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An Empirical Analysis of Resampled Efficiency

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An Empirical Analysis of Resampled Efficiency

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Abstract

Michaud introduced resampled efficiency as an alternative and improvement to Markowitz mean-variance efficiency. While resampled efficiency is far from becoming the standard paradigm of capital allocation amongst risky assets, it has nonetheless gained considerable ground in financial circles and become a fairly debated portfolio construction technique.

This thesis applies Michaud's techniques to a wide array of stocks and tries to validate claims of performance superiority of resampled portfolios. While there seems to be no conclusive advantage or disadvantage of using resampling as a technique to obtain better returns, resampled portfolios do seem to offer higher stability and lower transaction costs.

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1 Mean Variance Efficiency

1.1 Introduction to Markowitz Mean Variance Efficiency

Mean Variance Efficiency refers to the classical approach to solve the portfolio allocation problem as proposed by Harry Markowitz [2]. Given a portfolio with N assets with a fraction of total wealth x_i invested in each asset with return R_i (a random variable), the expected return on the portfolio is the weighted average of the individual expected returns:

$$E_p = E[R_p] = \sum_{i=1}^{N} x_i E[R_i] = \sum_{i=1}^{N} x_i \mu_i = x^T \mu .$$

Correspondingly, the portfolio risk is the variance (or standard deviation) of the return on the portfolio: [2]

$$V_p = Var[R_p] = \sum_{j=1}^{N} \sum_{i=1}^{N} x_i \sigma_{ij} x_j = x^T C x$$

Where C is the n x n covariance matrix with entries

$$\sigma_{ij} = E[(R_i - \mu_i)(R_j - \mu_j)]$$

Accordingly, given the expected returns, standard deviations of returns and correlations of returns on assets, Markowitz Mean Variance efficiency seeks to find portfolio weights which minimizes risk for a given level of return and maximizes return for a given level of risk. Such portfolios are called *efficient* and the set of all efficient (feasible) portfolios is called the *efficient frontier*. It reduces to a mathematical optimization problem which may be stated as:

a) Minimize V_p, subject to E_p = specified level of return and $\sum_{i=1}^{N} x_i = 1$, or

b) Maximize E_p, subject to V_p = specified level of risk and $\sum_{i=1}^{N} x_i = 1$, or

Additionally, we may impose constraints such as:

- $x_i \geq 0, \forall i$ (No asset may be shorted)
- $x_i \leq f, \forall i$ (No asset may have more than a certain fraction of total investment)

Any such constraint shrinks the feasible set and pulls the efficient frontier inwards in the risk return space.

Alternatively, the optimization problem may be stated as:

$$\text{Minimize}\, U = V_p - \lambda_E E_p \text{, subject to } \sum_{i=1}^{N} x_i = 1$$

Here, U is the *utility function* and the parameter λ_E is a measure of the *risk aversion* for the investor (reciprocal of the *risk tolerance*). This definition helps in customizing the portfolio allocation problem to the risk aversion of the individual investor. At the same time, however, no clear standards exist on the quantification of λ_E.

Standard algorithms (linear programming/quadratic programming) are available to compute the efficient frontier with or without short-selling/borrowing constraints. Markowitz extended the technique of quadratic programming to develop the “critical line algorithm” [7] to solve the optimization problem. The main drawback to this algorithm is that it does not solve for specific points on the efficient frontier, but rather provides a sample of the portfolios on the efficient frontier. It does not provide a portfolio for a specified return; therefore, very few commercial packages use this algorithm. Matlab’s

frontcon uses quadratic programming to solve the same problem by solving for optimal portfolios with specified returns along the efficient frontier and has been used for this project.

1.2 Criticisms of Mean-Variance Efficiency

There are a number of objections to MV Efficiency [1]

(1) Non Variance Risk Measures: Variance is not uniformly accepted as an appropriate measure of the risk of a portfolio. This criticism by itself raises serious doubts about the central role given to mean-variance efficient portfolios in both investment theory and practice. Various other risk definitions exist. Downside risk measures of variability such as mean-semivariance or mean - semi standard deviation of return, the mean absolute deviation and range measures could be good alternatives to the traditional risk measure variance or standard deviation [5].

Mean - semivariance is defined as the variance of returns on a portfolio below the mean return level. This may be generalized to *target semivariance* where returns below a target, such as zero or the risk free rate (and not just the mean) contribute towards calculation of risk.

Mean absolute deviation for a sample of N returns, $R_1 \ldots R_N$, is defined as

$$MD = \frac{1}{N}\sum_{i=1}^{N}\left|R_i - \mu\right|$$

Where, $\mu = \frac{1}{N}\sum_{i=1}^{N} R_i$.

Just as in the case of target semivariance, μ may be replaced by a specific return level.

