Quasi-static Fracture Evolution with Cohesive Energy

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Quasi-static Fracture Evolution with Cohesive Energy

by

Yiqing Li

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Abstract

The last fifteen years have seen much success in the analysis of quasi-static evolution for Griffith fracture, which is the mathematically natural starting point for studying fracture. At the same time, attempts have been made to show existence for similar models based on cohesive fracture rather than Griffith. These models are generally viewed as physically more realistic than Griffith, in that they are better models for crack nucleation. These attempts at existence proofs have been unsuccessful without very strong additional assumptions, for example, specifying the crack path a priori.

The main purpose of this thesis is to characterize as well as possible the mathematical difficulties in cohesive fracture, and to make progress toward an existence result without the prescribed crack path assumption. So far, the most powerful method for existence proofs is to build a sequence of approximate solutions, based on time discretization, and take the limit as the time steps go to zero. We show that there are mainly two complications on the cracks of these approximate solutions that we need to rule out in order to show existence. The first one is due to the potential oscillation of the crack path. The second is due to the potential splitting of a crack into two or more nearby cracks, with the same total jump in displacement.

We begin by first constructing an example illustrating how oscillations described above can affect the minimality of the limit. Then we prove that the splitting described above can be ruled out for any sequence of unilateral minimizers. With this result, we show how exactly oscillation affect the minimality on the limit of the sequence. We then move to the evolution problem and show the convergence of energy for almost every $t$. Based on this result we develop a method that allows us to analyze the problem using only a finite set of times. An application of this method is a proof of absolute continuity. Future work will be aimed at using the tools we developed to rule out oscillation and finally to prove existence results under more general assumptions.
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Introduction

In 1920, Griffith laid the foundation for brittle fracture in [17]. He considered materials’ toughness, denoted by $G_c$, and energy release rate, denoted by $G$, and regarded the propagation of fracture as the competition between $G$ and $G_c$. $G$ can be defined, in a two dimensional setting, as

$$ G := -\frac{dW}{dl} $$

where $W$ is the bulk energy and $l$ is the length of a crack. Propagation can take place if $G = G_c$ and cannot if $G < G_c$.

This idea is widely used to study crack propagation (e.g. [19]). However there are limitations on the model. It requires a pre-existing crack and it prescribes the crack path on the material. In 1998, Francfort A. G. and Marigo J.-J. addressed those issues by introducing a variational model of quasi-static growth for brittle cracks in [16]. They considered the formation and growth of a crack as the consequence of minimizing the sum of bulk energy and surface energy (fracture energy). Therefore the crack set is not prescribed, instead, the growth (direction and location) of cracks is determined by the minimization of the energy sum. To better illustrate the variational model introduced in [16], consider the quasi-static evolution dealt by Francfort G. A. and Larsen C. J. in [15], the first result for the Francfort-Marigo model without artificial restrictions on the crack set.

Let $\Omega$ be an open and bounded subset from $\mathbb{R}^N$ ($N \geq 2$) with Lipschitz boundary. Then consider the case of generalized antiplanar shear by assuming that the reference configuration is an infinite cylinder $\Omega \times \mathbb{R}$ and the displacement has the special form $(0, \ldots, 0, u(x_1, \ldots, x_N))$ where $u : \Omega \rightarrow \mathbb{R}$. The natural setting for $u$ is to let $u \in SBV(\Omega)$ where $SBV$ denotes the space of special functions of bounded variation. The space of $SBV$ functions was introduced by De Giorgi E. and Ambrosio L. [11] and the definition of $SBV$ functions is as follows

**Definition 1.** $u \in BV(\Omega)$ if $u \in L^1(\Omega)$ and the distributional derivative $Du$ is a finite Radon measure. $u \in SBV(\Omega)$ if $u \in BV(\Omega)$ and $Du$ can be split into two parts

$$ Du = \nabla u dx + [u] \nu \mathcal{H}^{N-1} |S_u| $$

Here $\nabla u dx$ equals the part of the distributional derivative $Du$ that is absolutely continuous with respect to Lebesgue measure. For $SBV$ functions, the singular part of $Du$ only contains the jump part $[u] \nu \mathcal{H}^{N-1} |S_u|$. $S_u$ denotes the jump set of $u$ and $[u]$ denotes the size of the jump on $S_u$. It can be shown that $S_u$ is countably $N-1$ rectifiable (see [14]). $\mathcal{H}^{N-1}$ is the $N-1$ dimensional Hausdorff measure and $|$ is the restriction of measures on sets. The vector $\nu$ is the unit normal to $S_u$. See [2], [13], [14] and [20] for more details on $SBV$ functions.
Then consider the total energy

\[ \int_{\Omega \setminus \Gamma} |\nabla u|^2 \, dx + G_c \mathcal{H}^{N-1}(\Gamma) \]

where \( \Gamma \) is the crack set (which can be empty). For simplicity the bulk energy consists only of elastic energy \( \int_{\Omega \setminus \Gamma} |\nabla u|^2 \, dx \). The fracture energy here is the toughness \( G_c \) times the \( \mathcal{H}^{N-1} \) measure or the size of crack set. Here can set \( G_c \) to be 1 as there’s no difference in terms of proof. This kind of fracture energy is in general regarded as Griffith energy. Let \( g \in W^{1,\infty}(\Omega) \) be the Dirichlet boundary condition and \( \Gamma \) be the pre-existing crack set, the displacement \( u \) will minimize the total energy in the following sense

\[ \int_{\Omega} |\nabla u|^2 \, dx + \mathcal{H}^{N-1}(\Gamma \cup S_u) \leq \int_{\Omega} |\nabla v|^2 \, dx + \mathcal{H}^{N-1}(\Gamma \cup S_v) \]

for all \( v = g \) on \( \partial \Omega \).

The existence of a minimizer \( u \) can be shown using \( SBV \) compactness which is due to Ambrosio L. [1], also see [12] for more details. Before we give the theorem of \( SBV \) compactness, let’s give the definition of \( SBV \) convergence.

**Definition 2.** A sequence \( u_n \in SBV(\Omega) \) converges in the sense of \( SBV \) to \( u \in SBV(\Omega) \) if \( u_n \to u \) in \( L^1(\Omega) \) and

\[ \nabla u_n \, dx \rightharpoonup \nabla u \, dx \]

\[ [u_n]_{\nu} \mathcal{H}^{N-1} |S_{u_n} \rightharpoonup [u]_{\nu} \mathcal{H}^{N-1} \]

where \( \rightharpoonup \) denotes the weak* convergence in measure.

Let \( \theta : [0, \infty) \to [0, \infty] \), \( \varphi : (0, \infty) \to (0, \infty] \) be lower semi-continuous increasing functions and assume that

\[ \lim_{t \to \infty} \frac{\theta(t)}{t} = \infty, \quad \lim_{t \to 0} \frac{\varphi(t)}{t} = \infty. \]

**Theorem 1.** (SBV compactness) Let \( \{u_n\}_{n=1}^\infty \subset SBV(\Omega) \) such that

\[ \sup_n \left\{ \int_{\Omega} \theta(|\nabla u_n|) \, dx + \int_{S_{u_n}} \varphi([u_n]) \, d\mathcal{H}^{N-1} \right\} < \infty \]

and \( \|u_n\|_{L^\infty} \) is uniformly bounded in \( n \), then there exists a subsequence \( \{u_{n_k}\}_{k=1}^\infty \) and an \( SBV \) function \( u \) such that

\[ u_{n_k} \rightharpoonup u \]

To include the pre-existing crack set in the energy reflects one of the most important features a fracture problem have—irreversibility. It says the crack set can only grow bigger or stay the same, it can not be reversed. Then consider the continuous-time evolution where change of boundary condition is slow such that at each time \( t \) the material is able to reach an equilibrium. The detailed definition of quasi-static evolution with Griffith energy, borrowed from [15], is as follows
Definition 3. (Quasi-static Evolution with Griffith Energy) Let \( g \in L^\infty([0, 1], L^\infty(\mathbb{R}^N)) \cap W^{1,1}([0, 1], H^1(\mathbb{R}^N)) \) be the Dirichlet boundary condition, a pair \( (u(t), \Gamma(t)) \), where \( u(t) \in SBV(\Omega) \) for each \( t \) and \( \Gamma(t) := \bigcup_{\tau \leq t} S_{u(\tau)} \), is said to be a quasi-static evolution if

   \[
   \int_\Omega |\nabla u(t)|^2 dx \leq \int_\Omega |\nabla v|^2 dx + \mathcal{H}^{N-1}(S_v \setminus \Gamma(t)).
   \]
   for all \( t \) and all \( v \in SBV(\Omega) \) s.t. \( v = g(t) \) on \( \partial\Omega \)

2. Irreversibility. \( \Gamma(t_1) \subset \Gamma(t_2) \) for all \( 0 \leq t_1 \leq t_2 \leq 1 \).

3. Energy balance. Define \( E(t) := \int_\Omega |\nabla u(t)|^2 dx + \mathcal{H}^{N-1}(S_{u(t)} \cup \Gamma(t)) \), then
   \[
   E(t) = E(0) + 2 \int_0^t \int_\Omega \nabla u(s) \cdot \nabla \dot{g}(s) dx ds.
   \]

\( u \) represents the evolution of the displacement of the material. \( \Gamma \) represents the evolution of cracks. Notice here irreversibility is automatically satisfied due to the definition of \( \Gamma(t) \). Global stability means the displacement of the material \( u(t) \) at each time \( t \) is always globally stable in the sense that \( u(t) \) minimizes the total energy with \( \Gamma(t) \) as pre-existing crack. It reflects the quasi-static property that says at each time the material is in equilibrium. Energy is balanced in the sense that the energy at each time equals the energy at the beginning plus the work done up to time \( t \).

Numerical implementations have been studied in [4], [5] and [6]. They were based on a finite time step approach. Meanwhile the continuous-time mathematical existence results were being studied. In [9] Dal Maso G. and Toader R. gave the first precise mathematical formulation of the model using a two-dimensional setting. They proved an existence result using the time discretization method introduced in [16] under the assumption that the bound on the number of connected components of cracks is set a priori. Later Chambolle A. [8] solved the planar elasticity setting under the same assumption on the number of connected components of crack sets. Time discretization, since its introduction, is widely used to approach quasi-static evolution problems. It provides an efficient way to approximate the continuous-time problem using discrete time solutions. One considers a sequence of minimizers as the discrete time step goes to 0. Then the problem becomes to show that the limit is also a minimizer.

A big success in showing the existence result for the continuous-time quasi-static evolution for Griffith energy is due to Francfort G. A. and Larsen C. J. in [15]. They showed the existence result with no other assumptions on the crack set than \( N - 1 \) rectifiability. As mentioned above, the key step is to show the sequence of approximating solutions converges to a limit that is also a solution in some sense. First they consider a sequence that satisfies the following minimality for its own jump set

\[
\int_\Omega |\nabla u_n|^2 dx \leq \int_\Omega |\nabla v|^2 dx + \mathcal{H}^{N-1}(S_v \setminus S_{u_n})
\]

for all \( v = u_n \) on \( \partial\Omega \) and for all \( n \). Then they show that the limit is also a minimizer for its own jump set. Use the same technique (jump transfer) they show that the discrete time solutions converge to a global minimizer and thus show the global stability. Once the global stability is shown, the rest follows easily. Another contribution of [15] is the jump transfer technique that
provides a method to alter crack set. This idea is widely used and extended in this thesis and in some of the proofs.

Meanwhile a cohesive model (see [3]) has been used to study crack propagation. For cohesive model the fracture energy also depends on the opening size of the crack. The fracture energy can be precisely written as

$$\int_{\Gamma} \varphi([u]) d\mathcal{H}^{N-1}$$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing, bounded and concave function with $\varphi(0) = 0$. Here $\varphi([u])$ is the energy density spent to create a crack with opening $[u]$ and $\varphi'(u)$ is the force density acting between the lips of crack. Typically the force decrease with distance and hence $\varphi$ is concave. By letting $\varphi \equiv G_c$ one obtains Griffith energy. The total energy for a cohesive model can be written

$$E(u) := \int_{\Omega} |\nabla u|^2 dx + \int_{\Gamma} \varphi([u]) d\mathcal{H}^{N-1}.$$

Dal Maso G. and Zanini C. [10] and Cagnetti F. and Toader R. [7] showed existence results for quasi-static evolution with cohesive energy, for the case of prescribed crack path. When the crack path is fixed and crack set is regular enough, $SBV$ functions are no longer needed to describe the displacement. One can define Sobolev functions on domain $\Omega \setminus \Gamma$ and study the behaviors of traces on two sides of the crack set. One of the biggest differences between [10] and [9] is the way they treat loading (increase) and unloading (decrease) of a crack. In [10], the unloading is constant and the problem is solved by using a special form of convergence. In [9], the unloading follows a convex function $\varphi(\cdot, z), z > 0$ where $z$ is the size to which the crack has been opened up previously. The solution is based on the use of Young measures.

In this thesis we consider a model that is very similar to the one in [15] where the crack path is free. But instead of considering the Griffith energy we consider a cohesive energy for the fracture part. We use the same time discretization method to approach continuous-time evolution. We show that the method used to prove a existence result for Griffith energy does not apply to the cohesive energy. Then we consider a sequence of unilateral minimizers and show that the limit is a minimizer that picks up the oscillation on the sequence. Later we move to the evolution problem and first show the convergence of energy. Based on this result we develop a so called little $o$ method that allows us to analyze the problem using only finite set of times.

Now we are in a position to give the details of the problem we are considering. As mentioned before for cohesive model the fracture energy depends on both the length and depth of a crack. The relation is described using a cohesive energy function $\varphi : [0, \infty) \to [0, \infty)$. For us, we consider a cohesive energy function that has the following properties.

1. $\varphi(0) = 0$.
2. $\varphi$ is differentiable, $\varphi' > 0$ and $\varphi'(0) = \infty$.
3. $\varphi$ is concave.
4. $\lim_{n \to \infty} \varphi(x) = M < \infty$.
5. $\frac{\varphi'(x) x}{\varphi(x)} \to C > 0$ as $x \to 0$. 


Here \( x \) represents the opening of a crack and \( \varphi(x) \) represents the fracture energy density due to that opening. It is increasing because the bigger opening on the crack the more energy loss due to that opening. The reason why \( \varphi \) is concave has been mentioned before. \( \varphi'(0) = \infty \) and \( \varphi'(x) \to C \) are mathematically convenient. The first one allows us to use SBV compactness and the second one allows us to bound some function of fracture defined on some small sets.

The energy form for cohesive model without pre-existing fracture is

\[
E(u) = \int_{\Omega} |\nabla u|^2 dx + \int_{S_u} \varphi([u]) d\mathcal{H}^{N-1}.
\]

A typical cohesive energy function may look like the graph in Figure 1. Next let’s consider the energy when there’s a pre-existing crack. For Griffith model we can measure the size of the union of current crack and pre-existing cracks by considering \( \mathcal{H}^{N-1}(S_u \cup \Gamma) \). But for cohesive model it is more complicated. First let’s give the cohesive energy function with history. Let \( z > 0 \), define cohesive energy function with history as follows

\[
\tilde{\varphi}(x,z) = \begin{cases} 
\varphi(z) - \varphi'(z)(z-x) & 0 \leq x \leq z \\
\varphi(x) & z < x
\end{cases}
\]

where \( z \) represents the opening size of pre-existing crack. In other words, the same crack may have different fracture energy if there was another crack at the same location before. Define \( \tilde{\varphi}(x,0) = \varphi(x) \) when \( z = 0 \). The graph of the function may look like Figure 2.

This extra definition covers the case when the crack closes up. If a crack is always increasing in the evolution, it will follow \( \varphi(x) \) straight. But if it starts to close up after the size reaches \( z \) it follows \( \tilde{\varphi}(x,z) \). For an example and all the important properties of \( \varphi \) please refer to Appendix A.

Let \( \Gamma \subset \Omega \) be \( \mathcal{H}^{N-1} \sigma \)-finite and let \( \gamma \in L^1(\Gamma, \mathcal{H}^{N-1}) \) denote the pre-existing cracks, our energy form will be

\[
E(u, \gamma) := \int_{\Omega} |\nabla u|^2 dx + \int_{S_u \cup \Gamma} \tilde{\varphi}([u], \gamma) d\mathcal{H}^{N-1}.
\]

Figure 3 shows another example of cohesive energy function others consider.
Figure 2: Graph of $\tilde{\varphi}(x, z)$.

Figure 3: Another example.
It says you won’t gain any energy back if you close up a crack, so the crack energy is fully irreversible. Another case is when you gain everything back, in which it follow $\varphi$ entirely. From now on we will only consider $\varphi$ as defined in (3).

Next let’s have a look at the way we deal with boundary condition. Let $\Omega'$ be open, bounded and with Lipschitz boundary such that $\Omega \subseteq \Omega'$. Let $g \in W^{1,\infty}(\Omega')$, we say an $SBV$ function $u$ satisfies the boundary condition $g$ on $\partial \Omega$ or $u = g$ on $\partial \Omega$ if $u = g$ on $\Omega' \setminus \Omega$. By treating the boundary condition this way we allow the jump occur on the boundary $\partial \Omega$.

We describe the boundary condition, let $g(t): [0, 1] \rightarrow W^{1,\infty}(\Omega'; \mathbb{R})$ s.t.

1. $\sup_t ||g(t)||_{W^{1,\infty}} < \infty$.
2. $\nabla \tilde{g}(t)$ exists and $\nabla \tilde{g}(t) \in L^2(\Omega) \ \forall t$, and $\sup_t ||\nabla \tilde{g}(t)||_{L^2} < \infty$.
3. $\nabla \tilde{g}(t)$ is continuous on $[0, 1]$.

**Definition 4. (Quasi-static Evolution with Cohesive Energy)** We say a pair $(u(t), \gamma(t)) : [0, 1] \rightarrow SBV(\Omega) \times L^1(\Gamma(t), \mathcal{H}^{N-1})$, where $u(t) \in SBV(\Omega)$ for each $t$ and $\gamma(t) := \sup_{\tau \leq t} [u(t)]$ defined on $\Gamma(t)$, is a quasi-static evolution that satisfies the boundary condition $g(t)$ if

1. $\forall t \in [0, 1]$

$$\int_{\Omega} |\nabla u(t)|^2 dx + \int_{S_u(t) \cup \Gamma(t)} \tilde{\varphi}([u(t)], \gamma(t)) d\mathcal{H}^{N-1} \leq \int_{\Omega} |\nabla v|^2 dx + \int_{S_v \cup \Gamma(t)} \tilde{\varphi}([v], \gamma(t)) d\mathcal{H}^{N-1}$$

$\forall v \in SBV(\Omega)$ s.t. $v = g(t)$ on $\partial \Omega$.

2. $\Gamma(t_1) \subseteq \Gamma(t_2)$ and $\gamma(t_1) \leq \gamma(t_2)$ for all $0 \leq t_1 \leq t_2 \leq 1$.

3.

$$E(u(t), \gamma(t)) = E(u(0), \gamma(0)) + 2 \int_0^t \int_\Omega \nabla u(s) \cdot \nabla \tilde{g}(s) dx ds$$

The existence of above evolution has been open for many years. The difficulty is to show that as the discrete time minimizers converge to a limit $u(t)$ in $SBV$, $u(t)$ also minimizes the energy. The way the cohesive energy is defined makes it sensitive to some of the complications in the fracture part of the sequence. We will illustrate exactly what those complications are in Chapter 1.

The following is the outline of the thesis.

*Chapter 1*

In Chapter 1, we first describes the procedure of time discretization. As the discrete time step goes to 0, we encounter a sequence of minimizers coupled with a sequence of pre-existing cracks and their limits. We then ask if the limit is also a minimizer. For Griffith model the answer is yes, in fact showing the limit is also a unilateral minimizer is the key part in showing the existence result for Griffith model. We show that for cohesive models the answer is no, by providing a counter example. We see that there are two potential complications on the fracture part of the sequence that will cause the limit not being a minimizer. One of them is the oscillation on the crack path, the other one is the splitting of one crack into two or more nearby cracks. We call the second one a “staircase”. Then we construct a sequence of unilateral minimizers who have oscillations of the
crack path. We show that the limit is not a unilateral minimizer. More interestingly, we show that
the limit does have a minimality, but it is a minimality that encodes the oscillations.

Chapter 2
In Chapter 2, we continue our discussion by reconsidering the counter example from Chapter 1.
We notice that the limit in the example picks up the oscillation of the sequence. We wonder if it
is true for any other cases. We then consider a more general sequence and show that the limit will
always pick up the oscillation, if there is oscillation of the sequence. To show this first we show the
sequence does not have the second complication, the staircase situation, mentioned in Chapter 1.
After excluding the staircase complication we show the oscillation will show up in the minimality
of the limit. For the counter example we did not have this extra step of excluding staircase be-
cause we constructed the sequence to only have oscillation. Even though the limit of a sequence
of a unilateral minimizers is not necessarily a unilateral minimizer, at least we show the staircase
complication can be ruled out. This leaves only the oscillation to be the complication we need to
rule out next. Further study indicates that it’s impossible to exclude oscillation by only looking at
a sequence of unilateral minimizers. So we move to the evolution problem to seek other ways to
exclude oscillation.

Chapter 3
In Chapter 3, we consider the evolution problem. First we use time discretization to approximate
the evolution. Then we formulate the energy and show the energy convergences for almost every
t. Showing the convergence of energy is almost always the first step in showing anything else. We
address the importance of convergence of energy by showing two applications of the result. First we
show that in terms of energy the whole evolution in discrete time can be approximated by finitely
many chosen times. It allows us to do analysis on finitely many minimizers. Using this idea we
then show the absolute continuity result for the fracture energy of the sequence of minimizers at
discrete time. As we can see in Chapter 2, the advantage of absolute continuity is to allow us apply
advanced techniques like covering theorem.
Chapter 1

The Difficulties

1.1 Time discretization

As mentioned before, let \( g(t) : [0, 1] \rightarrow W^{1,\infty}(\Omega; \mathbb{R}) \) be the boundary condition s.t.

1. \( \sup_t \| g(t) \|_{W^{1,\infty}} < \infty. \)

2. \( \nabla \dot{g}(t) \) exists and \( \nabla \dot{g}(t) \in L^2(\Omega) \) \( \forall t \), and \( \sup_t \| \nabla \dot{g}(t) \|_{L^2} < \infty. \)

3. \( \nabla \dot{g}(t) \) is continuous on \([0, 1]\).

Then split the time line \([0, 1]\) into \( 2^n \) pieces, denote \( t_i^n = \frac{1}{2^n}i \). When \( i = 0 \), let \( u_0^n \) be the solution to the following minimizing problem

\[
 u_0^n = \min_{v \in SBV(\Omega)} \left\{ \int_\Omega |\nabla v|^2 dx + \int_{S_v} \varphi([v]) dH^{N-1} \right\}
\]

When \( 0 \leq i < 2^n \), let

\[
 u_i^{i+1} = \min_{v \in SBV(\Omega)} \left\{ \int_\Omega |\nabla v|^2 dx + \int_{S_v \cup \Gamma_n} \tilde{\varphi}([v], \gamma_n^{i}) dH^{N-1} \right\}
\]

Where

\[
 \Gamma_n^i := \bigcup_{j=0}^i S_{u_j^n} \]
\[
 \gamma_n^{i} := \bigvee_{0 \leq j \leq i} [u_j^n]
\]

We see \( \gamma_n^i \) is defined on \( \Gamma_n^i \), and the existence of minimizers in each step can be derived from \( SBV \) compactness. Then define \( u_n(t) = u_i^n \) for \( t_n^i \leq t < t_n^{i+1} \) and for \( 0 \leq i < 2^n \). We see \( u_n(t) \) is well defined on \([0, 1]\) for \( \forall n > 0 \).

