2012-04-24

On Quasi-Volume-Filling Surfaces

Pan Liu

Worcester Polytechnic Institute

Follow this and additional works at: https://digitalcommons.wpi.edu/etd-theses

Repository Citation
https://digitalcommons.wpi.edu/etd-theses/259

This thesis is brought to you for free and open access by Digital WPI. It has been accepted for inclusion in Masters Theses (All Theses, All Years) by an authorized administrator of Digital WPI. For more information, please contact wpi-etd@wpi.edu.
ON QUASI-VOLUME-FILLING SURFACES

PAN LIU

A Thesis

Submitted to the Faculty

of the

WORCESTER POLYTECHNIC INSTITUTE

In partial fulfillment of the requirements for the

Degree of Master of Science

in

Applied Mathematics

by

April 24, 2012

APPROVED:

_______________________________
Professor Umberto Mosco, Thesis Advisor

_______________________________
Professor Bogdan M. Vernescu, Head of Department
Abstract

The purpose of this paper is to construct a Quasi-volume-filling surface and study its properties. We start with the construction of a volume-filling surface, the Pólya surface, based on Pólya’s curve, by rotating the Pólya’s curve in 3-dimensional space. Then we construct a Quasi-space-filling curve in 2-dimensions, the Quasi-Pólya curve, which approximates the Pólya’s curve and fills a triangle up to a residual small surface of arbitrary size. We prove that the Quasi-Pólya curve satisfies the open set condition, and there exists a unique invariant (self-similar) measure consistent with the normalized Hausdorff measure on it. Moreover, the energy form constructed on Quasi-Pólya curve is proved to be a closed & regular form, and we prove that the Quasi-Pólya curve is a variational fractal in the end. Next, we use the same idea, by rotating the Quasi-Pólya curve in 3-dimensional space, to construct the Quasi-Pólya surface, which is a Quasi-volume-filling surface and approximates to Pólya surface in some sense.
Acknowledgements

The completion of this thesis is due to the support and guidance of many people. First, I would like to express my gratitude to my thesis advisor, Professor Umberto Mosco, for his enthusiasm, his encouragements and his great efforts to explain things clearly and simply. Throughout my thesis-writing period, he provided sound advice, good teaching, and plenty of good ideas. I would have been lost without him.

I would like to thank Professor Bogdan M. Vernescu, Professor Mayer Humi, Professor Joseph D. Fehribach, Professor Christopher J. Larsen, and Professor William J. Martin for their kind assistance with writing letters, giving wise advice, helping with various applications, and so on.

I wish to thank Hongdong Liang for helping me collect all necessary research books, papers, and computation programs. I would like to thank all my friends colleagues in WPI for help me get through the difficult times, and for all the emotional support, camaraderie, entertainment, and caring they provided.

I wish to thank my girlfriend Pei Pang for her infinite love. Without her company and support, I can not image the life I would have spend on everyday.

Finally, but most importantly, I wish to pay my deepest gratitude and love to my parents, Milin Liu and Xin Zhu. They born me, raised me, supported me, taught me, and loved me. To them I dedicate this thesis.
CONTENTS

Part 0. Introduction 1
  1. Introduction 1

Part 1. The Quasi-Pólya Construction 2
  2. The Quasi-Pólya Curve Construction 2
     2.1. The Pólya Curve 2
     2.2. The Quasi-Pólya Curve 4
     2.3. The Convergence of Quasi-Pólya curve 6
     3. The Quasi-Pólya Surface Construction 9

Part 2. The Analysis on Quasi-Pólya Construction 11
  4. The Energy Form on Unit Interval 11
     4.1. The Construction of Energy Form & Closeness and Regularity Properties 11
     4.2. The Analysis of Energy form $E^p[u]$ 23
     4.3. The Scaled Poincaré inequalities on $I$ 30
     5. The Energy Forms on Quasi-Pólya Curve $Q$ 36
        5.1. The Invariant Set, Measure, Metrics 36
        5.2. The Metric Properties of $Q$ 38
        5.3. The Energy Forms 42
     6. The Laplacian on the Quasi-Pólya curve 50

Part 3. Future Work 51
  7. Future Work 51

Part 4. Appendix 1: A TWO-DIMENSIONAL PÓLYA-TYPE MAP FILLING A PYRAMID 52
  1. Introduction 52
  2. Pólya’s Function and Lax Result 53
  3. Self-Similarity of Pólya’s Curve 54
  4. Three Dimensional Constructions and Differentiability 61

Part 5. References 67
  References 67
Part 0. Introduction

1. INTRODUCTION

The space-filling curves were well studied by mathematicians since the beginning of the 20th century. Examples of such curves are the Peano curve [4], the Hilbert curve [2], the Sierpiński curve [21] and the Pólya’s curve [5]. In my recent paper in Appendix 1, we find a way to expand the Pólya’s curve to 3 dimensions and construct a Pólya surface, as a volume-filling surface.

However, as all space-filling constructions, the Pólya curve, and Pólya surface, are self-intersecting. The idea of this paper is to approximate Pólya curve with a family of curves with no self-intersections, that fill a triangle up to a residual small area of arbitrary size. We call this new curve the Quasi-Pólya Curve (QP curve). Since the QP curve is not self-intersecting, we prove that it satisfies the open set condition and then we proceed to build the energy forms, to obtain some important energy estimates.

This paper is organized as follows: In Part 1, we first briefly reviewed the basic construction map of Pólya’s curve. Then, we give the motivation and the construction details of Quasi-Pólya curve, as well as the dimension of Quasi-Pólya curve and the proof that the Quasi-Pólya curve approximates the Pólya’s curve. In the end of Part 1, by using a similar idea for constructing the Pólya surface, we construct the Quasi-Pólya surface.

In Part 2 we proceed to construct the energy form on Quasi-Pólya curve. Unlike the Koch curve and the Sierpiński Gaskets, the Pólya’s curve and Quasi-Pólya curve are not symmetrical. Thus, the coefficients in energy forms on Quasi-Pólya curve can not be uniquely determined by using the method introduced in [17], and it gives us some difficulties since the energy form should be unique. Therefore, in Section 4, we build an energy form first on an un-equal partitioned unit interval to simulate the asymmetrical property of Quasi-Pólya curve, and we prove that there exists one and only one pair of coefficient in energy form which satisfies some important properties that are required in our future study.

In Section 5 of Part 2, we start to build the energy form on Quasi-Pólya curve. Based on the observation in Section 4, we are able to build an unique energy form on Quasi-Pólya curve and we prove this form is closed & regular. Finally, we prove that the Quasi-Pólya curve is a variational fractal in the sense of [15].
Part 1. The Quasi-Pólya Construction

2. The Quasi-Pólya Curve Construction

2.1. The Pólya Curve.

We first briefly review the construction of Pólya’s curve. The details of the construction of Pólya’s curve can be found in Appendix 1, Section 2, as well as some nice and unique properties of Pólya’s curve.

Definition 2.1. Given \( \theta \in (0, \pi/4) \), we define the contractive similitudes \( \{ \phi_0, \phi_1 \} \) that operate on \( I = [0,1] \) for Pólya’s curve in the following way

\[
\phi_0(x) = \cos \theta \cdot O_0 x + \cos \theta \cdot \mu_0
\]

with

\[
O_0 = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}, \quad \mu_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]

and

\[
\phi_1(x) = \sin \theta \cdot O_1 x + \cos \theta \cdot \mu_1
\]

with

\[
O_1 = \begin{bmatrix} \sin \theta & -\cos \theta \\ -\cos \theta & -\sin \theta \end{bmatrix}, \quad \mu_1 = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}.
\]

Definition 2.2. Setting \( P^0 = \Gamma = \{0,1\} \), for arbitrary \( n \)-tuples of indices \( i_1 \ldots i_n \in \{0,1\} \), \( n = 1,2, \ldots \), we define

\[
\phi_{i_1 \ldots i_n} := \phi_{i_1} \circ \cdots \circ \phi_{i_n},
\]

\[
P_n = \bigcup_{i_1 \ldots i_n=0}^{1} \phi_{i_1 \ldots i_n}(\Gamma),
\]

and

\[
P^\infty = \bigcup_{n=1}^{\infty} P_n.
\]

Definition 2.3. We define the Pólya’s curve \( P \) to be the closure of \( P^\infty \) in \( \mathbb{R}^2 \), i.e.,

\[
P := \overline{P^\infty}.
\]

As we state in Appendix 1, the Pólya’s curve \( P \) fills the right triangle \( T = \Delta ABC \). That is, \( P \) has dimension 2. There is a major drawback to Pólya Curve, however, it violates the Open Set Condition, short as o.s.c, which defined in [9], Section 5.2, Definition (1). More precisely, let’s consider the triangle \( T^\circ := T \setminus \{A, B, C\} \), and we have

\[
\phi_0(T^\circ) \cup \phi_1(T^\circ) \subset T^\circ,
\]

but

\[
\phi_0(T^\circ) \cap \phi_1(T^\circ) \neq \emptyset,
\]
which violates the *o.s.c.*

Here we provide the graph of $P^2$ and $P^4$, and we will take a close look at $P^2$ to explain why $P^2$ violates the *o.s.c.*

\[
P^2 = \{ \phi_0 \circ \phi_0(\Gamma), \phi_0 \circ \phi_1(\Gamma), \phi_1 \circ \phi_0(\Gamma), \phi_1 \circ \phi_1(\Gamma) \}.
\]

Then, by (2.1) and (2.2), we obtain that $\phi_0 \circ \phi_1(\Gamma) = \phi_1 \circ \phi_0(\Gamma)$. In fact, they are all the red curve in Figure 1.

In the modern fractals analysis, however, *o.s.c.* is a very important aspect in order to proceed to future study. Therefore, motivated by the above reasons, we create a new type of curve, or a sequence of curves, which has the following properties:

**Note 2.4.** We wish to construct a new curve which has the following properties:

1. The new curve converges to Pólya’s curve in some sense.
2. The new curve satisfies *o.s.c.*
3. The dimension of new curve is smaller then 2.
2.2. The Quasi-Pólya Curve.

**Definition 2.5.** Let $T = \Delta ABC$ be a non-isosceles right triangle located at points $A = (0, 0)$, $B = (1, 0)$, and $C = (\cos^2 \theta, \cos \theta \sin \theta)$, where $\theta$ is the smallest angle in $T$. Given $h \in (0, \cos \theta \sin \theta)$, we pinpoint the point $H$ on the altitude of $T$ such that $CH = h$. By connecting point $H$ and two ending points on hypotenuse of $T$, we define a new non-isosceles obtuse triangle $L_h = \Delta ABH$ inside of $T$.

**Definition 2.6.** Given $\theta \in [0, \pi / 4]$ and $h \in [0, \sin \theta \cos \theta]$, we define the contractive similarities $\psi = \{\psi_0, \psi_1\}$ that operate on $I = [0, 1]$ for Quasi-Pólya curve in the following way:

$$\psi_0([x_1; x_2]) = r_0 \cdot \begin{bmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & -\cos \alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

and

$$\psi_1([x_1; x_2]) = r_1 \cdot \begin{bmatrix} \cos \beta & -\sin \beta \\ -\sin \beta & -\cos \beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + r_1 \cdot \begin{bmatrix} -\cos \beta \\ \sin \beta \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

where

$$\sin \alpha = \frac{\cos \theta \sin \theta - h}{\sqrt{\cos^4 \theta + (\cos \theta \sin \theta - h)^2}},$$

$$\sin \beta = \frac{\cos \theta \sin \theta - h}{\sqrt{\sin^4 \theta + (\cos \theta \sin \theta - h)^2}},$$

and

$$r_0 = \sqrt{\cos^4 \theta + (\cos \theta \sin \theta - h)^2},$$

$$r_1 = \sqrt{\sin^4 \theta + (\cos \theta \sin \theta - h)^2}. \quad (2.6)$$

For short, we write $\psi_0$ and $\psi_1$ as

$$\psi_0(x) = r_0 Q_0 x + r_0 q_0, \quad \psi_1(x) = r_1 Q_1 x + r_1 q_1, \quad (2.7)$$

where

$$q_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad q_1 = \begin{bmatrix} -\cos \beta + 1/r_1 \\ \sin \beta \end{bmatrix}, \quad (2.8)$$

and

$$Q_0 = \begin{bmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & -\cos \alpha \end{bmatrix}, \quad Q_1 = \begin{bmatrix} \cos \beta & -\sin \beta \\ -\sin \beta & -\cos \beta \end{bmatrix}.$$

**Definition 2.7.** Setting $Q^0 = \Gamma = \{0, 1\}$, for arbitrary $n$-tuples of indices $i_1 \ldots i_n \in \{0, 1\}$, $n = 1, 2, \ldots$, we define

$$\phi_{i_1 \ldots i_n} := \phi_{i_1} \circ \cdots \circ \phi_{i_n},$$

$$Q^n = \bigcup_{i_1 \ldots i_n = 0}^1 \phi_{i_1 \ldots i_n}(\Gamma), \quad (2.9)$$

and

$$Q^\infty = \bigcup_{n=1}^\infty Q^n.\quad (2.10)$$

**Definition 2.8.** We define the Quasi-Pólya curve $Q$ to be the closure of $Q^\infty$ in $\mathbb{R}^2$, i.e.,

$$Q := \overline{Q^\infty}.$$
Here we provide the graph of $Q^2$ and $Q^4$. Especially, we put both $P^2$ and $Q^2$ (blue) in Figure 3 to help understanding the difference between Pólya’s curve and Quasi-Pólya curve. Note that the length of red curve in Figure 3 is the value of $h$ which was given in Definition 2.6. Also, it is clear that the Quasi-Pólya curve is not self-intersecting.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{The Pólya’s curve and Quasi-Pólya curve with 2 iterations.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{The Quasi-Pólya curve with 4 iterations.}
\end{figure}

**Lemma 2.9.** $Q$ is invariant with respect to $\psi = \{\psi_0, \psi_1\}$.

**Proof.** The general case has been proved in [9], Section 3, Theorem and Definitions (3). Here we refer to this proof and conclude that $Q$ is invariant with respect to $\psi = \{\psi_0, \psi_1\}$. \hfill \Box

**Lemma 2.10.** $Q$ satisfies the Open Set Condition.

**Proof.** We proof this lemma by using triangle $L_h$ defined in Definition 2.5. By the construction of $L_h$, it follows that for $L_h^o$, the interior of $L_h$, we have

$$\psi(L_h^o) \subset L_h^o.$$  

Moreover, since

$$(\psi_0(L_h))^o \cap (\psi_1(L_h))^o = \emptyset,$$

we obtain

$$\psi_0(L_h^o) \cap \psi_1(L_h^o) = \emptyset.$$ \hfill \Box

**Remark 2.11.** Since $Q$ is invariant with respect to $\psi = \{\psi_0, \psi_1\}$, we conclude that for given $\theta \in [0, \pi/4]$ and $h \in [0, \sin \theta \cos \theta]$, $Q$ has similarity dimension $s$ which satisfies

$$\sum_{i=0}^{1} r^s_i = (\cos^4 \theta + (\cos \theta \sin \theta - h)^2)^{\frac{s}{2}} + (\sin^4 \theta + (\cos \theta \sin \theta - h)^2)^{\frac{s}{2}} = 1.$$  (2.11)

**Lemma 2.12.** Given $\theta \in (0, \pi/4)$ and $h \in (0, \sin \theta \cos \theta)$, the similarity dimension $s$ of $Q$ in (2.11) is unique and $1 < s < 2$. i.e., the equation

$$(\cos^4 \theta + (\cos \theta \sin \theta - h)^2)^{\frac{s}{2}} + (\sin^4 \theta + (\cos \theta \sin \theta - h)^2)^{\frac{s}{2}} - 1 = 0$$

is unique and $1 < s < 2$. i.e., the equation

$$(\cos^4 \theta + (\cos \theta \sin \theta - h)^2)^{\frac{s}{2}} + (\sin^4 \theta + (\cos \theta \sin \theta - h)^2)^{\frac{s}{2}} - 1 = 0$$

is unique and $1 < s < 2$. i.e., the equation
has an unique solution.

Proof. First, we notice that for
\[ r_1 = \sqrt{\cos^4 \theta + (\cos \theta \sin \theta - h)^2}, \]
we have
\[ 0 < r_0, r_1 < 1. \]
Therefore, for \( f(s) = r_0^s + r_1^s - 1 \), we have
\[ \frac{d}{ds} f(s) = r_0^s \log r_0 + r_1^s \log r_1 < 0 \]
for all \( s \).
That is, the function \( f(s) \) either has a unique solution \( s \), or has no solution. Moreover, for \( s = 1 \), we have that
\[ f(1) = r_0 + r_1 - 1 > 0, \]
and for \( s = 2 \), together with the properties of \( r_1 \) and \( r_1 \), we have
\[ f(2) = r_0^2 + r_1^2 - 1 < 0. \]
So, we know that the solution of \( f(s) \) is unique and it belongs to interval \((1, 2)\).

\[ \square \]

Example 2.13. Here we give some examples of the dimension \( s \) of Quasi-Pólya curve with \( \theta = \pi/6 \) and different value of \( h \).

1. For \( h = 0.1 \), by (2.11), we have
\[ s \approx 1.531892451. \]

2. For \( h = 0.2 \), by (2.11), we have
\[ s \approx 1.251372206 \]

3. For \( h = 0.05 \), by (2.11), we have
\[ s \approx 1.884080771 \]

2.3. The Convergence of Quasi-Pólya curve.

Theorem 2.14 (Theorem 3.4 in Appendix 1).

The Pólya’s mapping, \( P(t) = P(0.d_1d_2d_3\ldots d_n\ldots) \), can be represented for any \( t = 0.d_1d_2d_3\ldots d_n\ldots \), by
\[ P(t) = P(0.d_3d_2d_3\ldots) = \sum_{j=1}^{\infty} c^{Z_j} s^{V_j+1} O_{d_1} O_{d_2} \cdots O_{d_{j-1}} \mu, \] (2.12)
where \( c = \cos \theta_0 \), \( s = \sin \theta_0 \), and \( Z_i, V_i \) is the number of 0’s, 1’s preceding \( d_i \), respectively, and \( \mu \) is defined in (3.1) in Appendix 1.

Proof. Please refer to Theorem 3.4 for proof. Also, for the details of the binary form \( 0.d_1d_2d_3\ldots d_n\ldots \) which represents \( t \in [0, 1] \), please refer to Section 2 in Appendix 1. \( \square \)
**Lemma 2.15.** The function $P(t)$ is not homeomorphism.

*Proof.* It is clear that for $t_1 = 0.11$ and $t_2 = 0.01$, we have

$$P(t_1) = P(t_2),$$

and this shows that the function $PC$ is not homeomorphism. □

**Theorem 2.16.** The Quasi-Pólya mapping, $QP_h(t) = QP_h(0.d_1d_2d_3\ldots d_n\ldots)$, can be represented by

$$QP_h(t) = QP_h(0.d_1d_2d_3\ldots) = \sum_{j=1}^{\infty} r_0^{Z_j+1} r_1 V_j Q_{d_1} Q_{d_2} \cdots Q_{d_{j-1}} q_{d_j}, \quad (2.13)$$

where $Z_j, V_j$ is the number of 0’s, 1’s preceding $d_j$, respectively.

*Proof.* The details of the technique we used in this proof can be found in Theorem 3.4, Appendix 1. Here we only state the key steps.

Note that

$$QP_h(0.d_1d_2d_3\ldots) = \lim_{n \to \infty} \psi_{d_1} \psi_{d_2} \psi_{d_3} \cdots \psi_{d_n} L_h. \quad (2.14)$$

Since $0.d_1d_2d_3\ldots d_n = 0.d_1d_2d_3d_4\ldots d_n0000\ldots$, we have

$$P(0,d_1d_2d_3\ldots d_n) = \psi_{d_1} \psi_{d_2} \psi_{d_3} \cdots \psi_{d_n} \psi_0 \psi_0 \psi_0 \cdots L_h.$$

It follows that

$$\psi^n_0(x) = r_0^n Q_0^n x, \quad (2.15)$$

where $Q_0 \cdot Q_0 = I$ is a $2 \times 2$ identity matrix.

Moreover,

$$\lim_{n \to \infty} r_0^n \cdot Q_0^n = \begin{cases} \lim_{n \to \infty} r_0^n \cdot I, & \text{if } n \text{ is even} \\ \lim_{n \to \infty} r_0^n \cdot Q_0, & \text{if } n \text{ is odd} \end{cases} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore, we have

$$QP_h(0.d_1d_2d_3\ldots d_n) = \psi_{d_1} \psi_{d_2} \psi_{d_3} \cdots \psi_{d_n} \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Substitution of $Q_i$, $q_i$ for $\psi_i$ from (2.7) yields that

$$QP_h(t) = QP_h(0,d_1d_2d_3\ldots d_n) = \sum_{j=1}^{n} r_0^{Z_j+1} r_1 V_j Q_{d_1} Q_{d_2} \cdots Q_{d_{j-1}} q_{d_j},$$

where $Q_{d_0} = I$.

Moreover, by continuity, we have

$$QP_h(t) = QP_h(0.d_1d_2d_3\ldots) = \sum_{j=1}^{\infty} r_0^{Z_j+1} r_1 V_j Q_{d_1} Q_{d_2} \cdots Q_{d_{j-1}} q_{d_j},$$

as we wish. □
Theorem 2.17. Given $\theta \in (0, \pi/4)$ and $h \in (0, \cos \theta \sin \theta)$, the function $QP_h(t)$ converges uniformly to $P(t)$ as $h \to 0$.

Proof. Given $\theta \in (0, \pi/4)$, the Pólya function $P(t)$ has the expression such that

$$P(t) = P(0.d_1d_2d_3\cdots) = \sum_{j=1}^{\infty} (\cos \theta)^{Z_{j+1}}(\sin \theta)^{V_j}O_{d_1}O_{d_2}\cdots O_{d_j-1}\mu_{d_j},$$

(2.16)

where $O_i$ and $\mu_i$ are defined in (2.1) and (2.2).

We first prove that $r_0$ converges to $\cos \theta$ uniformly for $\theta \in (0, \pi/4)$. For $h \in (0, \cos \theta \sin \theta)$,

$$|r_0 - \cos \theta| = \left| \sqrt{\cos^4 \theta + (\cos \theta \sin \theta - h)^2} - \cos \theta \right|$$

$$= \left| \sqrt{\cos^4 \theta + (\cos \theta \sin \theta - h)^2} - \sqrt{\cos^2 \theta} \right|$$

$$= \left| \cos^4 \theta + (\cos \theta \sin \theta - h)^2 - \cos^2 \theta \right|$$

$$= \left| \frac{\cos^4 \theta + (\cos \theta \sin \theta - h)^2 - \cos^2 \theta^2}{\sqrt{\cos^4 \theta + (\cos \theta \sin \theta - h)^2}} \right|$$

$$= \left| \frac{h^2 - 2h \cos \theta \sin \theta}{\sqrt{\cos^4 \theta + (\cos \theta \sin \theta - h)^2}} \right|$$

$$\leq \frac{|h^2 - 2h \cos \theta \sin \theta|}{2}.$$

Therefore, we have

$$\lim_{h \to 0} |r_0 - \cos \theta| = 0,$$

(2.17)

for all $\theta \in (0, \pi/4)$.

That is, $r_0$ converges to $\cos \theta$ uniformly as $h \to 0$, for all $\theta \in (0, \pi/4)$. Similarly, we prove that $r_1$ converges to $\sin \theta$ uniformly for $\theta \in (0, \pi/4)$ as well.

Next, we prove that $\sin \alpha$ converges uniformly to $\sin \theta$ for all $\theta \in (0, \pi/4)$.

Taking

$$\left| \sin \alpha - \sin \theta \right|$$

$$= \left| \frac{\cos \theta \sin \theta - h}{\sqrt{\cos^4 \theta + (\cos \theta \sin \theta - h)^2}} - \sin \theta \right|$$

$$= \left| \frac{\cos \theta \sin \theta - h}{\sqrt{\cos^4 \theta + (\cos \theta \sin \theta - h)^2}} - \sin \theta \right|$$

$$= \left| \frac{\cos \theta \sin \theta - h}{\sqrt{\cos^4 \theta + (\cos \theta \sin \theta - h)^2}} \right|$$

$$\leq \left| \frac{\cos \theta}{\sqrt{\cos^4 \theta + (\cos \theta \sin \theta - h)^2}} \right| + \left| \frac{h}{\sqrt{\cos^4 \theta + (\cos \theta \sin \theta - h)^2}} \right|.$$
Recall in (2.17) we already proved that 
\[ |\sqrt{\cos^4 \theta + (\cos \theta \sin \theta - h)^2} - \cos \theta| \to 0 \] as \( h \to 0 \), it follows that 
\[ \lim_{h \to 0} |\sin \alpha - \sin \theta| = 0, \] 
for all \( \theta \in (0, \pi/4) \).

Similarly, we prove that \( \sin \beta \) converges to \( \cos \theta \) uniformly for all \( \theta \in (0, \pi/4) \).

Therefore, by (2.17) and (2.18), we conclude that the function 
\[ f_h(t) = r_{j+1} Z_j Q_d_1 Q_d_2 \cdots Q_d_{j-1} q_{d_j} \] 
converges uniformly to function 
\[ f(t) = (\cos \theta)^{Z_j+1} (\sin \theta)^{V_j} O_{d_1} O_{d_2} \cdots O_{d_{j-1}} \mu_{d_j}, \] 
as \( h \to 0 \).