One of the most commonly used alternative risk measure is *Value at Risk (VaR)*. It records the actual loss that would occur if the returns were in the worst x % of the distribution (where x is the threshold). More precisely *VaR* is an amount (say D dollars), where the probability of losing more than D dollars is p over some future time interval, T days.

(2) Utility Function Optimization: Markowitz MV efficiency is consistent with maximization of expected utility of terminal wealth, which acts as a rationale for financial investors to choose MV efficient portfolios. However, this is justified only in one of the following two conditions. In the first case utility of terminal wealth is maximized when returns are normally distributed. While the returns may be distributed symmetrically for diversified equity portfolio and index returns, the distribution is not precisely normal. Further, asset classes such as fixed income indexes are asymmetric. MV Efficiency also maximizes terminal wealth utility for quadratic utility functions. There is, however, a major limitation to the use of quadratic utility functions to mirror investment tendencies. This is because quadratic utility declines as a function of positive wealth increments beyond a certain point and therefore quadratic utility functions find better applicability in approximating maximum wealth in a given restricted range. Accordingly then, MV efficiency does not always achieve utility function maximization.

(3) Multiperiod Investment Horizons: Markowitz' mean-variance efficient paradigm is a one-period model. Most institutional and individual investors typically have a ten-

year, twenty-year, or even longer investment horizon. Markowitz himself has shown that mean-variance efficient portfolios need not be efficient in the long run [7]. Additionally, mean-variance efficient portfolios in the upper part of the efficient frontier are less efficient in the long run. To address these problems, some have suggested reformulating the mean-variance analysis using longer time-periods consistent with the investor's investment horizon. However, if one were to consider a ten-year or twenty- year return as one observation, we would have very few observations from the U.S. (or any other) capital markets for estimation (of the efficient frontier) purposes. Additionally, increasing historic data will reduce the accuracy of forecasting for a short term period.

(4) Instability and Ambiguity: In practice, mean-variance efficient portfolios have been found to be quite unstable: small changes in the estimated parameter inputs lead to large changes in the implied portfolio holdings (“Instability and Ambiguity”). The practical implementation of the mean-variance efficient paradigm requires determination of the efficient frontier. This requires three inputs: expected returns of the assets, expected correlation among these assets, and expected variance of these assets (individually). Typically, these input parameters are estimated using historical data. Researchers, as Jobson and Korkie [6], have found that estimation errors in these input parameters can overwhelm the theoretical benefits of the mean-variance paradigm. These estimation errors may result from uncharacteristically low or high recent returns for a certain set of securities which then results in much higher (or

lower) allocation. In effect, small changes in the inputs often lead to very different portfolio weights and accordingly wildly diverging efficient frontiers.

2 The Resampled Efficient Frontier

2.1 Introduction

While the above criticisms of MV efficiency are noteworthy, practitioners pay limited attention to them. To start off, efficient frontiers of non variance risk measures do not look significantly different from the Markowitz efficient frontiers except in cases of asset classes such as options, where returns are not approximately symmetric and MV efficiency is a bad allocation choice in the first place. Next, utility functions have practical limitations when it comes to using them as a basis for optimization definition. This is a consequence of lack of feasible and viable algorithms for computation of optimal portfolios that would conform to the utility function, since the utility function may very well require non linear optimization solutions. For these reasons, MV efficiency continues to remain a favorite allocation strategy. However, in 1998, Richard Michaud proposed “Resampling” to tackle at least one of the primary criticisms, i.e. “Instability and Ambiguity” [1]. This gave rise to the *resampled efficient frontier*.

2.2 Procedure to generate Resampled Efficient Frontier

The resampled efficient frontier is generated using the following procedure

- Estimate the expected returns (μ) and the variance – covariance matrix (C). Suppose there are *m* assets.
- Assuming no short selling is allowed, solve for the minimum-variance portfolio. Call the expected return of this portfolio L. Solve for the maximum return portfolio. Call the expected return of this portfolio H.

- Choose the number of discrete increments, in returns, for characterizing the frontier.
- For L=a and H=b, choose K increments and $\delta = (a-b)/K$. This means that we evaluate the frontier at expected return = $\{a, a + \delta, \ldots, b - \delta, b\}$, that is K different points.
- We will represent the 'frontier' as F_K, where 'F_K' consists of K row vectors containing weights of the K portfolios along the efficient frontier. So for m assets, F_K is K x m (rows represent the number of points on the frontier and columns are the asset weights).
- Now begin the Monte Carlo analysis. Assume a multivariate normal distribution with mean vector μ and variance – covariance matrix C, and draw m returns as many times as necessary so as to create a big sample which may be used as an approximation for the return distribution. With the generated data, calculate the simulated means (μ^*) and variance-covariance matrix (C^*).
- Now, C^* and C are "statistically equivalent" [6]. Using C^*, calculate the minimum variance portfolio (expected return L^*) and the maximum expected return portfolio (expected return H^*). Use these to determine the size of the expected return increments.
- Calculate the efficient portfolio weights at each of these K points
- With this information, we now have $F_{K,i}$. This is the same dimension, K x m.
- Repeat the simulations S times, so that we have S $F_{K,i}$s.
- Remove any portfolios that have an unusually high (or low) risk level in comparison to other statistically equivalent portfolios to get the modified $F_{K,i}$s

- Average the modified $F_{K,i}$s to get the Resampled weights. We can now draw the ***resampled efficient frontier***, by using the original means and variances combined with the new weights. (Refer to Figure 1.)