Define

\[
 E_n(t) := \int_\Omega |\nabla u_n(t)|^2 dx + \int_{S_{u_n(t)} \cup \Gamma_n(t)} \tilde{\varphi}([u_n(t)], \gamma_n(t)) dH^{N-1}
\]
As approximation steps become more and more, one encounters a sequence of minimizers $u_n(t)$, a limit of the sequence $u(t)$, a sequence of pre-existing fracture or history $\gamma_n(t)$ and a limit of the history $\gamma(t)$ such that

$$\int_{\Omega} |\nabla u_n(t)|^2 dx + \int_{S_{u_n(t)} \cup \Gamma_n(t)} \tilde{\varphi}([u_n(t)], \gamma_n(t)) d\mathcal{H}^{N-1} \leq \int_{\Omega} |\nabla v|^2 dx + \int_{S_v \cup \Gamma_n(t)} \tilde{\varphi}([v], \gamma_n(t)) d\mathcal{H}^{N-1}$$

for each $n$ and each $v = u_n(t)$ on $\Omega \setminus \Omega$. The question is that is the limit a minimizer with $\gamma(t)$ as history? If we can show it then the global stability can be shown and the rest follows easily.

As we will see, in the next section, that the limit might not be a minimizer if there's oscillation on the sequence of history. In fact, there are two complications on the sequence that we need to rule out if we want to show the limit is also a minimizer. The first one is due to the potential oscillation of the crack path. The second is due to the potential of splitting a crack into two or more nearby cracks, with the same total jump in displacement. (see Figure 1.1)

In the next section we construct an example to show how oscillation affects the minimality of the limit. In the next chapter we show how to prove the second complication won’t happen in a sequence of unilateral minimizers.

### 1.2 A Counter Example

#### 1.2.1 1-D domain

In this section we construct a sequence of minimizers whose limit is not a minimizer. Though the sequence will be constructed in 2-D domain, we begin with examples in 1-D domain. Consider domain $I := [0,1]$ and Dirichlet boundary condition $g(x) = x$ on $[0,1]$. Let $\alpha \geq 1$ and $\varphi$ be a cohesive energy function, let $u \in SBV(I)$ and define the energy

$$E_\alpha(u) := \int_0^1 |u'|^2 dx + \int_{S_u} \alpha \varphi([u]) d\mathcal{H}^0$$

Notice here $\mathcal{H}^0$ is the counting measure and $\int_{S_u} \alpha \varphi([u]) d\mathcal{H}^0 = \alpha \sum_{x_i \in S_u} \varphi([u](x_i))$. The existence of a minimizer of energy $E_\alpha$ can be shown using $SBV$ compactness. Let $u_\alpha$ be a minimizer of the energy satisfying the Dirichlet condition $g$, i.e. $g(0) = 0$ and $g(1) = 1$. We can conclude the following

1. If $u_\alpha$ has a jump, it can only have one jump.
2. $u'_\alpha$ is constant on $[0,1]$.
3. $2u'_\alpha = \alpha \varphi([u_\alpha])$ if $[u_\alpha] > 0$. 

Figure 1.1: Two difficulties illustrated by pictures.
If $u_\alpha$ has at least two jumps, we can move one jump to the other without changing the continuous part. By doing this we get a new SBV with smaller energy due to the strict concavity of $\varphi$. Since $u_\alpha$ has only one jump, let $h_\alpha$ denote the height of the jump. Notice that the uniqueness of the minimizer can not be guaranteed in the following sense. First of all, even though there is only one jump, the location of the jump can be anywhere on $[0,1]$. Second, even if the location is fixed, the height might not be unique. Here the location is not important since we can always translate the location of the jump to one fixed point, say $\frac{1}{2}$.

Lemma 1. There exists $\varphi$ and $\alpha_0$ such that for any $1 \leq \alpha \leq \alpha_0$, any minimizer of the energy $E_\alpha(v)$ satisfying $v = g$ on the boundary has a positive jump.

Proof. First pick any cohesive energy function $\phi$. Let $v$ be a SBV function that satisfies the following conditions

1. $v$ has only one jump at point $\{\frac{1}{2}\}$.
2. $v'$ is constant on $[0,1]$
3. $v(0) = 0$ and $v(1) = 1$.

Let $h$ denote the height of the jump of $v$, the graph of $v$ looks like Figure 1.2. Let $\beta \geq 0$ and due to the nature of $v$ the energy $\int_0^1 |v'|^2 dx + \int_{S_v} \beta \phi([v]) d\mathcal{H}^0$ can also be written in terms of $h$ as follows

$$f_\beta(h) = (h - 1)^2 + \beta \phi(h).$$

We see $f_\beta$ is continuous and $f_\beta(0) = 1$. Choose $\beta_0 > 0$ and $\beta_1 > 0$ such that

1. $\beta_0 < \beta_1$
2. $f_\beta(\frac{1}{2}) < \frac{1}{2}$ for all $\beta$ such that $\beta_0 \leq \beta \leq \beta_1$
Next define \( \alpha_0 := \frac{\beta_0}{\beta_1} > 1 \). We can see for any \( \alpha \) such that \( 1 \leq \alpha \leq \alpha_0 \), we have \( \beta_1 \leq \alpha \beta_1 \leq \beta_0 \), it follows

\[
f_{\alpha \beta_1} \left( \frac{1}{2} \right) < \frac{1}{2}
\]

Due to the definition of \( f_{\beta} \), there exists some \( SBV \) function \( v \) that satisfies the Dirichlet condition \( g \) and

\[
\int_0^1 |v'|^2 dx + \int_{S_v} \alpha \beta_1 \phi(|v|) dH^0 < \frac{1}{2}
\]

Let \( \varphi = \beta_1 \phi \), we have

\[
E_{\alpha}(v) < \frac{1}{2}
\]

This shows each minimizer of the energy has positive jump, since if there’s no jump the lowest energy it can get is 1. \( \square \)

The following lemma can be seen as concavity of \( \varphi \) with \( \alpha \) and history \( h \).

**Lemma 2.** \( \forall \varphi \) and \( 0 < h \leq 1 \), \( \exists \alpha_1 > 1 \) s.t. for any \( 1 < \alpha < \alpha_1 \) we have

\[
\alpha \tilde{\varphi}(a + b, h) < \alpha \tilde{\varphi}(b, h) + \varphi(a)
\]

\( \forall 0 \leq b \leq 1 \) and \( \forall 0 < a \leq 1 \).

**Proof.** First observe that the inequality (1.2) becomes equality when \( a = 0 \). Next we see

\[
0 < \tilde{\varphi}(a + b, h) - \tilde{\varphi}(b, h) \leq \tilde{\varphi}(a, h) - \tilde{\varphi}(0, h)
\]

It follows

\[
\frac{\varphi(a)}{\tilde{\varphi}(a + b, h) - \tilde{\varphi}(b, h)} \geq \frac{\varphi(a)}{\tilde{\varphi}(a, h) - \tilde{\varphi}(0, h)}
\]

We see \( \frac{\varphi(a)}{\tilde{\varphi}(a, h) - \tilde{\varphi}(0, h)} \to \infty \) as \( a \to 0 \) and \( \frac{\varphi(a)}{\tilde{\varphi}(a, h) - \tilde{\varphi}(0, h)} > 1 \) for all \( 0 < a < 1 \). Thus there exists \( \delta > 1 \) s.t. \( \frac{\varphi(a)}{\tilde{\varphi}(a, h) - \tilde{\varphi}(0, h)} > \delta > 1 \). Choose \( \alpha_1 = \delta \) and let \( 1 < \alpha < \alpha_1 \), we have \( \frac{\varphi(a)}{\tilde{\varphi}(a + h, h) - \tilde{\varphi}(b, h)} > \alpha \) and thus (1.2) is proved. \( \square \)

Let \( \varphi \) and \( \alpha_0 \) be chosen the same as in lemma 1. Since \( \varphi'(x) \to \infty \) as \( x \to 0 \), choose \( 0 < h < 1 \) small such that \( \varphi'(h) > 3 \). According to lemma 2, there exists \( \alpha_1 \) s.t. for any \( 1 < \alpha < \alpha_1 \), (1.2) is satisfied. Then fix an \( \alpha \) such that \( 1 < \alpha < \min\{\alpha_0, \alpha_1\} \). Let \( u_1 \) and \( u_{\alpha} \) be minimizers of energy \( E_{\alpha} \) that satisfies the boundary condition \( g \), define \( h_1 := [u_1] \) and \( h_{\alpha} := [u_{\alpha}] \). First we see \( 2u_{\alpha}' = \alpha \varphi'(h_{\alpha}) \) and \( 2u_1' = \varphi'(h_1) \). Since \( u_1' \leq 1 \) and \( u_{\alpha}' \leq 1 \), we have \( \varphi'(h_{\alpha}) \leq 2 \) and \( \varphi'(h_1) \leq 2 \). It follows \( h \leq h_{\alpha} \) and \( h \leq h_1 \) due to monotonicity of function \( \varphi' \). We further conclude the following properties

\[
h_1 > 0 \text{ and } h_{\alpha} > 0
\]

\[
\alpha \tilde{\varphi}(a + b, h_{\alpha}) < \alpha \tilde{\varphi}(b, h_{\alpha}) + \varphi(a)
\]

We see \( h_{\alpha} < 1 \) and \( h_1 < 1 \). The second estimate is due to the fact

\[
\alpha \tilde{\varphi}(a + b, h_{\alpha}) - \alpha \tilde{\varphi}(b, h_{\alpha}) \leq \alpha \tilde{\varphi}(a + b, h) - \alpha \tilde{\varphi}(b, h)
\]
for $h < h_\alpha$. From now on let’s fix $u_1$ and $u_\alpha$ by fixing the location of the jumps at middle point $\frac{1}{2}$.

### 1.2.2 2-D domain

At this point we’ve specifically picked $\alpha, \varphi, u_1, u_\alpha, h_1$ and $h_\alpha$. Next consider the 2-D square $[0,1] \times [0,1]$. Denote $L_1 = [0,1] \times \{1\}$, $L_2 = [0,1] \times \{0\}$, $L_3 = \{0\} \times [0,1]$ and $L_4 = \{1\} \times [0,1]$ as four boundaries of the square. Let $\Gamma$ denote the set $[0,1] \times \left\{ \frac{1}{2} \right\}$.

Define $\gamma : \Gamma \to \mathbb{R}$ such that $\gamma = h_\alpha$ on $\Gamma$. For each $n$ create a zigzag around $\Gamma$ with amplitude equal to $\frac{1}{n}$. Adjust the angle of zigzag such that the total length equals $\alpha$. Let $\Gamma_n$ denote the zigzag. See Figure 1.3 for the graph.

Define $\gamma_n = h_\alpha$ on $\Gamma_n$ such that $\gamma_n = h_\alpha$ and let $\tilde{u}_n$ minimizes the following

$$
\int_\Omega |\nabla \tilde{u}_n|^2 \, dx + \int_{S_{\tilde{u}_n} \cup \Gamma_n} \varphi([\tilde{u}_n], \gamma_n) d\mathcal{H}^1 \leq \int_\Omega |\nabla v|^2 \, dx + \int_{S_v \cup \Gamma_n} \varphi([v], \gamma_n) d\mathcal{H}^1
$$

over all $v$ such that $v = 1$ on $[0,1] \times \{1\}$, $v = 0$ on $[0,1] \times \{0\}$ and $\frac{\partial v}{\partial n} = 0$ on the other two sides. So $\tilde{u}_n$ is a minimizer with history $\gamma_n$ for each $n$. We see there exists a subsequence $\tilde{u}_n$ (not relabeled) and a $SBV$ limit $\tilde{u}_\infty$ s.t. $\tilde{u}_n \overset{SBV}{\longrightarrow} \tilde{u}_\infty$.

**Lemma 3.**

$$
\alpha \int_{\Gamma} \varphi((\tilde{u}_\infty), h_\alpha) d\mathcal{H}^1 + \int_{S_{\tilde{u}_\infty} \setminus \Gamma} \varphi((\tilde{u}_\infty)) d\mathcal{H}^1 \leq \liminf_{n \to \infty} \int_{S_{\tilde{u}_n} \setminus \Gamma_n} \varphi([\tilde{u}_n], h_\alpha) d\mathcal{H}^1
$$

**Proof.** Fix $\epsilon > 0$, first choose a small rectangular region $R_\epsilon$ containing $\Gamma$ s.t.

$$
\int_{S_{\tilde{u}_\infty} \setminus \Gamma \cap R_\epsilon} \varphi((\tilde{u}_\infty)) d\mathcal{H}^1 \leq \epsilon
$$

$$
\int_{\Omega \setminus R_\epsilon} \varphi((\tilde{u}_\infty)) d\mathcal{H}^1 \leq \liminf_{n \to \infty} \int_{\Omega \setminus R_\epsilon} \varphi([\tilde{u}_n]) d\mathcal{H}^1.
$$

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Next we see

\[
\text{LHS of (1.4)} = \alpha \int_{\Gamma} \bar{\varphi}([\bar{u}_\infty], h_\alpha) dH^1 + \int_{\partial_{\bar{u}_\infty} \cap \partial_{R_e}} \varphi([\bar{u}_\infty]) dH^1 + \int_{\Omega \setminus \partial_{R_e}} \varphi([\bar{u}_\infty]) dH^1
\]

\[
\text{RHS of (1.4)} = \int_{\Gamma} \bar{\varphi}([\bar{u}_n], h_\alpha) dH^1 + \int_{\partial_{\bar{u}_\infty} \cap \partial_{R_e}} \varphi([\bar{u}_n]) dH^1 + \int_{\Omega \setminus \partial_{R_e}} \varphi([\bar{u}_n]) dH^1
\]  

(1.6)

We constrain our domain on \( R_e \) and consider slices \( \bar{u}_n^l, \bar{u}_\infty^l \) for \( 0 < l < 1 \) along \( y \) axis (i.e. constrains of \( \bar{u}_n \) and \( \bar{u}_\infty \) on the set \( \{l\} \times [0, 1] \)). Let \( x_i \in \{l\} \times [0, 1] \cap \partial_{R_e} \) be the locations where \( \bar{u}_n^l \) has jumps. Let \( \Gamma^l \) and \( \Gamma_n^l \) denote the intersection of \( \Gamma \) and \( \Gamma_n \) with \( \{l\} \times [0, 1] \). We see for \( \mathcal{L}^1 \) a.e. \( l \)

\[
[\bar{u}_\infty^l(\Gamma^l)] \leq \liminf_{n \to \infty} \sum_i [\bar{u}_n^l(x_i)]
\]  

(1.7)

Combine with result from (1.3), we have

\[
\alpha \bar{\varphi}([\bar{u}_\infty^l(\Gamma^l)], h_\alpha) \leq \liminf_{n \to \infty} \alpha \bar{\varphi} \left( \sum_i [\bar{u}_n^l(x_i)], h_\alpha \right)
\]

\[
\leq \liminf_{n \to \infty} \left[ \alpha \bar{\varphi} \left( \sum_{x_i \in \Gamma^l_n} [\bar{u}_n^l(x_i)], h_\alpha \right) + \varphi(\sum_{x_i \not\in \Gamma^l_n} [\bar{u}_n^l(x_i)]) \right]
\]

\[
\leq \liminf_{n \to \infty} \left[ \alpha \int_{R_e \cap \Gamma^l_n} \bar{\varphi}([\bar{u}_n^l], h_\alpha) dH^0 + \int_{R_e \cap (\partial_{\bar{u}_\infty^l} \setminus \Gamma^l_n)} \varphi([\bar{u}_n^l]) dH^0 \right]
\]
According to Fatou’s lemma, we have
\[
\alpha \int_0^1 \tilde{\varphi}(\tilde{u}_\infty(\Gamma^l), h_\alpha) dl \leq \liminf_{n \to \infty} \left[ \alpha \int_0^1 \tilde{\varphi}(\tilde{u}_n, h_\alpha) dH^0 dl + \int_0^1 \int_{R_n \cap (S_{u_n} \cup \Gamma_n)} \varphi([u_n]) dH^0 dl \right]
\]

\[
\leq \liminf_{n \to \infty} \left[ \int_{R_n \cap \Gamma_n} \tilde{\varphi}(\tilde{u}_n, h_\alpha) d\mathcal{H}^1 + \int_{R_n \cap (S_{u_n} \cap \Gamma_n)} \varphi([u_n]) d\mathcal{H}^1 \right]
\]

And \( \alpha \int_0^1 \tilde{\varphi}(\tilde{u}_\infty(\Gamma^l), h_\alpha) dl = \alpha \int_\Gamma \tilde{\varphi}(\tilde{u}_\infty), h_\alpha) d\mathcal{H}^1 \), next combine results from (1.5) and (1.6) we conclude the proof.

\[\square\]

**Lemma 4.**

\[
\limsup_{n \to \infty} \left[ \int_\Omega |\nabla \tilde{u}_n|^2 dx + \int_{S_{u_n} \cup \Gamma_n} \tilde{\varphi}(\tilde{u}_n, h_\alpha) d\mathcal{H}^1 \right] \leq \int_\Omega |\nabla \tilde{u}_\alpha| dx + \alpha \int_\Gamma \varphi([\tilde{u}_\alpha]) d\mathcal{H}^1 \quad (1.8)
\]

where \( \tilde{u}_\alpha \) denotes the extension of \( u_\alpha \) into 2-D domain, i.e. each slice of \( \tilde{u}_\alpha \) equals \( u_\alpha \).

**Proof.** For each \( n \) define a small region \( R_n \) that closely contain set \( \Gamma_n \), as shown in Figure 1.5. Let \( u_\alpha^+ \) denote the right limit of \( u_\alpha(x) \) at \( \frac{1}{2} \) and \( u_\alpha^- \) the left limit. Let \( R_n^+ \) denote the horizontal boundary of \( R_n \) that is close to the set \( [0, 1] \times \{1\} \) and \( R_n^- \) the other horizontal boundary (see the above graph). Then define an SBV function \( w_n \) on \( \Omega \) that satisfies the following conditions

1. \( w_n = 1 \) on \( [0, 1] \times \{1\} \) and \( w_n = 0 \) on \( [0, 1] \times \{0\} \).
2. \( w_n = u_\alpha^+ \) on \( R_n^+ \) and linear between \( R_n^+ \) and \( [0, 1] \times \{1\} \).
3. \( w_n = u_\alpha^- \) on \( R_n^- \) and linear between \( R_n^- \) and \( [0, 1] \times \{0\} \).
4. For the values of \( w_n \) on \( R_n \), define one side of \( \Gamma_n \) to be equal to \( u_\alpha^+ \) and the other equal to \( u_\alpha^- \) such that \( w_n \) only has jumps on \( \Gamma_n \).
We can see that
\[
\int_{\Omega} |\nabla u_n|^2 dx + \int_{S_{\alpha n}} \tilde{\varphi}(|w_n|, h_\alpha) dH^1 \to \int_{\Omega} |\nabla \tilde{u}_\alpha|^2 dx + \alpha \int_{\Gamma} \varphi([\tilde{u}_\alpha]) dH^1.
\]
Since
\[
\int_{\Omega} |\nabla \tilde{u}_\alpha|^2 dx + \int_{S_{\alpha n} \cup \Gamma_n} \tilde{\varphi}(|\tilde{u}_\alpha|, h_\alpha) dH^1 \leq \int_{\Omega} |\nabla w_n|^2 dx + \int_{S_{\alpha n}} \tilde{\varphi}(|w_n|, h_\alpha) dH^1
\]
for each \(n\), we conclude the proof.

\[
\square
\]

Let \(\tilde{\gamma} : \Omega \to \mathbb{R}\) s.t. \(\tilde{\Gamma} := \text{supp}(\tilde{\gamma})\) is \(H^1\) measurable and \(\int_{\tilde{\Gamma}} \varphi(\tilde{\gamma}) dH^1 < \infty\).

**Lemma 5.** The limit \(\tilde{u}_\alpha\) has the following properties

1. \(\frac{\partial \tilde{u}_\alpha}{\partial x} = 0\).

2. \(\tilde{u}_\alpha^1\) is a minimizer of the energy \(E_\alpha\).

3. \(\exists \tilde{\omega} : \Omega \to \mathbb{R}\) that satisfies the boundary condition and

\[
\int_{\Omega} |\nabla \tilde{\omega}|^2 dx + \int_{S_{\alpha n} \cup \tilde{\Gamma}} \tilde{\varphi}(|\tilde{\omega}|, \tilde{\gamma}) dH^1 < \int_{\Omega} |\nabla \tilde{u}_\alpha|^2 dx + \int_{S_{\alpha n} \cup \tilde{\Gamma}} \tilde{\varphi}(|\tilde{u}_\alpha|, \tilde{\gamma}) dH^1
\]

for any \(\tilde{\gamma}\).

**Proof.** Combine lemma 3 and lemma 4 we get

\[
\int_{\Omega} |\nabla \tilde{u}_\alpha|^2 dx + \alpha \int_{\tilde{\Gamma}} \tilde{\varphi}(|\tilde{u}_\alpha|, h_\alpha) dH^1 + \int_{S_{\alpha n} \cup \tilde{\Gamma}} \varphi([\tilde{u}_\alpha]) dH^1 \leq \int_{\Omega} |\nabla \tilde{u}_\alpha|^2 dx + \alpha \int_{\Gamma} \varphi([\tilde{u}_\alpha]) dH^1
\]

Let \(v \in SBV([0,1])\) s.t. \(v(0) = 0\) and \(v(1) = 1\). Based on \(v\) let’s construct \(\hat{v}\) by moving all jumps of \(v\) to the point \(\frac{1}{2}\) and keeping \(\int_0^1 |v'|^2 dx = \int_0^1 |\hat{v}'|^2 dx\). Due to (1.3) we have

\[
\int_0^1 |v'|^2 dx + \alpha \int_{\{\frac{1}{2}\}} \varphi([v]) dH^0 \leq \int_0^1 |\hat{v}'|^2 dx + \alpha \int_{\{\frac{1}{2}\}} \varphi([\hat{v}]) dH^0
\]

\[
\leq \int_0^1 |v'|^2 dx + \alpha \int_{\{\frac{1}{2}\}} \varphi([v]) dH^0 + \int_{S \setminus \{\frac{1}{2}\}} \varphi([v]) dH^0
\]

Next we see

\[
\int_{\Omega} |\nabla \tilde{u}_\alpha|^2 dx + \alpha \int_{\Gamma} \tilde{\varphi}(|\tilde{u}_\alpha|, h_\alpha) dH^1 + \int_{S_{\alpha n} \setminus \Gamma} \varphi([\tilde{u}_\alpha]) dH^1
\]

\[
\leq \int_0^1 \int_0^1 |\nabla \tilde{u}_\alpha|^2 dxdl + \alpha \int_0^1 \int_{\{\frac{1}{2}\}} \tilde{\varphi}(|\tilde{u}_\alpha|, h_\alpha) dH^0 dl + \int_0^1 \int_{S_{\alpha n} \setminus \{\frac{1}{2}\}} \varphi([\tilde{u}_\alpha]) dH^0 dl
\]

This shows \(\int_{\Omega} |\nabla \tilde{u}_\alpha|^2 dx = \int_0^1 \int_0^1 |\nabla \tilde{u}_\alpha^l|^2 dxdl\) and thus \(\frac{\partial \tilde{u}_\alpha}{\partial x} = 0\).

It follows that each slice \(\tilde{u}_\alpha\) is the same. According to (1.9) we have

\[
\int_0^1 |\nabla \tilde{u}_\alpha|^2 dx + \alpha \int_{\{\frac{1}{2}\}} \tilde{\varphi}(|\tilde{u}_\alpha|, h_\alpha) dH^0 + \int_{S_{\alpha n} \setminus \{\frac{1}{2}\}} \varphi([\tilde{u}_\alpha]) dH^0 \leq \int_0^1 |\nabla u_\alpha|^2 dx + \alpha \int_{\{\frac{1}{2}\}} \varphi([u_\alpha]) dH^0
\]

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We see \( \int_{S^{u}_{l_{\infty}}} \varphi([\tilde{u}_l^{l_{\infty}}]) d\mathcal{H}^0 = 0 \) because otherwise we can construct a function that gives smaller energy than \( u_{\alpha} \). Then we conclude that \( \tilde{u}_{l_{\infty}} \) is also a minimizer to the energy \( E_{\alpha} \). It follows that \( \tilde{u}_{l_{\infty}} \) has following properties

1. \( \frac{d}{dx} \tilde{u}_{l_{\infty}} \) is constant.
2. \( 2 \frac{d}{dx} \tilde{u}_{l_{\infty}} = \alpha \varphi'([\tilde{u}_{l_{\infty}}]) \).
3. \( S_{\tilde{u}_{l_{\infty}}} = \{ \frac{1}{2} \} \).