Hence by properties of uniform convergence, we have for any \( t \in [0, 1) \)
\[ \lim_{h \to 0} QP_h(t) = \lim_{h \to 0} \sum_{j=1}^{\infty} r_j Z_j^{j+1} r_j V_j Q_d_1 Q_d_2 \cdots Q_d_{j-1} q_{d_j} \]
\[ = \sum_{j=1}^{\infty} \lim_{h \to 0} r_j Z_j^{j+1} r_j V_j Q_d_1 Q_d_2 \cdots Q_d_{j-1} q_{d_j} \]
\[ = \sum_{j=1}^{\infty} (\cos \theta)^{Z_j+1} (\sin \theta)^{V_j} O_{d_1} O_{d_2} \cdots O_{d_{j-1}} \mu_{d_j} \]
\[ = P(t). \]

That is, the function \( QP_h(t) \) converges uniformly to \( P(t) \) as \( h \to 0 \). \( \square \)

3. THE QUASI-PÓLYA SURFACE CONSTRUCTION

Now we introduce our three dimensional Quasi-Pólya Surface construction. We use the same method to create the Quasi-Pólya Surface as we create the Pólya Surface in Appendix 1, Section 4. So, here we only state the key definitions and steps. For details of notations and construction map, please refer to Appendix 1, Section 4.

**Definition 3.1.** The General Quasi-Pólya Function:
We define the function \( GQP(\theta, t) \), which maps the rectangle \([0, \frac{\pi}{4}] \times [0, 1]\) into a triangle \( L_\theta^h \) with vertices \((0, 0), (0, 1), \) and \((\tan \theta, 0)\), by
\[ GQP_h(\theta, t) := \sum_{j=1}^{\infty} r_j Z_j^{j+1} r_j V_j Q_d_1 Q_d_2 \cdots Q_d_{j-1} q_{d_j}, \] 
by treating \( \theta \), in equation (2.13), as a variable too.
**Definition 3.2.** The Quasi-Pólya Surface construction map:

Given $\theta_0 \in (0, \frac{\pi}{4})$ and $h \in [0, \sin \theta_0 \cos \theta_0]$, we define the function $QS_{h}^{\theta_0}(\alpha, t)$, which maps the rectangle $[0, \frac{\pi}{2}] \times [0, 1]$ into $R^{\theta_0}$ by

$$QS_{h}^{\theta_0}(\alpha, t) = (x, y, z),$$

where, by setting $(a, b) = GQP_h(\theta(\alpha), t)$, $x = a \cos \alpha$, $y = a \sin \alpha$ and $z = b$.

For a better understanding, and to help making a mental picture of Pólya surface and Quasi-Pólya surface, we provide the pictures of Pólya surface and Quasi-Pólya surface at iteration 2 and 3.
Part 2. The Analysis on Quasi-Pólya Construction

4. THE ENERGY FORM ON UNIT INTERVAL


Through Section 4 we are in the real line $\mathbb{R}^1$, equip with the Euclidean distance $|p - q|$, $(p, q \in \mathbb{R}^1)$. By $\bar{A}$ we denote the closure of a set $A$ in $\mathbb{R}^1$. By $C(A)$ we denote the space of real-valued, continuous functions on $A$, and by $C(A)'$ its dual.

**Definition 4.1.** We define two contractive similitudes that operate on $I = [0, 1]$ in the following way:

$$
\phi_0(x) = \alpha_0 x, \quad \phi_1(x) = \alpha_1 x + \alpha_0,
$$

for $1/2 \leq \alpha_0 < 1$ and $\alpha_1 = 1 - \alpha_0$.

**Definition 4.2.** Setting $I^0 = \{0, 1\}$, for arbitrary $n$-tuples of indices $i_1 \ldots i_n \in \{0, 1\}$, $n = 1, 2, \ldots$, we define

$$
\phi_{i_1 \ldots i_n} := \phi_{i_1} \circ \ldots \circ \phi_{i_n},
$$

$$
I^n = \bigcup_{i_1 \ldots i_n = 0}^1 \phi_{i_1 \ldots i_n}(I^0),
$$

and

$$
I^\infty = \bigcup_{n=1}^\infty I^n, \quad I = \overline{I^\infty}. \quad (4.1)
$$

Before proceeding we introduce few notations.

**Notation 4.3.** Given $p = \phi_{i_1 \ldots i_n}(0) \in I^n$ and constants $\alpha_0$, $\alpha_1$, $\rho_0$, and $\rho_1$, we define

$$
\alpha_{i_1 \ldots i_n} = \alpha_{i_1} \cdot \alpha_{i_2} \cdots \alpha_{i_n}, \quad \rho_{i_1 \ldots i_n} = \rho_{i_1} \cdot \rho_{i_2} \cdots \rho_{i_n}
$$

and for $p = 1$, we define

$$
\alpha_n(1) = \rho_n(1) = 0.
$$

It is easily seen that $2^n = 2^n + 1$, and every point $p$ in $I^n \setminus I^0$ has two neighbors $q \in I^n$, which are called $n$-neighbors of $p$. We say that $p$, $q \in I^n$ are $n$-neighbors, if there exists a $n$-tuple of indices $i_1 \ldots i_n \in \{0, 1\}$ such that $p, q \in \phi_{i_1 \ldots i_n}(I^0)$. For given $p \in I^n$, we define these two $n$-neighbors of $p$ as $q_1(p)$ and $q_2(p)$ such that

$$
q_1(p) < p < q_2(p),
$$

and we have the distance of $n$-neighbors of $p$ as the following:

$$
|q_1(p) - p| = \alpha_n(q_1(p)), \quad |q_2(p) - p| = \alpha_n(p).
$$
Also, we notice that the set $I$ can be obtained as the union of an increasing number of copies of smaller and smaller size. In fact, by iterating (4.1), for every $n = 1, 2, \ldots$, we get

$$I = \bigcup_{i_1 \ldots i_n=0}^1 I_{i_1 \ldots i_n}.$$ 

Moreover, we define the subset

$$I_{i_1 \ldots i_n} = \phi_{i_1 \ldots i_n}(I),$$

of $I$ are called $n$-intervals and the sets

$$\Gamma_{i_1 \ldots i_n} = \phi_{i_1 \ldots i_n}(\Gamma)$$

are called $n$-cells, where $\Gamma = I^0 = \{0, 1\}$.

**Lemma 4.4.** [17] Let $A, B, \rho_0,$ and $\rho_1$ be real constants, we have

$$\min_c (\rho_0 |A - c|^2 + \rho_1 |B - c|^2) = \frac{\rho_0 \rho_1}{\rho_0 + \rho_1} (A - B)^2,$$

and the minimizing $\bar{c}$ is obtained by

$$\bar{c} = \frac{\rho_0 A + \rho_1 B}{\rho_0 + \rho_1}.$$ (4.4)

**Proof.** Define

$$f(c) = \rho_0 (A - c)^2 + \rho_1 (c - B)^2.$$ 

Taking $f'(c) = 0$, we obtain that

$$c = \frac{\rho_0 A + \rho_1 B}{\rho_0 + \rho_1}.$$ (4.5)

Therefore,

$$\min_c (\rho_0 |A - c|^2 + \rho_1 |B - c|^2) = \rho_0 \left[ A - \frac{\rho_0 A + \rho_1 B}{\rho_0 + \rho_1} \right]^2 + \rho_1 \left[ \frac{\rho_0 A + \rho_1 B}{\rho_0 + \rho_1} - B \right]^2$$

$$= \frac{\rho_0 \rho_1}{\rho_0 + \rho_1} (A - B)^2.$$ 

Thus, we get

$$\min_c (\rho_0 |A - c|^2 + \rho_1 |B - c|^2) = \frac{\rho_0 \rho_1}{\rho_0 + \rho_1} (A - B)^2.$$ 

□

Now, our objective is to construct the Dirichlet integral

$$E[u] = \int_I u'^2 dx,$$

with domain $D_E = C(I)$, $I = [0, 1]$, without making use of the notion of derivative $u'$.

Let $u: I^\infty \to \mathbb{R}^1$, we first observe $u$ on $I^0$ by putting $A = u(0), B = u(1)$. We define

$$E_0[u] := |u(0) - u(1)|^2 = (A - B)^2.$$
We now define \( E_0[u] \) is interpreted as the energy of \( u \) on \( I^0 = \{0, 1\} \).

Next, we observe \( u \) on \( I^1 \), which gives us the new value \( c = u(\alpha_0) \), and we define

\[
E_1[u] := \rho_0|A - c|^2 + \rho_1|c - B|^2,
\]

where \( \rho_0, \rho_1 > 0 \) are factors to be suitable determined.

Now we apply the "Gauss variational principle", formulating

\[
\min_c \{\rho_0|A - c|^2 + \rho_1|c - B|^2\} = |A - B|^2,
\]

and by Lemma 4.5, we get that (4.7) hold by setting

\[
c = \frac{\rho_0A + \rho_1B}{\rho_0 + \rho_1}, \quad \text{and} \quad \frac{\rho_0\rho_1}{\rho_0 + \rho_1} = 1,
\]

So, here, by Gauss variational principle, we proved that the pair \( \rho = (\rho_0, \rho_1) \) has to satisfy the relation

\[
\frac{\rho_0\rho_1}{\rho_0 + \rho_1} = 1
\]

in order to make the energy stationary.

**Notation 4.5.** In the following we will use \( \rho \) to denote the pair \( (\rho_0, \rho_1) \) which satisfies relation (4.9). Note that there are infinite many pairs \( \rho \) which satisfies (4.9).

We define the energy form by

\[
E^n_\rho[u] := \sum_{i_1 \ldots i_n = 0}^1 \rho_{i_1 \ldots i_n} E_0[u \circ \phi_{i_1 \ldots i_n}],
\]

or more explicitly

\[
E^n_\rho[u] = \sum_{i_1 \ldots i_n = 0}^1 \rho_{i_1 \ldots i_n} \frac{1}{2} \sum_{\xi, \eta \in \Gamma} |u(\phi_{i_1 \ldots i_n}(\xi)) - u(\phi_{i_1 \ldots i_n}(\eta))|^2
\]

\[
= \frac{1}{2} \sum_{p \in I^n} \left[ \rho_n(q_1(p))(u(p) - u(q_1(p)))^2 + \rho_n(p)(u(p) - u(q_2(p)))^2 \right].
\]

where \( \rho \) in \( E^n_\rho \) indicates the pair \( \rho = (\rho_0, \rho_1) \) we are using.

Then, if we denote by \( \bar{u} \) the function on \( I^1 \) taking the values \( \bar{u}(0) = A, \bar{u}(1) = B \) and \( \bar{u}(\alpha) = (\rho_0A + \rho_1B)/(\rho_0 + \rho_1) \), by Lemma 4.5 we find that, for arbitrary \( u \) defined on \( I^1 \),

\[
E^n_0[u] = E^n_\rho[\bar{u}] \leq E^n_\rho[u].
\]

It can be verified that for arbitrary \( u : I^\infty \to \mathbb{R} \), we have

\[
E^n_0[u|I^n] \leq E^n_\rho[u|I^n] \leq \cdots \leq E^n_0[u] \leq \cdots
\]

and (4.12) holds everywhere if \( \bar{u} \) is a function obtained by starting with \( \bar{u}|I^0 = \{A, B\} \) and then by extending \( \bar{u} \) successively from \( I^{n-1} \) to \( I^n \), \( n \geq 1 \), by defining \( \bar{u}(p) \) at each point \( p \in I^n - I^{n-1} \) to be the value obtained by (4.8) where \( A, B \) replaced by the values of \( \bar{u} \) at the two rationals \( p_1 < p_2 \in I^{n-1} \) such that \( \phi_0(p_2) = \phi_1(p_1) = p \).

We now define

\[
E^n_\rho[u] := \sup_{n \geq 0} E^n_\rho[u|I^n]
\]
on the domain
\[ D_{E^\rho}: = \{ u: I^\infty \to \mathbb{R}: E^\rho[u] < +\infty \} . \]  \tag{4.14} 
Note that the existence of \( \bar{u} \), for all value \( A \) and \( B \) on \( I^\rho \), proves that \( D_{E^\rho} \neq \emptyset \). That is, the domain \( D_{E^\rho} \) is non-trivial.

**Definition 4.6.** Given \( \rho = (\rho_0, \rho_1) \), we define the harmonic extension \( h^\rho[u] \) of \( u|I^\infty \) on \( I^\infty \) as the function \( \bar{u} \) constructed above.

Moreover, by \( E(\cdot, \cdot) \) we denote the bilinear form by polarization, i.e.,
\[ E^\rho(u, v) = \frac{1}{2}(E^\rho[u + v] - E^\rho[u] - E^\rho[v]). \]  \tag{4.15} 
It follows that \( E^\rho(u, v) \) is the limit of sequence \( E_n^\rho(u, v) \) given by
\[ E_n^\rho(u, v) = \frac{1}{2} \sum_{p \in I^n} [\rho_n(q_1(p))(u(p) - u(q_1(p)))(v(p) - v(q_1(p))}
+ \rho_n(p)(u(p) - u(q_2(p)))(v(p) - v(q_2(p))). \]  \tag{4.16} 

**Lemma 4.7.** *The Energy Form* \( E^\rho[u] \) *in (4.10) has relation such that*
\[ E^\rho[u] = \sum_{i=0}^1 \rho_i E^\rho[u \circ \phi_i], \quad u \in D_{E^\rho}. \]

**Proof.** Recall that for \( n \geq 1 \), we define the Energy Form as
\[ E_n^\rho[u] = \sum_{i_1...i_n=0}^1 \rho_{i_1...i_n} E_0^\rho[u \circ \phi_{i_1...i_n}], \]
or more explicitly
\[ E_n^\rho[u] = \sum_{i_1...i_n=0}^1 \rho_{i_1...i_n} \frac{1}{2} \sum_{\xi, \eta \in I^n} [u(\phi_{i_1...i_n}(\xi)) - u(\phi_{i_1...i_n}(\eta))]^2. \]
Therefore, we have
\[ \sum_{i=0}^1 \rho_i E_n^\rho[u \circ \phi_i] = \sum_{i=0}^1 \rho_i \sum_{i_1...i_n=0}^1 \rho_{i_1...i_n} E_0^\rho[u \circ \phi_{i_1...i_n}]
\]
\[ = \sum_{i_1...i_n=0}^1 \rho_{i_1...i_n} E_0^\rho[u \circ \phi_{i_1...i_n+1}], \]
\[ = E_{n+1}^\rho[u] \]
That is, by taking \( \lim_{n \to \infty} E_n^\rho[u] = E^\rho[u] \), we obtain
\[ E^\rho[u] = \sum_{i=0}^1 \rho_i E[u \circ \phi_i], \quad u \in D_{E^\rho}, \]
or
\[ E^\rho[u] = \sum_{i_1...i_n=0}^1 \rho_{i_1...i_n} E[u \circ \phi_{i_1...i_n}], \quad u \in D_{E^\rho} \]
as we wish. \( \square \)
**Notation 4.8.** Given $\alpha_0$ and $\alpha_1$ in Definition 4.1, we define a special pair $\rho^* = \{\rho_0^*, \rho_1^*\}$ by

$$\rho_0^* = \frac{1}{\alpha_0}, \quad \rho_1^* = \frac{1}{\alpha_1}. \quad (4.17)$$

Clearly, the pair $\rho^*$ satisfies condition (4.9), and we have for $p \in I^n$,

$$\rho_n^*(p) = \frac{1}{\alpha_n(p)}. \quad (4.18)$$

**Lemma 4.9.** Given $\rho \neq \rho^*$, for any non-constant harmonic extension $h_\rho \in D_{E_\rho}^\infty$, $h_\rho$ is no-where differentiable on a dense subset $I_\infty \subset I$.

**Proof.** Let $p \in I_\infty$ and $n$ be the smallest integer such that $p \in I^n$. Then, by the construction of $I_\infty$, there exists two consecutive points $p_l < p_s \in I^{n-1}$ such that

$$\phi_1(p_l) = \phi_0(p_s) = p,$$

and we define

$$L := |p_l - p_s|.$$  

To study the derivative of harmonic extension $h_\rho$ at point $p$, we first construct a sequence of points $\{p_n\}$ which approaches to $p$ from left. It is easily seen that the construction

$$p_n = \phi_0^\infty \circ \phi_0(p_l)$$

works. That is

$$p_n \leq p \quad \text{for all } n \geq 1,$$

and

$$\lim_{n \to \infty} |p - p_n| \to 0.$$

Clearly, by (4.8), we have

$$|p - p_1| = L_0 \alpha_1, \quad h_\rho(p_1) = \frac{\rho_0 h_\rho(p_1) + \rho_1 h_\rho(p)}{\rho_0 + \rho_1};$$

$$|p - p_2| = L_0 \alpha_1^2, \quad h_\rho(p_2) = \frac{\rho_0 h_\rho(p_1) + \rho_1 h_\rho(p)}{\rho_0 + \rho_1};$$

$$\vdots$$

$$|p - p_n| = L_0 \alpha_1^n, \quad h_\rho(p_n) = \frac{\rho_0 h_\rho(p_{n-1}) + \rho_1 h_\rho(p)}{\rho_0 + \rho_1}. \quad (4.19)$$

By (4.19), we have

$$h_\rho(p) - h_\rho(p_n) = h_\rho(p) - \frac{\rho_0 h_\rho(p_{n-1}) + \rho_1 h_\rho(p)}{\rho_0 + \rho_1}$$

$$= h_\rho(p) - \frac{\rho_1}{\rho_0 + \rho_1} h_\rho(p) - \frac{\rho_0}{\rho_0 + \rho_1} h_\rho(p_{n-1})$$

$$= \frac{\rho_0}{\rho_0 + \rho_1} h_\rho(p) - \frac{\rho_0}{\rho_0 + \rho_1} h_\rho(p_{n-1})$$

$$= \frac{1}{\rho_1} [h_\rho(p) - h_\rho(p_{n-1})].$$
Therefore, by the recursive relation, we have
\[ h_\rho(p) - h_\rho(p_n) = \frac{1}{\rho^2} [h_\rho(p) - h_\rho(p_n)]. \tag{4.20} \]
\[ \text{Together with the result of (4.19), we have} \]
\[ \lim_{p_n \to p^-} \frac{h_\rho(p) - h_\rho(p_n)}{p - p_n} = \lim_{n \to \infty} \frac{h_\rho(p) - h_\rho(p_n)}{p - p_n} = \lim_{n \to \infty} \frac{1}{\rho^n \alpha_1^n} \frac{h_\rho(p) - h_\rho(p_n)}{L\alpha_0} = \lim_{n \to \infty} \left( \frac{\rho^n}{\rho_0} \right)^n \frac{h_\rho(p) - h_\rho(p_n)}{L\alpha_1}. \]
By a similar construction and calculation, we obtain
\[ \lim_{p_n \to p^+} \frac{h_\rho(p) - h_\rho(p_n)}{p - p_n} = \lim_{n \to \infty} \left( \frac{\rho^n}{\rho_0} \right)^n \frac{h_\rho(p) - h_\rho(p_n)}{L\alpha_1}. \]
Therefore, if \( \rho_0 < \rho_0^* \), we have
\[ \lim_{p \to p^-} \frac{h_\rho(p) - h_\rho(p_n)}{p - p_n} = 0, \quad \text{but} \quad \lim_{p \to p^+} \frac{h_\rho(p) - h_\rho(p_n)}{p - p_n} = \infty; \]
if \( \rho_0 > \rho_0^* \), we have
\[ \lim_{p \to p^-} \frac{h_\rho(p) - h_\rho(p_n)}{p - p_n} = \infty, \quad \text{but} \quad \lim_{p \to p^+} \frac{h_\rho(p) - h_\rho(p_n)}{p - p_n} = 0. \]
Hence we conclude that the harmonic extension function \( h_\rho \) for \( \rho \neq \rho^* \) is no-where differentiable on a dense subset \( I^\infty \subset I \).

\[ \square \]

**Lemma 4.10.** The interval \( I \) has the following properties:

i. There exists a positive number \( \alpha = \alpha_0 \) such that \( I_{i_1 \ldots i_m} \cap I_{j_1 \ldots j_m} = \emptyset \) implies \( \text{dist}(I_{i_1 \ldots i_m}, I_{j_1 \ldots j_m}) \geq \alpha^m \) for every integer \( m > 0 \).

ii. If \( i_1 \ldots i_m \neq j_1 \ldots j_m \), then \( I_{i_1 \ldots i_m} \cap I_{j_1 \ldots j_m} = \Gamma_{i_1 \ldots i_m} \cap \Gamma_{j_1 \ldots j_m} \).

**Lemma 4.11.** [6] There exists two constants \( c \) and \( \beta \) such that for every \( u : I^\infty \to \mathbb{R} \) and for arbitrary \( p \) and \( q \) in \( I^\infty \), the following estimate holds:
\[ |u(p) - u(q)| \leq c \sqrt{\sup_{n \geq 0} E_n[u|I^n]|p - q|^\beta}, \tag{4.21} \]
where
\[ \rho_s = \min\{\rho_0, \rho_1\}, \quad c = 2\rho_s/\rho_s^{1/2} - 1, \quad \beta = -\log \rho_s/2 \log \alpha_0. \]

**Proof.** For arbitrary \( p, q \in I^\infty \subset I \), we know that \( I = \cup^1_{i_1 \ldots i_m=0} I_{i_1 \ldots i_m} \). Therefore, we obtain \( p \in I_{i_1 \ldots i_m} \) and \( q \in I_{j_1 \ldots j_m} \) for some \( i_1 \ldots i_m \) and \( j_1 \ldots j_m \in \{0, 1\}^m \).

Suppose that \( 0 < |p - q| \leq 1 \), there exists a \( m \geq 0 \) such that
\[ \alpha^{(m+1)} \leq |p - q| < \alpha^m \tag{4.22} \]
Since we choose \( p \in I_{i_1...i_m} \) and \( q \in I_{j_1...j_m} \), we have
\[
\min_{p \in I_{i_1...i_m}} |p - q| = \text{dist}(I_{i_1...i_m}, I_{j_1...j_m}).
\] (4.23)
Together with (5.33), we obtain
\[
\text{dist}(I_{i_1...i_m}, I_{j_1...j_m}) \leq |p - q| < \alpha^m.
\] (4.24)
Then, by Lemma 5.24, part i, we conclude that
\[
I_{i_1...i_m} \cap I_{j_1...j_m} \neq \emptyset.
\] (4.25)
Moreover, by Lemma 5.24, part ii, we obtain
\[
\Gamma_{i_1...i_m} \cap \Gamma_{j_1...j_m} \neq \emptyset.
\] (4.26)
Thus, there exists a point \( \gamma \in \Gamma_{i_1...i_m} \cap \Gamma_{j_1...j_m} \) such that
\[
\gamma = \phi_{i_1...i_m}(\xi) = \phi_{j_1...j_m}(\eta)
\] (4.27)
where \( \xi, \eta \in \Gamma = I^0 = \{0, 1\} \).

For \( n \geq m \), since \( p, q \in I^\infty \), there exists the smallest \( n \geq m \) such that \( p, q \in I^n \). Then, \( p = \phi_{i_1...i_n}(\bar{\xi}) \) and \( q = \phi_{j_1...j_n}(\bar{\eta}) \), where \( \bar{\xi}, \bar{\eta} \in \Gamma \).

Now, we are going to construct a chain of points to connect \( p \) and \( q \) from two sides.
We start with
\[
x_n := p = \phi_{i_1...i_n}(\bar{\xi}) = \phi_{i_1i_2...i_{m+1}...i_n}(\bar{\xi}),
\] (4.28)
and we define
\[
x_{n-1} := \phi_{i_1...i_{n-1}}(\bar{\xi}) = \phi_{i_1i_2...i_{m+1}...i_{n-1}}(\bar{\xi});
x_{n-2} := \phi_{i_1...i_{n-2}}(\bar{\xi}) = \phi_{i_1i_2...i_{m+1}...i_{n-2}}(\bar{\xi});
\]
\[
\vdots
\]
\[
x_{n-k} := \phi_{i_1...i_{n-k}}(\bar{\xi}) = \phi_{i_1i_2...i_{m+1}...i_{n-k}}(\bar{\xi}),
\]
where \( 0 \leq k \leq n - m \).

Now we have a chain of points \( x_n, x_{n-1}, \ldots, x_m \). Then we insert point \( \gamma \) by defining \( x_{m-1} := \gamma = \phi_{i_1...i_m}(\eta) \).

Similarly, by defining \( y_n := q, y_{n-k} = \phi_{j_1...j_{n-k}}(\bar{\eta}) \), where \( 0 \leq k \leq n - m \), we obtain \( \gamma \) again by letting \( y_{m-1} = \gamma = \phi_{j_1...j_m}(\eta) \).

Thus, we have constructed a chain
\[
p = x_n, x_{n-1}, \ldots, x_m, x_{m-1} = \gamma = y_{m-1}, y_m, \ldots, y_{n-1}, y_n = q
\] (4.29)
with a property that two consecutive points in the chain belong to the same cell.

Now we study the case when \( k = 0 \). Let \( \bar{\xi} \) be the fixed point of \( \phi_{i_0} \), so \( x_{n-1} = \phi_{i_1 \ldots i_{n-1} i_0}(\bar{\xi}) \). If \( i_0 = i_n \), then \( x_n = x_{n-1} \). If \( i_0 \neq i_n \), then \( \phi_{i_0}(\bar{\xi}) = \phi_{i_0}(\bar{\eta}) \) for some \( \bar{\eta} \in \Gamma \). So \( x_n = \phi_{i_1 \ldots i_n}(\bar{\xi}) = \phi_{i_1 \ldots i_{n-1} i_0}(\bar{\eta}) \). Therefore \( x_n \), \( x_{n-1} \in \Gamma_{i_1 \ldots i_{n-1} i_0} \).