2.3 Statistically equivalent portfolios

The process of resampling generates K portfolios each time. Each of these portfolios corresponds to a return level on the original MV efficient frontier. Any two portfolios which correspond to the same return level on the original MV efficient frontier are termed *statistically equivalent* [6]. Alternatively, statistically equivalent portfolios may be defined as those that have the same risk – return trade-off [3]. This would mean that any two statistically equivalent portfolios minimize the utility function, $U = V_p - \lambda_E E_p$, for the same level of risk aversion, λ_E. It is important to note that two statistically equivalent portfolios, as defined by Michaud, are neither necessarily the same in terms of expected return nor in terms of risk. Typically, statistically equivalent resampled portfolios corresponding to MV efficient portfolios with returns in close proximity of least return MV efficient portfolio have similar risk and return levels. However, as one ventures towards higher returns, such similarities decrease.

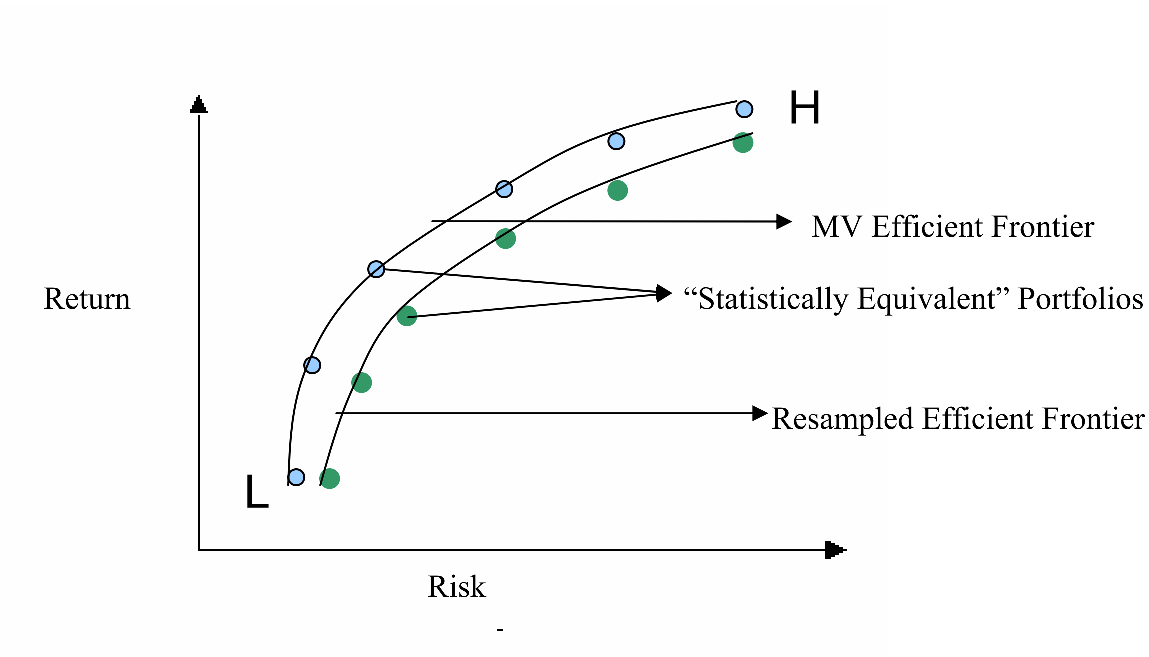

Figure 1: Resampled and MVE Frontiers

2.4 Advantages, Disadvantages

Advantages [1]**:**

The advantage of resampled efficiency is its use of available data to produce more intuitive portfolio allocations which are less sensitive to input perturbations. This is because the resampled efficient portfolio is more diversified and intuitively less risky than one on a corresponding Markowitz efficient frontier. Resampled efficiency therefore uses investment information in a more robust manner than MV efficiency. Also, because resampled efficiency is an averaging process, it is very stable. Small changes in the inputs are generally associated with only small changes in the optimized portfolios. The resampling process therefore provides protection against over fitting of data.

