0,T;H)\) and \(x \in D(A)\). Then the evolution problem

\[
\begin{cases}
\frac{du}{dt} + \partial \phi(u(t)) & \ni f, \quad 0 < t < T, \\
\quad u(0) = x,
\end{cases}
\]  

(5.2)

has a unique solution \(u \in C(0,T;H)\) which satisfies

\(u \in W^{1,2}(\delta,T;H)\) for every \(\delta \in (0,T)\), \(u(t) \in D(A)\) a.e. \(t \in (0,T)\), \(\sqrt{t} \frac{du}{dt} \in L^2(0,T;H)\) and \(\phi(u) \in L^1(0,T)\).

If in addition \(x \in D(\phi)\) then

\(\frac{du}{dt} \in L^2(0,T;H)\) and \(\phi(u) \in L^\infty(0,T)\).

Definition 6. Let \(A\) be a maximal monotone operator on a Hilbert space \(H\). We say that a sequence of maximal monotone operators \(A_n\) on \(H\) converges to \(A\) in the resolvent sense if

\[
\lim_{n \to \infty} (J_{A_n}^\lambda)^{-1}(x) = (J_{A_\lambda}^\lambda)^{-1}(x)
\]

for every \(\lambda > 0\) and every \(x \in H\).

The following theorem (see [62, 38, 2]) shows the connection between the convergence of convex functionals in the sense of Mosco and the convergence of their subdifferentials in the resolvent sense:

Theorem 28. Let \((\phi_n)_{n \in \mathbb{N}}\), \(\phi : H \to \mathbb{R} \cup \{+\infty\}\) be a sequence of proper, convex, lower semicontinuous functionals. Then the following are equivalent:

i) The sequence \(\phi_n\) \(M\)-converges to \(\phi\) in \(H\).

ii) For every \(\lambda > 0\) and any \(x \in H\), \((\phi_n)_\lambda(x)\) converges strongly to \(\phi_\lambda(x)\).
iii) $\partial \phi_n$ converges in the resolvent sense to $\partial \phi$ and there exists $(x, y) \in \partial \phi$ and $(x_n, y_n) \in \partial \phi_n$ such that $x_n \to x$, $y_n \to y$ and $\phi_n(x_n) \to \phi(x)$.

iv) For every $\lambda > 0$ and any $x \in H$, $(I + \lambda \partial \phi_n)^{-1}(x)$ converges weakly in $H$ to $(I + \lambda \partial \phi)^{-1}(x)$ and there exists $(x, y) \in \partial \phi$ and $(x_n, y_n) \in \partial \phi_n$ such that $x_n \to x$, $y_n \to y$ and $\phi_n(x_n) \to \phi(x)$.

Finally, for this chapter, we want to apply the following theorem, due to Attouch ([62, 2]):

**Theorem 29. (Attouch)** Let $\phi_n : H \to \mathbb{R} \cup \{+\infty\}$ be a sequence of proper, convex, lower semicontinuous functionals and assume $\partial \phi_n$ converges to $\partial \phi$ in the resolvent sense. Denote by $u_n, u$ the solutions for the following evolution problems, with initial condition $x_n \in D(\partial \phi_n), x \in D(\partial \phi)$

\[
\begin{aligned}
\frac{du_n}{dt} + \partial \phi_n(u_n(t)) &\ni f_n, \; 0 < t < T, \\
u_n(0) &= x_n,
\end{aligned}
\]  

(5.3)

and

\[
\begin{aligned}
\frac{du}{dt} + \partial \phi(u(t)) &\ni f, \; 0 < t < T, \\
u(0) &= x.
\end{aligned}
\]  

(5.4)

1) Assume that $x_n \to x$ in $H$ and $f_n \to f$ in $L^2(0, T; H)$. Then

$u_n \to u$ uniformly on $[0, T]$

and

\[
\int_0^T t \left\| \frac{du_n}{dt} - \frac{du}{dt} \right\|_H^2 dt \to 0 \text{ as } n \to \infty.
\]

2) Assuming moreover that $x_n \in D(\partial \phi_n)$, and $x \in D(\partial \phi)$ and $\phi_n(x_n) \to \phi(x)$ as $n \to \infty$, then

$\frac{du_n}{dt} \to \frac{du}{dt}$ strongly in $L^2(0, T; H)$

and

$\phi_n(u_n) \to \phi(u)$ uniformly on $[0, T]$. 

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5.2 Application to the Membrane Problem

We will chose for the application the case III from Chapter 3. Let 

\[ H = L^2(D), \]

and for any \( \epsilon > 0 \) let \( F^\epsilon \) be the convex functional on \( L^2(D) \), with \( D(F^\epsilon) = K^\epsilon \cap V \), defined by

\[
F^\epsilon(u) = \begin{cases} 
\frac{1}{2} a(u, u) + \int_{\Gamma^\epsilon} g|u_n|d\mathcal{L}^{n-1} & \text{for } u \in K^\epsilon \cap V, \\
+\infty & \text{for } u \notin K^\epsilon \cap V.
\end{cases}
\] (5.5)

Let us compute \( \partial F^\epsilon \). By the definition, for \( u \in D(F^\epsilon) \), \( \xi \in \partial F^\epsilon(u) \) if and only if for every \( v \in D(F^\epsilon) \)

\[
F^\epsilon(v) \geq F^\epsilon(u) + (\xi, v - u)_D \iff \frac{1}{2} a(v, v) + \int_{\Gamma^\epsilon} g|v_n|d\mathcal{L}^{n-1} \geq \frac{1}{2} a(u, u) + \int_{\Gamma^\epsilon} g|u_n|d\mathcal{L}^{n-1} + (\xi, v - u)_D.
\]