We are ready to estimate \( |u(p) - u(q)| \). By the chain constructed above, we have

\[
|u(p) - u(q)| \leq \sum_{k=0}^{n-m} [u(x_{n-k}) - u(x_{n-k-1})] + |u(y_{n-k}) - u(y_{n-k-1})|. \tag{4.30}
\]

Since \( \bar{\xi} = \phi_{i_0}(\bar{\xi}) \) with \( \phi_{i_{n-k}}(\bar{\xi}) = \phi_{i_0}(\bar{\eta}) \), we obtain

\[
|u(x_{n-k}) - u(x_{n-k-1})|^2 = |u(\phi_{i_1 \ldots i_{n-k-1} i_{n-k}}(\bar{\xi})) - u(\phi_{i_1 \ldots i_{n-k-1} i_0}(\bar{\xi}))|^2 \\ = |u(\phi_{i_1 \ldots i_{n-k-1} i_0}(\bar{\eta})) - u(\phi_{i_1 \ldots i_{n-k-1} i_0}(\bar{\xi}))|^2 \\ \leq \sum_{i_1 \ldots i_{n-k} = 0}^{1} |u(\phi_{i_1 \ldots i_{n-k}}(\bar{\eta})) - u(\phi_{i_1 \ldots i_{n-k}}(\bar{\xi}))|^2 \tag{4.31}
\]

Define \( \rho_s := \min\{\rho_0, \rho_1\} \) and multiply both sides by \( \rho_s^{n-k} \) to obtain

\[
\rho_s^{n-k}|u(x_{n-k}) - u(x_{n-k-1})|^2 \leq \rho_s^{n-k} \sum_{i_1 \ldots i_{n-k} = 0}^{1} \left\{ \frac{1}{2} \sum_{E', \eta'} |u(\phi_{i_1 \ldots i_{n-k}}(E')) - u(\phi_{i_1 \ldots i_{n-k}}(\eta'))|^2 \right\} \\
= \sum_{i_1 \ldots i_{n-k} = 0}^{1} \rho_s^{n-k} \left\{ \frac{1}{2} \sum_{E', \eta'} |u(\phi_{i_1 \ldots i_{n-k}}(E')) - u(\phi_{i_1 \ldots i_{n-k}}(\eta'))|^2 \right\} \\
\leq \sum_{i_1 \ldots i_{n-k} = 0}^{1} \rho_s^{n-k} \left\{ \frac{1}{2} \sum_{E', \eta'} |u(\phi_{i_1 \ldots i_{n-k}}(E')) - u(\phi_{i_1 \ldots i_{n-k}}(\eta'))|^2 \right\} \\
\leq E'_{n-k}[u] \\
\leq E[u]. \tag{4.32}
\]

So, we conclude that

\[
\rho_s^{n-k}|u(x_{n-k}) - u(x_{n-k-1})|^2 \leq E[u]. \tag{4.33}
\]

That is

\[
|u(x_{n-k}) - u(x_{n-k-1})|^2 \leq \rho_s^{\frac{k-n}{2}} \sqrt{E[n][u]}. \tag{4.34}
\]
Clearly, the same result holds for terms with $y$ and we obtain

$$|u(p) - u(q)| \leq 2 \sum_{k=0}^{n-m} \rho_s^{-\frac{k}{2}} \sqrt{E^\rho[u]}$$

$$\leq 2 \rho_s^{-\frac{n}{2}} \sqrt{E^\rho[u]} \sum_{k=0}^{n-m} \rho_s^{\frac{k}{2}}$$

$$\leq 2 \rho_s^{-\frac{n}{2}} \sqrt{E^\rho[u]} \frac{\rho_s^{\frac{n-m+1}{2}} - 1}{\rho_s^\frac{1}{2} - 1}$$

$$\leq \frac{2}{\rho_s^\frac{1}{2} - 1} \sqrt{E^\rho[u] |\rho_s^{\frac{n-m}{2}}}$$

$$= \frac{2 \rho_s}{\rho_s^\frac{1}{2} - 1} \sqrt{E^\rho[u] |\rho_s^{\frac{n-m}{2}}}.$$  \hspace{1cm} (4.35)

It is clear that

$$\rho_s^{-\frac{n-m-1}{2}} = \alpha_0^{-\frac{(m-1)(\log \alpha_1/\rho_s)}{2}},$$

and we define

$$\beta = -\frac{\log \rho_s}{2 \log \alpha_0}.$$  

Then by (5.33), we conclude

$$|u(p) - u(q)| \leq \frac{2 \rho_s}{\rho_s^\frac{1}{2} - 1} \sqrt{E^\rho[u]} |p - q|^\beta.$$  \hspace{1cm} (4.36)

Let $c = 2 \rho_s/(\rho_s^{1/2} - 1)$, finally we obtain

$$|u(p) - u(q)| \leq c \sqrt{E^\rho[u]} |p - q|^\beta.$$  \hspace{1cm} (4.37)

\[\square\]

**Corollary 4.12.** From Lemma 5.25 we know that every function $u \in D^\infty_{E^\rho}$ can be uniquely extended to a continuous function on $I$.

Therefore, every function $u \in D^\infty_{E^\rho}$ can be uniquely extended to an element of $C(I)$. We denote this extension still by $u$ and set

$$D_{E^\rho} = \{u \in C(I): E^\rho[u] < +\infty\},$$

where $E^\rho[u] = E^\rho[u|_{I=}].$  \hspace{1cm} (4.38)

**Definition 4.13.** Let $H$ denotes the Hilbert space $H = L^2(X, L^1(I))$, A form $E^\rho$ is closed in $H$ if its domain $D_{E^\rho}$ is complete under the inner product $E^\rho(u, v) + (u, v)_{L^2(I, L^1)}$.

**Lemma 4.14.** $D_{E^\rho}$ is complete under the norm

$$\|u\|_{D_{E^\rho}} = (\|u\|^2_{L^2(I, L^1)} + E^\rho[u])^{\frac{1}{2}}.$$  \hspace{1cm} (4.39)
Proof. We choose a Cauchy sequence \( \{ u_n \} \) in \( D_{E^\rho} \) such that
\[
\| u_n - u_m \|_{D_{E^\rho}} = \left( \| u_n - u_m \|_{L^2(I, \mathcal{L}^1)}^2 + E^\rho[u_n - u_m] \right)^{\frac{1}{2}} \to 0
\]
for \( n, m \to \infty \). Then we have
\[
\| u_n - u_m \|_{L^2(I, \mathcal{L}^1)}^2 \to 0,
\]
and
\[
E^\rho[u_n - u_m] \to 0.
\]
That is, there exist two constants \( C_1 \) and \( C_2 > 0 \) such that \( \| u_n \|_{L^2(I, \mathcal{L}^1)} \leq C_1 \) and \( E^\rho[u_n] \leq C_2 \), because Cauchy sequences are bounded.

Now we show that \( u_n(x) \) is uniformly bounded on \( I \).

For any \( x, y \in I \), we have
\[
|u_n(x)| \leq |u_n(x) - u_n(y)| + |u_n(y)|
\leq c\sqrt{E^\rho[u_n]}|x - y|^\beta + |u_n(y)|
\leq cC_2diam(I)^\beta + |u_n(y)|
\leq cC_2 + |u_n(y)|,
\]
where \( c, C_2 \) are constant. As \( \mu(I) = \int_I d\mathcal{L}^1 = 1 \), integrating on both sides in \( \mu(dy) \) gives
\[
|u_n(x)| \leq cC_2 + \int_I |u_n(y)|d\mu(y).
\]
(4.41)

By Schwarz inequality
\[
|u_n(x)| \leq cC_2 + \mu(I)^{\frac{1}{2}} \left( \int_I |u_n(y)|^2 d\mu(y) \right)^{\frac{1}{2}}
\leq cC_2 + C_1^{\frac{1}{2}},
\]
where \( C_1 \) is constant.

Additionally, it can be proved that the functions \( u_n(x) \) are equicontinuous, since for any \( x, y \in I \), we have
\[
|u_n(x) - u_n(y)| \leq c\sqrt{E^\rho[u_n]}|x - y|^\beta \leq cC_2diam(I)^\beta \leq cC_2.
\]
(4.42)

Hence, \( \{ u_n(x) \} \) is uniformly bounded and equicontinuous on \( I \). By Ascoli-Arzela theorem, there exists a subsequence \( \{ u_{n_k} \} \) of \( \{ u_n \} \) and \( u \in \mathcal{C}(I) \) such that
\[
|u_n(x) - u_{n_k}(x)| \to 0
\]
for \( k \to \infty \). It follows that \( u \in L^2(I, \mathcal{L}^1) \) as \( \mathcal{C}(I) \subset L^2(I, \mathcal{L}^1) \), and
\[
\| u_n - u \|_{L^2(I, \mathcal{L}^1)} \to 0
\]
(4.43)
for \( n \to \infty \).
Now we prove that \( u \in D_{E^p} \), and \( E^p[u_n - u] \to 0 \) as \( n \to 0 \).

Since \( E^p_k[u_n - u] \) is a finite sum, for a fixed \( n \), we have

\[
E^p_k[u_n - u] = \lim_{m \to \infty} E^p_k[u_n - u_m] \leq \lim_{m \to \infty} E^p[u_n - u_m] \tag{4.44}
\]

Let \( k \to \infty \), then

\[
E^p[u_n - u] = \lim_{m \to \infty} E^p[u_n - u_m],
\]

and

\[
\lim \sup_{m \to \infty} E^p[u_n - u_m] = \lim_{n,m \to \infty} E^p[u_n - u_m] = 0,
\]

which implies

\[
\lim_{n \to \infty} E^p[u_n - u] = 0.
\]

Therefore, we proved that there exists a \( u \in D_{E^p} \) such that

\[
\|u_n - u\|_{D_{E^p}} = \left( \|u_n - u\|^2_{L^2(I, L^1)} + E^p[u_n - u] \right)^{\frac{1}{2}} \to 0
\]

for \( n \to \infty \), i.e., the completeness of \( D_{E^p} \).

\( \square \)

**Lemma 4.15.** The form \( E^p(u, v) \) is closed in \( H \) with domain \( D_{E^p} \), under the norm

\[
\|u\|_{D_{E^p}} = (\|u\|^2_{L^2(I, L^1)} + E^p[u])^{\frac{1}{2}}.
\]

**Proof.** This is the result of Lemma 5.28. \( \square \)

**Lemma 4.16.** \( D_{E^p} \) is dense (uniform) in \( C(I) \).

**Proof.** Given any function \( f \in C(I) \), since \( I \) is compact, we know that \( f \) is uniformly continuous on \( I \).

Now, we extend the values \( f|_{I_i_1 \ldots i_n} \) to a harmonic function inside \( I_{i_1 \ldots i_n} \) by the average procedure given by Gauss variational principle. Noticing that by doing the harmonic extension in all \( I_{i_1 \ldots i_n} \), we construct a continuous function, named as \( f_n \), on \( I \). Next, by the maximum principle which comes from the Gauss minimization, we obtain that the extended harmonic function \( f_n \) will take max and min on \( \Gamma_{i_1 \ldots i_n} \).

It is clear that for any \( x \in I \), \( x \in I_{i_1 \ldots i_n} \) for some \( i_1 \ldots i_n \). Also, there exists a point \( y \) in \( \Gamma_{i_1 \ldots i_n} \) such that \( |x - y| \leq \alpha_{i_1 \ldots i_n} \) and

\[
\text{osc}_{I_{i_1 \ldots i_n}} f_n \leq \text{osc}_{\Gamma_{i_1 \ldots i_n}}.
\]

Moreover, we have

\[
f_n(x) = f_n(y) + f_n(x) - f_n(y) = f(y) + f_n(x) - f_n(y),
\]

since \( y \in \Gamma_{i_1 \ldots i_n} \).

Then, we have

\[
|f(x) - f_n(x)| \leq |f(x) - f(y)| + |f_n(x) - f_n(y)|.
\]

Therefore, we have

\[
\max_{x \in I} |f(x) - f_n(x)| \leq \max_{x,y \in I} |f(x) - f(y)| + \text{osc}_{I_{i_1 \ldots i_n}} f_n \leq \max_{x \in I} |f(x) - f(y)| \to 0. \tag{4.45}
\]
Hence we have shown that there exists a function \( f_n \in D_{E^n} \) such that
\[
\| f - f_n \|_\infty \to 0, \text{ as } n \to \infty.
\]

\( \square \)

**Definition 4.17.** A form \( E \) in \( H \) is regular if it possesses a core, a core being any subset \( C \) of \( D_E \cap C(X) \), which is dense both in \( C_0(X) \) with the uniform norm and in \( D[E] \) with the intrinsic norm \( (E(u,u) + \langle u, u \rangle)^{\frac{1}{2}} \).

**Lemma 4.18.** The form \( E^0 \) is regular in \( H \).

**Proof.** This is the result of Lemma 5.30. \( \square \)

**Lemma 4.19.** Given two pairs \( \rho^a = (\rho_0^a, \rho_1^a) \) and \( \rho^b = (\rho_0^b, \rho_1^b) \), where \( \rho_0^a \neq \rho_0^b \), we have
\[
D_{E^{\rho^a}} \cap D_{E^{\rho^b}} = D_0,
\]
where by \( D_0 \) we denote the set of constant functions on \( I \).

**Proof.** First of all, it is clear that for any constant functions \( d_0 \in D_0 \) will make the energy trivial. That is, \( E^{\rho^0}[d_0] = E^{\rho^0}[d_0] \equiv 0 \) all the time. So, we have \( D_0 \subset D_{E^{\rho^a}} \cap D_{E^{\rho^b}} \).

Now we prove that for any non-constant harmonic extension \( h_{\rho^a} \in D_{E^{\rho^a}}, h_{\rho^a} \notin D_{E^{\rho^b}} \).

It is clear that for \( h_{\rho^a}(0) = A \) and \( h_{\rho^a}(1) = B \), we have
\[
E_{\rho^a}[h_{\rho^a}] = (A - B)^2.
\]
Now we consider the energy form \( E_{\rho^b}[h_{\rho^a}] \). To estimate the value of \( E_{\rho^b}[h_{\rho^a}] \), we start from the beginning, as \( n = 1 \). By (4.10), we have
\[
E_{\rho^1}^{a}[h_{\rho^a}] = \rho_0^b \left| A - \frac{\rho_0^a}{\rho_0^a} A + \frac{\rho_1^a}{\rho_0^a} B \right|^2 + \rho_1^b \left| \frac{\rho_0^a}{\rho_0^a} A + \frac{\rho_1^a}{\rho_0^a} B - B \right|^2
\]
\[
= \rho_0^b \left| \frac{\rho_0^a}{\rho_0^a} (A - B) \right|^2 + \rho_1^b \left| \frac{\rho_0^a}{\rho_0^a} (A - B) \right|^2
\]
\[
= (A - B)^2 \frac{\rho_0^b (\rho_0^a)^2 + \rho_1^b (\rho_0^a)^2}{(\rho_0^a \rho_1^a)^2}.
\]

Then, for \( n = 2 \), omit some calculation details, we obtain
\[
E_{\rho^2}^{a}[h_{\rho^a}] = (\rho_0^b)^2 |A - h_{\rho^a}(\phi_0 \circ \phi_1(0))| + \rho_0^b \rho_1^a |h_{\rho^a}(\phi_0 \circ \phi_1(0)) - h_{\rho^a}(\phi_0 \circ \phi_0(0))| + \rho_0^b \rho_1^b |h_{\rho^a}(\phi_0 \circ \phi_1(0)) - h_{\rho^a}(\phi_1 \circ \phi_0(0))| + (\rho_1^b)^2 |h_{\rho^a}(\phi_1 \circ \phi_0(0)) - B|
\]
\[
= \rho_0^b (A - h_{\rho^a}(\phi_0 \circ \phi_0(0)))^2 \frac{\rho_0^b (\rho_0^a)^2 + \rho_1^b (\rho_0^a)^2}{(\rho_0^a \rho_1^a)^2} + \rho_1^b (B - h_{\rho^a}(\phi_1 \circ \phi_0(0)))^2 \frac{\rho_1^b (\rho_0^a)^2 + \rho_0^b (\rho_0^a)^2}{(\rho_0^a \rho_1^a)^2}
\]
\[
= (A - B)^2 \left[ \frac{\rho_0^b (\rho_0^a)^2 + \rho_1^b (\rho_0^a)^2}{(\rho_0^a \rho_1^a)^2} \right]^2.
\]

Generally, we have
\[
E_{\rho^n}^{a}[h_{\rho^a}] = (A - B)^2 \left[ \frac{\rho_0^b (\rho_0^a)^2 + \rho_1^b (\rho_0^a)^2}{(\rho_0^a \rho_1^a)^2} \right]^n. \tag{4.46}
\]
That is
\[ \lim_{n \to \infty} E_{\rho_n}^a[h_{\rho_n}] = \lim_{n \to \infty} (A - B)^2 \left[ \frac{\rho_0^b(\rho_1^b)^2 + \rho_1^b(\rho_0^a)^2}{(\rho_0^a \rho_1^b)^2} \right]^n = \infty. \]
Therefore, we have shown that \( h_{\rho_n} \notin D_{E_{\rho_b}} \), for all non-constant \( h_{\rho_n} \in D_{E_{\rho_a}} \).

Now we are going to show that for any non-constant function \( u \in D_{E_{\rho_a}} \), we have \( E_{\rho_b}[u] = \infty \).

For given \( u \in D_{E_{\rho_a}} \), we construct the function \( \bar{u}_n \) as the extension of function \( u|_{I^n} \) to harmonic function inside \( I_{i_1 \ldots i_n} \). Since \( u \in D_{E_{\rho_a}} \), we know that
\[ \lim_{n \to \infty} E_{\rho_{\bar{u}_n}}[u] = \lim_{n \to \infty} E_{\rho_{\bar{u}_n}}[u|_{I^n}] = E_{\rho_a}[u]. \]
However, since the function \( \bar{u}_n \) contains the harmonic extension \( h_{\rho_a} \) in \( D_{E_{\rho_a}} \), we know that
\[ E_{\rho_b}[\bar{u}_n] = \infty, \quad \text{for all } n \geq 1. \]
Hence we obtain that
\[ E_{\rho_b}[u] = \infty \]
as well, which implies \( u \notin D_{E_{\rho_b}} \).

Finally, we conclude that
\[ D_{E_{\rho_a}} \cap D_{E_{\rho_b}} = \emptyset. \]

\[ \square \]

4.2. The Analysis of Energy form \( E^\rho[u] \).

**Definition 4.20.** Given \( n \geq 1 \), we define a discrete measure \( \mu^n \) on \( I^n \) by
\[ \mu_n = \sum_{p \in I^n} \delta_{\{p\}} \cdot \alpha_n(p), \quad \text{(4.47)} \]
where \( \delta_{\{p\}} \) denotes the Dirac measure at point \( p \) and \( \alpha_n(p) \) is defined in Notation 4.3.

**Lemma 4.21.** The sequence \( \{\mu_n\}_{n \geq 1} \) is weakly convergent to the measure \( \mu \).

**Proof.** Taking \( \varphi \in C(I) \), we consider
\[ I \int \varphi d\mu_n = \sum_{p \in I^n} \alpha_n(p) I \int \varphi \, d\delta_{\{p\}} = \sum_{p \in I^n} \alpha_n(p) \varphi(p). \]
Therefore, we have
\[ \left| \int_I \varphi \, d\mu_n \right| = \left| \sum_{p \in I^n} \alpha_n(p) \varphi(p) \right| \leq \max_I |\varphi| \sum_{p \in I^n \setminus \{1\}} \alpha_n(p). \quad \text{(4.48)} \]
Notice that for $n = 1$, we have
\[ \sum_{p \in I^1} \alpha_n(p) = \alpha_0 + \alpha_1 = (\alpha_0 + \alpha_1)^1. \]
For $n = 2$, we have
\[ \sum_{p \in I^2} \alpha_n(p) = \alpha_0^2 + \alpha_0\alpha_1 + \alpha_1\alpha_0 + \alpha_1^2 = (\alpha_0 + \alpha_1)^2. \]
Generally, we have
\[ \sum_{p \in I^n} \alpha_n(p) = (\alpha_0 + \alpha_1)^n = 1. \tag{4.49} \]
Therefore, together with (4.48) and (4.49), we have
\[ \left| \int_I \varphi d\mu_n \right| \leq \max_I |\varphi| \sum_{p \in I^n} \alpha_n(p) \leq 2 \max_I |\varphi|. \]
That is, we have proven that \( \{\mu_n\}_{n \geq 1} \) is bounded in \( C(I)' \) and we conclude that there exists a subsequence convergence weakly to a \( \mu \in C(I)' \), which is the Lebesgue measure in 1-dimension restricted to the unit interval.

Now we take \( \varphi \equiv 1 \in C(I) \), repeat the calculation above, we obtain that
\[ \int_I \varphi d\mu_n = (\alpha_0 + \alpha_1)^n = 1. \]
Since \( \varphi \in C(I) \), we obtain
\[ \lim_{n \to \infty} \int_I d\mu_n = \lim_{n \to \infty} (\alpha_0 + \alpha_1)^n = 1 = \int_I dx, \]
and it concludes that \( \mu_n \) convergence weakly to \( \mu \) which is the 1 dimensional Lebesgue measure on \( I \). \qed

**Lemma 4.22.** The 1 dimensional Lebesgue measure on \( I \), denote as \( \mathcal{L}^1 \mid I \), is invariant with respect to \((\phi_i, \alpha_i)\) defined in Definition 4.1.

**Proof.** This proof is same to the proof for Lemma 5.5. \qed

**Example 4.23.** If we take \( \alpha_0 = \alpha_1 = 1/2 \), we have for (4.47)
\[ \mu_n = \frac{1}{2^n} \sum_{p \in I^n} \delta_{(p)}, \]
and we obtain,
\[ \lim_{n \to \infty} \int_0^1 \varphi(x) d\mu_n = \lim_{n \to \infty} \frac{1}{2^n} \sum_{p \in I^n} \varphi(p) = \int_I \varphi \, dx. \]

Next, we are going to construct the discrete Lagrangian on \( I^n \). One can find the basic concept and properties of Lagrangians operator in [16].

**Definition 4.24.** For every \( n \geq 0, p \in I^n \setminus \{1\} \), we define the classical discrete gradient for function \( u, v \in D_{E^p} \), by
\begin{align*}
\nabla_n u \cdot \nabla_n v(p) :&= \frac{1}{2} \left[ \frac{u(p) - u(q_2(p))}{|p - q_2(p)|} \frac{v(p) - v(q_2(p))}{|p - q_2(p)|} + \frac{u(p) - u(q_1(p))}{|p - q_1(p)|} \frac{v(p) - v(q_1(p))}{|p - q_1(p)|} \right] \\
&= \frac{1}{2} \left[ \frac{u(p) - u(q_2(p))}{\alpha_n(p)} \frac{v(p) - v(q_2(p))}{\alpha_n(p)} + \frac{u(p) - u(q_1(p))}{\alpha_n(q_1(p))} \frac{v(p) - v(q_1(p))}{\alpha_n(q_1(p))} \right]. \tag{4.50}
\end{align*}
where \( q_1(p) \) and \( q_2(p) \) are defined in Notation 4.3.

**Remark 4.25.** Here we choose \( u, v \in C^1_0(I) \) in order to be consistent with classical discrete gradient because

\[
\lim_{n \to \infty} \nabla_n u \cdot \nabla_n v(p) = u'(p)v'(p),
\]

if \( u, v \in C^1_0(I) \).

Following Notation 4.8, we can re-write (4.24) as

\[
\nabla_n u \cdot \nabla_n v(p) = \frac{1}{2} [\rho_n^*(p)]^2 (u(p) - u(q_2(p))) (v(p) - v(q_2(p)))
+ \rho_n^*(q_1(p))^2 (u(p) - u(q_1(p))) (v(p) - v(q_1(p))].
\]

**Definition 4.26.** Given \( \rho = \{ \rho_0, \rho_1 \} \) satisfies (4.9), for any point \( p \in I^n \), we define the function \( a_n^\rho(p) \) to be

\[
a_n^\rho(p) := \frac{\rho_0(p)}{\rho_n^*(p)},
\]

where \( \rho_n(p) \) is defined in Notation 4.3.

**Definition 4.27.** We define the Lagrangean form for \( u, v \in D_{E^\rho} \) on \( A \subset I \) as

\[
L_n^\rho(u, v) := \int_{A \cap I^n} a_n^\rho \nabla_n u \cdot \nabla_n v \, d\mu_n
\]

\[
= \int_{A \cap I^n} \frac{1}{2} [a_n^\rho(p)(\rho_n^*(p))^2 (u(p) - u(q_2(p))) (v(p) - v(q_2(p)))
+ a_n^\rho(q_1(p))(\rho_n^*(q_1(p))^2 (u(p) - u(q_1(p))) (v(p) - v(q_1(p)))] d\mu_n,
\]

where \( \mu_n \) defined in (4.47).