Disadvantages:

The biggest disadvantage of resampling comes from the fact that it does not have a sound theoretical foundation. Though the process creates "statistically equivalent" portfolios, it cannot be argued theoretically that the resampled portfolio outperforms the MV efficient portfolio. In fact, there is no statistical reasoning as to why the resampled portfolios are averaged (which is the only addition to the process of resampling as introduced by Jobson and Korkie [6]). Further, Michaud offers little reasoning for why the comparison of MV efficient portfolio and its statistically equivalent resampled portfolio is valid in the first place given that they are neither identical in risk nor return. Additionally, the so called definition of statistically equivalent portfolios is not a uniformly accepted one. Also, the process of resampling uses the original estimate of mean return vector, μ, and variance - covariance matrix C, to simulate μ^* and C^* and evaluate the resampled efficient frontier, thereby amplifying any errors in the original estimation (even if they are minor to begin with).

3 Empirical Study:

3.1 Procedure

This superiority of resampled efficiency is assessed by studying

(a) Return advantage

(b) Transaction cost advantage, for transaction costs resulting from periodic rebalancing

In calculating resampled efficient frontiers, it is assumed that no short selling is allowed and additionally any resampled portfolios which have unusually high or low risk associated with them (in comparison to other statistically equivalent resampled portfolios) are removed. The process of resampling then, leads to a large number of resampled efficient portfolios that are statistically equivalent to corresponding MV efficient portfolios. One should therefore be able to verify claims of performance superiority of resampled efficiency, on average, for any given resampled portfolio. An inexhaustible number of such portfolios are available for testing corresponding to each efficient frontier for any array of stocks. Performance for a resampled portfolio corresponding to the MV efficient portfolio with return equaling the average of the highest and lowest possible MV efficient returns is tested. This portfolio on the Markowitz efficient frontier may be defined as *Mean Return MVE* portfolio. The corresponding resampled statistical equivalent will be referred to as *Mean Resampled* portfolio (this is somewhat of a misnomer, but helps in reducing verbosity).

To test the return advantage, the above process is repeated for sixty portfolios of seventy five stocks each. In all, the theory is put to test for 4500 stocks using 600 days of data (January 2nd, 2001 to May 27th, 2003).

To test the transaction cost advantage, transaction costs are evaluated for rebalancing the Mean Return MVE and Mean Resampled portfolio. In all four portfolios of 20 stocks are chosen at random from the many stocks available.

3.2 Rationale behind the approach

There are a number of reasons for concentrating on the mean return MVE portfolio. At the very least, since resampled efficiency does not claim to perform in a restricted area of the efficient frontier, so hypothesis testing should work for any resampled portfolio. The resampled portfolios statistically equivalent to those in the lower part of the efficient frontier are however very similar to the MVE portfolios in terms of both return and risk level. The process of resampling would therefore offer little return advantage for these lower return levels and be of little interest. On the other extreme MVE portfolios with high risk and high return would need a very high level of risk tolerance and accordingly not conform to most utility functions. Again, this would lead to a study of portfolios of little practical investment value and wildly diverging behavior. This leaves us with resampled portfolios in close proximity to the statistical equivalent of mean return MVE portfolio that offer the best compromise in terms of risk and return.

Ideally, one may wish to specify an exact level of return or certain risk. However, that may not necessarily correspond to one of the portfolio points on the MVE frontier. This would need some adjustment and further approximation of results. Even if one was to obtain a MVE portfolio corresponding to a given return for one frontier, the same may not be possible for another. In fact, the specified return or risk may very well be beyond the feasible MVE portfolios for a given frontier. Indeed any portfolio which is close to the MVE portfolio may be used for the study. However, this is unlikely to yield dramatically different results.

3.3 In-sample testing

After going through rigorous data splitting, cleaning and recombining clean data (Refer to Appendix B: Matlab Code), an efficient frontier is calculated for a portfolio of 75 stocks chosen at random from the available stocks. This is done using the first 300 days of stock data (January 2nd, 2001 to March 18th, 2002), following which the resampled efficient frontier is constructed using the procedure mentioned in 3.1. The Mean Return MVE portfolio and mean resampled portfolio are then used for *In-sample testing*. The evolution of the portfolios is tracked over the next 300 days (March 19th, 2002 to May 27th, 2003, without rebalancing) in order to evaluate compare portfolio performance.

This process is repeated until all available stocks are exhausted, thereby constructing sixty such portfolios. In order to evaluate relative performance, multiple approaches are available. A naïve approach would simply give advantage to the higher value portfolio at the end of the second 300 day period. This study, however, makes use of Exponential

Weighted Moving Average of the value difference between the Mean MVE portfolio and its statistically equivalent resampled portfolio. A positive EWMA of the value difference between a mean resampled and mean MVE portfolio would indicate superiority of resampled efficiency.

In the second study, transaction costs are evaluated for rebalancing the Mean Return MVE and Mean Resampled portfolio. In all four portfolios of 20 stocks are chosen at random from the stocks available. Then the Mean Return MVE portfolio and Mean Resampled portfolio are evaluated for those portfolios on December 12th, 2002 and at subsequently at 30 day intervals (January 16th 2003, March 3rd 2003, April 14th 2003 and May 27th 2003). Transaction costs resulting from rebalancing the portfolios on those dates are calculated to assess which portfolio offers lower net transaction costs.