Let \( h = \lfloor \tilde{u}_{l_{\infty}} \rfloor \) and let Figure 1.6 be the graph of \( \tilde{u}_{l_{\infty}} \). We see \( \frac{d}{dx} \tilde{u}_{l_{\infty}} = 1 - h \) and thus \( 2(1 - h) = \alpha \varphi'(h) \) and further \( 2(1 - h) > \varphi'(h) \). Let \( \lambda > 0 \) and increase the right limit of \( \frac{d}{dx} \tilde{u}_{l_{\infty}} \) at \( \{ \frac{1}{2} \} \) by \( \lambda \) and decrease the left limit by \( \lambda \). Let \( w_{\lambda} \) denote the new function. Next define the elastic energy \( E^a(w_{\lambda}) := \int_0^1 |w_{\lambda}'|^2 dx \) and fracture energy \( E^s(w_{\lambda}) := \varphi([w_{\lambda}]) \). Further we see \( E^a(w_{\lambda}) \) is decreased comparing to \( E^a(\tilde{u}_{l_{\infty}}) \) and \( E^s(w_{\lambda}) \) is increased comparing to \( E^s(\tilde{u}_{l_{\infty}}) \). Next let \( \Delta E^a(\lambda) := E^a(\tilde{u}_{l_{\infty}}) - E^a(w_{\lambda}) \) denote the elastic energy decreased and let \( \Delta E^s(\lambda) := E^s(w_{\lambda}) - E^s(\tilde{u}_{l_{\infty}}) \) denote the fracture energy increased. Write down the details of \( \Delta E^a(\lambda) \) and \( \Delta E^s(\lambda) \)

\[
\Delta E^a(\lambda) = (1 - h)^2 - (1 - h - 2\lambda)^2 \\
\Delta E^s(\lambda) = \varphi(h + 2\lambda) - \varphi(h)
\]

We see both energies are differentiable with respect to \( \lambda \) and

\[
\frac{\partial}{\partial \lambda} \Delta E^a(0) = 4(1 - h) \\
\frac{\partial}{\partial \lambda} \Delta E^s(0) = 2\varphi'(h)
\]

Since \( 4(1 - h) > 2\varphi'(h) \), there exists positive small \( \lambda \) such that \( \Delta E^a(\lambda) > \Delta E^s(\lambda) \), it follows \( E(w_{\lambda}) < E(\tilde{u}_{l_{\infty}}) \). Next let \( \tilde{w} \) denote the extension of \( w_{\lambda} \) to 2D domain \( \Omega \), we see \( \tilde{w} \) satisfies the
boundary conditions on \( \partial \Omega \) and moreover \( S_{\bar{w}} = S_{\bar{u}_\infty} = \Gamma, \: [\bar{w}] > [\bar{u}_\infty] \) on \( \Gamma \) and

\[
\int_\Omega |\nabla \bar{w}|^2 dx + \int_{S_{\bar{w}}} \varphi([\bar{w}]) d\mathcal{H}^1 < \int_\Omega |\nabla \bar{u}_\infty|^2 dx + \int_{S_{\bar{u}_\infty}} \varphi([\bar{u}_\infty]) d\mathcal{H}^1
\]

It follows

\[
\int_\Omega |\nabla \bar{w}|^2 dx + \int_{S_{\bar{w}} \cup \Gamma} \hat{\varphi}([\bar{w}], \hat{\gamma}) d\mathcal{H}^1 < \int_\Omega |\nabla \bar{u}_\infty|^2 dx + \int_{S_{\bar{u}_\infty} \cup \hat{\Gamma}} \hat{\varphi}([\bar{u}_\infty], \hat{\gamma}) d\mathcal{H}^1
\]

for any \( \hat{\gamma} \). It is due to the fact that \( \hat{\varphi}(a, h) - \hat{\varphi}(b, h) \leq \varphi(a) - \varphi(b) \) if \( a \geq b \).

Let \( \eta: \Omega \to \mathbb{R} \) be defined

\[
\eta = \begin{cases} 
\alpha & \text{on } \Gamma \\
1 & \text{elsewhere}
\end{cases}
\]

It’s not hard to show that the limit \( \bar{u}_\infty \) has the following minimality

\[
\int_\Omega |\nabla \bar{u}_\infty|^2 dx + \int_{S_{\bar{u}_\infty}} \eta \varphi([\bar{u}_\infty]) d\mathcal{H}^{N-1} \leq \int_\Omega |\nabla v|^2 dx + \int_{S_v} \eta \hat{\varphi}([v], [\bar{u}_\infty]) d\mathcal{H}^{N-1} \tag{1.12}
\]

For Griffith model, the oscillation on the sequence won’t affect the minimality of the limit. But this example shows that, for cohesive model, the oscillation on the sequence will destroy the minimality of the limit. More interestingly, as shown in this example, the limit seems to pick up the oscillation according to (1.12). In the next chapter we will discuss a more general case and show the limit will always pick up the oscillation, if there is one.
Chapter 2

Limits of Unilateral Minimizers

2.1 Introduction

In Chapter 1, we considered a counter example where a sequence of unilateral minimizers converges to a limit that is not a unilateral minimizer. Moreover we showed the limit does have a minimality, that picks up the oscillation, as shown in (1.12). We wonder if it is true for any other cases. In this chapter, we consider a sequence of unilateral minimizers with itself as history and show that the limit will always pick up the oscillation, if there is oscillation.

Let $\Omega'$ be an open and bounded subset in $\mathbb{R}^N$ such that $\Omega \subseteq \Omega'$. Consider a sequence $\{g_n\}_{n=1}^\infty \subset W^{1,\infty}(\Omega')$ such that $g_n \rightarrow g$ in $W^{1,\infty}$.

Let $u_n$ be a unilateral minimizer that minimizes the following

$$
\int_\Omega |\nabla u_n|^2 \, dx + \int_{S(u_n)} \varphi([u_n]) \, d\mathcal{H}^{N-1} \leq \int_\Omega |\nabla v|^2 \, dx + \int_{S(v) \cup S(u_n)} \tilde{\varphi}([v], [u_n]) \, d\mathcal{H}^{N-1}
$$

$\forall \, v \, = \, g_n$ on $\Omega' \setminus \Omega$. In addition, assume the sequence $\{u_n\}_{n=1}^\infty$ is bounded from above in terms of the following energy

$$
E(u_n) = \int_\Omega |\nabla u_n|^2 \, dx + \int_{S(u_n)} \varphi([u_n]) \, d\mathcal{H}^{N-1}.
$$

Due to $SBV$ compactness we can extract a subsequence(not relabeled) and an $SBV$ function $u$ s.t. $u_n \rightharpoonup u$ in $SBV$. Then the following is true.

**Theorem 2.** $\exists \alpha(x) : \Omega' \rightarrow \mathbb{R}$ and $\alpha \geq 1$ such that

$$
\int_\Omega |\nabla u|^2 \, dx + \int_S \alpha \varphi([u]) \, d\mathcal{H}^{N-1} \leq \int_\Omega |\nabla v|^2 \, dx + \int_{S(v) \cup S_u} \alpha \tilde{\varphi}([v], [u]) \, d\mathcal{H}^{N-1}
$$

$\forall \, v \, = \, g$ on $\Omega' \setminus \Omega$.

**Remark 1.** Notice here $\exists b$ s.t. $\sup_n \|g_n\|_{W^{1,\infty}} \leq \frac{b}{2}$, by truncation argument the size of the jumps will stay below $b$, i.e.

$$
\sup_n [u_n] \leq b \\
\sup [u] \leq b.
$$
Also notice, due to the fact that $\varphi'(x) \geq \varphi'(b) > 0$ for $x \leq b$, $\varphi'([u_n]) \geq \varphi'(b)$ and $\varphi'(|u|) \geq \varphi'(b)$.

Remark 2. The introduce of $\Omega'$ is to deal with the case where jumps of $u_n$ converge to the boundary of $\Omega$.

2.2 Settings and Tools

Measure theory plays an important role here, let’s recall some of notions. For more details refer to [13] and [2]. Let $U \subset \mathbb{R}^N$ be open. If $\mu$ is a real or vector valued measures on $U$, $|\mu|$ denotes its total variation. Unless otherwise stated, when we say a measure or Radon measure it usually means positive measure.

Definition 5. Let $\mu$ be a positive measure and $\nu$ a real or vector valued measure on $U$. We say that $\nu$ is absolutely continuous with respect to $\mu$ and write $\nu \ll \mu$, if for every $B \subset U$ such that $\mu(B) = 0$ we have $|\nu|(B) = 0$.

Definition 6. Let $\mu$ and $\mu_n(n = 1, 2, \ldots)$ be vector Radon measures on $U$. We say $\mu_n$ converge weakly* to the measure $\mu$, written $\mu_n \rightharpoonup^* \mu$, if

$$\lim_{n \to \infty} \int_U f d\mu_n = \int_U f d\mu$$

for all $f \in C_c(U)$.

It’s well known that if $\mu_n \rightharpoonup^* \mu$, the followings are true:

1. $|\mu|(K) \geq \limsup_{n \to \infty} |\mu_n|(K)$ for all compact set $K \subset U$.

2. $|\mu|(A) \leq \liminf_{n \to \infty} |\mu_n|(A)$ for all open set $A \subset U$.

3. if $|\mu_n| \rightharpoonup^* \lambda$, $\lambda \geq |\mu|$ and for all bounded Borel set $B$ with $\lambda(\partial B) = 0$, we have $\mu(B) = \lim_{n \to \infty} \mu_n(B)$.

The following corollary of Besicovitch’s Theorem is well known, but since it will be used frequently we will list it here. See [13] for the proof of the theorem.

Theorem 3. Let $\mu$ be a Borel measure on $\mathbb{R}^N$, and $\mathcal{F}$ any collection of non-degenerate closed balls. Let $A$ denote the set of centers of the balls in $\mathcal{F}$. Assume $\mu(A) < \infty$ and $\inf\{r : B(a, r) \in \mathcal{F}\} = 0$ for each $a \in A$. Then for each open set $U \subset \mathbb{R}^N$, there exists a countable collection $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that

$$\bigcup_{B \in \mathcal{G}} B \subset U$$

and

$$\mu\left((A \cap U) \setminus \bigcup_{B \in \mathcal{G}} B\right) = 0.$$

Let $u \in SBV(\Omega)$ and $t \in \mathbb{R}$, define

$$E_t := \{x \in \Omega : u > t\}.$$
And the reduced boundary of $E_t$ is denoted by $\partial^* E_t$. Then we see $E_t$ is a set of finite perimeter for $\mathcal{L}$-a.e. $t \in \mathbb{R}$. And the coarea formula for $BV$ gives the following

$$|Du|(B) = \int_{-\infty}^{\infty} \mathcal{H}^{N-1}(\partial^* E_t \cap B) \, dt$$

for any Borel $B$. It follows

$$|D^s u|(\Omega) = \int_{-\infty}^{\infty} \mathcal{H}^{N-1}(\partial^* E_t \cap S_u) \, dt$$

and

$$\int_{\Omega} |\nabla u| \, dx = \int_{-\infty}^{\infty} \mathcal{H}^{N-1}(\partial^* E_t \\setminus S_u) \, dt$$

Another useful coarea formula

$$\int_{S_u} \varphi([u]) d\mathcal{H}^{N-1} = \int_{-\infty}^{\infty} \int_{\partial^* E_t} \frac{\varphi([u])}{[u]} d\mathcal{H}^{N-1} \, dt$$

see [18] Lemma 6.3, pick $f(x, v) = |v| \frac{\varphi([u])}{[u]} \chi_{S_u}$.

## 2.3 Minimality with $\alpha$

We delay the proof by introducing some of the preliminary results first. We split the proof of theorem 2 into three steps. First prove that the weak* limit of $\varphi([u_n]) \mathcal{H}^{N-1}[S_{u_n}]$ is absolutely continuous with respect to $\varphi([u]) \mathcal{H}^{N-1}[S_u]$. Then due to Radon-Nikodym theorem there exists a density function $\alpha'$ between the two measures, which gives us a candidate for $\alpha$. Next we show that the cracks will eventually all combine to one reduced boundary, locally. This enables us to pass $\alpha$ to the limit to show the minimality with density $\alpha$.

### 2.3.1 Absolute Continuity

If $u \in SBV$, let $\{[u] < h\}$ be the set $\{x \in S_u : [u](x) < h\}$. Define $\mu_n := \varphi([u_n]) \mathcal{H}^{N-1}[S_{u_n}]$ and $\mu := \varphi([u]) \mathcal{H}^{N-1}[S_u]$. Since $\mu_n$ is bounded, according to weak compactness in measure, there’s a subsequence $\mu_n$ (not relabeled) and a Radon measure $\mu_\infty$ on $\Omega'$ such that $\mu_n \rightharpoonup \mu_\infty$. We see that in general $\mu \lesssim \mu_\infty$ and thus $\mu \ll \mu_\infty$, but here due to unilateral minimality we are going to show that the opposite is also true.

**Remark 3.** Due to definition of $\mu$ and $\mu_n$ we see $\forall x \in \Omega'$

$$\mu(\partial B(x, r)) = 0$$

$$\mu_n(\partial B(x, r)) = 0, \forall \ n > 0$$

for $\mathcal{L}^1$-a.e. $r > 0$. Because $\mu(\partial B(x_1, r_1) \cap \partial B(x_2, r_2)) = 0$ for $(x_1, r_1) \neq (x_2, r_2)$, set $\{(x, r) \in (\Omega' \times \mathbb{R}) : \mu(\partial B(x, r)) > 0\}$ is at most countable. It follows set $\{x \in \Omega' : \exists r \text{ such that } \mu(\partial B(x, r)) > 0\}$ is at most countable. Similarly we can show $\{x \in \Omega' : \exists r \text{ and } n \text{ such that } \mu_n(\partial B(x, r)) > 0\}$ is at most countable.

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Immediately, from the first observation, we have \( \forall x \in \Omega' \)
\[
\mathcal{H}^{N-1}(S_u \cap \partial B(x,r)) = 0
\]
\[
\mathcal{H}^{N-1}(S_{u_n} \cap \partial B(x,r)) = 0, \forall n > 0
\] (2.1)
for \( \mathcal{L}^1 \)-a.e. \( r > 0 \). And from the second observation and the fact that \( N \geq 2 \), we have for \( \mathcal{H}^{N-1} \)-a.e. \( x \in S_u \)
\[
\mathcal{H}^{N-1}(S_u \cap \partial B(x,r)) = 0
\]
\[
\mathcal{H}^{N-1}(S_{u_n} \cap \partial B(x,r)) = 0, \forall n > 0
\] (2.2)
for \( \forall r > 0 \). This enables us to ignore the crack energy of \( u \) and \( u_n \) on boundary of \( B(x,r) \).

Let \( B(x,r) \Subset \Omega' \), use \( T^+_u \) to denote the trace of \( u \) inside the open ball. In other words we are regarding \( B(x,r) \) as a domain for \( u \). Similarly \( T^-_u \) denotes the trace of \( u \) on \( \partial B(x,r) \) with \( \Omega' \setminus B(x,r) \) as its domain,

Lemma 6. Fix \( x \in \Omega' \), let \( \{u_n\}_{n=1}^\infty \subset BV(\Omega') \) s.t. \( u_n \rightharpoonup u \) in \( L^1(\Omega') \). Then we can extract a subsequence \( \{n_k\}_{k=1}^\infty \) s.t.
\[
\lim_{n \to \infty} \int_{\partial B(x,r)} |T^+_{u_{n_k}} - T^-_u| d\mathcal{H}^{N-1} = 0
\]
\[
\lim_{n \to \infty} \int_{\partial B(x,r)} |T^-_{u_{n_k}} - T^+_u| d\mathcal{H}^{N-1} = 0
\]
for \( \mathcal{L}^1 \)-a.e. \( r > 0 \).

Proof. Let \( u \in BV \), we see
\[
\int_{\partial B(x,r)} |T^+_u - T^-_u| d\mathcal{H}^{N-1} = 0
\] (2.3)
for \( \mathcal{L}^1 \)-a.e. \( r > 0 \).

So
\[
\int_{\Omega'} u dx = \int_0^\infty \int_{\partial B(x,r)} u d\mathcal{H}^{N-1} dr = \int_0^\infty \int_{\partial B(x,r)} T^+_u d\mathcal{H}^{N-1} dr
\]
\[
= \int_0^\infty \int_{\partial B(x,r)} T^-_u d\mathcal{H}^{N-1} dr
\]

Let \( \{v_n\}_{n=1}^\infty \subset BV \) s.t. \( v_n \rightharpoonup 0 \) in \( L^1 \). We see
\[
\int_{\Omega'} |v_n| dx = \int_0^\infty \int_{\partial B(x,r)} T^+_{v_{n_k}} d\mathcal{H}^{N-1} dr \to 0.
\]

It’s well-known that we can extract a subsequence \( \{n_k\}_{k=1}^\infty \) s.t.
\[
\int_{\partial B(x,r)} T^+_{v_{n_k}} d\mathcal{H}^{N-1} \to 0
\]
for $L^1$-a.e. $r > 0$. Let $v_n = u_n - u$ we see

$$
\int_{\partial B(x,r)} |T_{u_n}^+ - T_u^+| d\mathcal{H}^{N-1} = \int_{\partial B(x,r)} T_{u_n}^+ - T_u^+ d\mathcal{H}^{N-1} \to 0.
$$

Since $\int_{\partial B(x,r)} |T_u^+ - T_u^-| d\mathcal{H}^{N-1} = 0$ for $L^1$-a.e. $r > 0$, we see $\lim_{n \to \infty} \int_{\partial B(x,r)} |T_{u_n}^+ - T_u^-| d\mathcal{H}^{N-1} = 0$ for $L^1$-a.e. $r > 0$. The proof of the second half of the lemma is similar. \qed}

Let $\{u_n\}_{n=1}^\infty$ be the sequence mentioned in the beginning of this section. Let $S_n \subset S_{u_n}$ be $\mathcal{H}^{N-1}$ measurable.

**Lemma 7.** We have $\forall \epsilon > 0$, $\exists \delta > 0$ s.t.

$$
\int_{S_n \cap S_{u_n}} \varphi([u_n]) d\mathcal{H} \leq \epsilon
$$

whenever $\int_{S_n \cap S_{u_n}} \varphi'([u_n])[u_n] d\mathcal{H}^{N-1} \leq \delta$.

**Proof.** Fix $\epsilon > 0$, as usual let’s assume $S_n \subset S_{u_n}$. First we see $\frac{\varphi'(x)x}{\varphi(x)} \to C$ as $x \to 0$. So we can find $\tau > 0$ s.t. $\frac{\varphi'(x)x}{\varphi(x)} > \frac{C}{2}, \forall x \leq \tau$. Since $\varphi'(x)x$ is continuous, positive and $\varphi'(x)x = 0$ if and only if $x = 0$ for $0 \leq x \leq b$. Thus we have $m := \min_{\tau \leq x \leq b} \varphi'(x)x > 0$.

Then pick $\delta$ s.t.

$$
\left(\frac{2\delta}{C} + \varphi(b)\frac{\delta}{m}\right) \leq \epsilon.
$$

Then consider the following

$$
\frac{C}{2} \int_{S_n \cap \{[u_n] \leq \tau\}} \varphi([u_n]) d\mathcal{H}^{N-1} \leq \int_{S_n \cap \{[u_n] \leq \tau\}} \varphi([u_n]) \frac{\varphi'([u_n])[u_n]}{\varphi([u_n])} d\mathcal{H}^{N-1}
$$

$$
\leq \int_{S_n \cap \{[u_n] \leq \tau\}} \varphi'([u_n])[u_n] d\mathcal{H}^{N-1}
$$

$$
\leq \delta.
$$

It follows $\int_{S_n \cap \{[u_n] \leq \tau\}} \varphi([u_n]) d\mathcal{H}^{N-1} \leq \frac{2\delta}{C}$.

Then

$$
m\mathcal{H}^{N-1}(S_n \cap \{[u_n] > \tau\}) \leq \int_{S_n \cap \{[u_n] > \tau\}} \varphi'([u_n])[u_n] d\mathcal{H}^{N-1} \leq \delta
$$

which gives $\mathcal{H}^{N-1}(S_n \cap \{[u_n] > \tau\}) \leq \frac{\delta}{m}$.

Finally

$$
\int_{S_n} \varphi([u_n]) d\mathcal{H}^{N-1} = \int_{S_n \cap \{[u_n] > \tau\}} \varphi([u_n]) d\mathcal{H}^{N-1} + \int_{S_n \cap \{[u_n] \leq \tau\}} \varphi([u_n]) d\mathcal{H}^{N-1}
$$

$$
\leq \frac{2\delta}{C} + \varphi(b)\frac{\delta}{m}
$$

$$
\leq \epsilon.
$$

\qed
We are in a position to show the following theorem.

**Theorem 4.** \( \mu_\infty \ll \mu \)

**Proof.** Fix \( x \in \Omega' \), let \( B(x, r) \subset \Omega' \) where lemma (6) and remark (3) hold, which is true for \( \mathcal{L}^1 \)-a.e. \( r > 0 \).

First, we claim that

\[
\limsup_{n \to \infty} \int_{S_{u_n} \cap B(x, r)} \varphi'(\{u_n\}) \phi^\prime(u_n) \, dH^{N-1} \leq \int_{S_u \cap B(x, r)} \varphi(u) \, dH^{N-1}.
\]

(2.4)

For each \( n \) consider the test function \( v_n \) constructed as follows

\[
v_n = \begin{cases} 
  u & \text{on } B(x, r) \\
  u_n & \text{on } \Omega' \setminus B(x, r).
\end{cases}
\]

Due to unilateral minimality

\[
\int_{\Omega'} |\nabla u_n|^2 \, dx + \int_{S_{u_n}} \varphi([u_n]) \, dH^{N-1} \leq \int_{\Omega'} |\nabla v_n|^2 \, dx + \int_{S_{u_n} \cup S_{v_n}} \varphi([v_n], [u_n]) \, dH^{N-1}.
\]

Considering the construction of \( v_n \), the above inequality can be written as

\[
\int_{B(x, r)} |\nabla u_n|^2 \, dx + \int_{B(x, r)} \varphi([u_n]) \, dH^{N-1} + \int_{\Omega' \setminus B(x, r)} |\nabla v_n|^2 \, dx \\
+ \int_{\Omega' \setminus B(x, r)} \varphi([u_n]) \, dH^{N-1} + \int_{\partial B(x, r)} \varphi([u_n]) \, dH^{N-1} \\
\leq \int_{B(x, r)} |\nabla u|^2 \, dx + \int_{B(x, r)} \varphi([u], [u_n]) \, dH^{N-1} + \int_{\Omega' \setminus B(x, r)} |\nabla u_n|^2 \, dx \\
+ \int_{\Omega' \setminus B(x, r)} \varphi([u_n]) \, dH^{N-1} + \int_{\partial B(x, r)} \varphi([v_n], [u_n]) \, dH^{N-1}.
\]

It follows

\[
\int_{B(x, r)} |\nabla u_n|^2 \, dx + \int_{B(x, r)} \varphi([u_n]) \, dH^{N-1} + \int_{\partial B(x, r)} \varphi([u_n]) \, dH^{N-1} \\
\leq \int_{B(x, r)} |\nabla u|^2 \, dx + \int_{B(x, r)} \varphi([u], [u_n]) \, dH^{N-1} + \int_{\partial B(x, r)} \varphi([v_n], [u_n]) \, dH^{N-1} \\
\leq \int_{B(x, r)} |\nabla u|^2 \, dx + \int_{\partial B(x, r)} \varphi([v_n], [u_n]) \, dH^{N-1} \\
+ \int_{B(x, r)} \varphi([u]) \, dH^{N-1} + \int_{B(x, r)} \varphi(0, [u_n]) \, dH^{N-1}.
\]

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The last inequality is due to the fact that \( \varphi(x, z) \leq \varphi(x) + \varphi(0, z) \). It follows

\[
\int_{B(x,r)} \varphi'([u_n]) [u_n] d\mathcal{H}^{N-1} \leq \int_{B(x,r)} \varphi([u]) d\mathcal{H}^{N-1} + \int_{B(x,r)} |\nabla u|^2 dx - \int_{B(x,r)} |\nabla u_n|^2 dx + \int_{\partial B(x,r)} \tilde{\varphi}([v_n], [u_n]) d\mathcal{H}^{N-1}.
\]

Due to the fact \( \int_{B(x,r)} |\nabla u|^2 dx \leq \liminf_{n \to \infty} \int_{B(x,r)} |\nabla u_n|^2 dx \) and lemma(6) we conclude the first claim.