Take \( v = u + \lambda \phi \), with \( \phi \in (K^\epsilon \cap V) - (K^\epsilon \cap V) \), the hyperplane parallel with \( D(F^\epsilon) \) and \( \lambda \in \mathbb{R} \) and obtain that

\[
a(u, \phi) = (\xi, \phi)_D,
\]

which by Remark 2 implies the existence of \( p \in L^2(D) \) such that \( -\Delta u + \nabla p = \xi \) in \( D^\epsilon \). From here we obtain after several calculations that

\[
\int_{\Sigma} -[\sigma(u, p)e_n] \cdot e_n \leq \int_{\Gamma^\epsilon} (g|u_n + \phi_n| - g|u_n|)d\mathcal{L}^{n-1},
\]

for every \( \phi \in K^\epsilon \cap V \) which will lead by similar arguments as in Theorem 3 to

\[-[\sigma(u, p)e_n] \cdot e_n \in g\partial|u_n| \text{ a.e. on } \Gamma^\epsilon.\]

We obtained that the domain of \( \partial F^\epsilon \) is the set of all \( u \in K^\epsilon \cap V \) such that there exists \( p \in L^2(D) \) and \( \xi \in L^2(D) \) with \( -\Delta u + \nabla p = \xi \) in \( D^\epsilon \) and \( -[\sigma(u, p)e_n] \cdot e_n \in g\partial|u_n| \) a.e. on \( \Gamma^\epsilon \). Also for every \( u \in D(\partial F^\epsilon) \), \( \partial F^\epsilon(u) \) is the set of all such \( \xi \).

The nonlinear evolution problem

\[
\begin{cases} 
\frac{du^\epsilon}{dt} + \partial F^\epsilon(u^\epsilon(t)) \ni f^\epsilon, \ 0 < t < T, \\
u^\epsilon(0) = x^\epsilon,
\end{cases}
\] (5.6)
for \( x^\epsilon \in K^\epsilon \cap V \) and \( f^\epsilon \in L^2(0,T;L^2(D)) \) is equivalent with the system

\[
\begin{aligned}
\frac{du^\epsilon}{dt} - \Delta u^\epsilon + \nabla p^\epsilon &= f^\epsilon, & & \text{in } (0,T) \times D^\epsilon, \\
\nabla \cdot u^\epsilon &= 0, & & \text{in } (0,T) \times D^\epsilon, \\
u^\epsilon &= b, & & \text{on } (0,T) \times \partial D, \\
u_i^\epsilon &= 0, & & \text{on } (0,T) \times \Gamma^\epsilon \text{ for } 1 \leq i \leq n - 1, \\
-\sigma^\epsilon e_n \cdot e_n &\in g_\partial |u^\epsilon|, & & \text{on } (0,T) \times \Gamma, \\
u^\epsilon(0) &= x^\epsilon, & & \text{in } D^\epsilon,
\end{aligned}
\]

which models the time dependent membrane problem. The equalities are understood almost everywhere \( t \in [0,T] \).

Let \( F \) be the convex functional defined on \( L^2(D) \)

\[
F(u) = \begin{cases} 
\frac{1}{2} a(u,u) + \frac{\beta}{2} \int_\Sigma C u \cdot v d\mathcal{L}^{n-1} & \text{for } u \in H^1_0(D) \cap V, \\
+\infty & \text{for } u \notin H^1_0(D) \cap V.
\end{cases}
\]

It is easy to see in this case that for \( u \in H^1_0(D) \cap V \), a vector \( \xi \in L^2(D) \) belongs to \( \partial F(u) \) if and only if there exists \( p \in L^2(D) \) such that \(-\Delta u + \nabla p = \xi \) in \( D \setminus \Sigma \) and \(-[\sigma(u,p)e_n] = Cu \) on \( \Sigma \). So the nonlinear evolution problem

\[
\begin{aligned}
\frac{du}{dt} + \partial F(u(t)) &\ni f, & & 0 < t < T, \\
u(0) &= x,
\end{aligned}
\]

will be equivalent with the system

\[
\begin{aligned}
\frac{du}{dt} - \Delta u + \nabla p &= f, & & \text{in } (0,T) \times D \setminus \Sigma, \\
\nabla \cdot u &= 0, & & \text{in } (0,T) \times D, \\
u &= b, & & \text{on } (0,T) \times \partial D, \\
-\sigma e_n &= Cu, & & \text{on } (0,T) \times \Sigma, \\
u(0) &= x, & & \text{in } D,
\end{aligned}
\]

where \( x \in H^1_0(D) \cap V \) and \( f \in L^2(0,T;L^2(D)) \).

The next theorem is a straight forward application of Theorem 29 and the \( M^- \) convergence of \( F^\epsilon \) to \( F \).

**Theorem 30.** Assume that \( x^\epsilon \to x \) in \( L^2(D) \) and \( f^\epsilon \to f \) in \( L^2(0,T;L^2(D)) \). Then the solution \( u^\epsilon \) of 5.7 converges uniformly on \([0,T]\) to \( u \), the solution of 5.10 and
\[ \int_0^T t \left\| \frac{d u^\epsilon}{d t} - \frac{d u}{d t} \right\|^2 d t \to 0. \] If in addition \( x^\epsilon \in D(F^\epsilon) \), \( x \in D(F) \) and \( F^\epsilon(x^\epsilon) \) converges to \( F(x) \) then \( \frac{d u^\epsilon}{d t} \) converges strongly in \( L^2(0,T;L^2(D)) \) to \( \frac{d u}{d t} \) and \( F^\epsilon(u^\epsilon) \) converges to \( F(u) \).
Bibliography