4 Results and Analysis:

4.1 Results

For EWMA of the value difference between the mean resampled and mean return MVE portfolio, the mean resampled portfolio outperforms in 23 of the 60 portfolios (Refer to Appendix C for detailed results).

In the worst case scenario, the mean resampled portfolio on average makes 42 cents less than the mean return MVE portfolio for every dollar invested (on a daily basis) and in the best case scenario, it makes 46 cents more. The mean of EWMAs is -0.0215367 (which implies that the mean resampled portfolio makes 2.15 cents less on a daily basis) and the median EWMA is -0.0170125 (which implies that the mean resampled portfolio makes 1.7 cents less on a daily basis). Figure 2 shows the distribution of EWMAs.

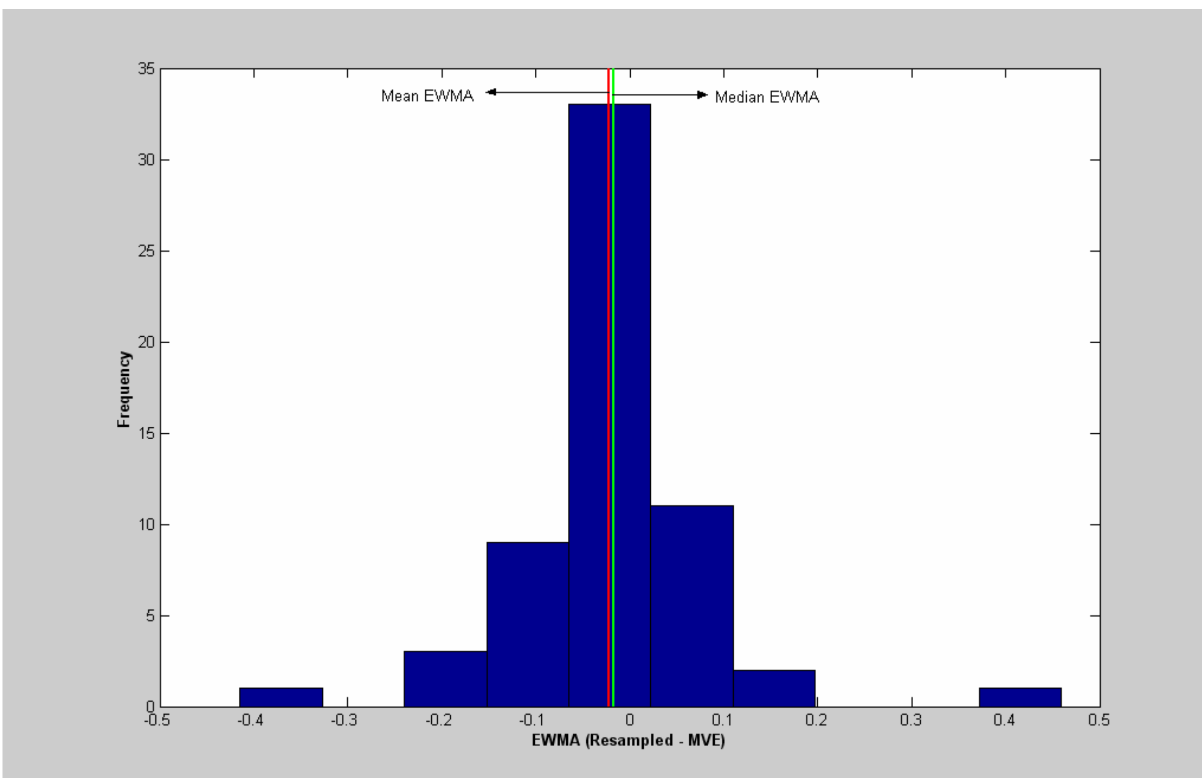

Figure 2: EWMA distribution

When the rebalancing costs are assessed (adding up transaction costs for each of the four rebalancing dates), the mean resampled portfolio offers lower net transaction costs for three of the four randomly chosen portfolios. In the one case that mean MVE portfolio does actually offer lower net transaction costs, the difference is of the order of one thousandth of the advantage offered by the mean resampled portfolio in the other three cases. This may very well be attributed to rounding errors and additionally the MVE portfolio being unusually diversified and stable to begin with (Details in Appendix C).

4.2 Analysis of resampled efficiency performance

Contrary to Michaud's claim, resampled efficiency seems to offer little advantage and ironically, on average, at a disadvantage when compared to the mean variance efficiency in terms of offering higher returns. Michaud's published claims rest on subjective arguments and exposition by using a very small data set. There is no theoretically sound explanation as to why resampling must work. It does not come across as a big surprise then, that resampled efficiency does not outperform Markowitz efficiency. To begin with, resampled efficiency does not offer a solution for MVE limitation of not being able to account for multi period investment horizons. In fact, it would be fair to say that there is no fool proof or even nearly fool proof mechanism to guarantee returns above the risk free rate for a long term period. In this respect resampling also fails.