For the sake of contradiction assume there exists a Borel set \( A \subset \Omega' \) s.t. \( \mu(A) = 0 \) but \( \mu_\infty(A) = \delta > 0 \). According to lemma (3) we can find \( \bar{\delta} > 0 \) s.t.

\[
\int_{S_n \cap S_{u_n}} \varphi([u_n]) d\mathcal{H}^{N-1} \leq \frac{\delta}{2} \tag{2.5}
\]

whenever

\[
\int_{S_n \cap S_{u_n}} \varphi'([u_n]) [u_n] d\mathcal{H}^{N-1} \leq \bar{\delta}.
\]

Let \( U \) be open s.t. \( A \subset U \subset \Omega' \) and \( \mu(U) \leq \frac{1}{2} \bar{\delta} \). Then consider the collection \( \mathcal{F} \) of balls \( B(x, r) \) that satisfy the following conditions

\[
x \in A \\
B(x, r) \subset U \\
\mu_\infty(\partial B(x, r)) = 0 \\
(2.4) \quad \text{holds}.
\]

According to Besicovitch covering theorem we can find a collection of countable disjoint closed balls \( \{\overline{B(x_i, r_i)}\}_{i=1}^\infty \) s.t. \( \mu_\infty(A \setminus \bigcup_{i=1}^\infty B(x_i, r_i)) = 0 \). Since \( \mu_\infty(\partial B(x_i, r_i)) = 0 \ \forall i \), we have \( \mu_\infty(A \setminus \bigcup_{i=1}^\infty B(x_i, r_i)) = 0 \). Then select a finite \( N \in \mathbb{N} \) s.t. \( \mu_\infty(A \setminus \bigcup_{i=1}^N B(x_i, r_i)) \leq \frac{1}{8} \delta \). We conclude that

\[
\mu\left(\bigcup_{i=1}^N B(x_i, r_i)\right) \leq \frac{1}{2} \bar{\delta}
\]

\[
\mu_\infty\left(\bigcup_{i=1}^N B(x_i, r_i)\right) \geq \frac{7}{8} \bar{\delta}.
\]

Again since \( \mu_\infty(\partial B(x_i, r_i)) = 0 \ \forall i \), we have

\[
\mu_\infty\left(\bigcup_{i=1}^N B(x_i, r_i)\right) = \lim_{n \to \infty} \int_{\bigcup_{i=1}^N B(x_i, r_i)} \varphi([u_n]) \geq \frac{7}{8} \bar{\delta}. \tag{2.6}
\]
Then consider (2.4) to get
\[
\liminf_{n \to \infty} \int_{\bigcup_{i=1}^{\infty} B(x_i, r_i)} \varphi'([u_n]) [u_n] d\mathcal{H}^{N-1}
\leq \limsup_{n \to \infty} \int_{\bigcup_{i=1}^{\infty} B(x_i, r_i)} \varphi'([u_n]) [u_n] d\mathcal{H}^{N-1}
\leq \int_{\bigcup_{i=1}^{\infty} B(x_i, r_i)} \varphi([u]) d\mathcal{H}^{N-1}
= \mu\left(\bigcup_{i=1}^{\infty} B(x_i, r_i)\right)
\leq \frac{1}{2} \bar{\delta}.
\]

Apply (2.5) to get
\[
\lim_{n \to \infty} \int_{\bigcup_{i=1}^{\infty} B(x_i, r_i)} \varphi([u_n]) d\mathcal{H}^{N-1} \leq \frac{\delta}{2}
\]
which contradicts to (2.6). Thus the theorem has been proved.

From the above theorem we see that \(\mu_{\infty}\) also concentrates on \(S_u\) and \(\mu_{\infty} \ll \mathcal{H}^{N-1}|_{S_u}\) since \(\mu \ll \mathcal{H}^{N-1}|_{S_u}\). It follows that \(D_{\mu\mu_{\infty}}\) exists and \(0 < D_{\mu\mu_{\infty}} \leq \infty\) for \(\mathcal{H}^{N-1}\)-a.e. \(x \in S_u\) and 0 else where. Also, regarding remark (3), we see for \(\mathcal{H}^{N-1}\)-a.e. \(x \in S_u\)
\[
D_{\mu\mu_{\infty}}(x) = \lim_{r \to 0} \frac{\mu_{\infty}(B(x, r))}{\mu(B(x, r))} = \lim_{r \to 0} \frac{\lim_{n \to \infty} \int_{S_{u_n} \cap B(x, r)} \varphi([u_n])}{\int_{S_u \cap B(x, r)} \varphi([u])}.
\]

(2.7)

2.3.2 No-staircase Lemma

In this section both \(u_n\) and \(u\) will be the same as mentioned in the beginning of this section. We are going to show the most important theorem of this section. We show that the jumps of the sequence will eventually be combined to one reduced boundary of some set, basically it is due to unilateral minimality and concavity of the cohesive function \(\varphi(x)\). The way we show it is something we call arguing from local. Fix \(x \in S_u\) we shrink the cube \(Q(x, r)\) to get nice results we want. Then we cover \(S_u\) with those carefully chosen cubes and sum over the errors to get global niceness.

We are switching from balls to cubes, technically there’s no difference in terms of proof. First let’s introduce some notations and show some results. Let \(x \in S_u\), let \(Q(x, r)\) be the cube with side length 2r and normal the same as \(\nu(x)\), then define
\[
Q^-(x, r) := \{y \in Q(x, r) : (y - x) \cdot \nu(x) < 0\}
Q^+(x, r) := \{y \in Q(x, r) : (y - x) \cdot \nu(x) > 0\}
H(x, r, s) := \{y \in Q(x, r) : (y - x) \cdot \nu(x) = s\}
R^b_{a}(x, r) := \{y \in Q(x, r) : a < (y - x) \cdot \nu(x) < b\}.
\]

The geometric meaning can be illustrated using Figure 2.1. Next we see \(\mathcal{H}^{N-1}(H(x, r, s)) = (2r)^{N-1}\) for \(-r \leq s \leq r\) and there exists \(C(N, \delta)\) that does not depend on \(r\) s.t. \(C(N, \delta) \to 0\) as
\[ Q^-(x, r) \quad \mu^+_{x, \xi} \quad Q^+(x, r) \]

\[ \nu(x) \]

\[ Q(x, r) \]

Figure 2.1: Illustration of \( Q(x, r) \).

\[ \delta \to 0 \text{ and} \]

\[ \mathcal{H}^{N-1}(\partial R_{-\delta r}(x, r) \setminus (H(x, r, -\delta r) \cup H(x, r, \delta r))) \leq C(N, \delta) r^{N-1} \quad \forall \ r. \quad (2.8) \]

Then we have

\[ \int_{S_u} \varphi([u])d\mathcal{H}^{N-1} \leq \varphi(b)\sqrt{\epsilon} r^{N-1} + \varphi(\sqrt{\epsilon})\mathcal{H}^{N-1}(S_u) \quad (2.9) \]

whenever \( \int_{S_u}[u]d\mathcal{H}^{N-1} \leq \epsilon r^{N-1} \). Indeed, first consider

\[ \mathcal{H}^{N-1}([u > \delta]) \delta \leq \int_{\{[u] > \delta\}} [u]d\mathcal{H}^{N-1} \leq \epsilon r^{N-1}. \]

It follows \( \mathcal{H}^{N-1}([u > \delta]) \leq \frac{\epsilon}{\delta} r^{N-1} \). Thus

\[ \int_{S_u} \varphi([u])d\mathcal{H}^{N-1} = \int_{\{[u] > \delta\}} \varphi([u])d\mathcal{H}^{N-1} + \int_{\{[u] \leq \delta\}} \varphi([u])d\mathcal{H}^{N-1} \leq \varphi(b)\frac{\epsilon}{\delta} r^{N-1} + \varphi(\delta)\mathcal{H}^{N-1}(S_u). \]

Pick \( \delta = \sqrt{\epsilon} \) to conclude (2.9).

Let \( t \in \mathbb{R} \), define

\[ E_t := \{ x \in \Omega : u > t \} \]
\[ E_t^n := \{ x \in \Omega : u_n > t \}. \]

We see \( E_t \) and \( E_t^n \) are sets with finite perimeter for \( L^1 \)-a.e. \( t \) and \( \forall \ n \). If a set \( E \) has finite perimeter, we use \( \partial^* E \) and \( \nu_E(x) \) to denote the reduced boundary of \( E \) and the generalized inner normal at \( x \in \partial^* E \).

**Lemma 8.** Let \( \beta(x) \) be a \( \mathcal{H}^{N-1} \) measurable function on \( S_u \) s.t. \( \beta(x) > 0 \) and \( \int_{S_u} \beta(x)d\mathcal{H}^{N-1} < \infty \),
then we have for all $\delta < 1$
\[
\lim_{r \to 0} \frac{\int_{S_u \cap Q(x,r)} \beta(y) d\mathcal{H}^{N-1}}{(2r)^{N-1}} = \beta(x) \quad (2.10a)
\]
\[
\lim_{r \to 0} \frac{\int_{S_u \cap R_\delta^{x,r}(x,r)} \beta(y) d\mathcal{H}^{N-1}}{(2r)^{N-1}} = \beta(x) \quad (2.10b)
\]
for $\mathcal{H}^{N-1}$-a.e. $x \in S_u$. As usual $Q(x,r)$ and $R_\delta^{x,r}(x,r)$ will be oriented by $\nu(x)$.

**Proof.** First let’s write $S_u = \bigcup_{i=1}^{\infty} A_i$ where $A_i$ is $\mathcal{H}^{N-1}$ rectifiable and
$\mathcal{H}^{N-1}(A_i) < \infty$ for $\forall i$. It suffices to show (2.10a) and (2.10b) are true in each $A_i$. Define Radon
measure $\mu := \beta \mathcal{H}^{N-1}|S_u$ and fix $A_i$ we see for $\mu$ a.e. $x \in A_i$ (or $\mathcal{H}^{N-1}$-a.e. since $\beta(y) > 0$)
$\lim_{r \to 0} \frac{\mu(A_i \cap Q(x,r))}{\mu(Q(x,r))} = 1$. That is
\[
\lim_{r \to 0} \frac{\int_{A_i \cap Q(x,r)} \beta(y) d\mathcal{H}^{N-1}}{\mathcal{H}^{N-1}(A_i \cap Q(x,r))} = \beta(x)
\]
for $\mathcal{H}^{N-1}$-a.e. $x \in A_i$. Then Besicovitch-Marstrand-Mattila theorem [2](page 83 theorem 2.63) says
\[
\lim_{r \to 0} \frac{\mathcal{H}^{N-1}(A_i \cap Q(x,r))}{(2r)^{N-1}} = 1
\]
for $\mathcal{H}^{N-1}$ a.e. $x \in A_i$ since $A_i$ is $\mathcal{H}^{N-1}$ rectifiable and $\mathcal{H}^{N-1}(A_i) < \infty$. Thus we conclude
\[
\lim_{r \to 0} \frac{\int_{S_u \cap Q(x,r)} \beta(y) d\mathcal{H}^{N-1}}{(2r)^{N-1}} = \beta(x)
\]
for $\mathcal{H}^{N-1}$ a.e. $x \in A_i$. And (2.10a) is proved.

Next consider (2.10b), the ”$\leq$” part is obvious. So let’s show the ”$\geq$” part. Let $D$ be a
countable dense set in $\mathbb{R}$ s.t.
\[
A_i = \bigcup_{t \in D} (A_i \cap \partial^* E_t)
\]
and $\mathcal{H}^{N-1}(\partial^* E_t) < \infty$ for $\forall t$. We see for all $t \in D$, $\mathcal{H}^{N-1}$ a.e. $x \in A_i \cap \partial^* E_t$ is a Lebesgue point
of $\chi_{A_i \cap \partial^* E_t}$ with Radon measure $\mathcal{H}^{N-1}|\partial^* E_t$ i.e.
\[
\lim_{r \to 0} \frac{\mathcal{H}^{N-1}(A_i \cap \partial^* E_t \cap Q(x,r))}{(2r)^{N-1}} = 1
\]
for $\mathcal{H}^{N-1}$ a.e. $x \in A_i \cap \partial^* E_t$.

It is known that
\[
\lim_{r \to 0} \frac{\mathcal{H}^{N-1}(\partial^* E_t \cap Q(x,r))}{(2r)^{N-1}} = 1,
\]
thus
\[
\lim_{r \to 0} \frac{C_{N-1}(A_i \cap \partial^* E_t \cap Q(x, r))}{(2r)^{N-1}} = 1.
\]
Next we see
\[
\lim_{r \to 0} \frac{C_{N-1}(A_i \cap Q(x, r))}{(2r)^{N-1}} = 1,
\]
and therefore
\[
\lim_{r \to 0} \frac{C_{N-1}((A_i \setminus \partial^* E_t) \cap Q(x, r))}{(2r)^{N-1}} = 0. \tag{2.11}
\]
From [20] (page 241 theorem 5.6.5) we see for \(\forall \delta < 1\)
\[
\frac{C_{N-1}((Q(x, r) \setminus R^\delta_{\delta r}(x, r)))}{(2r)^{N-1}} = 0 \tag{2.12}
\]
for \(C_{N-1}\) a.e. \(x \in \partial^* E_t\). Then we get
\[
\lim_{r \to 0} \frac{C_{N-1}(A_i \cap (Q(x, r) \setminus R^\delta_{\delta r}(x, r)))}{(2r)^{N-1}}
\leq \lim_{r \to 0} \left( \frac{C_{N-1}(A_i \setminus \partial^* E_t \cap (Q(x, r) \setminus R^\delta_{\delta r}(x, r)))}{(2r)^{N-1}} + \frac{C_{N-1}(\partial^* E_t \cap (Q(x, r) \setminus R^\delta_{\delta r}(x, r)))}{(2r)^{N-1}} \right)
\leq \lim_{r \to 0} \left( \frac{C_{N-1}(A_i \setminus \partial^* E_t \cap (Q(x, r)))}{(2r)^{N-1}} + \frac{C_{N-1}(\partial^* E_t \cap (Q(x, r) \setminus R^\delta_{\delta r}(x, r)))}{(2r)^{N-1}} \right)
= 0
\]
due to result (2.11) and (2.12). It follows
\[
\lim_{r \to 0} \frac{C_{N-1}(A_i \cap R^\delta_{\delta r}(x, r))}{C_{N-1}(A_i \cap Q(x, r))} = 1
\]
for \(C_{N-1}\) a.e. \(x \in A_i \cap \partial^* E_t\), and therefore for \(C_{N-1}\) a.e. \(x \in A_i\).
Next consider
\[
\lim_{r \to 0} \frac{\int_{A_i \cap Q(x, r)} |\beta(x) - \beta| dC_{N-1}}{C_{N-1}(A_i \cap Q(x, r))} = 0
\]
for \(C_{N-1}\) a.e. \(x \in A_i\). Consider
\[
\left| \frac{\int_{A_i \cap R^\delta_{\delta r}(x, r)} \beta dC_{N-1}}{C_{N-1}(A_i \cap R^\delta_{\delta r}(x, r))} - \beta(x) \right| \leq \frac{\int_{A_i \cap R^\delta_{\delta r}(x, r)} |\beta(x) - \beta| dC_{N-1}}{C_{N-1}(A_i \cap R^\delta_{\delta r}(x, r))} \leq \frac{\int_{A_i \cap R^\delta_{\delta r}(x, r)} |\beta(x) - \beta| dC_{N-1}}{C_{N-1}(A_i \cap Q(x, r))} \frac{C_{N-1}(A_i \cap Q(x, r))}{C_{N-1}(A_i \cap R^\delta_{\delta r}(x, r))}.
\]
So
\[
\lim_{r \to 0} \frac{\int_{A_i \cap R^\delta_{\delta r}(x, r)} \beta(y) dC_{N-1}}{C_{N-1}(A_i \cap Q(x, r))} = \beta(x),
\]
29
and thus
\[
\lim_{r \to 0} \frac{\int_{A_i \cap R_{-\delta r}(x,r)} \beta(y) d\mathcal{H}^{N-1}}{(2r)^{N-1}} = \beta(x).
\]

Then
\[
\lim_{r \to 0} \frac{\int_{S_u \cap R_{-\delta r}(x,r)} \beta(y) d\mathcal{H}^{N-1}}{(2r)^{N-1}} \geq \lim_{r \to 0} \frac{\int_{A_i \cap R_{-\delta r}(x,r)} \beta(y) d\mathcal{H}^{N-1}}{(2r)^{N-1}} = \beta(x).
\]
We conclude
\[
\lim_{r \to 0} \frac{\int_{S_u \cap R_{-\delta r}(x,r)} \beta(y) d\mathcal{H}^{N-1}}{(2r)^{N-1}} = \beta(x)
\]
for \(\mathcal{H}^{N-1}\) a.e. \(x \in A_i\). Therefore (2.10b) is proved.

**Remark 4.** The above lemma uses the fact that the fracture energy on \(S_u\) is mostly concentrated on some reduced boundary \(\partial^* E_t\) locally in the sense of measure. That’s why no matter how small \(\delta\) is, the measure of \(\partial^* E_t\) will stay mostly within \(R_{-\delta r}(x,r)\) as we shrink the cube \(Q(x,r)\).

During the following define
\[
E(v,u) = \int_{\Omega} |\nabla v|^2 \, dx + \int_{S_u \cup S_v} \tilde{\varphi}(v, [u]) d\mathcal{H}^{N-1}
\]

**Lemma 9.** Let \(u_n\) and \(u\) be from section §2.1. Then we have for \(\mathcal{H}^{N-1}\)-a.e. \(x \in S_u\)
\[
\lim_{r \to 0} \lim_{n \to \infty} \frac{\int_{Q(x,r)} |\nabla u_n| \, dx}{r^{N-1}} = 0.
\]

**Proof.** From (2.2) in remark (3), we can choose \(r\) s.t. \(\mathcal{H}^{N-1}(\partial Q(x,r) \cap S_u) = 0, \forall n\). Then define \(u'_n\) by
\[
u'_n = \begin{cases} u_n & \Omega \setminus Q(x,r) \\ u^+(x) & Q^-(x,r) \\ u^-(x) & Q^+(x,r) \end{cases}
\]
Due to unilateral minimality, we have

\[ E(u'_n, u_n) \]

\[ = \int_{\Omega \setminus Q(x,r)} |\nabla u_n|^2 \, dx + \int_{S_{u_n} \cup S_{u'_n}} \tilde{\varphi}([u'_n], [u_n]) \]

\[ = \int_{\Omega \setminus Q(x,r)} |\nabla u_n|^2 \, dx + \int_{S_{u_n} \setminus Q(x,r)} \varphi([u_n]) \]

\[ + \int_{\partial Q(x,r)} \tilde{\varphi}([u'_n], [u_n]) + \int_{Q(x,r) \cap (H(x,r,0) \cup S_{u_n})} \tilde{\varphi}([u'_n], [u_n]) \]

\[ \leq \int_{\Omega \setminus Q(x,r)} |\nabla u_n|^2 \, dx + \int_{S_{u_n} \setminus Q(x,r)} \varphi([u_n]) + \mathcal{H}^{N-1}(\partial Q(x,r)) \varphi(b) \]

\[ + \mathcal{H}^{N-1}(H(x,r,0)) \varphi(b) + \int_{Q(x,r) \cap S_{u_n}} \varphi([u_n]) \]

\[ \leq \int_{\Omega \setminus Q(x,r)} |\nabla u_n|^2 \, dx + \int_{S_{u_n} \setminus Q(x,r)} \varphi([u_n]) + \int_{Q(x,r) \cap S_{u_n}} \varphi([u_n]) + C r^{N-1} \]

for some constant \( C < \infty \). It follows

\[ E(u_n, u_n) - E(u'_n, u_n) \]

\[ \geq \int_{Q(x,r)} |\nabla u_n|^2 \, dx + \int_{Q(x,r)} |\nabla u_n|^2 \, dx + \int_{S_{u_n} \setminus Q(x,r)} \varphi([u_n]) + \int_{Q(x,r) \cap S_{u_n}} \varphi([u_n]) \]

\[ - \int_{\Omega \setminus Q(x,r)} |\nabla u_n|^2 \, dx - \int_{S_{u_n} \setminus Q(x,r)} \varphi([u_n]) - \int_{Q(x,r) \cap S_{u_n}} \varphi([u_n]) - C r^{N-1} \]

\[ \geq \int_{Q(x,r)} |\nabla u_n|^2 \, dx - C r^{N-1}. \]

Due to unilateral minimality, we have \( E(u_n, u_n) - E(u'_n, u_n) \leq 0 \). So

\[ \int_{Q(x,r)} |\nabla u_n|^2 \, dx \leq C r^{N-1}. \]

Then consider Cauchy-Schwartz inequality

\[ \int_{Q(x,r)} |\nabla u_n| \, dx \leq |Q(x, r)|^{\frac{1}{2}} \left( \int_{Q(x,r)} |\nabla u_n|^2 \, dx \right)^{\frac{1}{2}} \leq \sqrt{2C} 2^{\frac{N}{2}} r^{N-\frac{1}{2}}. \]

So

\[ \frac{\int_{Q(x,r)} |\nabla u_n| \, dx}{r^{N-1}} \leq \sqrt{2C} 2^{\frac{N}{2}} r^{\frac{1}{2}}. \]

Take the limit as \( r \to 0 \) we have proved the lemma.

During the following, let \( \{S_n\}_{n=1}^{\infty} \) be a sequence of \( \mathcal{H}^{N-1} \) measurable sets.

**Lemma 10.** Let \( u_n \) and \( u \) be from section §2.1. Let \( 0 < h \leq b \) and let \( g(y) := \varphi'(y)(h - y) - (\varphi(h) - \varphi(y)) \). For \( \mathcal{H}^{N-1} \)-a.e. \( x \in S_u \) and \( \forall \epsilon > 0 \), exists \( \delta > 0 \) and \( R \) s.t. \( \forall r < R \) there exists...
\[ N(r) \in \mathbb{N} \text{ s.t.} \]
\[ \sup_{n>N(r)} \int_{S_n \cap S_n \cap Q(x,r)} |\varphi(h) - \varphi([u_n])|d\mathcal{H}^{N-1} < \epsilon r^{N-1} \]

for any sequence of \( \mathcal{H}^{N-1} \) measurable set \( \{S_n\}_{n=1}^{\infty} \) s.t.
\[ \int_{S_n \cap S_n \cap Q(x,r)} g([u_n])d\mathcal{H}^{N-1} < \delta r^{N-1}. \]

**Proof.** First we can assume \( S_n \subset S_u \) always. We see \( g(y) \geq 0 \) and \( g(y) = 0 \) iff \( y = h \). \( g(y) \rightarrow \infty \) as \( y \rightarrow 0 \). And for \( \mathcal{H}^{N-1} \) a.e. \( x \in S_u \),
\[ \lim_{r \rightarrow 0} \frac{\int_{S_u \cap Q(x,r)} \varphi([u])}{(2r)^{N-1}} = \varphi([u](x)) \]
\[ D_{\mu \mu \infty}(x) = \lim_{r \rightarrow 0} \frac{\lim_{n \rightarrow \infty} \int_{S_u \cap Q(x,r)} \varphi([u_n])}{\int_{S_u \cap Q(x,r)} \varphi([u])} < \infty. \]

Fix \( \epsilon > 0 \), let \( \delta_0 < h \) s.t.
\[ \max\{\varphi(h + \delta_0) - \varphi(h), \varphi(h) - \varphi(h - \delta_0)\} \frac{5D_{\mu \mu \infty}(x)\varphi(b)2^{N-1}}{\varphi(h - \delta_0)} \leq \frac{\epsilon}{2}. \]

Then there exists \( \delta > 0 \) s.t.
\[ 2\varphi(b) \frac{\delta}{\min\{g(h - \delta_0), g(h + \delta_0)\}} \leq \frac{\epsilon}{2}. \]

Then there exists \( R > 0 \) s.t.
\[ \lim_{n \rightarrow \infty} \int_{S_u \cap Q(x,r)} \varphi([u_n]) \leq 2D_{\mu \mu \infty}(x) \int_{S_u \cap Q(x,r)} \varphi([u]) \]
\[ \int_{S_u \cap Q(x,r)} \varphi([u]) \leq 2\varphi([u](x))(2r)^{N-1} \]

for \( \forall r < R \). It follows
\[ \lim_{n \rightarrow \infty} \int_{S_u \cap Q(x,r)} \varphi([u_n]) \leq 4D_{\mu \mu \infty}(x)\varphi([u])r^{N-1} \leq 4D_{\mu \mu \infty}(x)\varphi(b)2^{N-1}(2r)^{N-1} \]

for all \( \forall r < R \).