**Lemma 4.28.** Let \( u, v \in D_{E^\rho} \), the Energy Form \( E_n^\rho \) defined in (4.16) can be obtained by integrating the Lagrangean \( L_n^\rho(u, v) \) over \( I^n \), that is

\[
E_n^\rho(u, v) = \int_I dL_n^\rho(u, v).
\]

**Proof.** Taking a pair \( \rho = (\rho_0, \rho_1) \) satisfies (4.9), we have

\[
\int_I dL_n^\rho(u, v) = \int_I a_n^\rho(x) \nabla_n u \cdot \nabla_n v(x) d\mu_n
\]

\[
= \int_I a_n^\rho(x) \nabla_n u \cdot \nabla_n v(x) \alpha_n(p) \delta_{(p)} dx
\]

\[
= \int_I \frac{1}{2} [a_n^\rho(x) \alpha_n(x)(\rho_n^*(x))^2 (u(x) - u(q_2(x))) (v(x) - v(q_2(x)))
+ a_n^\rho(q_1(x)) \alpha_n(q_1(x))(\rho_n^*(q_1(x))^2 (u(x) - u(q_1(x))) (v(x) - v(q_1(x)))] \delta_{(p)} dx
\]

\[
(4.54)
\]
That is
\[\int_I dL^\rho_n(u, v) = \int_I \frac{1}{2}[(\rho_n(x))(u(x) - u(q_2(x)))(v(x) - v(q_2(x))]
+ (\rho_n(q_1(x)))(u(x) - u(q_1(x)))(v(x) - v(q_1(x)))]d\delta_{p_1} dx\]
= \sum_{p \in I^n} \frac{1}{2}[(\rho_n(p))(u(p) - u(q_2(p)))(v(p) - v(q_2(p))]
+ (\rho_n(q_1(p)))(u(p) - u(q_1(p)))(v(p) - v(q_1(p)))],
\tag{4.55}
which is exactly the right-hand side of (4.16).
That is, we have proven that
\[E^\rho_n(u, v) = \int_I a^n(\rho, x)\nabla_n u \cdot \nabla_n v(x) d\mu_n = \int_I dL^\rho_n(u, v). \tag{4.56}\]
\[\square\]

**Lemma 4.29.** Let \(A\) be any subset of \(I\). For every \(u, v \in D_{E^\rho},\) the sequence of measures given by
\[L^\rho_n(u, v) := \int_A a^n(\rho_n, x)\nabla_n u \cdot \nabla_n v d\mu_n \quad (n \geq 0) \tag{4.57}\]
weakly converges in \(C(I)^{\prime}\) to a signed finite Radon measure \(L^\rho(u, v),\) the so-called Lagrangian measure, on \(I\) as \(n \to \infty.\) Moreover
\[E^\rho(u, v) = \int_I dL^\rho(u, v) \quad (u, v \in D_{E^\rho}).\]

**Proof.** Taking any \(\varphi(x) \in C^1(I),\) we have
\[\int_0^1 \varphi(x) dL^\rho_n(u, v) = \sum_{p \in I^n} \varphi(p) \frac{1}{2}[(\rho_n(p))(u(p) - u(q_2(p)))(v(p) - v(q_2(p))]
+ (\rho_n(q_1(p)))(u(p) - u(q_1(p)))(v(p) - v(q_1(p)))]
\leq ||\varphi||_{\infty} \sum_{p \in I^n} \frac{1}{2}[(\rho_n(p))(u(p) - u(q_2(p)))(v(p) - v(q_2(p))]
+ (\rho_n(q_1(p)))(u(p) - u(q_1(p)))(v(p) - v(q_1(p)))]
\leq ||\varphi||_{\infty} \left(\frac{1}{2} \sum_{p \in I^n} [\rho_n(p)(u(p) - u(q_2(p)))]^2 + \rho_n(q_1(p))(u(p) - u(q_1(p))]^2\right)^{\frac{1}{2}}
\leq \frac{1}{2} \sum_{p \in I^n} [\rho_n(p)(v(p) - v(q_2(p)))]^2 + \rho_n(q_1(p))(v(p) - v(q_1(p))]^2\right)^{\frac{1}{2}}
\leq ||\varphi||_{\infty} E^\rho[u]^{\frac{1}{2}} E^\rho[v]^{\frac{1}{2}} < \infty.
\]
This shows that
\[\left|\int_I \varphi dL^\rho_n(u, v)\right| < \infty \text{ uniformly,}\]
and hence we have proved that
\[\lim_{n \to \infty} \int_0^1 \varphi(x) dL^\rho_n(u, v) = \int_0^1 \varphi(x) dL^\rho(u, v), \tag{4.58}\]
\]
where $L^\rho(u, v)$ is a signed Radon measure.

Now I want to consider the density of the measure

$$L^\rho(u, u) := \lim_{n \to \infty} L^n_{\rho}(u, u),$$

where $u \in D_{E^\rho} = \{ u \in C(I); E^\rho[u] < +\infty\}$, with respect to the 1-d Lebesgue measure. That is, for $x \in I$, we are interesting in the limit

$$\lim_{r \to 0} \int_{B_r(x)} dL^\rho(u, u) \mu(B_r(x)),$$

where $\mu$ is the Lebesgue measure.

**Example 4.30.** Before we introduce the main Theorem of this section, we first do a simple example of equation (4.59) to see what is the meaning of it.

Simply, we take $\rho = \rho^*$ and $u = x^2 \in C^1(I)$. Clearly, we have

$$\int_{B_r(x)} dL^\rho(u, u) = \int_{B_r(x)} 4x^2 dx = \frac{4}{3} x^3|_x^{x+r} = \frac{4}{3} (2r^3 + 6x^2 r).$$

Then, we have

$$\lim_{r \to 0} \frac{\int_{B_r(x)} dL^\rho(u, u)}{\mu(B_r(x))} = \frac{4}{3} (2r^2 + 6x^2),$$

and hence

$$\lim_{r \to 0} \frac{\int_{B_r(x)} dL^\rho(u, u)}{\mu(B_r(x))} = \lim_{r \to 0} \frac{4}{3} (2r^2 + 6x^2) = 4x^2,$$

which is exactly $u''^2$, if $u'$ exists.

**Theorem 4.31** (Radon-Nikodym). Given a measurable space $X$, if a $\sigma$-finite measure $\nu$ on $X$ is absolutely continuous with respect to a $\sigma$-finite measure $\mu$ on $X$, then there is a measurable function $f$ on $X$ and taking values in $[0, \infty)$, such that

$$\nu(A) = \int_A fd\mu$$

for any measurable set $A$.

**Lemma 4.32.** Given a pair $\rho \neq \rho^*$, for any non-constant harmonic extension $h_\rho \in D_{E^\rho}$, the density of $L^\rho[h_\rho]$ with respect to 1-dimensional Lebesgue measure is singular.

**Proof.** Let $\mu$ denote the 1-dimensional Lebesgue measure. To show singularity, by Theorem 4.31, we have to show that there does not exist a function $f \in L^1(d\mu)$ such that

$$\int_{I_r} dL[h_\rho] = \int_{I_r} f d\mu, \text{ for every intervals } I_r \subset I.$$

We suppose such a function exists, and we define

$$f_n(x) = \sum_{i_1 \cdots i_n = 0}^1 \left( \frac{1}{\alpha_{i_1 \cdots i_n}} \int_{I_{i_1 \cdots i_n}} f d\mu \right) \chi_{I_{i_1 \cdots i_n}},$$

be the piecewise constant approximation to $f$ at step $n$. Then, we would have $f_n \to f$ in the $L^1$ norm. But we will prove that this is impossible.
Suppose there exists a function \( f \in L^1(d\mu) \) such that
\[
\int_{I_r} d\mathcal{L}[h_\rho] = \int_{I_r} f \, d\mu, \quad \text{for every interval } I_r \subset I.
\]
Then we obtain that, by taking \( I_r = I \),
\[
\int_I f \, d\mu = \int_I d\mathcal{L}[h_\rho] = E^\rho[h_\rho].
\]
Therefore, together with (4.60), we obtain that
\[
f_n(x) = \frac{1}{\alpha_{i_1 \cdots i_n}} E^\rho_{I_{i_1 \cdots i_n}}[h_\rho], \quad \text{for } x \in I_{i_1 \cdots i_n},
\]
and
\[
f_{n+1}(x) = \frac{1}{\alpha_{i_1 \cdots i_n} \alpha_i} E^\rho_{\phi_i(I_{i_1 \cdots i_n})}[h_\rho], \quad \text{for } x \in \phi_i(I_{i_1 \cdots i_n}),
\]
where by \( E_{I_{i_1 \cdots i_n}}[h_\rho] \) we denote the energy of \( h_\rho \) on interval \( I_{i_1 \cdots i_n} \).

It follows that
\[
\|f_n - f_{n+1}\|_{L^1} = \frac{1}{\alpha_{i_1 \cdots i_n}} \sum_{i_1 \cdots i_n=0}^1 \sum_{i=0}^1 \left| \alpha_i E^\rho_{I_{i_1 \cdots i_n}}[h_\rho] - E^\rho_{\phi_i(I_{i_1 \cdots i_n})}[h_\rho] \right|,
\]
and also
\[
\|f_n\|_{L^1} = \frac{1}{\alpha_{i_1 \cdots i_n}} \sum_{i_1 \cdots i_n=0}^1 |E^\rho_{I_{i_1 \cdots i_n}}[h_\rho]|.
\]

Next we claim that there exists a positive constant \( k > 0 \) such that
\[
\sum_{i=0}^1 \left| \alpha_i E^\rho_{I_{i_1 \cdots i_n}}[h_\rho] - E^\rho_{\phi_i(I_{i_1 \cdots i_n})}[h_\rho] \right| \geq k |E^\rho_{I_{i_1 \cdots i_n}}[h_\rho]|.
\]

Recall the definition of \( E^\rho|h_\rho| \) on (4.10), we obtain that
\[
E^\rho_{I_{i_1 \cdots i_n}}[h_\rho] = \frac{1}{\rho_{i_1 \cdots i_n}} E^\rho[h_\rho] = \frac{1}{\rho_{i_1 \cdots i_n}} E^\rho_0[h_\rho].
\]
Therefore, we have
\[
\left| \alpha_i E^\rho_{I_{i_1 \cdots i_n}}[h_\rho] - E^\rho_{\phi_i(I_{i_1 \cdots i_n})}[h_\rho] \right| = \left| \alpha_i \frac{1}{\rho_{i_1 \cdots i_n}} E^\rho_0[h_\rho] - \frac{1}{\rho_{i_1 \cdots i_n} \cdot \rho_i} E^\rho_0[h_\rho] \right|
\]
\[
= \frac{1}{\rho_{i_1 \cdots i_n} \cdot \rho_i} \left| \alpha_i E^\rho_0[h_\rho] - \frac{1}{\rho_i} E^\rho_0[h_\rho] \right|,
\]
for \( i \in \{0, 1\} \).

So, in order to prove (4.63), we only have to show
\[
\sum_{i=0}^1 \left| \alpha_i E^\rho_0[h_\rho] - \frac{1}{\rho_i} E^\rho_0[h_\rho] \right| \geq k |E^\rho_0[h_\rho]|.
\]
We extend the right hand side of (4.64) and obtain
\[
\left| \sum_{i=0}^{1} \frac{\alpha_i E_0^\rho[h_\rho]}{\rho_i} \right| - \left| \frac{1}{\rho_0} E_0^\rho[h_\rho] \right| + \left| \frac{1}{\rho_1} E_0^\rho[h_\rho] \right| = \left( \left| \alpha_0 - \frac{1}{\rho_0} \right| + \left| \alpha_1 - \frac{1}{\rho_1} \right| \right) E_0^\rho[h_\rho].
\]
(4.65)

Since the pair \( \rho \neq \rho^* \), we know that
\[
\left| \alpha_0 - \frac{1}{\rho_0} \right| \neq 0, \quad \left| \alpha_1 - \frac{1}{\rho_1} \right| \neq 0,
\]
and hence the constant \( k \), defined as
\[
k := \left| \alpha_0 - \frac{1}{\rho_0} \right| + \left| \alpha_1 - \frac{1}{\rho_1} \right| > 0,
\]
(4.66)

independent of \( n \).

Therefore, use the constant \( k \) in (4.66), we obtain the inequality (4.64), and hence (4.63).

Now, combining (4.63) with (4.61) and (4.62) yields
\[
\|f_n - f_{n+1}\|_{L^1} \geq k\|f_n\|_{L^1},
\]
and this contradicts to \( f_n \to f \neq 0 \) in \( L^1 \).

Therefore, together with Lebesgue Decomposition Theorem, we have proven that \( L^\rho[h_\rho] \) is singular with respect to \( \mu \), for any non-constant harmonic extension \( h_\rho \in D_{E^\rho} \).

**Theorem 4.33.** Given a pair \( \rho \neq \rho^* \), for any non-constant function \( u \in D_{E^\rho} \), the density of \( L^\rho[u] \) with respect to 1 dimensional Lebesgue measure is singular.

**Proof.** Given a non-constant function \( u \in D_{E^\rho} \), for each \( n \geq 1 \), we define a function \( \bar{u}_n \) as the extension of function \( u|_{I^n} \) to harmonic function inside \( I_{i_1...i_n} \). Since \( u \in D_{E^\rho} \), we know that
\[
\lim_{n \to \infty} E^\rho[\bar{u}_n] = \lim_{n \to \infty} E^\rho[u|_{I^n}] = E^\rho[u].
\]
(4.67)

Since the function \( \bar{u}_n \) contains the harmonic extension for each \( I_{i_1...i_n} \), by Lemma 4.32 we know that \( L^\rho[\bar{u}_n] \) is singular in each \( I_{i_1...i_n} \). Hence we conclude that \( L^\rho[\bar{u}_n] \) is singular on \( I \), i.e.
\[
E^\rho[\bar{u}_n] \text{ is singular for all } n \geq 1.
\]
(4.68)

To prove \( L^\rho[u] \) is singular on \( I \), we suppose that there exists a function \( f \in L^1(d\mu) \) such that
\[
E^\rho[u] = \int_I f d\mu,
\]
(4.69)

and we will show that this is impossible.

By (4.67), we know that there exists a constant \( N \) large enough such that for each \( n \geq N \), there exists a
constant $0 < \varepsilon_n < 1$ satisfies
\[ E^\rho[\overline{u}_n] = E^\rho[u] - \varepsilon_n. \]
Together with (4.69), we obtain that
\[ E^\rho[\overline{u}_n] = \int_I f d\mu - \varepsilon_n = \int_I f d\mu - \int_I \varepsilon_n d\mu = \int_I (f - \varepsilon_n) d\mu. \]
Since $\varepsilon_n$ is bounded, we know that $f - \varepsilon_n \in L^1$ as well. Hence this is contradict to the fact that $E^\rho[\overline{u}_n]$ is singular.

Finally, we have shown that the density of $L^\rho[u]$ with respect to 1 dimensional Lebesgue measure is singular, for all non-constant function $u \in D_{E^\rho}$.

**Remark 4.34.** Actually the inequality (4.64) gives us the fact that it is impossible to split the energy proportionally among two intervals $\phi_0(I)$ and $\phi_1(I)$ with respect to the ratio $\alpha_0/\alpha_1$. That is, the fact will hold
\[ \frac{E^\rho_{\phi_0(I)}}{E^\rho_{\phi_1(I)}} \neq \frac{\alpha_0}{\alpha_1}, \]
as long as $\rho \neq \rho^*$. But once we choose $\rho = \rho^*$, the constant $k = 0$ and we can not get the contradiction from above argument, which agrees with the fact that $E^{\rho^*}$ has the density with respect to $L^1$ as showed in Example 4.30.

4.3. The Scaled Poincaré inequalities on $I$. [15]

**Lemma 4.35.** Given $\rho \neq \rho^*$, there exists one and only one constant $\delta > 1$, such that (4.70) and (4.71) hold:
\[ d(x, y) \equiv |x - y|^{\delta} \quad (4.70) \]
\[ d^2(x, y) \equiv \sum_{i=0}^1 \rho_i d^2(\phi_i(x), \phi_i(y)) \quad (4.71) \]
for every $x, y \in I$. Such a $\delta$ is uniquely determined by the identity
\[ \sum_{i=0}^1 \rho_i \alpha_i^{2\delta} = 1. \quad (4.72) \]

**Proof.** We first prove that (4.72) has a unique solution $\delta > 1$.
Define
\[ f(\delta) = \sum_{i=0}^1 \rho_i \alpha_i^{2\delta} - 1 = \rho_0 \alpha_0^{2\delta} + \rho_1 \alpha_1^{2\delta} - 1, \]
then we have
\[ \frac{d}{d\delta} f(\delta) = 2\rho_0 \alpha_0^{2\delta} \ln \alpha_0 + 2\rho_1 \alpha_1^{2\delta} \ln \alpha_1 < 0 \quad \text{for all } \delta \in \mathbb{R}, \]
since both $\alpha_0, \alpha_1 < 1$.

That is, the function $f(\delta)$ has only one solution or no solutions.

Taking $\alpha = 1$, we consider

$$g(\alpha) = f(1) = \rho_0 \alpha_0^2 + \rho_1 \alpha_1^2 - 1.$$  \hspace{1cm} (4.73)

It follows that

$$\frac{d}{d\alpha_0} g(\alpha_0) = 2\rho_0 \alpha_0 - 2\rho_1 (1 - \alpha_0),$$

and

$$\frac{d^2}{d\alpha_0^2} g(\alpha_0) = 2\rho_0 + 2\rho_1 > 0.$$ 

Therefore, $g(\alpha_0)$ attains its minimum at $\alpha_0^*$ such that

$$\frac{d}{d\alpha_0} g(\alpha_0^*) = 2\rho_0 \alpha_0 - 2\rho_1 (1 - \alpha_0) = 0.$$ 

That is, when $\alpha_0^* = \frac{1}{\rho_0}$, the function $g(\alpha_0)$ reaches the minimum.

Since we define $\rho_0 \neq 1/\alpha_0$, we obtain that

$$f(1) > 0, \text{ for all } \alpha_0, \rho_0.$$ 

Moreover, by $\alpha_0, \alpha_1 < 1$, we know that

$$\lim_{\delta \to \infty} f(\delta) = \lim_{\delta \to \infty} \rho_0 \alpha_0^{2\delta} + \rho_1 \alpha_1^{2\delta} - 1 = -1 < 0.$$ 

Therefore, we conclude that the function $f(\delta)$ has the unique solution $\delta > 1$.

Finally, we conclude that (4.71) is true since there is a unique $\delta > 0$ such that (4.72) hold. \hfill \Box

**Notation 4.36.** For every integer $n \geq 1$ we have $\alpha_{i_1 \ldots i_n} < \alpha_{i_1 \ldots i_{n-1}}$. Let $0 < R \leq 1$, for every sequence $i_1 \ldots i_m \in \{0, 1\}^m$, there exists a least integer $m \geq 0$, such that $\alpha_{i_1 \ldots i_m} \leq R$. The set of all finite sequences $(i_1 \ldots i_m)$ obtained in this way, in correspondence of a given $0 < R \leq 1$, will be denoted by $I_R$.

**Lemma 4.37.** Given $0 < R \leq 1$ and $I_R$ be defined as above, then, for every $(i_1, \ldots, i_m) \in I_R$, we have

$$\alpha_1 R < \alpha_{i_1 \ldots i_m} \leq R,$$  \hspace{1cm} (4.74)

and

$$0 \leq m^* \leq m \leq m^*,$$

where $m^*$ is the least integer greater of equal to $\log_{\alpha_0}(R)$, $m^*$ is the largest integer smaller of equal to $\log_{\alpha_1}(R)$.

**Notation 4.38.** Given $0 < R \leq 1$, by $G_R$ we denote the set

$$G_R := \{I_{i_1 \ldots i_m} : (i_1, \ldots, i_m) \in I_R\}.$$ 

By $B_c(x, R), B_\delta(x, R)$ we denote the ball

$$B_c(x, R) := \{y \in I : |x - y| < R\},$$
and
\[ B_\delta(x, R) := \{ y \in I : |x - y|^\delta < R \}, \]
where \( \delta > 1 \) is fixed.

**Lemma 4.39.** Given a pair \( \rho \), let \( \mu \) denote the 1-dimensional Lebesgue measure on \( I \) obtained in Lemma 4.21 and \( \delta > 1 \) obtained from (4.72). Then, for \( \nu = 1/\delta \), we have
\[ \mu(B_\delta(x, r)) \geq c \left( \frac{r}{R} \right)^\nu \mu(B_\delta(x, R)) \]
for some constant \( c > 0 \).

**Proof.** It is clear that
\[ \mu(B_\delta(x, r)) = r^{1/\delta} = r^\nu \]
and
\[ \mu(B_\delta(x, R)) = R^{1/\delta} = R^\nu. \]
Therefore,
\[ \left( \frac{r}{R} \right)^\nu \mu(B_\delta(x, r)) = \left( \frac{r}{R} \right)^\nu \cdot R^\nu = r^\nu = \mu(B_\delta(x, r)). \]
Thus, for any \( 0 < c < 1 \), we have
\[ \mu(B_\delta(x, r)) \geq c \left( \frac{r}{R} \right)^\nu \mu(B_\delta(x, R)) \]
as we wish. \( \square \)

**Theorem 4.40.** There exists a constant \( M \) such that for every \( x \in I \) and \( 0 < R \leq 1 \), the family
\[ G_{x,R} = \{ I_{i_1 \ldots i_m} : I_{i_1 \ldots i_m} \in G_R, I_{i_1 \ldots i_m} \cap B_\delta(x, R) \neq \emptyset \}, \]
contains at most \( M \) distinct intervals and
\[ I \cap B_\delta(x, R) \subset \bigcup_{G_{x,R}} I_{i_1 \ldots i_m}. \]

**Proof.** Please refer to Theorem 2.1 in [15] for details. \( \square \)

**Lemma 4.41.** For any pair \( \rho = (\rho_0, \rho_1) \), the following Poincaré inequalities hold
\[ \int_I |u - u(z)|^2 d\mu \leq c_p \int_I dL^p[u] \]
for every \( u \in D_{E^p} \) and every \( z \in \Gamma \) with a constant \( c_p \) only depends on \( \rho \).

**Proof.** This is the result of Lemma 4.11. \( \square \)

**Lemma 4.42.** For arbitrary \( i_1 \ldots i_n \in \{0, 1\}^n \) and \( n \geq 0 \), we have
\[ \int_{I_{i_1 \ldots i_n}} |u - \bar{u}_{i_1 \ldots i_n}|^2 d\mu \leq c_p \alpha_{i_1 \ldots i_n} \rho_{i_1 \ldots i_n}^{-1} \int_{I_{i_1 \ldots i_n}} dL^p[u], \]
for every \( u \in D_{E^p} \).
Proof. It is clear that
\[
\int_{I_{i_1 \ldots i_n}} |u - \bar{u}_{I_{i_1 \ldots i_n}}|^2 \, d\mu = \int_I 1_{I_{i_1 \ldots i_n}} |u - \bar{u}_{I_{i_1 \ldots i_n}}|^2 \, d\mu
\]
\[
= \sum_{j_1, \ldots, j_n = 0}^1 \alpha_{i_1 \ldots i_n} \int_I (1_{I_{i_1 \ldots i_n}} |u - \bar{u}_{I_{i_1 \ldots i_n}}|^2) \circ \phi_{j_1 \ldots j_n} \, d\mu
\]
\[
= \alpha_{i_1 \ldots i_n} \int_I |u - \bar{u}_{I_{i_1 \ldots i_n}}|^2 \circ \phi_{i_1 \ldots i_n} \, d\mu
\]
\[
= \alpha_{i_1 \ldots i_n} \int_I |u \circ \phi_{i_1 \ldots i_n} - (u \circ \phi_{i_1 \ldots i_n})_{I_{i_1 \ldots i_n}}|^2 \, d\mu,
\]
where we have taken into account that $n$-intervals intersect on sets of $\mu$-measure zero. Applying (4.75) on $I$ at the right-hand side, we have
\[
\int_{I_{i_1 \ldots i_n}} |u - \bar{u}_{I_{i_1 \ldots i_n}}|^2 \, d\mu \leq c_p \alpha_{i_1 \ldots i_n} \int_I dL[u \circ \phi_{i_1 \ldots i_n}],
\]
(4.78)
By the scaling of Lagrangians, we obtain
\[
\int_I dL[u \circ \phi_{i_1 \ldots i_n}] \leq \rho_{i_1 \ldots i_n}^{-1} \int_{I_{i_1 \ldots i_n}} dL^\rho[u].
\]
(4.79)
In the end, we get the desired inequality by (4.77), (4.78), and (4.79) such that
\[
\int_{I_{i_1 \ldots i_n}} |u - \bar{u}_{I_{i_1 \ldots i_n}}|^2 \, d\mu \leq c_p \alpha_{i_1 \ldots i_n} \rho_{i_1 \ldots i_n}^{-1} \int_{I_{i_1 \ldots i_n}} dL^\rho[u].
\]
\[\square\]

Remark 4.43. We test this result by choosing $\rho = \rho^*$. When $\rho = \rho^*$, we have $\alpha_{i_1 \ldots i_n}^{-1} = \rho_{i_1 \ldots i_n}^*$ and (4.76) becomes
\[
\int_{I_{i_1 \ldots i_n}} |u - \bar{u}_{I_{i_1 \ldots i_n}}|^2 \, d\mu \leq c_p \alpha_{i_1 \ldots i_n} \int_{I_{i_1 \ldots i_n}} dL^\rho[u],
\]
which is exactly what we expect.