In fact, the resampled efficient frontier is no different from an *inefficient* mean variance frontier, in terms of risk and return levels. Superiority of resampled efficiency would then imply that the portfolio compositions, even though they offer similar return and risk, are better equipped to deal with market fluctuations. However, there are far too many factors which are impossible to ignore in favor of such a theory. Constant current events and company releases often necessitate that any portfolio, no matter how well thought or resampled, must be reallocated to conform to expected returns *based on key assumptions of the future*, and not past historical returns. Further, the averaging process may lead to the averaging of very high risk portfolios. Though this study removes such outliers (those with very high risk levels are removed) little benefit seems to have been derived in terms of obtained results as far as obtaining better returns is concerned.

However, when it comes to assessing transaction costs, the mean resampled portfolio offers a marked advantage over the MVE portfolio. Thus resampling is indeed an averaging process leading to increased diversification and somewhat incorporating mean variance efficiency, which gives it stability and accordingly better returns, this is simply not observed.

4.3 Observable trends

There are no observable trends in terms of returns obtained on the mean MVE and mean resampled portfolio. However, when 30 day periodic rebalancing is done, the mean resampled portfolio usually offers lower transaction costs.

5 Investment Recommendations

The most tangible benefit of resampling is that it helps in giving lower net transaction costs. This may be of significant advantage to any investor seeking to maintain a well balanced portfolio by incorporating any newly available information over pre specified intervals.

Even then, resampling must be combined with day to day market knowledge so as to bring the portfolio in sync with changing market conditions. A sudden and unexpected market fluctuation may, for example, necessitate immediate rebalancing. Additionally investors may need to tweak the amounts invested in individual securities based on their own intuition.

6 Conclusion

Resampling is an effective mechanism for portfolio construction which seems to provide, on average, lower transaction costs when rebalancing is done. It does not, however, decidedly point towards any higher (or lower) returns in comparison to the Markowitz portfolios.

Thus, any investor who seeks to reduce transaction costs in the process of rebalancing (as is the case with many actively trading firms) may look towards resampling as a way to do so, simultaneously taking into account other factors as company releases, press reports and other macroeconomic factors. It is important to note, though, that these results are based on a study of a limited number of stocks over a specific time frame and are not necessarily indicative of behavior of resampled efficiency over any future time frame.

Like any other investment optimization technology, however, resampling does not offer guaranteed results and must be used as one of the multiple components in investment allocation.

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Appendix A: Stock Data

600 days of stock data (January 2nd, 2001 to May 27th, 2003) was collected for 4500 stocks. This data is available at www.wpi.edu/~kohli/MSProject.