Then \( \forall r < R \), we can find a corresponding \( N(r) \in \mathbb{N} \) s.t.
\[ \int_{S_u \cap Q(x,r)} \varphi([u_n]) \leq 5D_{\mu \mu \infty}(x)\varphi(b)2^{N-1}r^{N-1} \] (2.13)

for all \( n > N(r) \).
Next let’s consider
\[
\int_{S_n \cap Q(x,r)} |\varphi(h) - \varphi([u_n])| d\mathcal{H}^{N-1}
\]
\[
= \int_{S_n \cap Q(x,r) \cap \{||u_n|-h| > \delta_0\}} |\varphi(h) - \varphi([u_n])| d\mathcal{H}^{N-1}
\]
\[
+ \int_{S_n \cap Q(x,r) \cap \{||u_n|-h| \leq \delta_0\}} |\varphi(h) - \varphi([u_n])| d\mathcal{H}^{N-1},
\]
we see
\[
\int_{S_n \cap Q(x,r) \cap \{||u_n|-h| \leq \delta_0\}} |\varphi(h) - \varphi([u_n])| d\mathcal{H}^{N-1}
\]
\[
\leq \max\{\varphi(h + \delta_0) - \varphi(h), \varphi(h) - \varphi(h - \delta_0)\} H^{N-1}(\{S_n \cap Q(x,r) \cap \{||u_n|-h| \leq \delta_0\}\}).
\]
Next consider
\[
\varphi(h - \delta_0) H^{N-1}(S_n \cap Q(x,r) \cap \{||u_n|-h| \leq \delta_0\})
\]
\[
\leq \int_{S_n \cap Q(x,r) \cap \{||u_n|-h| \leq \delta_0\}} \varphi([u_n]) d\mathcal{H}^{N-1}
\]
\[
\leq C_0 r^{N-1}
\]
where \(C_0 = 5D_{\mu\mu_{\infty}}(x)\varphi(b)2^{N-1}\). Therefore 
\(\mathcal{H}^{N-1}(S_n \cap Q(x,r) \cap \{||u_n|-h| \leq \delta_0\}) \leq \frac{C_0}{\varphi(h-\delta_0)} r^{N-1}\).
It follows
\[
\int_{S_n \cap Q(x,r) \cap \{||u_n|-h| \leq \delta_0\}} |\varphi(h) - \varphi([u_n])| d\mathcal{H}^{N-1}
\]
\[
\leq \max\{\varphi(h + \delta_0) - \varphi(h), \varphi(h) - \varphi(h - \delta_0)\} \frac{C_0}{\varphi(h-\delta_0)} r^{N-1}.
\]
Next due to the choice of \(S_n\) we have
\[
\int_{S_n \cap Q(x,r) \cap \{||u_n|-h| > \delta_0\}} g([u_n]) d\mathcal{H}^{N-1} + \int_{S_n \cap Q(x,r) \cap \{||u_n|-h| \leq \delta_0\}} g([u_n]) d\mathcal{H}^{N-1}
\]
\[
\leq \delta r^{N-1}.
\]
It follows
\[
\min\{g(h - \delta_0), g(h + \delta_0)\} \mathcal{H}^{N-1}(S_n \cap \{||u_n|-h| > \delta_0\})
\]
\[
\leq \int_{S_n \cap \{||u_n|-h| > \delta_0\}} g([u_n]) d\mathcal{H}^{N-1} \leq \delta r^{N-1}.
\]
Lemma 11. Let \( u_n \) and \( u \) be from section §2.1. Let \( 0 < h \leq b \). Then we have for \( H^{N-1} \text{-a.e.} \ x \in S_u \) and \( \forall \varepsilon > 0 \), exists \( \delta > 0 \) and \( R \) s.t. \( \forall r < R \) there exists \( N(r) \in \mathbb{N} \) s.t.

\[
\sup_{n > N(r)} \int_{S_u \cap [u_n] \geq h} (\varphi(h) - \varphi([u_n])) dH^{N-1} < \varepsilon r^{N-1}
\]

for any sequence of \( H^{N-1} \) measurable set \( \{S_n\}_{n=1}^{\infty} \) s.t.

\[
\int_{S_u \cap \{u_n\} \geq h} (\varphi([u_n]) - \varphi(h)) dH^{N-1} < \delta r^{N-1}.
\]

The following lemma can be proved using the exact technique in (7).

Lemma 12. For \( \forall \varepsilon > 0 \), \( \exists \delta > 0 \) s.t.

\[
\int_{S_u \cap S_n} \varphi([u_n]) dH^{N-1} \leq \varepsilon r^{N-1}
\]

whenever \( \int_{S_u \cap S_n} \varphi([u_n]) |u_n| dH^{N-1} \leq \delta r^{N-1} \).

Remark 5. The point of the above two results is to show that if we have value of one functional of \([u_n]\) can be as small as we want, then we can infer some other functional of \([u_n]\) can be as small as we want.

Notice that those results are based on the fact that \( u_n \) and \( u \) are from §2.1. They do not apply to general sequences of SBV functions.

The following theorem says, no matter how relatively small the region \( R_{\delta_r}^{\delta_r}(x, r) \) is, we can always shrink \( Q(x, r) \) to some degree such that the fracture energy of \( u_n \) mostly concentrates on \( R_{\delta_r}^{\delta_r}(x, r) \). And the reason why we consider a small region \( R_{\delta_r}^{\delta_r}(x, r) \) rather than the whole cube \( Q(x, r) \) is that later we are going to alter the values of \( u_n \) within the region \( R_{\delta_r}^{\delta_r}(x, r) \). Once we change the value of \( u_n \) on the region \( R_{\delta_r}^{\delta_r}(x, r) \), there could be new jumps created along the boundary \( \partial R_{\delta_r}^{\delta_r}(x, r) \). But we want the new jump created on the short side of \( \partial R_{\delta_r}^{\delta_r}(x, r) \) to be insignificant by reducing the 'length' of the short side.

Again, during the following we assume \( u_n \) and \( u \) are from section §2.1.

\[
\int_{S_u \cap Q(x, r)} \{ |u_n| - h > \delta_0 \} < \frac{\delta}{\min\{g(h - \delta_0), g(h + \delta_0)\}} r^{N-1}.
\]

Finally we have

\[
\int_{S_u \cap Q(x, r)} |\varphi(h) - \varphi([u_n])| dH^{N-1} \leq \left[ 2\varphi(b) \frac{\delta}{\min\{g(h - \delta_0), g(h + \delta_0)\}} + \max\{\varphi(h + \delta_0) - \varphi(h), \varphi(h) - \varphi(h - \delta_0)\} \frac{C_0}{\varphi(h - \delta_0)} \right] r^{N-1} \leq \varepsilon r^{N-1}.
\]

Take the sup to conclude the lemma.

The point of the above two results is to show that if we have value of one functional of \([u_n]\) can be as small as we want, then we can infer some other functional of \([u_n]\) can be as small as we want.

Notice that those results are based on the fact that \( u_n \) and \( u \) are from §2.1. They do not apply to general sequences of SBV functions.
Theorem 5. For $\mathcal{H}^{N-1}$ a.e. $x \in S_u$, for $\forall \epsilon > 0$, $\exists \delta > 0$ and $R > 0$ s.t. $\forall r < R$ there $\exists \{t_n\}_{n=1}^{\infty} \subset (u^-, u^+)$ and $\{s_n\}_{n=1}^{\infty} \subset (\delta r, 2\delta r)$ s.t.

\[
\lim_{n \to \infty} \int\limits_{S_u \cap R_{s_n}^n(x,r) \cap \partial^* E_t^n} |\varphi([u_n]) - \varphi([u](x))|dH^{N-1} < \epsilon r^{N-1}. \tag{2.14}
\]

Moreover

\[
\lim_{n \to \infty} \int\limits_{S_u \cap R_{s_n}^n(x,r) \cap \partial^* E_t^n} \varphi([u_n])dH^{N-1} < \epsilon r^{N-1} \tag{2.15}
\]

\[
\lim_{n \to \infty} \mathcal{H}^{N-1}(\partial^* E_t^n \cap \partial R_{s_n}^n(x,r)) < \epsilon r^{N-1} \tag{2.16}
\]

\[
\lim_{n \to \infty} \mathcal{H}^{N-1}(\partial R_{s_n}^n(x,r) \setminus (H(x,r,-s_n) \cup H(x,r,s_n))) \leq \epsilon r^{N-1} \tag{2.17}
\]

\[
\lim_{n \to \infty} \left| \int\limits_{S_u \cap R_{s_n}^n(x,r)} \varphi([u_n])dH^{N-1} - \mathcal{H}^{N-1}(\partial^* E_t^n \cap \partial R_{s_n}^n(x,r))\varphi([u](x)) \right| < \epsilon r^{N-1}. \tag{2.18}
\]
**Proof.** For $\mathcal{H}^{N-1}$-a.e. $x \in S_u$ we have

\[
\begin{align*}
\lim_{r \to 0} \int_{Q^-(x,r)} |u^+(x) - u(y)| dy &= 0 \\
\lim_{r \to 0} \int_{Q^+(x,r)} |u^-(x) - u(y)| dy &= 0 \\
\lim_{r \to 0} \frac{\int_{S_u \cap Q(x,r)} \varphi\left([u]\right)}{(2r)^{N-1}} &= \varphi\left([u](x)\right) \\
\lim_{r \to 0} \lim_{n \to \infty} \int_{Q(x,r)} |\nabla u_n| dx &= 0 \\
\mathcal{H}^{N-1}(\partial Q(x,r) \cap S_u) &= 0 \quad \text{and} \quad \mathcal{H}^{N-1}(\partial Q(x,r) \cap S_{u_n}) = 0
\end{align*}
\]

and

\[
D_{\mu \mu \infty}(x) = \lim_{r \to 0} \lim_{n \to \infty} \frac{\int_{S_{u_n} \cap Q(x,r)} \varphi\left([u_n]\right)}{\int_{S_u \cap Q(x,r)} \varphi\left([u]\right)} < \infty.
\]

Denote $h := u^+(x) - u^-(x)$. Fix $\epsilon > 0$, regarding lemma (12), $\exists \delta_1 \leq \epsilon$ s.t.

\[
\int_{S_u \cap S_{u_n}} \varphi\left([u_n]\right) \leq \epsilon r^{N-1}
\]

whenever $\int_{S_u \cap S_{u_n}} \varphi'(\left[\{u_n]\right])[u_n] \leq 3\delta_1 r^{N-1}$.

And for $\delta_1$, regarding lemma (10), $\exists \delta_2 \leq \delta_1$, $R_2$ and $N_2(r)$ s.t. $\forall r < R_2$

\[
\sup_{n > N_2(r)} \int_{S_n \cap S_{u_n} \cap Q(x,r)} |\varphi(h) - \varphi\left([u_n]\right)| d\mathcal{H}^{N-1} < \delta_1 r^{N-1}
\]

for any $\{S_n\}_{n=1}^\infty$ s.t. $\int_{S_n \cap S_{u_n} \cap Q(x,r)} \int g([u_n]) d\mathcal{H}^{N-1} < \delta_2 r^{N-1}$.

Then pick $\delta > 0$ and $\delta_0 > 0$ s.t.

\[
\delta_0 \leq \delta_2 \\
C(N,\delta)\varphi(b) \leq \frac{\delta_2}{4} \\
\varphi(b)\delta_0 \leq \frac{\delta_2}{4}
\]

Then pick $\epsilon_0 > 0$ s.t.

\[
\varphi(b) \sqrt{\frac{3\epsilon_0}{\delta}} + 2^N \varphi\left(\sqrt{\frac{3\epsilon_0}{\delta}}\right) \leq \frac{\delta_2}{4} \\
C_0 \frac{\epsilon_0}{\delta_0 h - \epsilon_0} \leq \frac{\delta_2}{4} \\
\delta_0 h - \epsilon_0 > 0
\]

where $C_0 = 5D_{\mu \mu \infty}(x)\varphi(b)2^{N-1}$. 

\[36\]
Considering (2.19), we have for the chosen $\epsilon_0$, $\exists R_1$ s.t.
\[
\int_{Q^-(x,r)} |u^+(x) - u(y)|dy \leq \epsilon_0 r^N
\]
\[
\int_{Q^+(x,r)} |u^-(x) - u(y)|dy \leq \epsilon_0 r^N
\]
\[
\int_{S_u \cap Q(x,r)} \varphi([u]) \leq 2\varphi([u])(2r)^{N-1}
\]
(2.24)
\[
\lim_{n \to \infty} \int_{Q(x,r)} |\nabla u_n|dx \leq \epsilon_0 r^{N-1}
\]
\[
\lim_{n \to \infty} \int_{S_u \cap Q(x,r)} \varphi([u_n]) \leq 2D_\mu \mu_\infty(x) \int_{S_u \cap Q(x,r)} \varphi([u])
\]
for all $r \leq R_1$.

Let $R = \min\{R_1, R_2\}$, we see $\forall r < R$, (2.20), (2.21) and (2.24) all hold. In particular $\forall r < R$, we can find $N_1(r) > N_2(r)$ s.t.
\[
\int_{S_u \cap Q(x,r)} \varphi([u_n]) \leq (5D_\mu \mu_\infty(x)\varphi(b)2^{N-1})r^{N-1} \quad \forall n > N_1(r).
\]
(2.25)

Since we are fixing $x$ and $r$ during the rest of the proof, let $R_b^a$ denote $R_b^a(x,r)$ and $H(s)$ denote $H(x,r,s)$. Because of $L^1$ convergence, we have
\[
\int_{R_{-\delta r}^a \cup R_{\delta r}^a} |u(y) - u_n(y)|dy \to 0
\]
as $n \to \infty$. Then consider
\[
\int_{R_{-\delta r}^a} |u^+(x) - u_n(y)|dy + \int_{R_{\delta r}^a} |u^-(x) - u_n(y)|dy
\]
\[
\leq \int_{R_{-\delta r}^a} |u^+(x) - u(y)|dy + \int_{R_{\delta r}^a} |u(y) - u_n(y)|dy + \int_{R_{\delta r}^a} |u(y) - u_n(y)|dy
\]
\[
\leq \int_{R_{-\delta r}^a} |u^+(x) - u(y)|dy + \int_{R_{\delta r}^a} |u^- (x) - u(y)|dy + \int_{R_{\delta r}^a} |u(y) - u_n(y)|dy
\]
\[
+ \int_{R_{-\delta r}^a \cup R_{\delta r}^a} |u(y) - u_n(y)|dy.
\]

It follows that we can find $N(r) > N_1(r)$ s.t.
\[
\int_{R_{-\delta r}^a} |u^+(x) - u_n(y)|dy + \int_{R_{\delta r}^a} |u^- (x) - u_n(y)|dy \leq 3\epsilon_0 r^N \quad \forall n > N(r).
\]

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But
\[
\int_{R_{-2δr}^{s}} |u^+(x) - u_n(y)|dy + \int_{R_{δr}^{s}} |u^-(x) - u_n(y)|dy
\]
\[
= \int_{δr}^{2δr} \left[ \int_{H(s)} |T_{u_n}^- - u^-(x)|d\mathcal{H}^{N-1} + \int_{H(-s)} |T_{u_n}^- - u^+(x)|d\mathcal{H}^{N-1} \right]ds. \tag{2.26}
\]

Pick \( s_n \in (δr, 2δr) \) s.t. \( \mathcal{H}^{N-1}(H(s_n) \cap S_{u_n}) + \mathcal{H}^{N-1}(H(-s_n) \cap S_{u_n}) = 0 \) and
\[
\int_{H(s_n)} |T_{u_n}^- - u^-(x)|d\mathcal{H}^{N-1} + \int_{H(-s_n)} |T_{u_n}^- - u^+(x)|d\mathcal{H}^{N-1} \leq \frac{3ε_0 r^N}{δr} = 3ε_0 r^{N-1}.
\]

Let \( t \in (u^-(x), u^+(x)) \), define the following \( u^n_t \) by
\[
u^n_t = \begin{cases} 
    u_n & \Omega \setminus R_{s_n}^u \\
    u^+(x) & R_{s_n}^u \cap E_t^n \\
    u^- & R_{-s_n}^u \setminus E_t^n
\end{cases}
\tag{2.27}
\]

We see \( u^n_t \) is defined everywhere on \( Ω \) and \( u^n_t \in SBV(Ω) \), moreover we have
\[
\int_{S_{u_n} \cup S_{u_n}^-} \dot{ϕ}([u^n_t], [u_n]) = \int_{Ω \setminus R_{s_n}^u} ϕ([u_n]) + \int_{R_{s_n}^u \cap E_t^n} ϕ([u^n_t]) + \int_{R_{-s_n}^u \setminus E_t^n} \dot{ϕ}([u^n_t], [u_n]).
\]

Then we see the new crack energy on the boundary of \( R_{s_n}^u \) looks like following
\[
\int_{\partial R_{s_n}^u} ϕ([u^n_t])d\mathcal{H}^{N-1}
\]
\[
= \int_{H(s_n)} ϕ([T_{u_n}^- - u^-(x)])d\mathcal{H}^{N-1} + \int_{H(-s_n)} ϕ([T_{u_n}^- - u^+(x)])d\mathcal{H}^{N-1}
+ \int_{\partial R_{s_n}^u \setminus (H(s_n) \cup H(-s_n))} ϕ([u^n_t])d\mathcal{H}^{N-1}
\leq \left[ ϕ(b) \sqrt{3ε_0 r^{N-1}} + ϕ(\sqrt{3ε_0})\mathcal{H}^{N-1}(H(s_n) \cup H(-s_n)) + C(N, δ)φ(b) \right] r^{N-1}
\leq \left[ ϕ(b) \sqrt{3ε_0 r^{N-1}} + 2^N φ(\sqrt{3ε_0}) + C(N, δ)φ(b) \right] r^{N-1}
\leq \frac{1}{2} δ_2.
\tag{2.28}
\]

The inequalities come from (2.8), (2.9), (2.22) and (2.23).

Next we see \( R_{s_n}^u \cap S_{u_n} = \partial E_t^n \cap R_{s_n}^u \) except for a \( \mathcal{H}^{N-1} \) measure 0 set and \([u^n_t] = u^+(x) \) -
\( u^-(x) = h \) on \( \partial E^n_t \cap R_{s_n}^{s_n} \), so
\[
\int_{R_{s_n}^{s_n} \cap (S_{u_n} \cup \partial^* E^n_t)} \tilde{\varphi}([u^n_t], [u_n])d\mathcal{H}^{N-1}
= \int_{R_{s_n}^{s_n} \cap (S_{u_n} \setminus \partial^* E^n_t)} \tilde{\varphi}(0, [u_n])d\mathcal{H}^{N-1} + \int_{R_{s_n}^{s_n} \cap (\partial^* E^n_t \setminus S_{u_n})} \varphi(h)d\mathcal{H}^{N-1} \\
+ \int_{R_{s_n}^{s_n} \cap (S_{u_n} \cap \partial^* E^n_t)} \tilde{\varphi}(h, [u_n])d\mathcal{H}^{N-1}.
\]

But we see
\[
\int_{R_{s_n}^{s_n} \cap (S_{u_n} \cap \partial^* E^n_t)} \tilde{\varphi}(h, [u_n])d\mathcal{H}^{N-1}
= \int_{R_{s_n}^{s_n} \cap (S_{u_n} \cap \partial^* E^n_t) \cap \{[u_n] > h\}} \varphi(h, [u_n])d\mathcal{H}^{N-1} \\
+ \int_{R_{s_n}^{s_n} \cap (S_{u_n} \cap \partial^* E^n_t) \cap \{[u_n] \leq h\}} \varphi(h, [u_n])d\mathcal{H}^{N-1} \\
= \int_{R_{s_n}^{s_n} \cap (S_{u_n} \cap \partial^* E^n_t) \cap \{[u_n] > h\}} \left[ \varphi([u_n]) - \varphi'([u_n])([u_n] - h) \right]d\mathcal{H}^{N-1} \\
+ \int_{R_{s_n}^{s_n} \cap (S_{u_n} \cap \partial^* E^n_t) \cap \{[u_n] \leq h\}} \varphi(h)d\mathcal{H}^{N-1}.
\]

For cleanness let’s drop domain \( R_{s_n}^{s_n} \),
\[
\int_{S_{u_n} \setminus \partial^* E^n_t} \varphi([u_n])d\mathcal{H}^{N-1}
- \int_{S_{u_n} \setminus \partial^* E^n_t} \varphi'([u_n])[u_n]d\mathcal{H}^{N-1} \\
+ \int_{\partial^* E^n_t \setminus S_{u_n}} \varphi(h)d\mathcal{H}^{N-1} + \int_{S_{u_n} \cap (\partial^* E^n_t \cap \{[u_n] > h\})} \varphi([u_n])d\mathcal{H}^{N-1} \\
- \int_{S_{u_n} \cap (\partial^* E^n_t \cap \{[u_n] > h\})} \varphi'([u_n])([u_n] - h)d\mathcal{H}^{N-1} \\
+ \int_{S_{u_n} \cap (\partial^* E^n_t \cap \{[u_n] \leq h\})} \varphi(h)d\mathcal{H}^{N-1}.
\]
Let $-\Delta := \int_{R_{-s_n}^t} \varphi([u_n]) - \int_{R_{-s_n}^t} \varphi([u_n^t], [u_n])$, we have

$$-\Delta = \int_{\partial E_t \cap \partial E_t^m} \varphi([u_n]) + \int_{\partial E_t \cap \partial E_t^m} \varphi([u_n]) - \int_{\partial E_t \cap \partial E_t^m} \varphi([u_n]) + \int_{\partial E_t \cap \partial E_t^m} \varphi([u_n]) [u_n]$$

$$= \int_{\partial E_t \cap \partial E_t^m} \varphi([u_n]) + \int_{\partial E_t \cap \partial E_t^m} \varphi([u_n]) - \int_{\partial E_t \cap \partial E_t^m} \varphi([u_n]) + \int_{\partial E_t \cap \partial E_t^m} \varphi([u_n]) ([u_n] - h)$$

$$-\int_{\partial E_t \cap \partial E_t^m} \varphi([u_n]) + \int_{\partial E_t \cap \partial E_t^m} \varphi([u_n]) (h - [u_n])$$

$$= \int_{\partial E_t \cap \partial E_t^m} \varphi([u_n]) [u_n] - \int_{\partial E_t \cap \partial E_t^m} \varphi([u_n]) h - \int_{\partial E_t \cap \partial E_t^m} \varphi([u_n])$$

$$+ \int_{\partial E_t \cap \partial E_t^m} \varphi([u_n]) (h - [u_n]) - (\varphi(h) - \varphi([u_n]))] .$$

Let’s fix $t^* \in (u^-(x), u^+(x))$, put back the integral domain $R_{-s_n}^t$ and split $-\Delta$ to two pieces, $-\Delta = A + B$ where

$$A := \int_{R_{-s_n}^t \cap \partial E_t \cap \partial E_t^m \cap \{[u_n] \leq h\}} \left[ \varphi([u_n]) [u_n] - (\varphi(h) - \varphi([u_n])) \right] dH^{N-1}$$

(2.29)

and

$$B := \int_{R_{-s_n}^t \cap \partial E_t \cap \partial E_t^m \cap \{[u_n] \leq h\}} \left[ \varphi([u_n]) [u_n] - (\varphi(h) - \varphi([u_n])) \right] dH^{N-1}$$

$$\int_{\partial E_t \cap \partial E_t^m \cap \{[u_n] \leq h\}} \varphi([u_n]) [u_n] - \int_{\partial E_t \cap \partial E_t^m \cap \{[u_n] \leq h\}} \varphi([u_n]) dH^{N-1}$$

$$- \int_{\partial E_t \cap \partial E_t^m \cap \{[u_n] \leq h\}} \varphi([u_n]) [u_n] - \int_{\partial E_t \cap \partial E_t^m \cap \{[u_n] \leq h\}} \varphi([u_n]) [u_n]$$

$$\int_{\partial E_t \cap \partial E_t^m \cap \{[u_n] \leq h\}} \varphi([u_n]) [u_n]$$

$$\int_{\partial E_t \cap \partial E_t^m \cap \{[u_n] \leq h\}} \varphi([u_n]) [u_n]$$

Considering the co-area formula, $B$ has the following form

$$B = \int_{-\infty}^{+\infty} \int_{\partial E_t \cap \partial E_t^m \cap \{[u_n] \leq h\}} \varphi([u_n]) dH^{N-1} dt$$

$$- \int_{-\infty}^{+\infty} \int_{\partial E_t \cap \partial E_t^m \cap \{[u_n] \leq h\}} \varphi([u_n]) dH^{N-1} dt$$

Notice that $A \geq 0$ since $\varphi([u_n]) (h - [u_n]) - (\varphi(h) - \varphi([u_n])) \geq 0$ for all $[u_n] \leq h$.