Lemma 4.44. For arbitrary $i_1 \ldots i_n$, $j_1 \ldots j_n \in \{0, 1\}$, $n \geq 0$, with $I_{i_1 \ldots i_n} \cap I_{j_1 \ldots j_n} \neq \emptyset$, if $Q = I_{i_1 \ldots i_n} \cup I_{j_1 \ldots j_n}$, then
\[
\int_Q |u - \bar{u}_Q|^2 \, d\mu \leq 2c_p \max \left\{ \alpha_{i_1 \ldots i_n} \rho_{i_1 \ldots i_n}^{-1}, \alpha_{j_1 \ldots j_n} \rho_{j_1 \ldots j_n}^{-1} \right\} \cdot \max \left\{ \int_{I_{i_1 \ldots i_n}} dL^\rho[u], \int_{I_{j_1 \ldots j_n}} dL^\rho[u] \right\}.
\]
(4.80)
Proof. We have $I_{i_1 \ldots i_n} \cap I_{j_1 \ldots j_n} = \Gamma_{i_1 \ldots i_n} \cap \Gamma_{j_1 \ldots j_n} \neq \emptyset$. Therefore, there exists $x \in I_{i_1 \ldots i_n} \cap I_{j_1 \ldots j_n}$ and $x = \phi_{i_1 \ldots i_n}(p) = \phi_{j_1 \ldots j_n}(1)$ with $p, q \in \Gamma$.
Proceeding as at the beginning of the proof of Lemma 4.42, we have
\[
\int_{I_{i_1 \ldots i_n}} |u - u(x)|^2 \, d\mu(y) = \alpha_{i_1 \ldots i_n} \int_I |u \circ \phi_{i_1 \ldots i_n} - u \circ \phi_{i_1 \ldots i_n}(p)|^2 \, d\mu(dx),
\]
and by the Poincaré inequality on $I$, we have
\[
\int_I |u \circ \phi_{i_1 \ldots i_n} - u \circ \phi_{i_1 \ldots i_n}(p)|^2 \, d\mu(dx) \leq c_p \int_I dL[u \circ \phi_{i_1 \ldots i_n}].
\]
Then, it is clear that
\[
\int_{I_{1\ldots i_n}} |u - u(x)|^2 d\mu \leq c_p \alpha_{i_1 \ldots i_n} \rho_{i_2 \ldots i_n}^{-1} \int_I |u \circ \phi_{i_1 \ldots i_n} - u \circ \phi_{i_1 \ldots i_n}(p)|^2 d\mu,
\]
as well as
\[
\int_{I_{1\ldots j_n}} |u - u(x)|^2 d\mu \leq c_p \alpha_{j_1 \ldots j_n} \rho_{j_1 \ldots j_n}^{-1} \int_I |u \circ \phi_{j_1 \ldots j_n} - u \circ \phi_{j_1 \ldots j_n}(p)|^2 d\mu.
\]
What is important here is that in both Poincaré inequalities above the constant \(u(x)\) occurring on the left-hand side is the same. As a consequence, we have
\[
\int_Q |u - \bar{u}_Q|^2 d\mu \leq \int_Q |u - u(x)|^2 d\mu \leq \int_{I_{1\ldots i_n}} |u - u(x)|^2 d\mu + \int_{I_{1\ldots j_n}} |u - u(x)|^2 d\mu \leq 2c_p \max \left\{ \alpha_{i_1 \ldots i_n} \rho_{i_2 \ldots i_n}^{-1} \int_{I_{i_2 \ldots i_n}} dL^\rho[u], \alpha_{j_1 \ldots j_n} \rho_{j_1 \ldots j_n}^{-1} \int_{I_{j_1 \ldots j_n}} dL^\rho[u] \right\}
\]
\[
\leq 2c_p \max \left\{ \alpha_{i_1 \ldots i_n} \rho_{i_2 \ldots i_n}^{-1}, \alpha_{j_1 \ldots j_n} \rho_{j_1 \ldots j_n}^{-1} \right\} \cdot \max \left\{ \int_{I_{1\ldots i_n}} dL^\rho[u], \int_{I_{1\ldots j_n}} dL^\rho[u] \right\},
\]
as we wish. \(\square\)

**Lemma 4.45.** For arbitrary \(i_1 \ldots i_n\) and \(n \geq 0\), let \(m \geq 2\) be an integer, and for \(j = 1, \ldots, m\) let \(i_1^j \ldots i_n^j \in \{0, 1\}^n\) with \(i_1^j \ldots i_n^j \neq i_1^{j+1} \ldots i_n^{j+1}\) and \(I_{i_1^j \ldots i_n^j} \cap I_{i_1^{j+1} \ldots i_n^{j+1}} \neq \emptyset\) for all \(1 \leq j \leq m - 1\). If \(Q = I_{i_1^1 \ldots i_n^1} \cup \cdots \cup I_{i_1^{m-1} \ldots i_n^{m-1}}\), then
\[
\int_Q |u - \bar{u}_Q|^2 d\mu \leq 2mc_p \max_{j=1,\ldots,m} \left\{ \alpha_{i_1^j \ldots i_n^j} \rho_{i_2^j \ldots i_n^j}^{-1} \right\} \cdot \max_{j=1,\ldots,m} \left\{ \int_{I_{i_1^j \ldots i_n^j}} dL^\rho[u] \right\}. \quad (4.81)
\]

**Proof.** Since we are only in the unit interval, this is the direct result for Lemma 4.44. \(\square\)

Now we try to prove the following inequalities on \(B_\delta(x, r)\).

**Theorem 4.46.** Given a pair \(\rho\), let \(\delta\) be obtained by (4.72), \(B_\delta(x, r)\) as defined in Notation 4.38, then we have
\[
\int_{I_{r^{1/\delta}}} |u - \bar{u}_B| d\mu \leq c \max_{i_1 \ldots i_n \in I_{r^{1/\delta}}} \left\{ \alpha_{i_1 \ldots i_n} \rho_{i_2 \ldots i_n}^{-1} \right\} \int_{I_{r^{1/\delta}}} dL^\rho[u], \quad (4.82)
\]
where \(I_{r^{1/\delta}}\) is defined in Notation 4.36 and Lemma 4.37 and \(u \in D_{E^\varphi}\).

**Proof.** By the definition of \(I_{r^{1/\delta}}\), we know that
\[
\alpha_1 r^{1/\delta} < \alpha_{i_1 \ldots i_n} \leq r^{1/\delta},
\]
for all \(i_1 \ldots i_n \in I_{r^{1/\delta}}\).

Therefore, we have
\[
B_\delta(x, \alpha_1 r) \subset B_c(x, \alpha_{i_1 \ldots i_n}), \text{ for all } i_1 \ldots i_n \in I_{r^{1/\delta}}.
\]
Moreover, by Theorem 4.40 we know that the family
\[ G_{x,r^{1/\delta}} = \{ I_{i_1 \ldots i_m} : I_{i_1 \ldots i_m} \subseteq G_r, r^{1/\delta}, I_{i_1 \ldots i_m} \cap B_{e}(x,r^{1/\delta}) \neq \emptyset \}, \]
contains at most \( M \) distinct intervals and
\[ I \cap B_{e}(x,\alpha_{i_1 \ldots i_m}) \subseteq I \cap B_{\delta}(x,r) \subseteq \bigcup_{G_{x,r^{1/\delta}}} I_{i_1 \ldots i_m}. \]

By Lemma 4.45, we obtain that
\[
\int_{I \cap B_{\delta}(x,r)} |u - \bar{u}_{B_{\delta}(x,r)}|^2 \, d\mu \leq \int_{I \cap B_{e}(x,\alpha_{i_1 \ldots i_m})} |u - \bar{u}_{B_{e}(x,\alpha_{i_1 \ldots i_m})}|^2 \, d\mu \\
\leq 2MC_p \max_{I_{i_1 \ldots i_m} \subseteq I \cap B_{\delta}(x,r)} \{ \alpha_{i_1 \ldots i_m} \cdot \rho_{i_1 \ldots i_m}^{-1} \} \cdot \max_{I_{i_1 \ldots i_m} \subseteq G_{x,r^{1/\delta}}} \left\{ \int_{I_{i_1 \ldots i_m}} dL^p[u] \right\} \\
\leq \tilde{c}_p \max_{I_{i_1 \ldots i_m} \subseteq I \cap B_{\delta}(x,r)} \{ \alpha_{i_1 \ldots i_m} \cdot \rho_{i_1 \ldots i_m}^{-1} \} \cdot \int_{I \cap B_{\delta}(x,r)} dL^p[u],
\]
as we wish.

\( \square \)

**Remark 4.47.** As we did before, here we test the result by choosing \( \rho = \rho^* \). Clearly, when \( \rho = \rho^* \), we have \( \delta = 1 \) and \( \rho_{i_1 \ldots i_m}^{-1} = \alpha_{i_1 \ldots i_m} \leq r, \) (4.82) will become
\[
\int_{I \cap B_{\delta}(x,\alpha_{i_1 \ldots i_m})} |u - \bar{u}_{B_{\delta}(x,\alpha_{i_1 \ldots i_m})}|^2 \, d\mu \leq c r^2 \int_{I \cap B_{\delta}(x,r)} dL^p[u],
\]
which is the normal Poincaré inequalities on \( I \) as we expect.

**Remark 4.48.** In (4.82), it is clear that the term
\[ \alpha_{i_1 \ldots i_m} \cdot \rho_{i_1 \ldots i_m}^{-1} = \alpha_{i_1 \ldots i_m}^2 \cdot \rho_{i_1 \ldots i_m}^* \leq r^{2/\delta} \cdot \rho_{i_1 \ldots i_m}^* \rho_{i_1 \ldots i_m}^{-1}. \]
The question is that whether we could control the value of the term \( \rho_{i_1 \ldots i_m}^* / \rho_{i_1 \ldots i_m} \), or, in other words, is it possible to prove that
\[ \frac{\rho_{i_1 \ldots i_m}^*}{\rho_{i_1 \ldots i_m}} < 1, \]
for all \( i_1 \ldots i_m \in I_{1/\delta}. \)

Unfortunately, the answer is no. Clearly, we have
\[
\frac{\rho_{i_1 \ldots i_m}}{\rho_{i_1 \ldots i_m}} = \frac{\rho_{i_1}}{\rho_{i_1}} \cdots \frac{\rho_{i_m}}{\rho_{i_m}} = \left( \frac{\rho_0}{\rho_1} \right)^a \left( \frac{\rho_1}{\rho_0} \right)^b,
\]
where \( a + b = n \), and either
\[ \frac{\rho_0}{\rho_0} < 1, \ \frac{\rho_1}{\rho_1} > 1, \]
or
\[ \frac{\rho_0}{\rho_0} > 1, \ \frac{\rho_1}{\rho_1} < 1, \]
depends on the value of \( \rho_0 \) and \( \rho_1 \).

However, the value of \( a, b, \rho_0, \) and \( \rho_1 \) are varied all the time, especially the value \( \rho_{0}^* / \rho_0 \) and \( \rho_{1}^* / \rho_1 \) can not be controlled by the value of \( r \), and hence the term \( \rho_{i_1 \ldots i_m}^* / \rho_{i_1 \ldots i_m} \) can not be bounded by the value of
It might be true that for some \(a, b, \rho_0,\) and \(\rho_1\) the inequality (4.83) might hold, but generally it won’t be true.

5. The Energy Forms on Quasi-Pólya Curve \(Q\)

In this Section we are in 2 dimensions and we build the energy forms on Quasi-Pólya curve.

5.1. The Invariant Set, Measure, Metrics.

**Definition 5.1.** We define the contractive similitudes that operate on \(I = [0, 1]\) in the following way
\[
\phi_0(x) = r_0^s x, \quad \phi_1(x) = r_1^s x + r_0^s,
\]
where \(r_0\) and \(r_1\) are defined in (2.6), and \(s\) is defined in (2.11).

**Definition 5.2.** Setting \(I_0^0 = \{0, 1\}\), for arbitrary \(n\)-tuples of indices \(i_1 \ldots i_n \in \{0, 1\}\), \(n = 1, 2, \ldots\), we define
\[
\phi_{i_1 \ldots i_n} := \phi_{i_1} \circ \ldots \circ \phi_{i_n},
\]
\[
I_r^n = \bigcup_{i_1 \ldots i_n = 0}^1 \phi_{i_1 \ldots i_n}(I_0^0),
\]
and
\[
I_r^\infty = \bigcup_{n=1}^\infty I_r^n. \quad (5.1)
\]

**Remark 5.3.** It is clear that \(I = I_r^\infty\), where \(I = [0, 1]\).

**Definition 5.4.** We define the map \(\xi(t)\) by
\[
\xi(t) := \phi_{d_1} \circ \phi_{d_2} \circ \ldots \circ \phi_{d_n}(0) := \phi_{d_1} \circ \phi_{d_2} \circ \ldots \circ \phi_{d_n}(0),
\]
for \(t \in [0, 1]\) and \(0.d_1d_2d_3 \ldots d_n \ldots\) is the binary form which represents \(t\).

**Lemma 5.5.** The measure \(L^1|_I\) is invariant with respect to \((\phi_1^s, \phi_1^s)\).

**Proof.** Let the segment \(A \subset I = [0, 1]\), we consider the equation
\[
r_0^s L^1|_I(\phi_0^{-1}(A)) + r_1^s L^1|_I(\phi_1^{-1}(A)).
\]
Here we have three different cases: \(A \subset [0, r_0^s]\), \(A \subset [r_0^s, 1]\), and \(r_0^s \in A\).

For case 1, we have
\[
r_0^s L^1|_I(\phi_0^{-1}(A)) + r_1^s L^1|_I(\phi_1^{-1}(A)) = r_0^s \frac{1}{r_0} L^1|_I(A) = L^1|_I(A).
\]
Similarly, in case 2,
\[
r_0^s L^1|_I(\phi_0^{-1}(A)) + r_1^s L^1|_I(\phi_1^{-1}(A)) = r_1^s \frac{1}{r_1} L^1|_I(A) = L^1|_I(A).
\]
In case 3, we can partition the interval \( A = [a, b] \) into \([a, r_0^s]\) and \([r_0^s, b]\). Then apply case 1 and case 2 we again reach the situation

\[
\mathcal{L}^1 [ I(A) = r_0^s \mathcal{L}^1 [ I (\phi_0^{-1}(A)) + r_1^s \mathcal{L}^1 [ I(\phi_1^{-1}(A))],
\]

which gives the Lemma. □

**Definition 5.6.** We define the map \( \zeta(t) \) by

\[
\zeta(t) := \psi_{d_1} \circ \psi_{d_2} \circ \cdots \circ \psi_{d_n}(0),
\]

for \( t \in [0, 1] \) and \( 0.d_1d_2d_3\cdots d_n\) is the binary form which represents \( t \).

**Lemma 5.7.** The map \( \zeta^{-1} : Q^\infty \to I^\infty \) extends to a homeomorphism \( \overline{\zeta^{-1}} : Q \to [0, 1] \).

**Proof.** The proof can be found in [11]. □

**Definition 5.8.** We define the map \( \eta : I_r^\infty \to Q^\infty \) by

\[
\eta(x) := \zeta \circ \xi^{-1},
\]

for \( x = \phi_{i_1} \circ \phi_{i_2} \circ \phi_{i_3} \circ \cdots \circ \phi_{i_n}(0) \in I_r^\infty \).

**Lemma 5.9.** The map \( \eta^{-1} : Q^\infty \to I_r^\infty \) extends to a homeomorphism \( \overline{\eta^{-1}} : Q \to [0, 1] \).

**Proof.** This is clear from the result of Lemma 5.7. □

**Definition 5.10.** The measure \( \mu \) obtain on \( Q \) by transforming the 1-d Lebesgue measure on \([0, 1]\) by means of the homeomorphism between \([0, 1]\) and \( Q \) is defined as following:

For any \( B \subset \mathbb{R}^2 \),

\[
\mu(B) := \mathcal{L}^1 [ I (\eta^{-1}(B \cap Q)).
\]

**Theorem 5.11.** The measure \( \mu \) defined in Definition 5.10 is invariant with respect to \((\psi_i, r_i)\).

**Proof.** We want to show that for any ball \( B \) in Definition 5.10 satisfies

\[
\mu(B) = \sum_{i=0}^{1} r_i^s \mu(\psi_i^{-1}(B)) = r_0^s \mu(\psi_0^{-1}(B)) + r_1^s \mu(\psi_1^{-1}(B)). \tag{5.2}
\]

First of all, it has been proved in Appendix I that

\[
\eta \circ \phi_i(A) = \psi_i \circ \eta(A),
\]

for all \( A \subset I \).

That is

\[
\phi_i^{-1} \circ \eta^{-1}(B) = \eta^{-1} \circ \psi_i^{-1}(B),
\]

for all \( B \subset Q \).
Thus, by Lemma 5.5, we have
\[
 r^*_0 \mu(\psi_0^{-1}(B)) + r^*_1 \mu(\psi_1^{-1}(B)) = r^*_0 | I(\eta^{-1}(\psi_0^{-1}(B) \cap Q)) + r^*_1 | I(\eta^{-1}(\psi_1^{-1}(B) \cap Q))
\]
\[
= r^*_0 | I(\eta^{-1} \circ \psi_0^{-1}(B) \cap Q) + r^*_1 | I(\eta^{-1} \circ \psi_1^{-1}(B) \cap Q)
\]
\[
= r^*_0 | I(\phi_0^{-1} \circ \eta^{-1}(B \cap Q)) + r^*_1 | I(\phi_1^{-1} \circ \eta^{-1}(B \cap Q))
\]
\[
= \mathcal{L}^1 | I(\eta^{-1}(B \cap Q)) = \mu(B).
\]
Therefore, we have proven that
\[
\mu(B) = \sum_{i=0}^{1} r^*_i \mu(\psi_i^{-1}(B)),
\]
which shows that the measure \( \mu \) is invariant with respect to \((\psi_i, r_i)\).

\[\square\]

**Remark 5.12.** Since we have shown in Lemma 2.10 that \( Q \) satisfies the Open Set Condition, we refer to Theorem (1) in [9], Section 5.3, that the Invariant Measure with respect to \((\psi_i, r_i)\) on \( Q \) is unique and the Hausdorff Measure
\[
[\mathcal{H}^D(Q)]^{-1} \mathcal{H}^D[Q],
\]
with \( D = s \), is an invariant measure with respect to \((\psi_i, r_i)\) on \( Q \). Therefore, by the uniqueness of invariant measure, we conclude that
\[
\mu(B) = [\mathcal{H}^D(Q)]^{-1} \mathcal{H}^D[Q(B)],
\]
for all \( B \subset \mathbb{R}^2 \).

### 5.2. The Metric Properties of \( Q \)

In Section 5.1 we introduce and prove that \( \eta \) is a homeomorphism, and we use the Lebesgue measure on \([0, 1]\) to induce a measure on \( Q \) by
\[
\mu(B) := \mathcal{L}^1 | I(\eta^{-1}(B \cap Q)).
\]
Note that integrability of a function \( w \) on \( Q \) is then equivalent to integrability of \( w \circ \eta \) on \([0, 1]\). Thus we have
\[
(\eta^{-1} \circ \psi_0 \circ \eta)(t) = r^*_0 t,
\]
and
\[
(\eta^{-1} \circ \psi_1 \circ \eta)(t) = 1 - r^*_1 t.
\]
Then, for any integrable function \( w \), we get
\[
\int_Q w \, d\mu = \int_0^1 w \circ \eta(t) \, dt
\]
\[
= r^*_0 \int_0^1 w \circ \eta(r^*_0 t) \, dt + r^*_1 \int_0^1 w \circ \eta(1 - r^*_1 t) \, dt
\]
\[
= r^*_0 \int_0^1 (w \circ \eta) \circ (\eta^{-1} \circ \psi_0 \circ \eta) \, dt + r^*_1 \int_0^1 (w \circ \eta) \circ (\eta^{-1} \circ \psi_1 \circ \eta) \, dt
\]
\[
= r^*_0 \int_Q w \circ \psi_0 d\mu + r^*_1 \int_Q w \circ \psi_1 d\mu.
\]
We introduce a measure-valued bilinear form \( L \), to be called a Lagrangean, defined for \( u, v \) in the space \( C \) of real valued functions on \( Q \) that can be represented as \( w \circ \eta^{-1} \) with \( w \in C^1((0, 1)) \).

**Definition 5.13.** We define the measure \( L_Y \) on \( Y = (Q, \mu) \), where \( \mu \) defined in (5.2), by

\[
L_Y(u, v) := L_X(u \circ \eta, v \circ \eta) \circ \eta^{-1} = \left\{ (u \circ \eta)'(\cdot)(v \circ \eta)'(\cdot) \right\} \circ \eta^{-1} \mu,
\]

with domain

\[
C_Y := C \circ \eta^{-1},
\]

where \( C \) is a dense sub-algebra of \( C_0(X) \).

We observe that

\[
[(u \circ \psi_0 \circ \eta)'(\cdot)(v \circ \psi_0 \circ \eta)'(\cdot)] \circ \eta^{-1} = [(u \circ \eta \circ (\eta^{-1} \psi_0 \circ \eta))'(\cdot)(v \circ \eta \circ (\eta^{-1} \psi_0 \circ \eta))'(\cdot)] \circ \eta^{-1}
\]

\[
= [(u \circ \eta (r_1^*)')'(v \circ \eta (r_1^*))'] \eta^{-1}
\]

\[
= r_0^{2s}[(u \circ \eta)'(r_1^*)'(v \circ \eta)'(r_1^*)] \circ \eta^{-1}
\]

\[
= r_0^{2s}[(u \circ \eta)' \circ \eta^{-1} \circ \psi_0 \circ \eta(\cdot)(v \circ \eta)' \circ \eta^{-1} \circ \psi_0 \circ \eta(\cdot)] \circ \eta^{-1}
\]

\[
= r_0^{2s}[(u \circ \eta)'(\cdot)(v \circ \eta)'(\cdot)] \circ \eta^{-1} \circ \psi_0.
\]

A similar calculation holds also with \( \psi_1 \) such as

\[
[(u \circ \psi_1 \circ \eta)'(\cdot)(v \circ \psi_1 \circ \eta)'(\cdot)] \circ \eta^{-1} = r_1^{2s}[(u \circ \eta)'(\cdot)(v \circ \eta)'(\cdot)] \circ \eta^{-1} \circ \psi_1.
\]

Therefore, by using (5.10), and (5.11), we have

\[
L_Y(u, v) = r_0^{2s}L_Y(u, v) \circ \psi_1 \mu + r_1^{2s}L_Y(u, v) \circ \psi_1 \mu
\]

\[
= r_0^{2s}L_Y(u \circ \psi_0, v \circ \psi_0) \mu + r_1^{2s}L_Y(u \circ \psi_1, v \circ \psi_1) \mu
\]

\[
= r_0^{2s}L_Y(u \circ \psi_0, v \circ \psi_0) + \frac{1}{r_1^{2s}}L_Y(u \circ \psi_1, v \circ \psi_1).
\]

Then, we define the total energy form \( E \) by

\[
E(u, v) := \int_Q dL(u, v), \quad u, v \in H^1(Q) := H^1([0, 1]) \circ \eta^{-1}.
\]

**Lemma 5.14.** The measure \( L_Y \) is a Lagrangian on \( Y \).

**Proof.** The definition of Lagrangian can be found in [14], Section 3.3.

We want to prove the equality such that

\[
\int_Q dL_Y(\varphi(u), v) = \int_Q \varphi'(u) dL_Y(u, v),
\]

for all \( \varphi \in C_0^1(Q) \).
We start from the right hand side of (5.14), writing
\[
\int_Q L_Y (\varphi(u), v) \, dy = \int_Q \varphi' (u) \varphi (u) \, dx,
\]
for \( y = \eta(x) \)
\[
= \int_I \varphi' (u) \varphi (u) \, dx
\]
\[
= \int_Q \varphi' (u) \varphi (u) \, dx
\]
\[
= \int_Q \varphi' (u) dL_Y (u, v).
\]
That is
\[
L_Y (\varphi(u), v) = \varphi' (u) L_Y (u, v),
\]
for all \( \varphi \in C^1_0 (Q) \).

**Definition 5.15.** We define the arc \( \tilde{xy} \) between \( x, y \in Q^n \) by
\[
\tilde{xy} := \eta ([\min \{ \eta^{-1}(x), \eta^{-1}(y) \}, \max \{ \eta^{-1}(x), \eta^{-1}(y) \}]).
\]

**Definition 5.16.** We define three different metrics on \( Q \) by
\[
(1) \quad d_1 (x, y) := |\eta^{-1}(x) - \eta^{-1}(y)|;
\]
\[
(2) \quad d_2 (x, y) := |x - y|^s;
\]
\[
(3) \quad d_3 (x, y) := [\mathcal{H}^D (Q)]^{-1} \mathcal{H}^D (Q \ (\tilde{xy})),
\]
where \( x, y \in Q \).

**Theorem 5.17.** The metrics (5.16), (5.17), and (5.18) in Definition 5.16 are equivalent.

**Proof.** It is easily seen that the metric \( d_1 \) is carried on by the measure \( \mu \) introduced in Definition 5.10. Therefore, by Remark 5.12 we conclude that the metric \( d_1 \) is equivalent to \( d_3 \).

We next prove that \( d_1 \) is equivalent to \( d_2 \). That is, we want to show there exists two constants \( C_1 \) and \( C_2 \) such that for all \( x, y \in Q \),
\[
C_1 |x - y|^s \leq d_1 (x, y) \leq C_2 |x - y|^s.
\]

Let's start with the first inequality of (5.19). For any \( x, y \in Q^n \), let \( n \) be the smallest number such that \( x \) and \( y \) are contained in \( Q^n \). If \( \tilde{xy} \) is a segment, which means there exists a sequence \( i_1, i_2, \ldots, i_n \) such that \( \psi_{i_1 \ldots i_n}(0) = x \) and \( \psi_{i_1 \ldots i_n}(1) = y \). Then by the construction maps of unit interval \( I \) and \( Q \), it carries out that
\[
d_1 (x, y) = \mathcal{L}^1 [\phi_{i_1 \ldots i_n}(0), \phi_{i_1 \ldots i_n}(1)] = r_{i_1}^s r_{i_2}^s \cdots r_{i_n}^s,
\]
and
\[
|x - y| = |\psi_{i_1 \ldots i_n}(0) - \psi_{i_1 \ldots i_n}(0)| = r_{i_1} r_{i_2} \cdots r_{i_n}.
\]
Then we have

$$|x - y|^s = d_1(x, y).$$

(5.22)

Note that for any $\overline{xy}$ in $Q^n$, we can decompose it as

$$\overline{xy} = \overline{x_1y_1} \cup \overline{y_1y_2} \cup \cdots \cup \overline{y_ly}.$$

Then, by the construction of $d_1$ we have

$$d_1(x, y) = |x - y_1|^s + |y_1 - y_2|^s + \cdots + |y_l - y|^s,$$

(5.23)

and it proves the first inequality of (5.19) with $C_1 = 1$.