Appendix B: Matlab Code

Code for comparing mean MVE portfolio and mean resampled portfolio returns.

```
% Tests performance of MVE Resampled portfolio
% Updated: 3/16/05

% Read Stock Data, IDs
StockData=dlmread('StockData.txt',' ');
ID=dlmread('StockData.txt',' ');

% Add IDs to Stock Data
% Now StDatAndID2 - Same as StockData but additionally the first row
% has the IDs for stocks
StDatAndID2=[ID;StockData];

% Data Cleaning
% Split Stock Data into smaller dat files which can then be read from
% excel
% Then, in excel, remove any columns with Nan!
for i=1:floor(length(StDatAndID2)/200)
    dlmwrite(strcat('RetTemp',int2str(i),'.txt'), StDatAndID2(:,200*(i-
1)+1:200*i), ' ');
end
i=i+1;
dlmwrite(strcat('RetTemp',int2str(i),'.txt'), StDatAndID2(:,200*(i-
1)+1:length(StDatAndID2)), ' ');

% RetTemp*.txts are cleaned in Excel
% Read peices of cleaned data and combine them

StAndIDClean=dlmread('RetTemp1.txt',' ');
for i=2:25
    Temp=dlmread(strcat('RetTemp',int2str(i),'.txt'),' ');
    StAndIDClean=[StAndIDClean Temp];
end

% Store Intermediate Clean Data
dlmwrite('StAndIDClean.txt', StAndIDClean, ' ');

% Convert Stock Data to Return Data

StClean=StAndIDClean(2:601,:);
% Return Data Matrix
RetData=zeros(600,length(StClean));

for i=600:-1:2
    for j=1:length(StClean)
        RetData(i,j)=StClean(i,j)/StClean(i-1,j)-1;
    end
end

RetData(1,:)=StAndIDClean(1,:);

% Store Intermediate Return Data
dlmwrite('RetData.txt', RetData, ' ');
```

```
% Split Return Data Matrices into matrices of 75 stocks a peice
% For each resulting 75 stock matrix, split it into 2 components of 300
dates each
% The first will be used for construction of efficient and resampled
frontiers
% The second will be used for in-sample testing
RetData=dlmread('RetData.txt', ' ');
div=75;

for i=1:floor(length(RetData)/div)
    dlmwrite(strcat('Ret_A',int2str(i),'.txt'), RetData(2:300,div*(i-
1)+1:div*i), ' ');
    dlmwrite(strcat('Ret_B',int2str(i),'.txt'), RetData(301:600,div*(i-
1)+1:div*i), ' ');
end
i=i+1;
dlmwrite(strcat('Ret_A',int2str(i),'.txt'), RetData(2:300,div*(i-
1)+1:length(RetData)), ' ');
dlmwrite(strcat('Ret_B',int2str(i),'.txt'), RetData(301:600,div*(i-
1)+1:length(RetData)), ' ');


% -------- Construct Resampled Portfolios for In-sample Testing -------

% Estimate Mean Returns and Covariance Matrix Based on First 300 dates
% of Data
for master=1:61

Data = dlmread(strcat('Ret_A',int2str(master),'.txt'), ' ');
mu = mean(Data)';
C = cov(Data);

% Number of Portfolios along the efficient frontier

Nports= 100;

[EffRisk, EffReturn, EffW] = frontcon(mu,C, Nports);

% Resampling ...

A = chol(C);

dim = size(Data);
B=100;
Port50Wts=zeros(B,dim(1,2));
Port50Ret=zeros(B,1);
Port50Risk=zeros(B,1);

for a = 1:B

Gamma2 = randn(size(mu,1),Nports);
R = A'*Gamma2;

simMu = mu' + (mean(R'));
simC = cov(R');
```

```
[PortRisk, PortReturn, PortWts] = frontcon(simMu,simC, Nports);


Port50Wts(a,:)=PortWts(50,:);
Port50Ret(a)=PortWts(50,:)*mu;
Port50Risk(a)=sqrt(PortWts(50,:)*C*PortWts(50,:)');

end


m= mean(Port50Risk);
s=std(Port50Risk);

% Remove Outliers

dyn=B;
co=0;
for r=2:B
        r=r-co;
        if ((Port50Risk(r)> m + 2*s)|(Port50Risk(r) < m - 2*s))
                Port50Wts=[Port50Wts(1:r-1,:);Port50Wts(r+1:dyn,:)];
                Port50Risk=[Port50Risk(1:r-1,:);Port50Risk(r+1:dyn,:)];
                Port50Ret=[Port50Ret(1:r-1,:);Port50Ret(r+1:dyn,:)];
                dyn=dyn-1;
                co=co+1;
        end
 end

        if ((Port50Risk(1)> m + 2*s)|(Port50Risk(1) < m - 2*s))
                Port50Wts=Port50Wts(2:dyn,:);
                Port50Risk=Port50Risk(2:dyn,:);
                Port50Ret=Port50Ret(2:dyn,:);
                dyn=dyn-1;
                co=co+1;
        end

% Display Efficient at 50th point and Resampled Weights
ReSamPort=sum(Port50Wts/length(Port50Wts));
EffP=EffW(50,:);

dlmwrite(strcat('ReSam_',int2str(master),'.txt'), ReSamPort,' ');
dlmwrite(strcat('EffP_',int2str(master),'.txt'), EffP,' ');
end;

% The Mean MVE portfolios and corresponding resampled portfolios can
% now be used for in-sample testing
```

Code for assessing rebalancing transaction costs of mean return MVE and mean resampled portfolios.

```
% Estimate Mean Returns and Covariance Matrix Based on First 480 dates
% of Data

for master=1:4
% Repeat this process 4 times, for 4 portfolios

Data = dlmread(strcat('Ret_',int2str(master),'.txt'), ' ');

% Calculate mean resampled and mean return MVE portfolio 5 times for
% each portfolio
for anoth=1:5

    mu = mean(Data(2:480 + 30*(anoth-1),:))';
    C = cov(Data(2:480 + 30*(anoth-1),:));

% Number of Portfolios along the efficient frontier
    Nports= 30;
    [EffRisk, EffReturn, EffW] = frontcon(mu,C, Nports);

% Resampling ...

    A = chol(C);
    dim =  20;
    B=100;
    PortMidWts=zeros(B,dim);
    % PortMidRet=zeros(B,1);
    % PortMidRisk=zeros(B,1);

for a = 1:B

Gamma2 = randn(size(mu,1),Nports);
R = A'*Gamma2;

simMu = mu' + (mean(R'));
simC = cov(R');

[PortRisk, PortReturn, PortWts] = frontcon(simMu,simC, Nports);


PortMidWts(a,:)=PortWts(Nports/2,:);
% PortMidRet(a)=PortWts(Nports/2,:)*mu;
% PortMidRisk(a)=sqrt(PortWts(Nports/2,:)*C*PortWts(Nports/2,:)');

end



% Display Efficient at Midth point and Resampled Weights
ReSamPort=sum(PortMidWts/length(PortMidWts));
EffP=EffW(Nports/2,:);
```

```
dlmwrite(strcat('ReSam_',int2str(master),'_',int2str(anoth),'.txt'),
ReSamPort,' ');
dlmwrite(strcat('EffP_',int2str(master),'_',int2str(anoth),'.txt'),
EffP,' ');
end

end

% Rebalancing transaction costs can now be evaluated using available
% portfolios.
```