Then define

$$T_{\delta_0}^+ := \{ u^- < t < u^+ : H^{N-1}(R_{-s_n}^t \cap \partial E_t \cap \partial E_t^m \cap \{[u_n] \leq h\}) \geq \delta_0 r^{N-1} \}$$

$$T_{\delta_0}^- := \{ u^- < t < u^+ : H^{N-1}(R_{-s_n}^t \cap \partial E_t \cap \partial E_t^m \cap \{[u_n] \leq h\}) < \delta_0 r^{N-1} \}$$

Notice $T_{\delta_0}^+$ is defined differently when $\delta_0$ is a scaler. Since $\int_{R_{-s_n}^t} |\nabla u_n| dx \leq \epsilon_0 r^{N-1}$, after applying
co-area formula again we get

\[
\int_{T_{\delta_0}}^{+} \mathcal{H}^{N-1}(R_{-s_{n}}^{+} \cap (\partial^* E_{t}^{n}) \cap (S_{u_{n}})) \, dt + \int_{T_{\delta_0}}^{-} \mathcal{H}^{N-1}(R_{-s_{n}}^{+} \cap (\partial^* E_{t}^{n}) \cap (S_{u_{n}})) \, dt
\]

\[
= \int_{u_{-}}^{u_{+}} \mathcal{H}^{N-1}(R_{-s_{n}}^{+} \cap (\partial^* E_{t}^{n}) \cap (S_{u_{n}})) \, dt
\]

\[
< \epsilon_{0} r^{N-1}.
\]

It follows \(|T_{\delta_{0}}^{+}| \leq \frac{\epsilon_{0}}{\delta_{0}}, |T_{\delta_{0}}^{-}| > h - \frac{\epsilon_{0}}{\delta_{0}}\). And since \(h - \frac{\epsilon_{0}}{\delta_{0}} > 0\), we have \(|T_{\delta_{0}}^{+}| \leq \frac{\epsilon_{0}}{\delta_{0}} h - \epsilon_{0} \).

For fixed \(\delta_{0}\) and \(n\) we always let \(t^*(\delta_{0}, n) \in T_{\delta_{0}}^{-}\) be specifically chosen s.t.

\[
\int_{T_{\delta_{0}}^{-}}^{+} \left[ \int_{R_{-s_{n}}^{+} \cap (\partial^* E_{t}^{n})} \varphi'[([u_{n}])] \mathcal{H}^{N-1} - \int_{R_{-s_{n}}^{+} \cap (\partial^* E_{t}^{n})} \varphi'([u_{n}]) \mathcal{H}^{N-1} \right] \, dt \geq 0.
\]

We see that it’s possible. From now on let \(t^*\) always be chosen that way.

Because of that, we have

\[
\int_{T_{\delta_{0}}^{-}}^{+} \int_{R_{-s_{n}}^{+} \cap (\partial^* E_{t}^{n})} \varphi'([u_{n}]) \mathcal{H}^{N-1} \, dt \geq |T_{\delta_{0}}^{-}| \int_{R_{-s_{n}}^{+} \cap (\partial^* E_{t}^{n})} \varphi'([u_{n}]) \mathcal{H}^{N-1}
\]

and thus

\[
\frac{\int_{T_{\delta_{0}}^{-}}^{+} \int_{R_{-s_{n}}^{+} \cap (\partial^* E_{t}^{n})} \varphi'([u_{n}]) \mathcal{H}^{N-1} \, dt}{|T_{\delta_{0}}^{-}|} \geq \int_{R_{-s_{n}}^{+} \cap (\partial^* E_{t}^{n})} \varphi'([u_{n}]) \mathcal{H}^{N-1}.
\]

It follows

\[
|T_{\delta_{0}}^{+}| \int_{R_{-s_{n}}^{+} \cap (\partial^* E_{t}^{n})} \varphi'([u_{n}]) \mathcal{H}^{N-1}
\]

\[
\leq |T_{\delta_{0}}^{+}| \int_{T_{\delta_{0}}^{-}}^{+} \int_{R_{-s_{n}}^{+} \cap (\partial^* E_{t}^{n})} \varphi'([u_{n}]) \mathcal{H}^{N-1} \, dt
\]

\[
\leq |T_{\delta_{0}}^{+}| \int_{u_{-}}^{u_{+}} \int_{R_{-s_{n}}^{+} \cap (\partial^* E_{t}^{n})} \varphi'([u_{n}]) \mathcal{H}^{N-1} \, dt
\]

\[
\leq |T_{\delta_{0}}^{+}| \int_{R_{-s_{n}}^{+} \cap (S_{u_{n}})} \varphi'([u_{n}]) \mathcal{H}^{N-1}
\]

\[
\leq \frac{\epsilon_{0}}{\delta_{0}} h - \epsilon_{0} \int_{S_{u_{n}} \cap Q(x, r)} \varphi'([u_{n}])
\]

\[
\leq C_{0} \frac{\epsilon_{0}}{\delta_{0}} h - \epsilon_{0} \, r^{N-1}.
\]
where \( C_0 = 5D_\mu \mu_\infty(x)\varphi(b)2^{N-1} > 0 \), regarding (2.25). Next consider

\[ B \geq \int_{u_-}^{u_+} \int_{R^*_{-s_n} \cap S_{u_n} \cap \partial^* E_t^*} \varphi([u_n])d\mathcal{H}^{N-1} dt - \int_{u_-}^{u_+} \int_{R^*_{-s_n} \cap S_{u_n} \cap \partial^* E_t^*} \varphi'([u_n])d\mathcal{H}^{N-1} dt \]

\[ - \int_{R^*_{-s_n} \cap \partial^* E_t^* \setminus S_{u_n}} \varphi(h)d\mathcal{H}^{N-1} \]

\[ \geq \int_{T^-_{0_1}} \left[ \int_{R^*_{-s_n} \cap S_{u_n} \cap \partial^* E_t^*} \varphi'([u_n])d\mathcal{H}^{N-1} \right] dt + \int_{T^-_{0_0}} \left[ \int_{R^*_{-s_n} \cap S_{u_n} \cap \partial^* E_t^*} \varphi'([u_n])d\mathcal{H}^{N-1} \right] dt \]

\[ - \int_{R^*_{-s_n} \cap \partial^* E_t^* \setminus S_{u_n}} \varphi(h)d\mathcal{H}^{N-1} \]

\[ \geq - \int_{T^-_{0_1}} \left[ \int_{R^*_{-s_n} \cap S_{u_n} \cap \partial^* E_t^*} \varphi'([u_n])d\mathcal{H}^{N-1} \right] dt - \int_{R^*_{-s_n} \cap \partial^* E_t^* \setminus S_{u_n}} \varphi(h)d\mathcal{H}^{N-1}. \]

According to (2.30) and the fact that \( t^* \) is always chosen in \( T^-_{0_1} \), we have

\[ B \geq -C_0 \frac{\epsilon_0}{\delta_0 h - \epsilon_0} r^{N-1} - \varphi(h)\delta_0 r^{N-1} \geq \left( -C_0 \frac{\epsilon_0}{\delta_0 h - \epsilon_0} - \varphi(b)\delta_0 \right) r^{N-1} \]

(2.31)

\[ \geq -\frac{\delta_2}{2} r^{N-1}. \]

Then consider the energy drop \( E(u_n, u_n) - E(u^n_{t^*}, u_n) \) where

\[ E(u_n, u_n) = \int_{\Omega} |\nabla u_n|^2 dx + \int_{S_{u_n}} \varphi([u_n])d\mathcal{H}^{N-1} \]

and

\[ E(u^n_{t^*}, u_n) \]

\[ = \int_{\Omega} |\nabla u^n_{t^*}|^2 dx + \int_{S_{u_n} \cup S_{u^n_{t^*}}} \tilde{\varphi}([u^n_{t^*}], [u_n])d\mathcal{H}^{N-1} \]

\[ = \int_{\Omega \setminus R^*_{-s_n}} |\nabla u^n|_2 dx + \int_{\Omega \setminus R^*_{-s_n}} \varphi([u_n]) + \int_{\partial R^*_{-s_n}} \varphi([u_n]) + \int_{R^*_{-s_n}} \tilde{\varphi}([u^n_{t^*}], [u_n]) \]

\[ \leq \int_{\Omega \setminus R^*_{-s_n}} |\nabla u^n|^2 dx + \int_{\Omega \setminus R^*_{-s_n}} \varphi([u_n]) + \int_{R^*_{-s_n}} \tilde{\varphi}([u^n_{t^*}], [u_n]) + \frac{1}{2} \delta_2 r^{N-1}. \]
Thus we have

\[
E(u_n, u_n) - E(u^*_n, u_n) \\
\geq \int_{R^{n}_{\leq s_n}} \varphi([u_n]) - \int_{R^{n}_{\leq s_n}} \tilde{\varphi}([u^*_n], [u_n]) - \frac{1}{2} \delta_2 r^{N-1} \\
= - \Delta - \frac{1}{2} \delta_2 r^{N-1} \\
= A + B - \frac{1}{2} \delta_2 r^{N-1}
\]

(2.32)

considering the definition \(-\Delta\). Then, according to unilateral minimality, we see \(E(u_n, u_n) - E(u^*_n, u_n) \leq 0\), and thus

\[
A \leq -B + \frac{1}{2} \delta_2 r^{N-1} \leq \delta_2 r^{N-1}
\]

considering (2.31). That is

\[
\int_{R^{n}_{\leq s_n} \cap S_{u_n} \cap \partial^* E^n_{t_n} \cap \{[u_n] \leq h\}} \left[ \varphi'([u_n])(h - [u_n]) - (\varphi(h) - \varphi([u_n])) \right] d\mathcal{H}^{N-1} \leq \delta_2 r^{N-1}.
\]

Due to (2.20) we have

\[
\int_{R^{n}_{\leq s_n} \cap S_{u_n} \cap \partial^* E^n_{t_n} \cap \{[u_n] \leq h\}} (\varphi(h) - \varphi([u_n])) d\mathcal{H}^{N-1} \leq \delta_1 r^{N-1} \leq \epsilon r^{N-1}.
\]

Therefore one direction has been proved.

Considering (2.32) and the fact that \(A \geq 0\), we have

\[
B \leq \frac{1}{2} \delta_2 r^{N-1}.
\]

For clearness let’s again drop the integral domain \(R^{n}_{\leq s_n}\) for now. Consider the following

\[
B = \int_{S_{u_n}} \varphi'([u_n])[u_n] - \int_{S_{u_n} \cap \partial^* E^n_{t_n}} \varphi'([u_n])h - \int_{\partial^* E^n_{t_n} \setminus S_{u_n}} \varphi(h)
\]

\[
= \int_{S_{u_n} \cap \partial^* E^n_{t_n} \cap \{[u_n] > h\}} \varphi'([u_n])[u_n] - \int_{S_{u_n} \cap \partial^* E^n_{t_n} \cap \{[u_n] \leq h\}} \varphi'([u_n])[u_n] - \int_{S_{u_n} \cap \partial^* E^n_{t_n} \cap \{[u_n] > h\}} \varphi'(h)
\]

\[
+ \int_{S_{u_n} \cap \partial^* E^n_{t_n} \cap \{[u_n] \leq h\}} \varphi'([u_n])[u_n] - \int_{S_{u_n} \cap \partial^* E^n_{t_n} \cap \partial^* E^n_{t_n} \setminus S_{u_n}} \varphi(h)
\]

\[
\geq \int_{S_{u_n} \cap \partial^* E^n_{t_n} \cap \{[u_n] \leq h\}} \varphi'([u_n])[u_n] - \int_{S_{u_n} \cap \partial^* E^n_{t_n} \cap \partial^* E^n_{t_n} \setminus S_{u_n}} \varphi(h).
\]
First we see \( \int_{\partial^* E_t^n \setminus S_{u_n}} \varphi(h) \leq \varphi(b)\delta_0 r^{N-1} \leq \frac{3}{4} r^{N-1} \). With a bit of manipulation we get
\[
\int_{S_{u_n} \setminus \partial^* E_t^n} \varphi'([u_n]) [u_n] d\mathcal{H}^{N-1} \\
\leq \int_{S_{u_n} \cap \partial^* E_t^n \cap \{[u_n] \leq h\}} \varphi'([u_n])(h - [u_n]) + \frac{3}{4} \delta_2 r^{N-1} \\
\leq \int_{S_{u_n} \cap \partial^* E_t^n \cap \{[u_n] \leq h\}} [\varphi'([u_n])(h - [u_n]) - (\varphi(h) - \varphi([u_n]))] \\
+ \int_{S_{u_n} \cap \partial^* E_t^n \cap \{[u_n] > h\}} \varphi(h) - \varphi([u_n])] + \frac{3}{4} \delta_2 r^{N-1} \\
\leq 3\delta_1 r^{N-1}.
\]

(2.20) implies
\[
\int_{S_{u_n} \setminus \partial^* E_t^n} \varphi([u_n]) \leq \epsilon r^{N-1}.
\]

Thus (2.15) has been proved. Next consider
\[
-\Delta \\
= \int_{S_{u_n} \cap \partial^* E_t^n} \varphi([u_n]) + \int_{S_{u_n} \setminus \partial^* E_t^n} \varphi([u_n]) - \int_{S_{u_n} \setminus \partial^* E_t^n} \varphi([u_n]) + \int_{S_{u_n} \setminus \partial^* E_t^n} \varphi'([u_n]) [u_n] \\
- \int_{\partial^* E_t^n \setminus S_{u_n}} \varphi(h) - \int_{S_{u_n} \cap \partial^* E_t^n \cap \{[u_n] > h\}} \varphi([u_n]) + \int_{S_{u_n} \cap \partial^* E_t^n \cap \{[u_n] > h\}} \varphi'([u_n])([u_n] - h) \\
- \int_{S_{u_n} \cap \partial^* E_t^n \cap \{[u_n] \leq h\}} \varphi(h) \\
= \int_{S_{u_n} \cap \partial^* E_t^n \cap \{[u_n] > h\}} \varphi([u_n]) + \int_{S_{u_n} \cap \partial^* E_t^n \cap \{[u_n] \leq h\}} \varphi([u_n]) + \int_{S_{u_n} \setminus \partial^* E_t^n} \varphi([u_n]) \\
- \int_{\partial^* E_t^n \setminus S_{u_n}} \varphi(h) - \int_{S_{u_n} \cap \partial^* E_t^n \cap \{[u_n] > h\}} \varphi([u_n]) + \int_{S_{u_n} \cap \partial^* E_t^n \cap \{[u_n] > h\}} \varphi'([u_n])([u_n] - h) \\
- \int_{S_{u_n} \cap \partial^* E_t^n \cap \{[u_n] \leq h\}} \varphi(h) \\
= \int_{S_{u_n} \cap \partial^* E_t^n \cap \{[u_n] \leq h\}} \varphi([u_n]) + \int_{S_{u_n} \setminus \partial^* E_t^n} \varphi'(h)[u_n] - \int_{\partial^* E_t^n \setminus S_{u_n}} \varphi(h) \\
+ \int_{S_{u_n} \cap \partial^* E_t^n \cap \{[u_n] > h\}} \varphi'([u_n])([u_n] - h) - \int_{S_{u_n} \cap \partial^* E_t^n \cap \{[u_n] \leq h\}} \varphi(h) \\
= \int_{S_{u_n} \setminus \partial^* E_t^n} \varphi'([u_n])([u_n]) + \int_{S_{u_n} \cap \partial^* E_t^n \cap \{[u_n] > h\}} \varphi'([u_n])([u_n] - h) \\
- \int_{S_{u_n} \cap \partial^* E_t^n \cap \{[u_n] \leq h\}} \varphi(h) - \varphi([u_n]) - \int_{\partial^* E_t^n \setminus S_{u_n}} \varphi(h).
Again consider

\[ 0 \geq E(u_n, u_n) - E(u_n^0, u_n) \geq -\Delta - \frac{1}{2} \delta_2 r^{N-1} \]

we have

\[
\int_{\partial^* E_1^\alpha \cap \{u_n \geq 0\}} \varphi'(\{u_n\})(\{u_n\} - h) \\
\leq \int_{\partial^* E_1^\alpha \cap \{u_n \leq 0\}} \varphi(h) - \varphi(\{u_n\}) + \int_{\partial^* E_1^\alpha \setminus S_{u_n}} \varphi(h) + \frac{1}{2} \delta_2 r^{N-1} \\
\leq \frac{7}{4} \delta_1 r^{N-1}. \tag{2.33}
\]

Next consider

\[
\int_{\partial^* E_1^\alpha \cap \{u_n \geq 0\}} (\varphi(\{u_n\}) - \varphi(h)) \\
\leq \int_{\partial^* E_1^\alpha \cap \{u_n \leq 0\}} \frac{\varphi(\{u_n\})}{h}[u_n] - \varphi(h)) \\
\leq \int_{\partial^* E_1^\alpha \cap \{u_n \geq 0\}} \frac{\varphi(h)}{\varphi'(b)h} [u_n] - \varphi(h)) \\
\leq \frac{\varphi(h)}{\varphi'(b)h} \int_{\partial^* E_1^\alpha \cap \{u_n \geq 0\}} \varphi'(b)(\{u_n\} - h) \\
\leq \frac{\varphi(h)}{\varphi'(b)h} \frac{7}{4} \delta_1 r^{N-1} \\
\leq \frac{\varphi(h)}{\varphi'(b)h} \frac{7}{4} \epsilon r^{N-1}.
\]

The last estimate comes from (2.33). Thus the other direction has been proved. (2.15), (2.16), (2.17) and (2.18) should follow directly. \[\square\]

### 2.3.3 Proof of Minimality with \( \alpha \)

Define

\[ \alpha(x) = \begin{cases} 
D_\mu \mu_{\infty}(x) & x \in S_u \\
1 & \text{elsewhere} 
\end{cases} \]

**Lemma 13.** Let \( v \in SBV(\Omega') \). For \( \mathcal{H}^{N-1} \)-a.e. \( x \in S_u \) and \( \forall \epsilon > 0, \exists \delta \) and \( R \) s.t. \( \forall r < R, \exists \{v_n\}_{n=1}^\infty \subset SBV(\Omega') \) and \( \{s_n\}_{n=1}^\infty \subset (\delta r, 2\delta r) \) s.t.

\[
\lim_{n \to \infty} \int_{(S_{u_n} \cup S_{u_n}) \setminus R_{2\delta r}(x,r)} \tilde{\varphi}([v_n],[u_n]) \leq \int_{(S_u \cup S_v) \setminus R_{2\delta r}(x,r)} \alpha \tilde{\varphi}([v],[u]) + \epsilon r^{N-1}. \tag{2.34}
\]

\[45\]
Moreover
\[ \nabla v_u = 0 \quad \text{on } R_{s_n}^\infty (x, r) \]  
\[ v_n = v \quad \text{on } \Omega \setminus R_{s_n}^\infty (x, r) \]  
\[ \lim_{n \to \infty} \int_{\partial R_{s_n}^\infty (x, r)} \varphi([v_n]) dH^{N-1} \leq r^{N-1}. \]  
(2.37)

Proof. For \( \mathcal{H}^{N-1} \)-a.e. \( x \in S_u \)
\[ \alpha(x) = \lim_{r \to 0} \frac{\int_Q(x, r) \alpha \varphi([u])}{\int_Q(x, r) \varphi([u])} = \lim_{r \to 0} \frac{\int_Q(x, r) \alpha \varphi([u])}{\varphi([u](x))(2r)^{N-1}} = \lim_{r \to 0} \frac{\int_{R_{s_n}^\infty (x, r)} \alpha \varphi([u])}{\varphi([u](x))(2r)^{N-1}} \]  
(2.38)
and
\[ \alpha(x) \tilde{\varphi}([u](x), [u(x)]) = \lim_{r \to 0} \frac{\int_{R_{s_n}^\infty (x, r)} \alpha \tilde{\varphi}([u], [u]) d\mathcal{H}^{N-1}}{(2r)^{N-1}} \]  
(2.39)
for \( \forall \delta < \frac{1}{2} \).

Thus \( \forall \delta < \frac{1}{2} \) and \( \forall \epsilon > 0 \), \( \exists R \) s.t.
\[ \left| \alpha(x) - \lim_{n \to \infty} \int_{R_{s_n}^\infty (x, r)} \varphi([u_n]) \right| \leq \epsilon \]  
\[ \left| \alpha(x) - \lim_{n \to \infty} \int_{R_{s_n}^\infty (x, r)} \varphi([u_n]) \right| \leq \epsilon \]  
\[ \left| \alpha(x) \tilde{\varphi}([u(x), [u(x)]) - \int_{R_{s_n}^\infty (x, r)} \alpha \tilde{\varphi}([u], [u]) d\mathcal{H}^{N-1} \right| \leq \epsilon \] 
(2.40)
for all \( r < R \).

Then consider lemma (5), \( \forall \epsilon > 0 \exists \delta > 0 \) and \( R > 0 \) s.t. \( \forall < R \) there exists \( \{t_n\}_{n=1}^\infty \subset (u^-, u^+) \)
and \( \{s_n\}_{n=1}^\infty \subset (\delta r, 2\delta r) \) s.t.

\[
\lim_{n \to \infty} \left| \int_{S_{un} \cap R_{s_n}^{\infty} (x, r)} |\varphi([u_n]) - \varphi([u](x))| \, d\mathcal{H}^{-1} \right| \leq \varepsilon r^{N-1} \tag{2.41}
\]

\[
\lim_{n \to \infty} \int_{(S_{un} \setminus \partial^* E_{t_n}^n) \cap R_{s_n}^{\infty} (x, r)} |\varphi([u_n])| \, d\mathcal{H}^{-1} \leq \varepsilon r^{N-1} \tag{2.42}
\]

\[
\lim_{n \to \infty} \mathcal{H}^{-1}((\partial^* E_{t_n}^n \setminus S_{un}) \cap R_{s_n}^{\infty} (x, r)) \leq \varepsilon r^{N-1} \tag{2.43}
\]

\[
\lim_{n \to \infty} \mathcal{H}^{-1}(\partial R_{s_n}^{\infty} (x, r) \setminus (H(x, r, -s_n) \cup H(x, r, s_n))) \leq \varepsilon r^{N-1} \tag{2.44}
\]

\[
\lim_{n \to \infty} \left| \int_{S_{un} \cap R_{s_n}^{\infty} (x, r)} \varphi([u_n]) \, d\mathcal{H}^{-1} - \mathcal{H}^{-1}(\partial^* E_{t_n}^n \cap R_{s_n}^{\infty} (x, r)) \varphi([u](x)) \right| \leq \varepsilon r^{N-1}. \tag{2.45}
\]

\[
\lim_{n \to \infty} \left( \int_{H(x, r, -s_n)} \varphi(|T_v^- - v^+(x)|) + \int_{H(x, r, s_n)} \varphi(|T_v^- - v^-(x)|) \right) \leq \varepsilon r^{N-1}. \tag{2.46}
\]

We see (2.46) can be shown using the same argument we used in (2.26). Regarding lemma (11) we also have

\[
\lim_{n \to \infty} \int_{S_{un} \cap \{[u] \geq [u](x)\} \cap R_{s_n}^{\infty} (x, r)} \left( \varphi'([u](x)) - \varphi'( [u_n] ) \right) \leq \varepsilon r^{N-1}. \tag{2.47}
\]

Next consider

\[
\begin{align*}
\mathcal{H}^{-1}(\partial^* E_{t_n}^n \cap R_{s_n}^{\infty} (x, r)) &= \frac{\mathcal{H}^{-1}(\partial^* E_{t_n}^n \cap R_{s_n}^{\infty} (x, r))}{(2r)^{N-1}} - \alpha(x) \\
&= \frac{\mathcal{H}^{-1}(\partial^* E_{t_n}^n \cap R_{s_n}^{\infty} (x, r))}{(2r)^{N-1}} - \frac{\int_{R_{2s_n}^{\infty} (x, r)} \varphi([u_n])}{\varphi([u](x))(2r)^{N-1}} \\
&\quad + \frac{\int_{R_{2s_n}^{\infty} (x, r)} \varphi([u_n])}{\varphi([u](x))(2r)^{N-1}} - \alpha(x) \\
&\leq \frac{\mathcal{H}^{-1}(\partial^* E_{t_n}^n \cap R_{s_n}^{\infty} (x, r))}{(2r)^{N-1}} - \frac{\int_{R_{2s_n}^{\infty} (x, r)} \varphi([u_n])}{\varphi([u](x))(2r)^{N-1}} \\
&\quad + \frac{\int_{R_{2s_n}^{\infty} (x, r)} \varphi([u_n])}{\varphi([u](x))(2r)^{N-1}} - \alpha(x) \\
&\leq \frac{1}{\varphi([u](x))} 2^{N-1} \varepsilon + \varepsilon. \tag{2.48}
\end{align*}
\]

The last inequality comes respectively from (2.45) and (2.40).