To prove the second inequality of (5.19), we first inspect the case for $n = 2$. Clearly, for $x, y \in Q^2$, the smallest distance of $\overline{xy}$ is $(1 - r_0^2 - r_1^2)$, and the corresponding $d_1$ distance of $\overline{xy}$ is $2(r_0r_1)^s$. Therefore, we have

$$d_1(x, y) \leq \frac{2(r_0r_1)^s}{(1 - r_0^2 - r_1^2)^s}|x - y|^s,$$

for all $x, y \in Q^2$.

It can be checked that if the segment $\overline{xy} \subset \mathbb{R}^2$, $x, y \in Q$, intersects $Q$ only with $x$ and $y$, then

$$d_1(x, y) \leq C_2|x - y|^s,$$

where $C_2 := 2(r_0r_1)^s/(1 - r_0^2 - r_1^2)^s$.

Now, for arbitrary $x, y \in Q$, decompose $\overline{xy}$ as

$$\overline{xy} = \overline{x_1y_1} \cup \overline{y_1y_2} \cup \cdots \cup \overline{y_ly},$$

where $y_m \in \overline{xy} \cap Q$ for every $1 \leq m \leq l$.

Then we have

$$d_1(x, y) \leq d_1(x, y_1) + \cdots + d_1(y_l, y).$$

Note that

$$|y_{m-1} - y_m|^s + |y_m - y_{m+1}|^s \leq (|y_{m-1} - y_m| + |y_m - y_{m+1}|)^s \leq |y_{m-1} - y_{m+1}|^s.$$

(5.24)

Thus, we have

$$d_1(x, y) \leq C_2|x - y|^s,$$

and we proved (5.19).

Therefore, by setting $C_1 = 1$ and $C_2 = 2(r_0r_1)^s/(1 - r_0^2 - r_1^2)^s$, we proved that $d_1, d_2$ are equivalent. Hence, all three metrics $d_1, d_2, d_3$ are equivalent. □

**Remark 5.18.** Note that the inequality in (5.23) is due to the triangle inequality of metric space. But in (5.24), the inequality is based on the convex properties of power function.

**Remark 5.19.** For $a, b \in I = [0, 1]$, we use the term $d_0(a, b)$ to denote the usual Euclidean distance. It is clear that the quintuple

$$(I, d_0, L^1 | I, L_X, C)$$
is a metric fractal.

Then, for quintuple

\[(Q, d_1, \mu, L_Y, C_Y),\]

where \(Q\) is Quasi-Pólya curve defined in Definition 2.8, \(d_1\) is a metric on \(Q\) defined in (5.16), \(\mu\) is an invariant measure defined in Definition 5.10, \(L_Y\) is a Lagrangian defined in (5.8), and \(C_Y\) is defined in (5.9), is a metric fractal due to Theorem 5.17, Theorem 5.11 and Lemma 5.14.

5.3. The Energy Forms.

From here we start to build the Energy Forms on \(Q\). We will use the method which has been developed in [14], and we choose two factors \(\rho_0\) and \(\rho_1\) to be

\[
\rho_0 = \frac{1}{r_0^s}, \quad \rho_1 = \frac{1}{r_1^s},
\]

(5.25)
due to the result of Theorem 4.33.

**Definition 5.20.** For arbitrary \(u: Q^n \to \mathbb{R}\), we define the Energy Form on \(Q^n\) by

\[
E_n[u] = \sum_{i_1...i_n=0}^1 \rho_{i_1...i_n} E_0[u \circ \psi_{i_1...i_n}],
\]

(5.26)

where

\[
E_0[u] = \frac{1}{2} \sum_{\xi, \eta \in \Gamma} |u(\xi) - u(\eta)|^2,
\]

(5.27)

for \(\Gamma = \{(0,0), (1,0)\}\). Moreover, by the choosing of the constants \(\rho_i > 0\) in (5.25), we have

\[
\min_{u|(Q^1 - Q^0)} E_1[u] = E_0[u].
\]

(5.28)

**Definition 5.21.** We define the Energy Form on \(Q\) by

\[
E[u] := \sup_{n \geq 0} E_n[u|Q^n]
\]

(5.29)
on the domain

\[
D^\infty_E := \left\{ u: Q^\infty \to \mathbb{R}: \sup_{n \geq 0} E_n[u|Q_n] < +\infty \right\}.
\]

(5.30)

**Remark 5.22.** Note that the equality of (5.28) holds everywhere if \(\bar{u}\) is a function obtained by starting with \(\bar{u}|Q^0 = \{A, B\}\) and extending \(\bar{u}\) from \(Q^0\) to \(Q^1\), by defining \(\bar{u}(p)\) at each dyadic \(p \in Q^1 - Q^0\) to be the "average values"

\[
\frac{\rho_0 A + \rho_1 B}{\rho_0 + \rho_1}.
\]
Do the same extension from \(Q^{n-1}\) to \(Q^n\), by defining \(\bar{u}\) at each new point, which belongs to the same triangle with vertices \(\Gamma_{i_1 \ldots i_{n-1}}\), to be the "average values" of \(u\) at \(\Gamma_{i_1 \ldots i_{n-1}}\). We say that such a \(\bar{u}\) on \(I^\infty\) is the harmonic extension of \(u|Q^0\), which keeps energy stationary. Hence, \(D_E^\infty \neq \emptyset\), as it contains the harmonic extension of \(u|Q^0\).

**Lemma 5.23.** The Energy Form \(E[u]\) in Definition 5.21 has the relation such that

\[
E[u] = \sum_{i=0}^{1} \rho_i E[u \circ \psi_i], \quad u \in D_E^\infty.
\]

**Proof.** Recall that for \(n \geq 1\), we define the Energy Form as

\[
E_n[u] = \sum_{i_1 \ldots i_n=0}^{1} \rho_{i_1 \ldots i_n} E_0[u \circ \psi_{i_1 \ldots i_n}],
\]

or more explicitly

\[
E_n[u] = \sum_{i_1 \ldots i_n=0}^{1} \rho_{i_1 \ldots i_n} \frac{1}{2} \sum_{\xi, \eta \in \Gamma} |u(\psi_{i_1 \ldots i_n}(\xi)) - u(\psi_{i_1 \ldots i_n}(\eta))|^2.
\]  

(5.31)

Therefore, by (5.26), we have

\[
\sum_{i=0}^{1} \rho_i E_n[u \circ \psi_i] = \sum_{i=0}^{1} \rho_i \sum_{i_1 \ldots i_n=0}^{1} \rho_{i_1 \ldots i_n} E_0[u \circ \psi_{i_1 \ldots i_n}]
\]

\[
= \sum_{i_1 \ldots i_n=0}^{1} \rho_{i_1 \ldots i_n} E_0[u \circ \psi_{i_1 \ldots i_n}]
\]

\[
= E_{n+1}[u].
\]

That is, by \(\lim_{n \to \infty} E_n[u] = E[u]\), we obtain

\[
E[u] = \sum_{i=0}^{1} \rho_i E[u \circ \psi_i], \quad u \in D_E^\infty,
\]

as we wish. \(\Box\)

**Lemma 5.24.** [6] The Quasi-Pólya Curve has the following properties:

i. There exists a positive number \(\alpha\) such that \(Q_{i_1 \ldots i_m} \cap Q_{j_1 \ldots j_m} = \emptyset\) implies \(\text{dist}(Q_{i_1 \ldots i_m}, Q_{j_1 \ldots j_m}) \geq \alpha^m\) for every \(m\), where

\[
\alpha = \frac{3}{2} - 2[\cos^4 \theta + (\cos \theta \sin \theta - h)^2].
\]

ii. If \(i_1 \ldots i_m \neq j_1 \ldots j_m\), then \(Q_{i_1 \ldots i_m} \cap Q_{j_1 \ldots j_m} = \Gamma_{i_1 \ldots i_m} \cap \Gamma_{j_1 \ldots j_m}\).

**Lemma 5.25.** There exists a constant \(c\) and \(\beta\) such that for every \(u: Q^\infty \to \mathbb{R}\) and for arbitrary \(p\) and \(q\) in \(Q^\infty\), the following estimate holds:

\[
|u(p) - u(q)| \leq c \sqrt{\sup_{n \geq 0} E_n[u|Q^n]}|p - q|^\beta,
\]  

(5.32)

where

\[
\rho_0 = \min\{\rho_0, \rho_1\}, \quad c = 2\rho_0 / \rho_0^{1/2} - 1, \quad \beta = -\log \rho_0 / 2 \log \alpha.
\]
Proof. For arbitrary \( p, q \in Q^\infty \subset Q \), we know that \( Q = \bigcup_{i_1 \ldots i_m = 0}^1 Q_{i_1 \ldots i_m} \). Therefore, we obtain \( p \in Q_{i_1 \ldots i_m} \) and \( q \in Q_{j_1 \ldots j_m} \) for some \( i_1 \ldots i_m, j_1 \ldots j_m \in \{0, 1\}^m \).

Suppose that \( 0 < |p - q| \leq 1 \). Then, there exists a \( m \geq 0 \) such that
\[
\alpha^{m+1} < |p - q| < \alpha^m.
\] (5.33)

Since we choose \( p \in Q_{i_1 \ldots i_m} \) and \( q \in Q_{j_1 \ldots j_m} \), we have
\[
\min_{p \in Q_{i_1 \ldots i_m}, q \in Q_{j_1 \ldots j_m}} |p - q| = \text{dist}(Q_{i_1 \ldots i_m}, Q_{j_1 \ldots j_m}).
\] (5.34)

Together with (5.33), we obtain
\[
\text{dist}(Q_{i_1 \ldots i_m}, Q_{j_1 \ldots j_m}) \leq |p - q| < \alpha^m.
\] (5.35)

Then, by Lemma 5.24, part i, we conclude that
\[
Q_{i_1 \ldots i_m} \cap Q_{j_1 \ldots j_m} \neq \emptyset.
\] (5.36)

Moreover, by Lemma 5.24, part ii, we obtain
\[
\Gamma_{i_1 \ldots i_m} \cap \Gamma_{j_1 \ldots j_m} \neq \emptyset.
\] (5.37)

Thus, there exists a point \( \gamma \in \Gamma_{i_1 \ldots i_m} \cap \Gamma_{j_1 \ldots j_m} \) such that
\[
\gamma = \psi_{i_1 \ldots i_m}(\xi) = \psi_{j_1 \ldots j_m}(\eta),
\] (5.38)
where \( \xi, \eta \in \Gamma \).

Now, we consider the case \( n \geq m \). Since \( p, q \in Q^\infty \), there exists the smallest \( n \geq m \) such that \( p, q \in Q^n \). Then, we have \( p = \psi_{i_1 \ldots i_n}(\xi) \) and \( q = \psi_{j_1 \ldots j_n}(\eta) \), where \( \xi, \xi \in \Gamma \).

Now, we construct a chain of points to connect \( p \) and \( q \) from two sides. We start with
\[
x_n =: p = \phi_{i_1 \ldots i_n}(\xi) = \phi_{i_1 i_2 \ldots i_m i_{m+1} \ldots i_n}(\xi)
\] (5.39)

Then we define
\[
x_{n-1} := \psi_{i_1 \ldots i_{n-1}}(\xi) = \psi_{i_1 i_2 \ldots i_m i_{m+1} \ldots i_{n-1}}(\xi)
x_{n-2} := \psi_{i_1 \ldots i_{n-2}}(\xi) = \psi_{i_1 i_2 \ldots i_m i_{m+1} \ldots i_{n-2}}(\xi)
\]
\[
\vdots
\]
\[
x_{n-k} := \psi_{i_1 \ldots i_{n-k}}(\xi) = \psi_{i_1 i_2 \ldots i_m i_{m+1} \ldots i_{n-k}}(\xi),
\]
where \( 0 \leq k \leq n - m \).

Now we have a chain of points \( x_n, x_{n-1}, \ldots, x_m \). Next, we insert a point \( \gamma \) such that \( x_{m-1} = \gamma = \psi_{i_1 \ldots i_m}(\eta) \).
Similarly, by defining \( y_n = q \), \( y_{n-k} = \psi_{j_1\ldots j_n-k}(\bar{\eta}) \), where \( 0 \leq k \leq n-m \), we obtain \( \gamma \) again by letting \( y_{n-1} = \gamma = \psi_{j_1\ldots j_m}(\eta) \).

Thus, we have constructed a chain
\[
p = x_n, x_{n-1}, \ldots, x_m, x_{m-1} = \gamma = y_{m-1}, y_m, \ldots, y_{n-1}, y_n = q,
\] (5.40)
with a property that two consecutive points in the chain belong to the same cell.

Now we study the case when \( k = 0 \). Let \( \bar{\xi} \) be the fixed point of \( \psi_{i_0} \), so \( x_{n-1} = \psi_{i_1\ldots i_n-i_0}(\bar{\xi}) \). If \( i_0 = i_n \), then \( x_n = x_{n-1} \). If \( i_0 \neq i_n \), then \( \psi_{i_0}(\bar{\xi}) = \psi_{i_0}(\bar{\eta}) \) for some \( \bar{\eta} \in \Gamma \). So \( x_n = \psi_{i_1\ldots i_n}(\bar{\xi}) = \psi_{i_1\ldots i_{n-1}i_0}(\bar{\eta}) \). Therefore \( x_n, x_{n-1} \in \Gamma_{i_1\ldots i_n-i_0} \).

Now we are ready to estimate \( |u(p) - u(q)| \). By the chain constructed above, we have
\[
|u(p) - u(q)| \leq \sum_{k=0}^{n-m} \left[ |u(x_{n-k}) - u(x_{n-k-1})| + |u(y_{n-k}) - u(y_{n-k-1})| \right].
\] (5.41)

Since \( \bar{\xi} = \psi_{i_0}(\xi) \) with \( \psi_{i_{n-k}}(\bar{\xi}) = \psi_{i_0}(\bar{\eta}) \), we obtain
\[
|u(x_{n-k}) - u(x_{n-k-1})|^2 = |u(\psi_{i_1\ldots i_{n-k-1}i_0}(\bar{\xi})) - u(\psi_{i_1\ldots i_{n-k-1}i_0}(\bar{\eta}))|^2
\] 
\[= |u(\psi_{i_1\ldots i_{n-k-1}}(\bar{\eta})) - u(\psi_{i_1\ldots i_{n-k-1}}(\bar{\xi}))|^2
\] 
\[\leq \sum_{i_1\ldots i_{n-k}-0}^1 \left[ \frac{1}{2} \sum_{\xi',\eta'} |u(\psi_{i_1\ldots i_{n-k}i_0}(\bar{\xi}')) - u(\psi_{i_1\ldots i_{n-k}i_0}(\bar{\eta}'))|^2 \right].
\] (5.42)

Recall \( \rho_s = \min\{\rho_0, \rho_1\} \) and multiply both sides by \( \rho_s^{n-k} \) to obtain
\[
\rho_s^{n-k}|u(x_{n-k}) - u(x_{n-k-1})|^2 \leq \rho_s^{n-k} \sum_{i_1\ldots i_{n-k}=0}^1 \left[ \frac{1}{2} \sum_{\xi',\eta'} |u(\psi_{i_1\ldots i_{n-k}}(\bar{\xi}')) - u(\psi_{i_1\ldots i_{n-k}}(\bar{\eta}'))|^2 \right]
\] 
\[= \sum_{i_1\ldots i_{n-k}=0}^1 \rho_s^{n-k} \left[ \frac{1}{2} \sum_{\xi',\eta'} |u(\psi_{i_1\ldots i_{n-k}}(\bar{\xi}')) - u(\psi_{i_1\ldots i_{n-k}}(\bar{\eta}'))|^2 \right]
\] 
\[\leq \sum_{i_1\ldots i_{n-k}=0}^1 \sum_{\xi',\eta'} \rho_s^{n-k} \left[ \frac{1}{2} \sum_{\xi',\eta'} |u(\psi_{i_1\ldots i_{n-k}}(\bar{\xi}')) - u(\psi_{i_1\ldots i_{n-k}}(\bar{\eta}'))|^2 \right]
\] 
\[\leq E_{n-k}[u]
\] 
\[\leq E[u].
\] (5.43)

So, we conclude that
\[
\rho_s^{n-k}|u(x_{n-k}) - u(x_{n-k-1})|^2 \leq E[u],
\] (5.44)
That is
\[ |u(x_{n-k}) - u(x_{n-k-1})|^2 \leq \rho_s^{2n} \sqrt{E[u]}, \quad (5.45) \]
Clearly, the same result holds for terms with \( y \) and we obtain
\[ |u(p) - u(q)| \leq 2 \sum_{k=0}^{n-m} \rho_s^{\frac{n}{2}} \sqrt{E[u]} \leq 2 \rho_s^{\frac{n}{2}} \sqrt{E[u]} \sum_{k=0}^{n-m} \rho_s^k \]
\[ \leq 2 \rho_s^{\frac{n}{2}} \sqrt{E[u]} \rho_s^{\frac{n-m+1}{2}} \left( \frac{\rho_s^{2m}}{\rho_s^{2n}} - 1 \right)^{\frac{1}{2}} \]
\[ \leq \frac{2 \rho_s^{\frac{n}{2}} \sqrt{E[u]} \rho_s^{\frac{n-m}{2}}}{\rho_s^{2} - 1} \]
\[ = \frac{2 \rho_s^{\frac{n}{2}} \sqrt{E[u]} \rho_s^{\frac{n-m}{2}}}{\rho_s^{2} - 1}. \]
Define
\[ \beta := -\frac{\log \rho_s}{2 \log \alpha_0}. \]
Since
\[ \rho_s^{\frac{n-m}{2}} = \frac{\alpha_0^{-(m+1)(\log \alpha_0 1/\rho_s))}}{2}, \]
by (5.33), we conclude
\[ |u(p) - u(q)| \leq \frac{2 \rho_s^{\frac{n}{2}} \sqrt{E[u]} |p - q|^\beta}{\rho_s^{2} - 1}. \]
(5.47)
Let \( c = 2 \rho_s/((\rho_s^{1/2} - 1)) \), finally we obtain
\[ |u(p) - u(q)| \leq c \sqrt{E[u]} |p - q|^\beta. \]
(5.48)

\[ \square \]

**Corollary 5.26.** From Lemma 5.25 we get every function \( u \in D_E^\infty \) can be uniquely extended to a continuous function on \( Q \).

Therefore, every function \( u \in D_E^\infty \) can be uniquely extended to an element of \( C(Q) \). We denote this extension still by \( u \) and set
\[ D_E = \{ u \in C(Q); E[u] < +\infty \}, \]
(5.49)
where \( E[u] = E[u|_{Q^\infty}] \).
Moreover, for every \( u \in D_E \), the estimate in Lemma 5.25 will hold, by which we conclude that \( D_E \subset C^{0,\beta}(Q) \).

**Definition 5.27.** Let \( H \) denotes the Hilbert space \( H = L^2(X, \mu(Q)) \). A form \( E \) is closed in \( H \) if its domain \( D_E \) is complete under the inner product \( E(u,v) + (u,v)_{L^2(Q,\mu)} \).

**Lemma 5.28.** \( D_E \) is complete under the norm
\[ \|u\|_{D_E} = (\|u\|_{L^2(Q,\mu)}^2 + E[u])^{\frac{1}{2}}. \]
(5.50)
Proof. Choose a Cauchy sequence \( \{u_n\} \) in \( D_E \) such that
\[
\|u_n - u_m\|_{D_E} = \left( \|u_n - u_m\|^2_{L^2(Q,\mu)} + E[u_n - u_m] \right)^{\frac{1}{2}} \to 0, \tag{5.51}
\]
for \( n, m \to 0 \).
Then we have
\[
\|u_n - u_m\|^2_{L^2(Q,\mu)} \to 0,
\]
and
\[
E[u_n - u_m] \to 0.
\]
Thus we have \( \|u_n\|_{L^2(Q,\mu)} \leq C_1 \) and \( E[u_n] \leq C_2 \), because Cauchy sequences are bounded.

Now show that \( u_n(x) \) is uniformly bounded on \( Q \).
For any \( x, y \in Q \), we have
\[
|u_n(x)| \leq |u_n(x) - u_n(y)| + |u_n(y)|
\leq c \sqrt{E[u_n]|x - y|^\beta + |u_n(y)|}
\leq cC_2diam(Q)^\beta + |u_n(y)|
\leq cC_2 + |u_n(y)|.
\]
where \( c, C_2 \) are constant. As \( \mu(Q) = \int_Q d\mu = 1 \), integrating on both sides in \( \mu(dy) \) gives
\[
|u_n(x)| \leq cC_2 + \int_Q |u_n(y)|d\mu(y). \tag{5.52}
\]
By Schwarz inequality
\[
|u_n(x)| \leq cC_2 + \mu(Q)^{\frac{1}{2}} \left( \int_Q |u_n(y)|^2d\mu(y) \right)^{\frac{1}{2}}
\leq cC_2 + C_1^{\frac{1}{2}}.
\]
where \( C_1 \) is constant.
Additionally, it can be proved that the functions \( u_n(x) \) are equicontinuous, since for any \( x, y \in Q \), we have
\[
|u_n(x) - u_n(y)| \leq c \sqrt{E[u_n]|x - y|^\beta} \leq cC_2diam(Q)^\beta \leq cC_2. \tag{5.53}
\]
Hence, \( \{u_n(x)\} \) is uniformly bounded and equicontinuous on \( Q \). By Ascoli-Arzela theorem, there exists a subsequence \( \{u_{n_k}\} \) of \( \{u_n\} \) and \( u \in C(Q) \) such that
\[
\|u_{n_k} - u\| \to 0
\]
for \( k \to \infty \). It follows that \( u \in L^2(Q,\mu) \) as \( C(Q) \subset L^2(Q,\mu) \), and
\[
\|u_n - u\|_{L^2(Q,\mu)} \to 0 \tag{5.54}
\]
for \( n \to \infty \).
Now we want to prove that $u \in D_E$, and $E[u_n - u] \to 0$ as $n \to 0$.

Since $E_k[u_n - u]$ is a finite sum, that for a fixed $n$, we have

$$E_k[u_n - u] = \lim_{m \to \infty} E_k[u_n - u_m] \leq \lim_{m \to \infty} E[u_n - u_m]$$  \hspace{1cm} (5.55)

Let $k \to \infty$, then

$$E[u_n - u] \leq \lim_{m \to \infty} E[u_n - u_m]$$

$$\limsup_{m \to \infty} E[u_n - u_m] \leq \lim_{n,m \to \infty} E[u_n - u_m] = 0,$$

which implies

$$\lim_{n \to \infty} E[u_n - u] = 0.$$  

Therefore, we proved that there is a $u \in D_E$ such that

$$\|u_n - u\|_{D_E} = \left(\|u_n - u\|^2_{L^2(Q,\mu)} + E[u_n - u]\right)^{\frac{1}{2}} \to 0$$

for $n \to \infty$, i.e., the completeness of $D_E$. \hfill \Box

**Lemma 5.29.** The form $E(u, v)$ is closed in $H$ with domain $D_E$, under the norm

$$\|u\|_{D_E} = (\|u\|^2_{L^2(Q,\mu)} + E[u])^{\frac{1}{2}}.$$  

**Proof.** This is the result of Lemma 5.28. \hfill \Box

**Lemma 5.30.** $D_E$ is dense in $C(Q)$ with respect to uniform norm.

**Proof.** Given any function $f \in C(Q)$, since $Q$ is compact, we know that $f$ is uniformly continuous on $Q$. Now, we extend the values $f|_{\Gamma_{i_1 \ldots i_n}}$ to a harmonic function inside $Q_{i_1 \ldots i_n}$ by the average procedure given by Gauss variational principle. Noticing that by doing the harmonic extension in all $Q_{i_1 \ldots i_n}$, we construct a continuous function, named as $f_n$, on $Q$. Next, by the maximum principle which comes from the Gauss minimization, we obtain that the extended harmonic function $f_n$ will take $\max$ and $\min$ on $\Gamma_{i_1 \ldots i_n}$.

It is clear that for any $x \in Q$, there exists a point $y$ in $\Gamma_{i_1 \ldots i_n}$ such that $|x - y| \leq \alpha_{i_1 \ldots i_n}$ and

$$\text{osc}_{Q_{i_1 \ldots i_n}} f_n \leq \text{osc}_{\Gamma_{i_1 \ldots i_n}}.$$  

Moreover, we have

$$f_n(x) = f_n(y) + f_n(x) - f_n(y)$$

$$= f(y) + f_n(y) - f_n(y),$$

since $y \in \Gamma_{i_1 \ldots i_n}$.

Then, we have

$$|f(x) - f_n(x)| \leq |f(x) - f(y)| + |f_n(x) - f_n(y)|.$$  

Therefore, we conclude that

$$\max_{x \in Q} |f(x) - f_n(x)| \leq \max_{x,y \in Q_{i_1 \ldots i_n}, |x-y| \leq \alpha_{i_1 \ldots i_n}} |f(x) - f(y)| + \text{osc}_{Q_{i_1 \ldots i_n}} f_n \leq \max |f(x) - f(y)| \to 0.$$  \hspace{1cm} (5.56)

Hence we have shown that there exists a function $f_n \in D_E$ such that

$$\|f - f_n\|_{\infty} \to 0,$$  

as $n \to \infty$.
Definition 5.31. A form $E$ in $H$ is regular if it possesses a core, a core being any subset $C$ of $D_E \cap C(X)$, which is dense both in $C_0(X)$ with the uniform norm and in $D[E]$ with the intrinsic norm $(E(u, u) + (u, u))^{\frac{1}{2}}$.

Lemma 5.32. The form $E$ is regular in $H$.