Appendix C: Detailed Results

The following values of the EWMA of difference of value between mean resampled portfolio and mean return MVE portfolio were obtained:

Portfolio Number	EWMA
1	-0.135912
2	-0.042282
3	-0.089215
4	-0.015510
5	-0.020797
6	-0.018515
7	0.025401
8	-0.004618
9	-0.018843
10	-0.026886
11	-0.007078
12	-0.219134
13	0.049935
14	-0.030767
15	-0.127331
16	0.006624
17	0.026846
18	0.069111
19	-0.106697
20	0.013148
21	0.018866
22	-0.049970
23	-0.000003
24	0.041080
25	0.132408
26	-0.227719
27	-0.024827
28	0.033706
29	0.024453
30	0.063506
31	0.053577
32	0.007248
33	-0.414574
34	-0.128260
35	-0.039560
36	0.459628
37	-0.065657
38	0.058052

39	-0.045645
40	-0.025780
41	-0.044659
42	-0.005642
43	-0.039348
44	0.019454
45	0.005688
46	0.058406
47	-0.028010
48	-0.034785
50	0.006304
51	0.120574
52	-0.054716
53	-0.075233
54	-0.131489
55	0.021278
56	-0.075607
57	0.005695
58	-0.008658
59	-0.003071
60	-0.185110
61	-0.041282

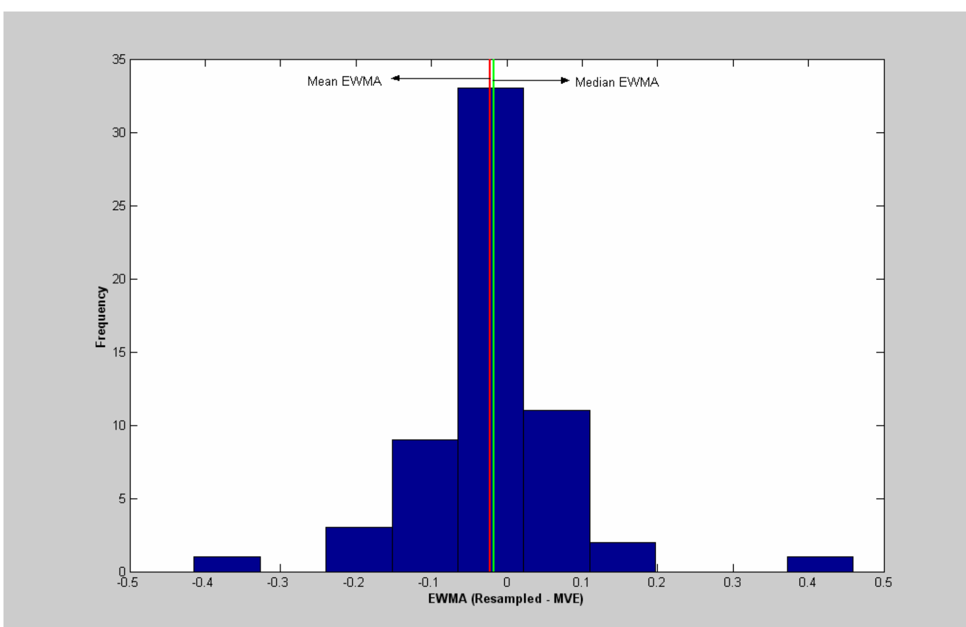

Figure 3: EWMA distribution

(Details of stocks used for individual portfolios are in Appendix A).

Following were the differences in transaction costs for rebalancing the mean return MVE portfolio and mean rebalanced portfolio.

	Cost of Rebalancing Difference				
Portfolio\Date	1/16/2003	3/3/2003	4/14/2003	5/27/2003	Total Dollar Cost (Per Dollar Invested)
Portfolio 1	0.011070865	-0.0011	0.032973	-0.02321	0.019733
Portfolio 2	0.000376819	-0.00375	-0.00567	0.008172	-0.00087
Portfolio 3	-0.000806262	0.006837	0.012046	-0.0069	0.011178
Portfolio 4	0.013606127	-0.00287	-0.01058	0.03439	0.034549

(Details of stocks used for individual portfolios are in Appendix A).