Next construct SBV function \( v_n \) as follows

\[
\begin{align*}
v_n &= \begin{cases} 
v & \text{on } \Omega \setminus R_{s_n}^{\infty} (x, r) \\
v^+(x) & \text{on } R_{s_n}^{\infty} (x, r) \cap E_{t_n}^n \\
v^-(x) & \text{on } R_{s_n}^{\infty} (x, r) \setminus E_{t_n}^n \end{cases}
\end{align*}
\]
First we see, according to (2.46) and (2.44), that
\[
\int_{\partial R_{\epsilon_n}(x,r)} \phi([v_n]) \leq (\epsilon \varphi(b) + \epsilon) r^{N-1},
\]
which proves (2.37).

Second we see
\[
\int_{(S_{\epsilon_n} \cup S_{u_n}) \cap R_{\epsilon_n}(x,r)} \tilde{\phi}([v_n], [u_n]) \]
\[
= \int_{(\partial^* E_{\epsilon_n} \cup S_{u_n}) \cap R_{\epsilon_n}(x,r)} \phi([v^+(x) - v^-(x)], [u_n]) \]
\[
= \int_{(\partial^* E_{\epsilon_n} \cup S_{u_n}) \cap R_{\epsilon_n}(x,r)} \tilde{\phi}([v](x), [u_n]) \]
\[
= \int_{(\partial^* E_{\epsilon_n} \cup S_{u_n}) \cap R_{\epsilon_n}(x,r)} \varphi([v](x)) + \int_{(S_{\epsilon_n} \cap S_{u_n}) \cap R_{\epsilon_n}(x,r)} \tilde{\phi}(0, [u_n]) + \int_{(\partial^* E_{\epsilon_n} \cap S_{u_n}) \cap R_{\epsilon_n}(x,r)} \tilde{\phi}([v](x), [u_n]) \]
\[
\leq (\varphi(b) \epsilon + \epsilon) r^{N-1} + \int_{(\partial^* E_{\epsilon_n} \cap S_{u_n}) \cap R_{\epsilon_n}(x,r)} \tilde{\phi}([v](x), [u_n]).
\]

Denote \( A_n = (\partial^* E_{\epsilon_n} \cap S_{u_n}) \cap R_{\epsilon_n}(x,r) \), let's consider
\[
\Delta := \int_{A_n} \tilde{\phi}([v](x), [u_n]) - \int_{A_n} \tilde{\phi}([v](x), [u](x)) \]
\[
\leq \int_{A_n \cap \{[u_n] > [u](x)\}} \tilde{\phi}([v](x), [u_n]) - \int_{A_n \cap \{[u_n] > [u](x)\}} \tilde{\phi}([v](x), [u](x)).
\]

If \([v](x) \leq [u](x)\)
\[
\Delta \leq \int_{A_n \cap \{[u_n] > [u](x)\}} (\varphi([u_n]) - \varphi'([u_n])([u_n] - [v](x)) - \varphi([u](x)) + \varphi'([u](x))([u](x) - [v](x))) \]
\[
\leq \int_{A_n \cap \{[u_n] > [u](x)\}} (\varphi([u_n]) - \varphi([u](x))) + ([u](x) - [v](x)) \int_{A_n \cap \{[u_n] > [u](x)\}} [\varphi'([u](x)) - \varphi'([u_n])] \]
\[
\leq (\epsilon + b\epsilon) r^{N-1},
\]
the last inequality is due to (2.47).
If \([v](x) > [u](x)\),

\[
\Delta \leq \int_{A_n \cap \{[u_n] > [u](x)\}} \tilde{\varphi}([v](x), [u_n]) - \int_{A_n \cap \{[u_n] > [u](x)\}} \varphi([v](x))
\]

\[
\leq \int_{A_n \cap \{[u_n] > [v](x)\}} (\tilde{\varphi}([v](x), [u_n]) - \varphi([v](x)))
\]

\[
\leq \int_{A_n \cap \{[u_n] > [v](x)\}} (\varphi([u_n]) - \varphi([v](x)))
\]

\[
\leq \int_{A_n \cap \{[u_n] > [v](x)\}} (\varphi([u_n]) - \varphi([u](x)))
\]

\[
\leq \epsilon r^{N-1}.
\]

It follows

\[
\int_{(S_{v_n} \cup S_u) \cap \partial_n R_{\delta_n}} \tilde{\varphi}([v_n], [u_n])
\]

\[
- \int_{(\partial^* E_t) \cap (S_{v_n} \cup S_u) \cap \partial_n R_{\delta_n}} \tilde{\varphi}([v](x), [u](x))
\]

\[
\leq \int_{(\partial^* E_t) \cap (S_{v_n} \cup S_u) \cap \partial_n R_{\delta_n}} \tilde{\varphi}([v_n], [u_n])
\]

\[
- \int_{(\partial^* E_t) \cap (S_{v_n} \cup S_u) \cap \partial_n R_{\delta_n}} \tilde{\varphi}([v](x), [u](x))
\]

\[
+ (\varphi(b)\epsilon + \epsilon)r^{N-1}
\]

\[
\leq (2\epsilon + b\epsilon + \varphi(b)\epsilon + \epsilon)r^{N-1}
\]

\[
= (3\epsilon + b\epsilon + \varphi(b)\epsilon)r^{N-1}.
\]

It follows

\[
\int_{(S_{v_n} \cup S_u) \cap \partial_n R_{\delta_n}} \tilde{\varphi}([v_n], [u_n]) - \int_{(S_u \cup S_v) \cap \partial_n R_{\delta_n}} \alpha\tilde{\varphi}([v], [u])
\]

\[
= \int_{(S_{v_n} \cup S_u) \cap \partial_n R_{\delta_n}} \tilde{\varphi}([v_n], [u_n]) - \mathcal{H}^{N-1}(\partial^* E_t \cap \partial_n R_{\delta_n})\tilde{\varphi}([v](x), [u](x))
\]

\[
+ \mathcal{H}^{N-1}(\partial^* E_t \cap \partial_n R_{\delta_n})\tilde{\varphi}([v](x), [u](x)) - \alpha(x)\tilde{\varphi}([v](x), [u](x)) (2r)^{N-1}
\]

\[
\leq (3\epsilon + b\epsilon + \varphi(b)\epsilon)r^{N-1} + \left[ \frac{\tilde{\varphi}([v](x), [u](x))}{\varphi([u](x))} \epsilon + \epsilon \tilde{\varphi}([v](x), [u](x)) 2^{N-1} \right] r^{N-1}
\]

\[
\leq O(\epsilon)r^{N-1}
\]

regarding (2.49), (2.48) and (2.40). Thus we have proved the main result. (2.35) and (2.36) are due to construction of \(v_n\). \(\square\)
Once lemma (13) has been proved, it is straightforward to show the following result using Besicovitch’s covering theorem.

**Theorem 6.**

\[
\int_{\Omega'} |\nabla u|^2 dx + \int_{\Omega} \alpha \varphi([u]) d\mathcal{H}^{N-1} \leq \int_{\Omega'} |\nabla v|^2 dx + \int_{\Omega} \alpha \tilde{\varphi}([v], [u]) d\mathcal{H}^{N-1}
\]

for all \( v \in SBV(\Omega) \) with \( v = u \) on \( \Omega' \setminus \Omega \).

**Proof.** Recall that \( \mu_n := \varphi([u_n]) |S_{u_n}| \rightharpoonup \mu_\infty \), the weak limit is absolutely continuous with respect to \( \mathcal{H}^{N-1} |S_u \). Let \( \mu_\infty = \xi |S_u \) where \( \xi \) is \( \mathcal{H}^{N-1} \) measurable on \( S_u \). Then define Radon measure

\[
w := \alpha \tilde{\varphi}([v], [u]) |S_u| + \xi |S_u|.
\]

Fix \( \epsilon > 0 \), let \( U \) be open s.t. \( S_u \subset U \) and \( |U| < \epsilon \). Then consider the collection of closed cubes \( \mathcal{F} : \{ Q(x, r) : Q(x, r) \) satisfies the following \}

1. \( x \in S_u \).
2. \( Q(x, r) \) is oriented by \( \nu(x) \).
3. \( Q(x, r) \subset U \).
4. \( \int_{Q(x, r) \setminus R^{\delta, \epsilon}(x, r)} \varphi\big([v]\big) \leq \epsilon \varphi\big([u](x)\big) r^{N-1} \).
5. \( \int_{Q(x, r) \setminus R^{\delta, \epsilon}(x, r)} \xi \leq \epsilon \varphi\big([u](x)\big) r^{N-1} \).

and

\[
\lim_{n \to \infty} \int_{(S_{v_n} \cup S_{u_n}) \setminus R_{\epsilon, \epsilon}(x, r)} \tilde{\varphi}\big([v_n]\big, [u_n]\big) \\
\leq \int_{(S_u \cup S_v) \setminus R_{2\epsilon, 2\epsilon}(x, r)} \alpha \tilde{\varphi}([v], [u]) + \epsilon \varphi([u](x)) r^{N-1}.
\]

(2.50)

Here \( v_n, s_n \) and \( \delta \) are the same as in lemma (13). Then according to Bescovitch covering theorem, we can find a countable family of disjoint closed cubes \( \bigcup_{i=1}^{\infty} Q(x_i, r_i) \subset \mathcal{F} \) s.t. \( w(S_u \setminus \bigcup_{i=1}^{\infty} Q(x_i, r_i)) = 0 \). Pick an integer \( J \) s.t.

\[
w(S_u \setminus \bigcup_{i=1}^{J} Q(x_i, r_i)) < \epsilon.
\]

(2.51)

Since \( Q(x_i, r_i) \) are disjoint, let’s define

\[
\tilde{v}_n = \begin{cases} \\
v_n & \text{on each } Q(x_i, r_i) \\
v & \text{on } \Omega' \setminus \bigcup_{i=1}^{J} Q(x_i, r_i).
\end{cases}
\]
During the following we let \( n \) be big enough. It follows
\[
\int_{\Omega'} |\nabla \tilde{v}_n|^2 \, dx + \int_{S_{\partial \Omega} \cup S_{\partial u}} \tilde{\varphi}([\tilde{v}_n], [u_n]) - \int_{\Omega'} |\nabla v|^2 \, dx - \int_{S_{\partial \Omega} \cup S_u} \alpha \tilde{\varphi}([v], [u]) \\
\leq \int_{S_{\partial \Omega} \cup S_{\partial u}} \tilde{\varphi}([\tilde{v}_n], [u_n]) - \int_{S_{\partial \Omega} \cup S_u} \alpha \tilde{\varphi}([v], [u]) \\
\leq \int_{S_{\partial \Omega} \cup S_{\partial u} \cap (\Omega' \setminus \bigcup_i Q(x_i, r_i))} \tilde{\varphi}([v], [u_n]) - \int_{S_{\partial \Omega} \cup S_u \cap (\Omega' \setminus \bigcup_i Q(x_i, r_i))} \alpha \tilde{\varphi}([v], [u]) \\
+ \int_{S_{\partial \Omega} \cup S_{\partial u} \cap \bigcup_i Q(x_i, r_i)} \tilde{\varphi}([\tilde{v}_n], [u_n]) - \int_{S_{\partial \Omega} \cup S_u \cap \bigcup_i Q(x_i, r_i)} \alpha \tilde{\varphi}([v], [u]).
\]

We see
\[
\int_{S_{\partial \Omega} \cup S_{\partial u} \cap (\Omega' \setminus \bigcup_i Q(x_i, r_i))} \tilde{\varphi}([v], [u_n]) - \int_{S_{\partial \Omega} \cup S_u \cap (\Omega' \setminus \bigcup_i Q(x_i, r_i))} \alpha \tilde{\varphi}([v], [u]) \\
\leq \int_{S_{\partial \Omega} \cap (\Omega' \setminus \bigcup_i Q(x_i, r_i))} \varphi([v]) + \int_{S_{\partial \Omega} \cap (\Omega' \setminus \bigcup_i Q(x_i, r_i))} \varphi([u_n]) \\
- \int_{S_{\partial \Omega} \cup S_u \cap (\Omega' \setminus \bigcup_i Q(x_i, r_i))} \alpha \tilde{\varphi}([v], [u]) \\
\leq \int_{S_{\partial \Omega} \cup S_{\partial u} \cap (\Omega' \setminus \bigcup_i Q(x_i, r_i))} \varphi([u_n]).
\]

Due to (2.51), \( \lim_{n \to \infty} \int_{S_{\partial \Omega} \cap (\Omega' \setminus \bigcup_i Q(x_i, r_i))} \varphi([u_n]) \leq \epsilon. \)

Then we see
\[
\int_{S_{\partial \Omega} \cup S_{\partial u} \cap \bigcup_i Q(x_i, r_i)} \tilde{\varphi}([\tilde{v}_n], [u_n]) - \int_{S_{\partial \Omega} \cup S_u \cap \bigcup_i Q(x_i, r_i)} \alpha \tilde{\varphi}([v], [u]) \\
= \int_{\bigcup_i Q(x_i, r_i) \setminus \bigcup_j R_{-\delta r_i}^h} \tilde{\varphi}([v], [u_n]) - \int_{\bigcup_i Q(x_i, r_i) \setminus \bigcup_j R_{-\delta r_i}^{2\delta r_i}} \alpha \tilde{\varphi}([v], [u]) \\
+ \int_{\bigcup_j R_{-\delta r_i}^h} \tilde{\varphi}([\tilde{v}_n], [u_n]) - \int_{\bigcup_j R_{-\delta r_i}^{2\delta r_i}} \alpha \tilde{\varphi}([v], [u]).
\]

It follows
\[
\int_{\bigcup_j R_{-\delta r_i}^h} \tilde{\varphi}([\tilde{v}_n], [u_n]) - \int_{\bigcup_j R_{-\delta r_i}^{2\delta r_i}} \alpha \tilde{\varphi}([v], [u]) \\
= \sum_{i=1}^J \int_{R_{-\delta r_i}^h} \tilde{\varphi}([v_i^h], [u_n]) - \sum_{i=1}^J \int_{R_{-\delta r_i}^{2\delta r_i}} \alpha \tilde{\varphi}([v], [u]) \\
\leq \epsilon \sum_{i=1}^J \varphi([u](x_i)) r_i^{N-1}.
\]

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The inequality is due to (2.50). And

\[
\begin{align*}
\int_{\bigcup_i Q(x_i,r_i) \setminus R_{-s_h}^i} \varphi([v], [u]) &\leq \int_{\bigcup_i Q(x_i,r_i) \setminus R_{-s_h}^i} \varphi([v]) - \int_{\bigcup_i Q(x_i,r_i) \setminus R_{-2s_h}^i} \alpha\varphi([v], [u]) \\
&\leq \int_{\bigcup_i Q(x_i,r_i) \setminus R_{-s_h}^i} \varphi([v]) + \int_{\bigcup_i Q(x_i,r_i) \setminus R_{-s_h}^i} \varphi([u]) \\
&\leq \int_{\bigcup_i Q(x_i,r_i) \setminus R_{-s_h}^i} \varphi([v]) + \int_{\bigcup_i Q(x_i,r_i) \setminus R_{-s_h}^i} \varphi([u]) \\
&\leq \sum_{j=1}^{J} \int_{Q(x_j,r_j) \setminus R_{-s_h}^{x_j}} \varphi([v]) + \sum_{j=1}^{J} \int_{Q(x_j,r_j) \setminus R_{-s_h}^{x_j}} \varphi([u]) \\
&\leq \epsilon \sum_{i=1}^{J} \varphi([u](x_i)) r_i^{N-1} + \epsilon \sum_{i=1}^{J} \varphi([u](x_i)) r_i^{N-1}.
\end{align*}
\]

The last inequality is due to condition (4) and (5) of the chosen cubes.

To sum up all the estimates, we see

\[
\begin{align*}
\int_{\Omega'} |\nabla \tilde{v}_n|^2 dx + \int_{S_{\tilde{v}_n} \cup S_u} \varphi([\tilde{v}_n], [u]) &\leq \int_{\Omega'} \nabla v|^2 dx - \int_{S_{\tilde{v}} \cup S_u} \alpha\varphi([v], [u]) \\
&\leq \epsilon + 3\epsilon \sum_{i=1}^{J} \varphi([u](x_i)) r_i^{N-1} \\
&\leq \mathcal{O}(\epsilon).
\end{align*}
\]

We see \( \sum_{i=1}^{J} \varphi([u](x_i)) r_i^{N-1} \to \int_{S_u} \varphi([u]) d\mathcal{H}^{N-1} \) as we choose finer cover of \( S_u \).

According to lower-semi-continuity and unilateral minimality

\[
\begin{align*}
\int_{\Omega'} |\nabla u|^2 dx + \int_{S_u} \alpha\varphi([u]) d\mathcal{H}^{N-1} &\leq \liminf_{n \to \infty} \left[ \int_{\Omega'} |\nabla u_n|^2 dx + \int_{S_{u_n}} \varphi([u_n]) d\mathcal{H}^{N-1} \right] \\
&\leq \liminf_{n \to \infty} \left[ \int_{\Omega'} |\nabla \tilde{v}_n|^2 dx + \int_{S_{\tilde{v}_n} \cup S_{u_n}} \varphi([\tilde{v}_n], [u_n]) d\mathcal{H}^{N-1} \right] \\
&\leq \int_{\Omega'} |\nabla u|^2 dx + \int_{S_u \cup S_v} \alpha\varphi([v], [u]) d\mathcal{H}^{N-1} + \mathcal{O}(\epsilon).
\end{align*}
\]
This concludes our main result of the section. We show that the crack energy of $u_n$ concentrates on reduced boundary of some level sets. And due to this condition, the density function $\alpha$ only depends on the oscillation, if there’s any. Moreover we can show that the density $\alpha$ can be passed to the minimality independent of the choose of test function $v$. 
Chapter 3

The Evolution Problem

3.1 Convergence of the Energy

Just by looking at a sequence of unilateral minimizers is not enough to exclude the oscillation complication. We then move on to the evolution problem because the history after all is union of all cracks from the previous minimizers at the discrete time. The problem is as approximating steps becomes more and more the minimizers become hard to control. So we hope to find finitely many minimizers that could represent the whole evolution to make our analysis easier.

Follow the time discretization procedure described in section §1.1, we see $\gamma_i^n$ is defined on $\Gamma_i^n$, and the existence of minimizers in each step can be derived from $SBV$ compactness. Then define $u_n(t) = u_n^i$ for $t^n_i \leq t < t^{i+1}_n$ and for $\forall 0 \leq i < 2^n$. We see $u_n(t)$ is well defined on $[0, 1]$ for $\forall n > 0$.

Define

$$E_n(t) := \int_{\Omega} |\nabla u_n(t)|^2 dx + \int_{S_{u_n(t)} \cup \Gamma_n(t)} \tilde{\varphi}([u_n(t)], \gamma_n(t)) d\mathcal{H}^{N-1}$$

At time $t^{i+1}_n$, pick $v = u_n(t^n_i) + g(t^{i+1}_n) - g(t^n_i)$ as a test function, we get

$$E_n(t^{i+1}_n) \leq \int_{\Omega} |\nabla u_n(t^{i+1}_n) - \nabla g(t^{i+1}_n)|^2 dx + \int_{S_{u_n(t^n_i)} \cup \Gamma_n(t^n_i)} \tilde{\varphi}([u_n(t^n_i)], \gamma_n(t^n_i)) d\mathcal{H}^{N-1}$$

$$\leq E_n(t^n_i) + \int_{\Omega} |\nabla g(t^{i+1}_n) - \nabla g(t^n_i)|^2 dx + 2 \int_{\Omega} \nabla u_n(t^n_i) \cdot (\nabla g(t^{i+1}_n) - \nabla g(t^n_i)) dx$$

$$\leq E_n(t^n_i) + \int_{\Omega} \left( \int_{t^n_i}^{t^{i+1}_n} |\nabla g(s)| ds \right)^2 dx + 2 \int_{\Omega} \nabla u_n(t^n_i) \cdot \left( \int_{t^n_i}^{t^{i+1}_n} \nabla g(s) ds \right) dx$$

$$\leq E_n(t^n_i) + \Delta t_n \int_{t^n_i}^{t^{i+1}_n} |\nabla g(s)|^2 ds \frac{1}{2} (\Delta t_n)^{\frac{1}{2}} \left( \int_{t^n_i}^{t^{i+1}_n} \nabla u_n(s) \cdot \nabla g(s) ds \right) dx$$

$$\leq E_n(t^n_i) + \Delta t_n \int_{t^n_i}^{t^{i+1}_n} |\nabla g(s)|^2 dx + 2 \int_{t^n_i}^{t^{i+1}_n} \nabla u_n(s) \cdot \nabla g(s) dx ds$$
Then sum from $i$ to $j$ where $j > i$ we get

$$E_n(t_n^i) \leq E_n(t_n^j) + \Delta t_n \int_{t_n^i}^{t_n^j} \int_\Omega |\nabla \tilde{g}(s)|^2 dx ds + 2 \int_{t_n^i}^{t_n^j} \int_\Omega \nabla u_n(s) \cdot \nabla \tilde{g}(s) dx ds$$

On the other hand for $t = t_n^i$, take $v = u_n^{i+1} + g(t_n^i) - g(t_n^{i+1})$ as test function to get

$$E_n(t_n^i) \leq E_n(t_n^{i+1}) + \int_\Omega |\nabla g(t_n^i) - \nabla g(t_n^{i+1})|^2 dx + 2 \int_\Omega \nabla u_n(t_n^{i+1}) \cdot (\nabla g(t_n^i) - \nabla g(t_n^{i+1})) dx$$

$$\leq E_n(t_n^{i+1}) + \Delta t_n \int_{t_n^i}^{t_n^{i+1}} \int_\Omega |\nabla \tilde{g}(s)|^2 dx ds - 2 \int_{t_n^i}^{t_n^{i+1}} \int_\Omega \nabla u_n(s + \Delta t_n) \cdot \nabla \tilde{g}(s) dx ds$$

So sum from $i$ to $j$ to get

$$E_n(t_n^i) \leq E_n(t_n^j) + \Delta t_n \int_{t_n^i}^{t_n^j} \int_\Omega |\nabla \tilde{g}(s)|^2 dx ds - 2 \int_{t_n^i}^{t_n^j} \int_\Omega \nabla u_n(s + \Delta t_n) \cdot \nabla \tilde{g}(s) dx ds$$

It follows

$$E_n(t_n^i) - \Delta t_n \int_{t_n^i}^{t_n^j} \int_\Omega |\nabla \tilde{g}(s)|^2 dx ds + 2 \int_{t_n^i}^{t_n^j} \int_\Omega \nabla u_n(s + \Delta t_n) \cdot \nabla \tilde{g}(s) dx ds \leq E_n(t_n^j) \leq E_n(t_n^i) + \Delta t_n \int_{t_n^i}^{t_n^j} \int_\Omega |\nabla \tilde{g}(s)|^2 dx ds + 2 \int_{t_n^i}^{t_n^j} \int_\Omega \nabla u_n(s) \cdot \nabla \tilde{g}(s) dx ds$$

Set $i = 0$ and let $t \in I_\infty$, we see for sufficiently large $n$, $t = t_n^j$ for some $n$ and $j$. Then we have

$$E(0) - \Delta t_n \int_0^t \int_\Omega |\nabla \tilde{g}(s)|^2 dx ds + 2 \int_0^t \int_\Omega \nabla u_n(s + \Delta t_n) \cdot \nabla \tilde{g}(s) dx ds \leq E_n(t) \leq E(0) + \Delta t_n \int_0^t \int_\Omega |\nabla \tilde{g}(s)|^2 dx ds + 2 \int_0^t \int_\Omega \nabla u_n(s) \cdot \nabla \tilde{g}(s) dx ds$$

(3.1)

Now we have bounds for the sequence $E_n(t)$, next is to show the bounds converge. The following results are pretty straight forward.