Proof. This is the result of Lemma 5.30. □

Definition 5.33. A variational fractal - embedded in the space $X \equiv (X, d_0)$ - is a triple $K \equiv (K, \mu, W)$, where $K$ is the invariant set of a given family $\{\psi_1, \ldots, \psi_N\}$; $\mu$ is the invariant measure on $K$; moreover, $W$ is a irreducible, strongly local, regular Dirichlet form in the Hilbert space $H = L^2(K, \mu)$ - with domain $D_W$ - such that

$$W[u] = \sum_{i=1}^{N} \rho_i W[u \circ \psi_i], \quad u \in D_E,$$

where the constants $\rho_1, \ldots, \rho_N$ satisfy

$$\rho_i \equiv \mu(K_i)^{\sigma},$$

(5.57)

for some constant $\sigma < 1$, independent of $i = 1, \ldots, N$.

By irreducible, we mean above that $W[u] = 0$ implies $u =$ constant, that is, potentials with null energy are constants.

Theorem 5.34. The Triple $Q \equiv (Q, \mu, E)$ is a variational fractal.

Proof. First of all, it is clear that $Q$ is an invariant set of a given family $\{\psi_0, \psi_1\}$ satisfying

$$d_0(\psi_1(x), \psi_1(y)) = r_1 d_0(x, y), \quad 0 < r_1 < 1,$$

and satisfies the o.s.c; also, $\mu$ is the invariant measure on $Q$.

Now we prove the form $E$ is irreducible. Recall that in Lemma 5.25, we have for any $p, q \in Q^\infty$,

$$|u(p) - u(q)| \leq c \sqrt{\sup_{n \geq 0} E_n[u|Q^n]|p - q|^\beta.}$$

(5.58)

Therefore, when $E \equiv 0$, we have $u(q) = u(p)$ for all $p, q \in Q^\infty$.

That is, $u$ is a constant on $Q$.

Moreover, since $\mu(Q_i) = r_i^s$, where $s$ is defined in (2.11), we have

$$\rho_i \equiv (r_i^s)^{-1} \equiv \mu(Q_i)^{\sigma},$$

(5.59)

which is exactly what we want in (5.25) by choosing $\sigma = -1$. Therefore, we conclude that the triple Q is a variational fractal.

Remark 5.35. Since $Q$ is a variational fractal, all results proved in [14] and [15] will go through, such as the Scaled Poincaré inequalities, Nash inequalities, and Morrey-Sobolev imbeddings.
Example 5.36. Here we give an example of $\rho_0$ and $\rho_1$ by setting $\theta = \pi/6$ and for several different value of $h$.

First of all, let $h = 0.1$, by (2.11), we know that

$$\left[\frac{9}{16} + \left(\frac{\sqrt{3}}{4} - 0.1\right)^2\right]^\frac{1}{2} + \left[\frac{1}{16} + \left(\frac{\sqrt{3}}{4} - 0.1\right)^2\right]^\frac{1}{2} = 1.$$

Calling Matlab, we have

$s \approx 1.531892451$.

Then, we reach the value of $\rho_0$ and $\rho_1$ in (5.25) such that

$\rho_0 \approx 1.353738613$ \hspace{1cm} $\rho_1 \approx 3.826946121$.

Now we try $h = 0.2$, similarly, we have

$s \approx 1.251372206$, \hspace{1cm} $\rho_0 \approx 1.3530255$ \hspace{1cm} $\rho_1 \approx 3.832656$.

More, try $h = 0.05$, we have

$s \approx 1.884080771$, \hspace{1cm} $\rho_0 \approx 1.339778912$ \hspace{1cm} $\rho_1 \approx 3.943090236$.

6. The Laplacian on the Quasi-Pólya Curve

We now define the Laplace operator on the fractal $Q$, with homogeneous Dirichlet boundary conditions. Below we will use the symbol $\Delta_Q$ to denote the Laplace operator.

As the form $Q$ is closed in $L^2(Q, \mu)$, there exists a non-positive self-adjoint operator $\Delta_Q$ in $L^2(Q, \mu)$, with dense domain $D_E$ in $L^2(Q, \mu)$, such that

$$E(u, v) = -\int_Q (\Delta_Q u)v d\mu, \text{ for every } v \in D_0(Q), \quad (6.1)$$

where $D_0(Q) = \{v \in D_E : v = 0 \text{ on } Q^0\}$. 

Part 3. Future Work

7. Future Work

Following the investigations described in this thesis, a number of future works could be taken up:

1. We have constructed the energy forms on Quasi-Pólya curve and finished the geometric construction of Quasi-Pólya surface. Then we want to construct the energy forms on Quasi-Pólya surface as well. However, we may have to use a technique other than the one we used in Section 6.3 since the Quasi-Pólya surface lose the self-similarity.

2. As we state in Part 1, the Quasi-Pólya curve approximates the Pólya’s curve as \( h \to 0 \). Thus, it might expect that the Laplacian operator on Quasi-Pólya curve, generated by using energy form on Quasi-Pólya curve, converges to the two dimensional Laplacian operator as \( h \to 0 \). Also, the Laplacian operator on Quasi-Pólya surface converges to the three dimensional Laplacian operator as \( h \to 0 \).

3. Since we proved the Quasi-Pólya curve is a variational fractal, the \textit{Scaled Poincaré inequalities, Nash inequalities, and Morrey-Sobolev imbeddings} theorems will be hold on Quasi-Pólya curve. Then, it is natural to consider these theorems on Quasi-Pólya surface.
Part 4. Appendix 1: A TWO-DIMENSIONAL PÓLYA-TYPE MAP FILLING A PYRAMID

Accepted: *Numerical Functional Analysis and Optimization*

Pan Liu

Department of Mathematical Sciences, Worcester Polytechnic Institute, 100 Institute Road, Worcester MA 01609-2280, USA

**Abstract.** We construct a family of Pólya-type volume-filling continuous maps from a rectangle onto a right triangular solid pyramid. The family is indexed by a parameter $\theta_0$ between $0$ and $\pi/4$, which is the largest of the smallest angles of the vertical sections of the pyramid. We extend to these maps the differentiability results obtained by Peter Lax for the Pólya’s map for different values of $\theta_0$.

1. Introduction

The space-filling curves were well studied by mathematicians since the beginning of the 20th century. Examples of such curves are the Peano curve [4], the Hilbert curve [2], the Sierpiński curve [21] and the Pólya curve [5]. Later on, it was proved that (the coordinate functions of) the Peano, Hilbert, and Sierpiński curves are nowhere differentiable. So, it was tacitly assumed that the Pólya curve will be nowhere differentiable too. However, in 1973, Peter Lax proved that Pólya’s curve is differentiable on a subset of the domain that depends on the smallest angle $\theta_0$ of the triangle, and he further described the properties of this differentiability set in the three regions defined by two special angles, namely $15^\circ$ and $30^\circ$, as showed in Fig.9.

Volume-filling constructions have rarely been studied. An example of volume-filling curve was given by Sagan, who extended the Hilbert construction to 3-dimensional space, and this 3D curve is also nowhere differentiable.

The idea of this paper is to extend Pólya’s curve to a map from a rectangle onto a right pyramid preserving, at the same time, Lax differentiability properties. The construction of such a map, however, cannot be based on altitude projections, like in the case of Pólya’s curve, because there is no geometrical construction of such a kind that partitions the pyramid into smaller similar pyramids. We carry out the construction of a continuous map from the rectangle $Q = \{(t, \alpha) \in \mathbb{R}^2: 0 \leq t \leq 1, 0 \leq \alpha \leq \pi/2\}$ onto the right pyramid of $\mathbb{R}^3$, with vertices $(0, 0, 0)$, $(0, \tan \theta_0, 0)$, $(0, 0, 1)$, and $(\tan \theta_0, 0, 0)$. We observe that now $\theta_0 \in (0, \pi/4)$ is the largest of the smallest angles of the vertical sections of the pyramid $R^{\theta_0}$. We denote this map by $L^{\theta_0} = L^{\theta_0}(t, \alpha)$ and call it Pólya surface.

We then show that, though not differentiable, the map $L^{\theta_0}$ possesses both partial derivatives $\frac{\partial}{\partial \alpha} L^{\theta_0}$ and $\frac{\partial}{\partial t} L^{\theta_0}$ on a subset of the domain that can be described as a function of the angle $\theta_0$ in a similar way as in Lax result. In addition to the two special angles single out by Lax, that is $15^\circ$ and $30^\circ$, the fine tuning of the differentiability property brings to light two more relevant angles, namely $20.36^\circ$ and $39.25^\circ$. 
In Lax’s result, the two angles $15^\circ$ and $30^\circ$ distinguish three regions for the parameter $\theta_0 \in (0, \frac{\pi}{4})$, which are denoted by $\ast$, $\ast\ast$ and $\ast\ast\ast$ in Fig.9. Pólya’s curve has the best differentiability behavior in the region $\ast\ast\ast$ and the worse in $\ast$, as described more precisely in Theorem 2.4.

In the case of Pólya surface, the worse region $\ast$ is still the one corresponding to $\theta_0$ near $45^\circ$ (flat pyramids). The differentiability behavior, however, improves to the status $\ast\ast$ before the angle $\theta_0$, decreasing from $45^\circ$, reaches the value $30^\circ$, more precisely, as soon as $\theta_0$ crosses the new special angle, $39.25^\circ$. In fact, for $\theta_0$ strictly less than $39.25^\circ$, an intermediate vertical sector appears in the pyramid where the differentiability upgrades to the status $\ast\ast$. The opening of such a sector increases as $\theta_0$ decreases, and the sector fills the whole pyramid when $\theta_0$ reaches the values $30^\circ$. A similar effect takes place when $\theta_0$ decreases further from $30^\circ$ to $15^\circ$ (pointed pyramids). As $\theta_0$ decreases further and crosses the second additional angle $20.36^\circ$, again a central section of the pyramid shows upgraded differentiability of status $\ast\ast\ast$, and this central sector becomes the whole pyramid when $\theta_0$ reaches the value $15^\circ$. This new phenomenon is described in Theorem 4.7 and illustrated in Fig.12.

As all space-filling constructions, Pólya surface, like Pólya’s curve, is self-intersecting. However, it is possible to approximate Pólya surface with surfaces with no self-intersections, that fill a pyramid up to a residual small volume of arbitrary size $\varepsilon$. These new fractal surfaces are the boundary of an open domain. The result presented in this paper is therefore related to the recent literature on boundary value problems in small domains with large boundary, as the ones considered in [13]. However, we do not develop this aspect of our research in present paper.

Our paper is organized as follows: In Section 2 we describe the construction of Pólya’s curve and the differentiability results of Lax. In Section 3 we prove the self-similarity of Pólya’s map, shown recently by Ramsay and Terry in [1], and the representation formulas original given by Sagan in [8]. By relying on these properties, we study Pólya’s map as a function of both $t$ and $\theta$, and prove, Lemma 3.7, the existence of the partial derivative in $\theta$. In Section 4, we give the construction of Pólya surface and our main result, the partial differentiability properties in Theorem 4.7.

2. PÓLYA’S FUNCTION AND LAX RESULT

In 1913 Pólya presented an iterative geometric construction of a map, $P$, mapping the interval $[0, 1]$ onto a non-isosceles right triangle $T$, [5]. In 1973 Lax proved that the Pólya’s map is differentiable on a subset of $[0, 1]$ that depends on the smallest angle of $T$. To build Pólya’s map, we need to follow the process illustrated below.

Let $t$ be any number in the unit interval and consider it to be a binary fraction:

$$t = .d_1d_2 \cdots$$

where the $n-th$ digit $d_n$ is either 0 or 1.
**Definition 2.1.** Let $T$ be a non-isosceles right triangle, we define the Pólya’s function $P(t)$, which maps a point $t \in [0, 1]$ into $T$, by the following process:

i. We subdivide $T$ into two smaller and similar triangles in $T$ by drawing the altitude of $T$. Since $T$ is non-isosceles, these two triangles are unequal; call the larger of the two $T_0$, the smaller $T_1$.

ii. We define $T_1$ to be $T_0$, if $d_1 = 0$; $T_1$ to be $T_1$, if $d_1 = 1$; $T^2$, $T^3$... are defined recursively, with $T^{n-1}$ taking the place of $T$ and $d_n$ replacing $d_1$.

iii. We denote the $n$-th triangle assigned to the number $t = 0.d_1d_2d_3...d_n$ by $T^n(t)$. The sequence $T^n(t)$ is nested, and the hypotenuse of $T^n(t)$ goes to 0 as $n \to \infty$. The triangles $T^n(t)$ have exactly one point in common, and we define this point to be $P(t)$.

**Remark 2.2.**

We know that there can be two binary representations of the same real number $t$. Lax, however, has shown that the procedure described in Definition 2.1 will assign the same point $P(t)$ to different binary representations of $t$.

**Theorem 2.3.** [5] Pólya’s Theorem.

i. The function $P$ maps the interval $I = [0, 1]$ continuously onto the triangle $T$.

ii. If one chooses the ratio of the shorter side to the hypotenuse to be a transcendental number, then every point in $T$ has, at most, three pre-images in $I$.

**Theorem 2.4.** [12] Lax’s Differentiability Theorem.

Let $\theta_0$ denote the smaller angle of the (non-isosceles) right triangle $T$. Then, the function $P(t)$ has the following properties:

* If $30^\circ < \theta_0 < 45^\circ$, $P$ is nowhere differentiable

** If $15^\circ < \theta_0 < 30^\circ$, $P$ is not differentiable on a set of Lebesgue measure 1, but has derivative zero on a non-denumerable set

*** If $\theta_0 < 15^\circ$, $P' = 0$ on a set of measure 1.

In the picture below (Fig.9) we outline the 3 cases.

\[
\begin{array}{cccc}
\theta_0 : 2D & \ast\ast\ast & \ast\ast & \ast \\
0^\circ & 15^\circ & 30^\circ & 45^\circ
\end{array}
\]

**Figure 9.** The differentiability regions of Pólya’s curve

3. **Self-Similarity of Pólya’s Curve**

In order to generate Pólya’s curve, we divide the triangle $T$ by its altitude and create two similar small triangles. In other words, we map the interval $[0, \frac{1}{2}]$ onto the triangle $T_0$ and the interval $[\frac{1}{2}, 1]$ onto the triangle $T_1$. We then iterate this procedure further. This suggests that the Pólya’s map must possess some self-similarity property. This has been shown by Ramsay and Terry, who proved that Pólya’s map is indeed *self-similar* in the sense of the Definition 3.1 below, [1].
Now we define the contractive similitudes, for every $\pi$ the smallest angle of $T$:

$$S^i = [0, \pi].$$

Now, to show that Pólya’s function is self-similar, we first define the contractive similitudes that operate on $S$. Let $\chi$ be a surjective mapping $\chi : S \to S'$ such that

$$F(\chi_{i/n}(S)) = \omega_{i/n}(S'),$$

for every $i/n$. Then we say that the mapping $F$ is self-similar.

Now, to show that Pólya’s function is self-similar, we first define the contractive similitudes that operate on $I = [0, 1]$ in the following way:

$$\phi_0(x) = \frac{x}{2}, \quad \phi_1(x) = \frac{x + 1}{2}.$$ 

Clearly, the interval $I = [0, 1]$ is self-similar with respect to $\phi_0$ and $\phi_1$. In particular

$$I = \bigcup_{i/n} \phi_{i/n}(I),$$

for every $n$. The length of every interval $\phi_{i/n}(I)$ equal to $2^{-n}$.

Now we define the contractive similitudes, $\psi_0$ and $\psi_2$, that operate on a triangle $T$. Assign $\theta_0$ as the smallest angle of $T$, $a = (a_1, a_2)$ as the vertex of $T$ at the angle $\theta_0$, and $b = (b_1, b_2)$ as the vertex of $T$ at the angle $\frac{\pi}{2} - \theta_0$:

- $\psi_0$ is the similarity which maps $T$ to the larger of the two similar triangles, named as $T_0$, formed by the altitude of $T$. For an arbitrary point $x = (x_1, x_2) \in T$:

$$\psi_0(x) = \psi_0(x_1, x_2) = \left[\begin{array}{c}
a_1 + \cos \theta_0[(a_1 - x_1) \cos \theta_0 + (a_2 - x_2) \sin \theta_0] \\
a_2 + \cos \theta_0[(a_1 - x_1) \sin \theta_0 + (x_2 - a_2) \cos \theta_0]
\end{array}\right].$$

- $\psi_1$ is the similarity which maps $T$ to the smaller of the two similar triangles, named as $T_1$, formed by the altitude of $T$. For an arbitrary point $x = (x_1, x_2) \in T$:

$$\psi_1(x) = \psi_1(x_1, x_2) = \left[\begin{array}{c}
b_1 + \sin \theta_0[(x_1 - b_1) \sin \theta_0 + (b_2 - x_2) \cos \theta_0] \\
b_2 + \sin \theta_0[(b_1 - x_1) \cos \theta_0 + (b_2 - x_2) \sin \theta_0]
\end{array}\right].$$
For convenience, we define the triangle $T$ occurring in Pólya construction to be the following triangle of $\mathbb{R}^2$:

- The point of smallest angle $\theta_0$ in $T$ is the point $(0, 1)$, that is, $a = (a_1, a_2) = (0, 1)$
- The point of angle $\pi/2 - \theta_0$ is the point $(\tan \theta_0, 0)$, that is, $b = (b_1, b_2) = (\tan \theta_0, 0)$.

Then, the maps $\psi_0$ and $\psi_1$ can be written in the following way:

$$
\psi_0(x) = \cos \theta_0 \cdot O_0 x + \sin \theta_0 \cdot \mu
$$

with

$$
O_0 = \begin{bmatrix} -\cos \theta_0 & -\sin \theta_0 \\ -\sin \theta_0 & \cos \theta_0 \end{bmatrix}, \quad \mu = \begin{bmatrix} \cos \theta_0 \\ \sin \theta_0 \end{bmatrix},
$$

and

$$
\psi_1(x) = \sin \theta_0 \cdot O_1 x + \sin \theta_0 \cdot \mu
$$

with

$$
O_1 = \begin{bmatrix} \sin \theta_0 & -\cos \theta_0 \\ -\cos \theta_0 & -\sin \theta_0 \end{bmatrix}, \quad \mu = \begin{bmatrix} \cos \theta_0 \\ \sin \theta_0 \end{bmatrix}.
$$

Note that in this way, the two legs of $T$ lay on the positive $x_1$- and $x_2$-axis.

The larger sub-triangle $T_0$ in Definition 1 is now obtained as $T_0 = \psi_0(T)$, and similarly, the smaller sub-triangle $T_1 = \psi_1(T)$. At the next subdivision, we obtain the sub-triangles $T_{00} = \psi_0 \circ \psi_0(T)$, $T_{01} = \psi_0 \circ \psi_1(T)$, $T_{10} = \psi_1 \circ \psi_0(T)$, and $T_{11} = \psi_1 \circ \psi_1(T)$, and we get the decomposition $T = T_{00} \cup T_{01} \cup T_{10} \cup T_{11}$. After $n$ iterations, we get the decomposition

$$
T = \bigcup_{i/n} \psi_{i/n}(T).
$$

This shows that the triangle $T$ is self-similar with respect to $\psi_0$ and $\psi_1$.

**Theorem 3.2.** [1] Let $P : I \rightarrow T$ be Pólya’s map. Then for every $i/n$,

$$
P(\phi_{i/n}(I)) = \psi_{i/n}(T).
$$

Moreover,

$$
P(\phi_1(I)) = \psi_1(T)
$$

for every $i$. In particular, $P$ is self-similar.

**Proof.** Define

$$
I_{i/n} = \phi_{i/n}([0, 1]) = \left[ \frac{k}{2^n}, \frac{k + 1}{2^n} \right].
$$

Since every point in $I_{i/n}$ has the same first $n$ digits, these points will all be mapped to the same subtriangle of $T$. We can find which subtriangle of $T$ this is if we take $\psi_{i/n}(T)$. We note that the action of the maps $\psi_0$ and $\psi_1$ reproduces the process carried out by Pólya’s function, i.e., the division of $T$ by the altitude and the choice of the smaller or larger triangle.
Therefore, we have that
\[ P(\phi_{i/n}(I)) = \psi_{i/n}(T). \]
For a given \( i \), by taking the limit as \( n \to \infty \) in the previous identity, we obtain by the continuity of \( P \)
\[ P(\phi_{i}(I)) = \psi_{i}(T). \]
\( \square \)

Now that we have established the Self-Similarity of Pólya function, we introduce the Pólya Interpolates, which are suitable Polygonal trajectories based on Pólya construction.

First of all, we partition the unite interval \([0, 1]\) into \( n \) small intervals, \([t_j, t_{j+1}]\), \( j = 0, 1, 2, \ldots, n \), by letting
\[ 0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = 1, \tag{3.5} \]
where \(|t_j - t_{j+1}| = 2^{-n}\).
Clearly, by the construction of \([t_j, t_{j+1}]\), there exists a \( i/n \) such that
\[ P^j := [t_j, t_{j+1}] = \phi_{i/n}(I), \tag{3.6} \]
and by using the same \( i/n \), we get the triangle \( T^j := \psi_{i/n}(T) \).

**Lemma 3.3.** For any interval \([t_j, t_{j+1}]\) of the partition introduced in (3.5) and (3.6), there exists a \( t^*_j \in [t_j, t_{j+1}] \) such that the point
\[ G_j = P(t^*_j) \tag{3.7} \]
is the geometric center of triangle \( T^j = T_{i/n} = \psi_{i/n}(T) \).

**Proof.** By the self-similarity of \( I = [0, 1] \), we have
\[ [t_j, t_{j+1}] = \phi_{i/n}(I) = \bigcup_{k \in K} \phi_{i/n} \circ \phi_k(I), \tag{3.8} \]
where \( k = d_1d_2\ldots d_m \ldots \) is a infinite sequence with \( d_m \in \{0, 1\} \) and \( K \) is the collection of all possible sequences \( k \). Therefore,
\[ P(\phi_{i/n}(I)) = \bigcup_{k \in K} P(\phi_{i/n} \circ \phi_k(I)). \tag{3.9} \]
By (3.3), we have
\[ P(\phi_{i/n}(I)) = \psi_{i/n}(T) = T_{i/n} \]
and by (3.4),
\[ P(\phi_{i/n} \circ \phi_k(I)) = \psi_{i/n} \circ \psi_k(T). \]
Therefore, from (3.9), we obtain

\[ T_{i/n} = \bigcup_{k \in K} \psi_{1/n} \circ \psi_k(T). \]

So, there exists a sequence \( k^* \in K \) such that \( \psi_{1/n} \circ \psi_{k^*}(T) \) is the geometric center \( G_j \) of the triangle \( T_{i/n} \). We then define \( t_j^* \) to be

\[ t_j^* := \phi_{1/n} \circ \phi_{k^*}(I). \] (3.10)

By (3.8), we know that

\[ t_j^* \in [t_j, t_{j+1}]. \] (3.11)

Moreover, by the self-similarity of \( P \)

\[ G_j = P(t_j^*), \]

and this concludes our lemma.

For every partition as in Lemma 3.3, we obtain the sequence of \( t_j^* \in [t_j, t_{j+1}] \), such that

\[ t_0 < t_0^* < t_1 < t_1^* < t_2 < \cdots < t_{n-1} < t_n^* < t_n \]

for \( j = 0, 1, 2, \ldots \), and we now define the Pólya interpolate, associated with that partition, as the oriented polygonal curve in \( T \) that connects each \( P(t_j^*) \) to the successive one, as showed in Fig.??.

Moreover, since

\[ |t_j - t_{j+1}| = 2^{-n} \to 0, \text{ as } n \to \infty, \] (3.12)

we obtain that \( t_j = t_j^* = t_{j+1} \) in the limit as \( n \to \infty \). Therefore, the Pólya interpolate converges eventually to the Pólya curve as \( n \to \infty \), in an obvious sense.

**Theorem 3.4.** [8] The Pólya’s mapping, \( P(t) = P(0.d_1d_2d_3\ldots d_n\ldots) \), can be represented for any \( t = 0.d_1d_2d_3\ldots d_n\ldots \), by

\[ P(t) = P(0.d_1d_2d_3\ldots) = \sum_{j=1}^{\infty} e^{2\pi j s V_j} O_{d_1} O_{d_2} \cdots O_{d_j-1} \mu, \] (3.13)

where \( c = \cos \theta_0, \ s = \sin \theta_0, \) and \( Z_i, V_i \) is the number of 0’s, 1’s preceding \( d_i \), respectively, and \( \mu \) is defined in equation (3.1).

**Proof.** Recall the equation (3.4) in Theorem 3.1, that is

\[ P(\phi(I)) = \psi(T). \]

It is clear that

\[ \phi(I) = 0.d_1d_2d_3\ldots d_n\ldots \]

represents the parameter \( t \in [0, 1] \) in binary form. Obviously, for any binary fraction \( t \), we have

\[ 0.d_1d_2d_3\ldots d_n = 0.d_1d_2d_3\ldots d_n0000000\ldots, \]
and, for convenience, we define \( \psi^n_0(x) \) to be \( \psi_0 \) applied \( n \) times on \( x \). i.e., \( \psi^n_0(x) = \psi_0 \circ \psi_0 \circ \cdots \circ \psi_0(x) \) for \( n \) \( \psi_0(x) \)’s.

Taking any \( x = (x_1, x_2) \in T \), we have

\[
\psi^n_0(x) = (\cos \theta_0 \cdot O_0 + \sin \theta_0 \cdot \mu) \circ \cdots \circ (\cos \theta_0 \cdot O_0 x + \sin \theta_0 \cdot \mu) \\
= \cos^n \theta_0 \cdot O^n_0 \cdot x + \sin \theta_0 [\cos^{n-1} \theta_0 \cdot O^{n-1}_0 \cdot \mu + \\
+ \cos^{n-3} \theta_0 \cdot O^{n-3}_0 \cdot \mu + \cdots + \cos^2 \theta_0 \cdot O^2_0 \mu + \cos \theta_0 \cdot O_0 \mu + \mu].
\]

(3.14)

Notice that \( O_0 \cdot O_0 = I \) is a \( 2 \times 2 \) identity matrix, and we obtain

\[
\lim_{n \to \infty} \cos^n \theta_0 \cdot O^n_0 = \begin{cases} 
\lim_{n \to \infty} \cos^n \theta_0 \cdot I, & \text{if } n \text{ is even} \\
\lim_{n \to \infty} \cos^n \theta_0 \cdot O_0, & \text{if } n \text{ is odd}
\end{cases} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

Also, we have

\[
\lim_{n \to \infty} [\sin \theta_0 \cdot (\cos^{n-1} \theta_0 \cdot O^{n-1}_0 \mu + \cdots + \cos \theta_0 \cdot O_d \mu + \mu)] \\
= \sin \theta_0 [1 + \cos^2 \theta_0 + \cos^4 \theta_0 + \cdots + \cos \theta_0 + \cos^2 \theta_0 + \cdots] O_0 \mu \\
= \frac{1}{\sin \theta_0} [\cos \theta_0 \cos \theta_0 - \cos \theta_0 \sin \theta_0 - \sin \theta_0 \cos \theta_0] \begin{bmatrix} \cos \theta_0 \\ \sin \theta_0 \end{bmatrix} \\
= \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

Therefore, by (3.14), we get

\[
\lim_{n \to \infty} \psi^n_0(T) = \psi_0 \circ \psi_0 \cdots \circ \psi_0 \circ \cdots (T) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

(3.15)

Thus

\[
P(0.d_1d_2d_3 \ldots d_n) = P(0.d_1d_2d_3 \ldots d_n000 \ldots) \\
= \lim_{k \to \infty} \psi_{d_1d_2d_3 \ldots d_n}^k \circ \psi_0^k T = \psi_{d_1d_2d_3 \ldots d_n}^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

(3.16)

By replacing the expression of \( \psi_0 \) and \( \psi_1 \) into right side of (3.16), we obtain

\[
P(0.d_1d_2d_3 \ldots d_n) = e^{Z_j \cdot S^j V_i} O_{d_1} O_{d_2} \cdots O_{d_n} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sum_{j=1}^{n} e^{Z_j \cdot S^j V_j+1} O_{d_1} \cdots O_{d_{j-1}} \mu,
\]

(3.17)

where

\[
\mu = \begin{bmatrix} \cos \theta_0 \\ \sin \theta_0 \end{bmatrix}
\]

as defined in (3.1).

To go to the limit as \( n \to \infty \), we need the property that the Pólya’s mapping is continuous. We could just accept this property from the original Pólya Theorem 2.3, or we can show this by taking \( |t_1 - t_2| < \frac{1}{\pi} \).
Hence, by (3.17) and the continuity of Pólya map, we obtain

\[ P(t) = P(0.d_1d_2d_3\cdots) = \sum_{j=1}^{\infty} c^{Z_j} s^{V_j+1} O_{d_1} O_{d_2} \cdots O_{d_{j-1}} \mu, \tag{3.18} \]

since

\[ \lim_{n \to \infty} c^{Z_j} s^{V_j} O_{d_1} O_{d_2} \cdots O_{d_n} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \]

\[\square\]

**Remark 3.5.**

By using a formula similar to equation (3.4), Sagan proved that the Pólya’s coordinate functions have the same differentiability shown in Theorem 2.4. Moreover, Sagan proved that the Pólya’s function is nowhere differentiable when the smallest angle \( \theta_0 \) is 30°, and has derivative zero on a non-denumerable set if the smallest angle \( \theta_0 \) is equal to 15°.

**Definition 3.6.** The General Pólya Function:

We define the function \( P_G(\theta, t) \), which maps the rectangle \([0, \frac{\pi}{4}] \times [0, 1]\) onto a right triangle \( T^\theta \) with vertices \((0, 0)\), \((0, 1)\), and \((\tan \theta, 0)\), by

\[ P_G(\theta, t) := \sum_{j=1}^{\infty} c^{Z_j} s^{V_j+1} O_{d_1} O_{d_2} \cdots O_{d_{j-1}} \mu, \tag{3.19} \]

by treating \( \theta_0 \), in equation (3.13), as a variable too.

**Lemma 3.7.** The General Pólya’s Function \( P_G(\theta, t) \) is continuous on \([0, \frac{\pi}{4}] \times [0, 1]\). Moreover, for every \( t \in [0, 1] \), the partial derivative \( \frac{\partial}{\partial \theta} P_G(\theta, t) \) does exist for all \( \theta \in (0, \frac{\pi}{4}) \).

**Proof.** Since the equation (3.19), after we fix \( t \), is only the composition of summations and multiplications of simple functions \( \sin \theta \) and \( \cos \theta \) and these two simple functions are both continuous and differentiable, we know that \( P(\theta, t_0) \) is continuous and differentiable with respect to \( \theta \). \[\square\]
4. Three Dimensional Constructions and Differentiability

The very first and natural idea to extend Pólya’s mapping to a pyramid would be trying to partition a triangular pyramid into $2^n$ small and similar triangular pyramids all similar to the original pyramid. But, unfortunately, we know of no construction leading to such kind of partition. In this regard, we refer to the recent paper [3], where similar packing constructions are carried out that provide only a partial partition of the initial pyramid up to 85.63% of the total volume.

Now, we introduce our three dimensional Solid Triangular Pyramid Construction. The idea is if we take the Pólya interpolate and rotate it around the vertical leg of $T$, we have a solid cone, as showed in Fig.10. Let $\alpha \in [0, 2\pi]$ denote the rotation angle and $t \in [0, 1]$ denote the variable in Pólya’s Function. Clearly, this operation maps a rectangle, $[0, 2\pi] \times [0, 1]$, to a solid cone from $\mathbb{R}^2$ to $\mathbb{R}^3$. But this is not good enough. We are looking for a map which can map the rectangle to a solid triangular pyramid.

\[ O_p^{\theta_0} := \{ (x, y) \in \mathbb{R}^2, (|x|^p + |y|^p)^{\frac{1}{p}} = \tan \theta_0 \}, \]

for $p > 0$.

- Given $\theta_0 \in (0, \frac{\pi}{4})$, we define

\[ C_p^{\theta_0} := \{ (x, y, z) : x, y \geq 0, 0 \leq z \leq 1, (|x|^p + |y|^p)^{\frac{1}{p}} \leq \tan \theta_0(1 - z) \}. \]
For example, $C_{2}^{\theta_{0}}$ is a quarter of the Solid Cone in Fig.10, and $C_{1}^{\theta_{0}}$ is a right triangular pyramid, as showed in Fig.11.

- Given $\theta_{0} \in (0, \frac{\pi}{4})$, we define $T^{\theta_{1}(\alpha)}$ to be a right triangle generated by the intersection of the vertical plane $y = (\tan \alpha)x$ and $C_{1}^{\theta_{0}}$.

**Lemma 4.2.** Given $\theta_{0} \in (0, \frac{\pi}{4})$, for each $\alpha \in [0, \frac{\pi}{2}]$, the smallest angle $\theta_{1}(\alpha)$ of the right triangle $T^{\theta_{1}(\alpha)}$, has the formula:

$$\theta_{1}(\alpha) = \arcsin \left[ \frac{\tan^{2} \theta_{0}(\tan^{2} \alpha + 1)}{\tan^{2} \theta_{0}(\tan^{2} \alpha + 1) + (1 + \tan \alpha)^2} \right]^{\frac{1}{2}}.$$  

Moreover, the function $\sin(\theta_{1}(\alpha))$ is continuous for all $\alpha \in [0, \frac{\pi}{2}]$.

**Proof.** Let $(x_{0}, y_{0}) \in O_{1}^{\theta_{0}}$, that is

$$\begin{align*}
x_{0} + y_{0} &= \tan \theta_{0} \\
\tan \alpha &= x_{0}/y_{0}
\end{align*}$$

(4.1)

Since the height of right triangle $T^{\theta_{1}(\alpha)}$ is 1, we have

$$\sin(\theta_{1}(\alpha)) = \left[ \frac{x_{0}^2 + y_{0}^2}{x_{0}^2 + y_{0}^2 + 1} \right]^{\frac{1}{2}},$$  

(4.2)

Together with (4.1), we obtain

$$\sin(\theta_{1}(\alpha)) = \left[ \frac{\tan^{2} \theta_{0}(\tan^{2} \alpha + 1)}{\tan^{2} \theta_{0}(\tan^{2} \alpha + 1) + (1 + \tan \alpha)^2} \right]^{\frac{1}{2}}.$$  

(4.3)

It is clear that $\sin(\theta_{1}(\alpha))$ is well defined for all $\alpha \in [0, \pi/2]$. For $\alpha = \pi/2$, though the function $\tan \alpha$ is not defined at $\alpha = \pi/2$, the function $\sin(\theta_{1}(\alpha))$ still makes sense and continuous at $\alpha = \pi/2$ by setting $\sin(\theta_{1}(\pi/2)) = \sin(\theta_{1}(0))$, because

$$\lim_{\alpha \to \frac{\pi}{2}} \sin(\theta_{1}(\alpha)) = \sin(\theta_{1}(0)).$$  

(4.4)

Therefore, $\theta_{1}(\alpha)$ is continuous in an obvious sense.

**Notation 4.3.** In the following we will use:

- By $\sin(\theta(\alpha))$ and $\theta(\alpha)$, we denote $\sin(\theta_{1}(\alpha))$ and $\theta_{1}(\alpha)$, respectively.
- By $R^{\theta_{0}}$, we denote $C_{1}^{\theta_{0}}$, a right triangular pyramid with vertices $(0, 0, 0), (0, \tan \theta_{0}, 0), (0, 0, 1)$, and $(\tan \theta_{0}, 0, 0)$.
- By $T^{\theta(\alpha)}$, we denote the right triangle given by the intersection of the vertical plane $y = (\tan \alpha)x$ with $R^{\theta_{0}}$.

**Definition 4.4.** The 3D construction map:

Given $\theta_{0} \in (0, \frac{\pi}{4})$, we define the function $L^{\theta_{0}}(\alpha, t)$, which maps the rectangle $Q = [0, \frac{\pi}{2}] \times [0, 1]$ onto $R^{\theta_{0}}$ by

$$L^{\theta_{0}}(\alpha, t) = (x, y, z),$$

where, by setting $(a, b) = P_{G}(\theta(\alpha), t)$, $x = a \cos \alpha$, $y = a \sin \alpha$ and $z = b$.

**Remark 4.5.**
As the definition above of $L^6(\alpha, t)$ is rather abstract and hard to visualize, it is useful to describe the function $L^6(\alpha, t)$ in a geometric way. For a better understanding, and to help building a mental picture of $L^6(\alpha, t)$, we can fix the rotation angle $\alpha$ first. We then consider the right triangle $T^{6(\alpha)}$ obtained by intersecting the pyramid $R^6$ with the vertical plane $y = (\tan \alpha)x$. Next, we build the Pólya’s curve inside $T^{6(\alpha)}$. Finally, we move the rotation angle. The function $L^6(\alpha, t)$ is the function that describes the rotation in $\alpha$ of the point $P(t)$ on the Pólya’s curve in $T^{6(\alpha)}$.

**Theorem 4.6.** Let $0 < \theta_0 < \frac{\pi}{4}$ be fixed, the map $L^6(\alpha, t)$ is surjective and continuous for all $\alpha \in [0, \frac{\pi}{2}]$ and $t \in [0, 1]$.

**Proof.** We first prove $L^6(\alpha, t)$ is surjective. Taking any point $m = (x, y, z) \in R^6$, there exists a $\alpha_0 \in [0, \frac{\pi}{2}]$, such that $m \in T^{6(\alpha_0)}$. Then, since the Pólya map is surjective, there exists a point $t_0 \in [0, 1]$ such that $P_G(\theta(\alpha_0), t_0) = (a, b)$ where $a = x/\cos(\alpha_0)$ and $b = z$. Therefore, for any point $m \in R^6$, there exists $\alpha_0$ and $t_0$ such that $L^6(\alpha_0, t_0) = m$. That is, the map $L^6(\alpha, t)$ is surjective.

To prove the continuity of $L^6(\alpha, t)$ in $t$ for fixed $\alpha$, it suffices to note that this map is the map $P(t)$ acting on $T^{6(\alpha)}$, which we know to be a continuous map from Theorem 2.3.

Next, to prove the continuity of function $L^6(\alpha, t)$ with respect to $\alpha$, we define an inter-median function

$$r(\alpha, x, z) := (x \cos \alpha, x \sin \alpha, z),$$

which rotates the point $(x, y, x) \alpha$ degrees around $z$-axis.

It is clear that the function $r(\alpha, x, z)$ is continuous and we have

$$L^6(\alpha, t) = (r \circ P_G)(\theta(\alpha), t).$$

We have shown in Lemma 1 that the function $P_G(\theta, t)$ is continuous for all $\theta \in [0, \frac{\pi}{2}]$, and in Lemma 4.2 that $\theta(\alpha)$ is also continuous. Therefore, the composite function $L^6(\alpha, t)$ is continuous for all $\alpha \in [0, \frac{\pi}{2}]$. \hfill \Box

Now, we are ready to introduce our main result, the **Solid Triangular Pyramid Differentiability Theorem.** First of all, for future convenience, we define

$$\theta_1 := \arcsin \sqrt{\frac{2}{5}}, \quad \theta_2 := \arcsin \sqrt{\frac{14 - 8\sqrt{3}}{15 - 8\sqrt{3}}}, \quad \theta(\alpha_1) := \frac{\pi}{6}, \quad \theta(\alpha_2) := \frac{\pi}{12}.$$ 

Moreover, to be consistent with Lax’s paper we convert angles measured in radians into angles measured in degrees. Approximately, $\theta_1 \approx 0.6847 = 39.25^\circ$ and $\theta_2 \approx 0.3622 = 20.36^\circ$.

**Theorem 4.7.** Solid Triangular Pyramid Partial Differentiability Theorem.

Given $\theta_0 \in (0, \frac{\pi}{4})$, the map $L^6(\alpha, t)$ defined for $\alpha \in [0, \frac{\pi}{2}]$, $t \in [0, 1]$ has the following partial differentiability properties:

i. Given $\theta_0 \in (0, \frac{\pi}{4})$, $\frac{\partial}{\partial \alpha} L^6(\alpha, t)$ does exist for all $\alpha \in (0, \frac{\pi}{2})$ and $t \in [0, 1]$;

ii. Given $\theta_0 \in [\theta_1, \frac{\pi}{4})$, for all $\alpha \in [0, \frac{\pi}{2}]$, $\frac{\partial}{\partial \alpha} L^6(\alpha, t)$ has property $*$;

iii. Given $\theta_0 \in [\theta_2, \frac{\pi}{6})$, for all $\alpha \in [0, \frac{\pi}{2}]$, $\frac{\partial}{\partial \alpha} L^6(\alpha, t)$ has property $**$;
iv. Given \( \theta_0 \in (0, \frac{\pi}{7}) \), for all \( \alpha \in [0, \frac{\pi}{2}] \), \( \frac{\partial}{\partial t} L^{\theta_0}(\alpha, t) \) has property \( ** \).

Moreover,

v. Given \( \theta_0 \in [\frac{\pi}{12}, \theta_1] \), when \( \alpha \in [0, \alpha_1] \cup [\frac{\pi}{2} - \alpha_1, \frac{\pi}{2}] \), \( \frac{\partial}{\partial t} L^{\theta_0}(\alpha, t) \) has property \( * \); when \( \alpha \in (\alpha_1, \frac{\pi}{2} - \alpha_1) \), \( \frac{\partial}{\partial t} L^{\theta_0}(\alpha, t) \) has property \( ** \).

vi. Given \( \theta_0 \in [\frac{\pi}{12}, \theta_2] \), when \( \alpha \in [0, \alpha_2] \cup [\frac{\pi}{2} - \alpha_2, \frac{\pi}{2}] \), \( \frac{\partial}{\partial t} L^{\theta_0}(\alpha, t) \) has property \( ** \); when \( \alpha \in (\alpha_2, \frac{\pi}{2} - \alpha_2) \), \( \frac{\partial}{\partial t} L^{\theta_0}(\alpha, t) \) has property \( ** \).

\[
\begin{array}{cccccc}
\theta_0 : & 2D & & & & \\
0^\circ & 15^\circ & 30^\circ & 45^\circ & & \\
\end{array}
\]

\[
\begin{array}{ccccccc}
\theta_0 : & 3D & & & & & \\
0^\circ & 15^\circ & 20.36^\circ & 30^\circ & 39.25^\circ & 45^\circ & \\
\end{array}
\]

**Figure 12.** The 3D partial differentiability properties regions, compared with the 2D case.

**Proof. Case 1:**

By (4.6), we have

\[
L^{\theta_0}(\alpha, t) = (r \circ P_G)(\theta(\alpha), t).
\]

It is clear that the partial derivative \( \frac{\partial}{\partial \alpha} r(\alpha, x, z) \) and \( \frac{\partial}{\partial \alpha} P_G(\theta(\alpha), t) \) both exist for all \( \alpha \in (0, \pi/2) \). Therefore, the partial derivative \( \frac{\partial}{\partial \alpha} L^{\theta_0}(\alpha, t) \) does exist for all \( \alpha \in (0, \frac{\pi}{2}) \) by Composite function theorem.

**Case 2:**

Recall that the Pólya function is nowhere differentiable when \( \theta_0 \in [\frac{\pi}{6}, \frac{\pi}{4}] \). Therefore, we want to find a set \( S \) such that when the given angle \( \theta_0 \in S \), \( \theta(\alpha) \in [\frac{\pi}{6}, \frac{\pi}{4}] \) for all \( \alpha \in [0, \frac{\pi}{2}] \).

By the properties of \( \theta(\alpha) \), we have

\[
\min_{\alpha \in [0, \frac{\pi}{2}]} \theta(\alpha) = \theta(0) = \theta \left( \frac{\pi}{2} \right), \quad \max_{\alpha \in [0, \frac{\pi}{2}]} \theta(\alpha) = \theta \left( \frac{\pi}{4} \right).
\]

Moreover, since \( \sin x \) is monotone increasing for \( x \in [0, \frac{\pi}{2}] \), we obtain

\[
\min_{\alpha \in [0, \frac{\pi}{2}]} \sin(\theta(\alpha)) = \sin(\theta(0)) = \sin \left( \theta \left( \frac{\pi}{2} \right) \right), \quad \max_{\alpha \in [0, \frac{\pi}{2}]} \sin(\theta(\alpha)) = \sin \left( \theta \left( \frac{\pi}{4} \right) \right).
\]

Therefore, we need to determine the value of \( \theta_0 \) such that

\[
\sin \left( \theta \left( \frac{\pi}{4} \right) \right) = \sin \left( \theta \left( \frac{\pi}{6} \right) \right) = \frac{1}{2}.
\]

By (4.3) we have

\[
\sin \left( \theta \left( \frac{\pi}{4} \right) \right) = \left[ \frac{2 \tan^2(\theta_0)}{2 \tan^2(\theta_0) + 4} \right]^{\frac{1}{2}} = \sin \left( \theta \left( \frac{\pi}{6} \right) \right) = \frac{1}{2}.
\]

Thus

\[
\theta_0 = \arcsin \sqrt{\frac{2}{5}} \approx 0.6847 \approx 39.25^\circ,
\]
and we use \( \theta_1 \), defined in (4.7), to denote the \( \theta_0 \) in (4.8).

Therefore, we conclude that given any \( \theta_0 \in [\theta_1, \frac{\pi}{4}] \), \( \theta(\alpha) \in [\frac{\pi}{6}, \frac{\pi}{4}] \) for all \( \alpha \in [0, \frac{\pi}{2}] \). Based on Theorem 2.4, the partial derivative \( \frac{\partial}{\partial t} L^{\theta_0}(\alpha, t) \) does not exist for all \( \theta_0 \in [\theta_1, \frac{\pi}{4}] \).

**Case 3:**

Similar to Case 2, we would like to find out the set \( S \) such that when \( \theta_0 \in S \), \( \theta(\alpha) \in [\frac{\pi}{12}, \frac{\pi}{6}] \) for all \( \alpha \in [0, \frac{\pi}{2}] \).

Setting

\[
\sin \left( \theta \left( \frac{\pi}{4} \right) \right) = \sin \left( \frac{\pi}{12} \right),
\]

we obtain

\[
\theta_0 = \arcsin \left[ \frac{14 - 8\sqrt{3}}{15 - 8\sqrt{3}} \right]^2 \approx 0.3622 = 20.36^\circ,
\]

(4.9)

and we use \( \theta_2 \), defined in (4.7), to denote the \( \theta_0 \) in (4.9).

Therefore, by Theorem 2.4, for \( \theta_0 \in [\theta_2, \frac{\pi}{6}] \), the partial derivative \( \frac{\partial}{\partial t} L^{\theta_0}(\alpha, t) \) does not exist for all \( t \) in a set of measure 1, but it exists when \( t \) is in a non-denumerable set.

**Case 4:**

Recall that the function \( \theta(\alpha) \) attains the maximal value at \( \alpha = 0 \). So, for any \( \theta_0 \in (0, \frac{\pi}{6}) \), \( \theta(\alpha) \in (0, \frac{\pi}{6}) \) for all \( \alpha \in [0, \frac{\pi}{2}] \).

Therefore, for all \( \theta_0 \in (0, \frac{\pi}{12}) \), the partial derivative \( \frac{\partial}{\partial t} L^{\theta_0}(\alpha, t) \) does exist when \( t \) on a set of Lebesgue measure 1.

**Case 5:**

Based on Case 2, we know that when \( \theta_0 \in [\theta_1, \frac{\pi}{4}] \)

\[
\frac{\pi}{6} \leq \min_{\alpha \in [0, \frac{\pi}{4}]} \theta(\alpha) = \theta \left( \frac{\pi}{4} \right) < \max_{\alpha \in [0, \frac{\pi}{4}]} \theta(\alpha) = \theta \left( \frac{\pi}{2} \right) = \theta(0) \leq \frac{\pi}{4},
\]

and for \( \theta_0 = \theta_1 \),

\[
\min_{\alpha \in [0, \frac{\pi}{4}]} \theta(\alpha) = \theta \left( \frac{\pi}{4} \right) = \frac{\pi}{6}.
\]

Thus, \( \theta \left( \frac{\pi}{4} \right) \) will be less than \( \pi/6 \) when \( \theta_0 < \theta_1 \) because \( \theta(\alpha) \) is strictly decreasing for \( \alpha \in [0, \frac{\pi}{4}] \).

Therefore, for \( \theta_0 \in \left[ \frac{\pi}{6}, \theta_1 \right) \), there exists an \( \alpha_1 \in [0, \frac{\pi}{6}] \) such that \( \alpha_1 \) satisfies the equation

\[
(3 \tan^2 \theta_0 - 1) \tan^2 \alpha_1 - 2 \tan \alpha_1 + (3 \tan^2 \theta_0 - 1) = 0,
\]

(4.10)

where \( \alpha_1 \) is defined in (4.7).

Clearly, when \( \theta_0 \in [\pi/6, \theta_1) \), the equation (4.10) has two solutions and one of it satisfies

\[
\theta(\alpha_2) = \theta \left( \frac{\pi}{2} - \alpha_1 \right) = \frac{\pi}{6}, \quad \alpha_2 \rightarrow 0 \text{ as } \theta_0 \rightarrow \frac{\pi}{6}.
\]

So, we conclude that for given \( \theta_0 \in \left[ \frac{\pi}{6}, \theta_1 \right) \), there exists a \( \alpha_1 \in [0, \frac{\pi}{6}] \) with \( \theta(\alpha_1) = \pi/6 \) such that

(i) When \( \alpha \in [0, \alpha_1] \cup \left[ \frac{\pi}{2} - \alpha_1, \frac{\pi}{2} \right] \), the partial derivative \( \frac{\partial}{\partial t} L^{\theta_0}(\alpha, t) \) does not exist for any \( t \in [0, 1] \);
When $\alpha \in (\alpha_1, \frac{\pi}{2} - \alpha_1)$, the partial derivative $\frac{\partial}{\partial t} L^{\theta_0}(\alpha, t)$ does not exist for $t$ in a set of measure 1, but it exists for $t$ in a non-denumerable set of $[0, 1]$.

Since $\alpha_1 \to 0$ as $\theta_0 \to \pi/6$, the interval $\alpha \in (\alpha_1, \frac{\pi}{2} - \alpha_1)$ will expand and fill the area $(0, \frac{\pi}{2})$ when $\theta_0 = \pi/6$. As mentioned in Section 1, we have proved that when $\theta_0$ crosses the new special angle $\theta_1$, an intermediate vertical sector starts to appear in the pyramid where the differentiability property upgrades to the status $\ast\ast$ and this sector will fill the whole pyramid when $\theta_0$ reaches the values $\pi/6$.

**Case 6:**
Similar as in Case 5, we conclude that for given $\theta_0 \in \left[\frac{\pi}{12}, \theta_2\right]$, there exists an angle $\alpha_2$ with $\theta(\alpha_2) = \pi/12$ such that

(i) When $\alpha \in [0, \alpha_2] \cup [\frac{\pi}{2} - \alpha_2, \frac{\pi}{2}]$, the partial derivative $\frac{\partial}{\partial t} L^{\theta_0}(\alpha, t)$ does not exist for $t$ in a set of measure 1, but it exists for $t$ in a non-denumerable set of $[0, 1]$;

(ii) When $\alpha \in (\alpha_2, \frac{\pi}{2} - \alpha_2)$, the partial derivative $\frac{\partial}{\partial t} L^{\theta_0}(\alpha, t)$ does exist for all $t$ in a set of measure 1 in $[0, 1]$. 

$\square$
Part 5. References

REFERENCES

[20] Sabot, C., Density of states of diffusions on self-similar sets and holomorphic dynamics in $\mathbb{P}^k$: the example of the interval $[0,1]$.