**Lemma 14.**

$$\sup_{t,n} \int_\Omega |\nabla u_n(t)|^2 dx < \infty$$

$$\sup_{t,n} E_n(t) < \infty$$

$$\sup_{t,n} \int_{\Gamma_n(t)} \tilde{\varphi}(0, \gamma_n(t)) d\mathcal{H}^{N-1} < \infty$$

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Proof. At each time $t$, pick $g(t)$ as test function to get
\[
\int_{\Omega} |\nabla u_n(t)|^2 dx + \int_{S_{u_n(t)} \cup \Gamma_n(t)} \tilde{\varphi}(u_n(t), \gamma_n(t)) d\mathcal{H}^{N-1} \\
\leq \int_{\Omega} |\nabla g(t)|^2 dx + \int_{S_{u_n(t)} \cup \Gamma_n(t)} \tilde{\varphi}(0, \gamma_n(t)) d\mathcal{H}^{N-1}
\]

Since
\[
\int_{S_{u_n(t)} \cup \Gamma_n(t)} \tilde{\varphi}(u_n(t), \gamma_n(t)) d\mathcal{H}^{N-1} \geq \int_{S_{u_n(t)} \cup \Gamma_n(t)} \tilde{\varphi}(0, \gamma_n(t)) d\mathcal{H}^{N-1}
\]
we get $\int_{\Omega} |\nabla u_n(t)|^2 dx \leq \int_{\Omega} |\nabla g(t)|^2 dx$, due to definition of $g(t)$ the first result is shown.

Considering (3.1),
\[
E_n(t) \leq E(0) + \Delta t_n \int_0^t \int_{\Omega} |\nabla \tilde{g}(s)|^2 dx ds + 2 \int_0^t \int_{\Omega} \nabla u_n(s) \cdot \nabla \tilde{g}(s) dx ds
\]
Due to Cauchy-Schwartz inequality and uniform boundedness of $\|\nabla \tilde{g}(t)\|_{L^2}$, the second result is proved.

The third result is straight forward. □

Let $I_n := \bigcup_{i=0}^n t_i$ and $I_\infty := \bigcup_{n=1}^\infty I_n$, we see $I_\infty$ is a dense and countable subset of $[0,1]$. For $a_1 \in I_\infty$ we have $\sup_n E_n(a_1) < \infty$, by SBV compactness we can extract a subsequence \(\{u_n(a_1)\}_{n=1}^\infty\) (not relabeled) and $u(a_1) \in SBV(\Omega)$ s.t. $u_n(a_1) \xrightarrow{SBV} u(a_1)$. Then apply the diagonal argument we can extract a subsequence \(\{u_n(t)\}_{n=1}^\infty\) (not relabeled) s.t.
\[
u_n(t) \xrightarrow{SBV} u(t) \quad \forall \ t \in I_\infty
\]
We see $u(t)$ is well defined on the dense and countable subset $I_\infty$.

Define $\gamma_n(t) := \bigvee_{\tau \leq t} [u_n(\tau)]$ and $\Gamma_n(t) := \bigcup_{\tau \leq t} S_{u_n(\tau)}$. We have at each $n$ the global minimality
\[
\int_{\Omega} |\nabla u_n(t)|^2 dx + \int_{S_{u_n(t)} \cup \Gamma_n(t)} \tilde{\varphi}(u_n(t), \gamma_n(t)) d\mathcal{H}^{N-1} \leq \int_{\Omega} |\nabla v|^2 dx + \int_{S_v \cup \Gamma(t)} \tilde{\varphi}(v, \gamma_n(t)) d\mathcal{H}^{N-1}
\]
for $\forall v = g(t)$ on $\partial \Omega$, for all $t = t_i(0 \leq i \leq 2^n)$. Define $\gamma(t) := \bigvee_{\tau \leq t} [u(\tau)]$ and $\Gamma(t) := \bigcup_{\tau \leq t} S_{u(\tau)}$.

Our first question is if it’s true $\forall t \in I_\infty$
\[
\int_{\Omega} |\nabla u(t)|^2 dx + \int_{S_u(t) \cup \Gamma(t)} \tilde{\varphi}(u(t), \gamma(t)) d\mathcal{H}^{N-1} \leq \int_{\Omega} |\nabla v|^2 dx + \int_{S_v \cup \Gamma(t)} \tilde{\varphi}(v, \gamma(t)) d\mathcal{H}^{N-1}
\]
$\forall v \in SBV(\Omega)$ s.t. $v = g(t)$ on $\partial \Omega$. If the above minimality could be proved, the next step would be to extend the minimality to the whole time interval $[0,1]$ and to show the global stability in quasi-static evolution. However, due to possible complications in the sequence of minimizers, it is
Lemma 15. If $t < 0$, we have $\forall \epsilon > 0$, there $\exists \Delta t > 0$ and $N \in \mathbb{N}$ s.t.

$$\| \nabla u_n(t_1) - \nabla u_n(t_2) \|_2 < \epsilon$$

$\forall t_1, t_2$ s.t. $t - \Delta t < t_1, t_2 < t + \Delta t$ and $\forall n > N$.

Proof. Let $\epsilon > 0$, first we see $\exists \Delta t > 0$ s.t. $m_\infty(t + \Delta t) - m_\infty(t - \Delta t) < \epsilon$. Then let $N \in \mathbb{N}$ be big s.t.

$$|m_\infty(t + \Delta t) - m_n(t + \Delta t)| < \epsilon$$

$$|m_\infty(t - \Delta t) - m_n(t - \Delta t)| < \epsilon$$

$\forall n > N$.

Let $t_1, t_2$ be s.t. $t - \Delta t < t_1 < t_2 < t + \Delta t$, we have $|m_n(t_2) - m_n(t_1)| < 3\epsilon$, $\forall n > N$, or equivalently

$$\int \tilde{\phi}(0, \gamma_n(t_2))d\mathcal{H}^{N-1} - \int \tilde{\phi}(0, \gamma_n(t_1))d\mathcal{H}^{N-1} < 3\epsilon$$

Next consider test function

$$u = \frac{1}{2}u_n(t_1) + \frac{1}{2}u_n(t_2) - \frac{1}{2}g(t_1) - \frac{1}{2}g(t_2) + g(t_1)$$

We see $u = g(t_1)$ on $\partial \Omega$ and according to minimality in discrete time we have

$$\int_\Omega |\nabla u_n(t_1)|^2 dx + \int \tilde{\phi}([u_n(t_1)], \gamma_n(t_1))d\mathcal{H}^{N-1}$$

$$\leq \int_\Omega |\nabla u|^2 dx + \int \tilde{\phi}([u], \gamma_n(t_1))d\mathcal{H}^{N-1}$$

$$\leq \int_\Omega |\nabla u|^2 dx + \int \tilde{\phi}([u], \gamma_n(t_2))d\mathcal{H}^{N-1}$$

$$\leq \int_\Omega \frac{1}{2}|\nabla u_n(t_1) + \frac{1}{2}\nabla u_n(t_2)|^2 dx + \frac{1}{2} \int \tilde{\phi}([u_n(t_1)], \gamma_n(t_2))d\mathcal{H}^{N-1}$$

$$+ \frac{1}{2} \int \tilde{\phi}([u_n(t_2)], \gamma_n(t_2))d\mathcal{H}^{N-1} + O(|t_2 - t_1|)$$

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Similarly we can deduce

\[
\int_{\Omega} |\nabla u_n(t_2)|^2 dx + \int_{\Omega} \tilde{\varphi}([u_n(t_2)], \gamma_n(t_2)) d\mathcal{H}^{N-1}
\leq \int_{\Omega} \frac{1}{2} |\nabla u_n(t_1)|^2 dx + \frac{1}{2} \int_{\Omega} \tilde{\varphi}([u_n(t_1)], \gamma_n(t_2)) d\mathcal{H}^{N-1}
\]

Summing the above two inequalities we get

\[
\int_{\Omega} |\nabla u_n(t_1)|^2 dx + \int_{\Omega} |\nabla u_n(t_2)|^2 dx + \int_{\Omega} \tilde{\varphi}([u_n(t_1)], \gamma_n(t_1)) d\mathcal{H}^{N-1}
\]

It follows

\[
\frac{1}{2} \int_{\Omega} |\nabla u_n(t_2) - \nabla u_n(t_1)|^2 dx
\]=\int_{\Omega} |\nabla u_n(t_1)|^2 dx + \int_{\Omega} |\nabla u_n(t_2)|^2 dx - 2 \int_{\Omega} \frac{1}{2} \nabla u_n(t_1) + \frac{1}{2} \nabla u_n(t_2)|^2 dx
\]

\[
\leq \int_{\Omega} \tilde{\varphi}([u_n(t_1)], \gamma_n(t_2)) d\mathcal{H}^{N-1} - \int_{\Omega} \tilde{\varphi}([u_n(t_1)], \gamma_n(t_1)) d\mathcal{H}^{N-1} + O(|t_2 - t_1|)
\]

This concludes the proof.

For \( t \in [0, 1] \), let’s define

\[
\theta_n(t) = \int_{\Omega} \nabla u_n(t) \cdot \nabla \tilde{g}(t) dx
\]

and the first obvious result we can see is

\[
\sup_{n,t} \theta_n(t) < \infty.
\]

According to weak convergence in \( L^2 \), \( \theta_n(t) \to \int_{\Omega} \nabla u(t) \cdot \nabla \tilde{g}(t) dx \) for \( \forall t \in I_\infty \). Then we show the sequence \( \theta_n(t) \) is Cauchy a.e. on \([0, 1] \).

**Lemma 16.** For \( \forall t \in [0, 1] \setminus D \), \( \{\theta_n(t)\}_{n=1}^\infty \) is Cauchy.
Proof. Let $\epsilon > 0$, according to lemma (15) we have there exists $\Delta t > 0$ and $N$ s.t.
\[
\|\nabla u_n(t_1) - \nabla u_n(t_2)\|_2 < \epsilon
\]
for $\forall t_1, t_2 \in [t - \Delta t, t + \Delta t]$ and $\forall n > N$. Let $\tau \in [t - \Delta t, t + \Delta t] \cap I_\infty$, we can find $N_1 > N$ s.t.
\[
|\theta_k(\tau) - \theta_l(\tau)| < \epsilon
\]
for $\forall k, l > N_1$. According to the definition of $\nabla u_n(t)$ on $[0, 1]$, $\nabla u_n(t) = \nabla u_n(t_{n}^{i(n,t)})$ where $i(n,t)$, depending on $n$ and $t$, is the largest integer s.t. $t_{n}^{i(n,t)} < t$. Then let $N_2 > N_1$ be s.t. $t_{n}^{i(n,t)} > t - \Delta t$ for $\forall n > N_2$. It follows
\[
|\theta_k(t) - \theta_l(t)| = \left| \int_{\Omega} \nabla u_k(t_{k}^{i(k,t)}) \cdot \nabla \hat{g}(t) dx - \int_{\Omega} \nabla u_l(t_{l}^{i(l,t)}) \cdot \nabla \hat{g}(t) dx \right|
\]
\[
\leq \left| \int_{\Omega} \nabla u_k(t_{k}^{i(k,t)}) \cdot \nabla \hat{g}(t) dx - \int_{\Omega} \nabla u_k(t_{k}^{i(k,t)}) \cdot \nabla \hat{g}(\tau) dx \right|
\]
\[
+ \left| \int_{\Omega} \nabla u_k(t_{k}^{i(k,t)}) \cdot \nabla \hat{g}(\tau) dx - \int_{\Omega} \nabla u_k(\tau) \cdot \nabla \hat{g}(\tau) dx \right|
\]
\[
+ \left| \int_{\Omega} \nabla u_k(\tau) \cdot \nabla \hat{g}(\tau) dx - \int_{\Omega} \nabla u_l(t_{l}^{i(l,t)}) \cdot \nabla \hat{g}(\tau) dx \right|
\]
\[
+ \left| \int_{\Omega} \nabla u_l(t_{l}^{i(l,t)}) \cdot \nabla \hat{g}(\tau) dx - \int_{\Omega} \nabla u_l(t_{l}^{i(l,t)}) \cdot \nabla \hat{g}(t) dx \right|
\]
\[
\leq O(|t - \tau|) + O(\epsilon) + \epsilon + O(\epsilon) + O(|\tau - t|)
\]
for $\forall k, l > N_2$. This concludes the proof.

Remark 6. We see there exists $\theta(t) : [0, 1] \setminus D \to \mathbb{R}$ s.t. $\theta_n(t) \to \theta(t)$ for all $t \in [0, 1] \setminus D$. Since $D$ is at most countable and apply the compactness in $\mathbb{R}$ and diagonal argument we can find a subsequence (not labeled) and $\theta(t)$ s.t. $\theta_n(t) \to \theta(t)$ for $\forall t \in [0, 1]$. From there we can show the convergence of $E_n(t)$.

Lemma 17. Let $t \in I_\infty$, the following is true
\[
\int_{0}^{t} \int_{\Omega} \nabla u_n(s + \Delta t_n) \cdot \nabla \hat{g}(s) dx ds \to \int_{0}^{t} \theta(s) ds
\]

Proof. It suffices to show
\[
\left| \int_{0}^{t} \int_{\Omega} \nabla u_n(s + \Delta t_n) \cdot \nabla \hat{g}(s) dx ds - \int_{0}^{t} \int_{\Omega} \nabla u_n(s) \cdot \nabla \hat{g}(s) dx ds \right| \to 0
\]
First of all we see
\[
\left| \int_0^t \int_\Omega \nabla u_n(s + \Delta t_n) \cdot \nabla \dot{g}(s + \Delta t_n) dx ds - \int_0^t \int_\Omega \nabla u_n(s) \cdot \nabla \dot{g}(s) dx ds \right| \\
= \left| \int_{\Delta t_n}^{t + \Delta t_n} \int_\Omega \nabla u_n(s) \cdot \nabla \dot{g}(s) dx ds - \int_0^t \int_\Omega \nabla u_n(s) \cdot \nabla \dot{g}(s) dx ds \right| \\
\leq \int_0^{\Delta t_n} \theta_n(s) ds + \int_t^{t + \Delta t_n} \theta_n(s) ds \\
\leq \int_0^{\Delta t_n} |\theta_n(s)| ds + \int_t^{t + \Delta t_n} |\theta_n(s)| ds \to 0 \text{ as } n \to \infty
\]

The convergence is due to uniform boundedness of \( \theta_n(t) \). Then consider
\[
\left| \int_0^t \int_\Omega \nabla u_n(s + \Delta t_n) \cdot \nabla \dot{g}(s + \Delta t_n) dx ds - \int_0^t \int_\Omega \nabla u_n(s + \Delta t_n) \cdot \nabla \dot{g}(s) dx ds \right| \\
\leq \int_0^t \left( \int_\Omega |\nabla u_n(s + \Delta t_n)|^2 dx \right)^{\frac{1}{2}} \left( \int_\Omega |\nabla \dot{g}(s + \Delta t_n) - \nabla \dot{g}(s)|^2 dx \right)^{\frac{1}{2}} ds
\]

The above estimate goes to 0 as \( n \to \infty \) because \( \left( \int_\Omega |\nabla u_n(s + \Delta t_n)|^2 dx \right)^{\frac{1}{2}} \) is bounded over \( s \) and \( \left( \int_\Omega |\nabla \dot{g}(s + \Delta t_n) - \nabla \dot{g}(s)|^2 dx \right)^{\frac{1}{2}} \) goes to 0 for \( \forall s \), apply again D.C.T. Thus the lemma has been proved.

\textbf{Remark 7.} Thus we have \( E_n(t) \to E(0) + 2 \int_0^t \theta(s) ds, \forall t \in [0, 1] \).

### 3.2 Little o Method and Its Applications

The idea here is to see if we can find finitely many fixed times that can roughly represent the whole process. The convergence of energy shows that maybe we have a way.

Fix \( t \), let’s look at the energy at a small time step \( \Delta t \) further. Let \( \Delta t > \Delta t_n \), we have

\[
E_n(t + \Delta t) \leq E_n(t) + \Delta t_n \int_t^{t + \Delta t} \int_\Omega |\nabla \dot{g}(s)|^2 dx ds + 2 \int_t^{t + \Delta t} \int_\Omega \nabla u_n(s) \cdot \nabla \dot{g}(s) dx ds
\]

Then let \( v(t + \Delta t) \) be any SBV function s.t. \( v(t + \Delta t) = g(t + \Delta t) \) on \( \partial \Omega \), we have

\[
E_n(t) \leq \int_\Omega |\nabla v(t + \Delta t) - \nabla g(t + \Delta t) + \nabla g(t)|^2 dx + \int \check{\varphi}([v(t + \Delta t)], \gamma_n(t)) \\
\leq \int_\Omega |\nabla v(t + \Delta t)|^2 dx + \int \check{\varphi}([v(t + \Delta t)], \gamma_n(t)) + \Delta t \int_t^{t + \Delta t} \int_\Omega |\nabla \dot{g}(s)|^2 dx ds \\
- 2 \int_t^{t + \Delta t} \int_\Omega (\nabla v(t + \Delta t)) \cdot \nabla \dot{g}(s) dx ds
\]
Let $3.2.1$ Finitely many minimizers

Let $p \in \mathbb{N}$, split time interval $[0, 1]$ evenly into $2^p$ pieces and we have $2^p + 1$ times. To choose $2^p$ because after we fix $p$ those finitely chosen times will be a subset of $I_n$ as $n > p$. During the following we always assume $n > p$. It follows, from above estimate, that

$$\int_{S_n(t_i^{i+1}) \cup \Gamma_n(t_i^{i+1})} \varphi([u_n(t_i^{i+1})], \gamma_n(t_i^{i+1})) dH^{N-1} - \int_{S_n(t_i^{i+1}) \cup \Gamma_n(t_i^{i+1})} \varphi([v(t_i^{i+1})], \gamma_n(t_i^{i+1})) dH^{N-1}$$

$$\leq (\Delta t_p + \Delta t_i) \int_{t_i^{i+1}}^{t_i^{i+1}} \int_{\Omega} |\nabla \hat{g}(s)|^2 dx ds + 2 \int_{t_i^{i+1}}^{t_i^{i+1}} \int_{\Omega} (\nabla u_n(s) - \nabla v(t_i^{i+1})) \cdot \nabla \hat{g}(s) dx ds$$

$$+ \int_{\Omega} |\nabla v(t_i^{i+1})|^2 dx - \int_{\Omega} |\nabla u_n(t_i^{i+1})|^2 dx$$

for any SBV function $v(t_i^{i+1}) = g(t_i^{i+1})$ on $\partial \Omega$. We see immediately, by letting $v(t_i^{i+1}) = u_n(t_i^{i+1})$, that

$$\int_{S_n(t_i^{i+1}) \cup \Gamma_n(t_i^{i+1})} \varphi([u_n(t_i^{i+1})], \gamma_n(t_i^{i+1})) dH^{N-1} - \int_{S_n(t_i^{i+1}) \cup \Gamma_n(t_i^{i+1})} \varphi([u_n(t_i^{i+1})], \gamma_n(t_i^{i+1})) dH^{N-1}$$

$$\leq (\Delta t_p + \Delta t_i) \int_{t_i^{i+1}}^{t_i^{i+1}} \int_{\Omega} |\nabla \hat{g}(s)|^2 dx ds + 2 \int_{t_i^{i+1}}^{t_i^{i+1}} \int_{\Omega} (\nabla u_n(s) - \nabla u_n(t_i^{i+1})) \cdot \nabla \hat{g}(s) dx ds$$

(3.6)
Let $t \in I_{\infty}$, we are going to sum the above estimate over all $t_{i_p}^j$ s.t. $t_{i_p}^j \leq t$.

\[
\sum_{i=0}^{t_{i_p}^j+1 \leq t} \left[ \int \tilde{\varphi}(u_n(t_{i_p}^j)), \gamma_n(t_{i_p}^j) d\mathcal{H}^{N-1} - \int \tilde{\varphi}(v(t_{i_p}^j)), \gamma_n(t_{i_p}^j) d\mathcal{H}^{N-1} \right]
\]

\[
\leq (\Delta t_n + \Delta t_p) \int_0^t \int_\Omega |\nabla \dot{g}(s)|^2 dx ds + \sum_{i=0}^{t_{i_p}^j \leq t} \int_\Omega \left( |\nabla u_n(s) - \nabla v_n(t_{i_p}^j)| \cdot \nabla \dot{g}(s) \right) dx ds + \sum_{i=0}^{t_{i_p}^j \leq t} \int_\Omega |\nabla v_n(t_{i_p}^j)|^2 dx - \int_\Omega |\nabla u_n(t_{i_p}^j)|^2 dx
\]

\[
\leq (\Delta t_n + \Delta t_p) \int_0^t \int_\Omega |\nabla \dot{g}(s)|^2 dx ds + \sum_{i=0}^{t_{i_p}^j \leq t} \int_\Omega \left( |\nabla u_n(s) - \nabla u_n(t_{i_p}^j)| \cdot \nabla \dot{g}(s) \right) dx ds + \sum_{i=0}^{t_{i_p}^j \leq t} \int_\Omega |\nabla v_n(t_{i_p}^j)|^2 dx - \int_\Omega |\nabla u_n(t_{i_p}^j)|^2 dx
\]

Next define

\[G_n^p(t) = (\Delta t_n + \Delta t_p) \int_0^t \int_\Omega |\nabla \dot{g}(s)|^2 dx ds + \sum_{i=0}^{t_{i_p}^j \leq t} \int_\Omega \left( |\nabla u_n(s) - \nabla u_n(t_{i_p}^j)| \cdot \nabla \dot{g}(s) \right) dx ds\]

Before introducing the next lemma let $f : [0, 1] \to \mathbb{R}$ s.t. $\sup_{0 \leq x \leq 1} |f(x)| < \infty$. Define it's step function with $2^n$ equi-length partitions as follows

\[f^{(n)}(x) = \begin{cases} f(0) & x = 0 \\ f\left(\frac{i+1}{2^n}\right) & \frac{i}{2^n} < x \leq \frac{i+1}{2^n}, \ 0 \leq i \leq 2^n - 1 \end{cases}\]

**Lemma 18.** Let $t \in I_{\infty}$, we have

\[
\lim_{p \to \infty} \lim_{n \to \infty} \int_0^t \int_\Omega (\nabla u_n(s) - \nabla u_n^{(p)}(s)) \cdot \nabla \dot{g}(s) dx ds = 0
\]

**Proof.** First we see $\theta_n(s) = \int_\Omega \nabla u_n(s) \cdot \nabla \dot{g}(s) dx \to \theta(s)$ for $\forall 0 < s < 1$ and $\int_0^t |\theta_n(s) - \theta(s)| ds \to 0$ \forall $t \in I_{\infty}$. Then show $\theta(s)$ is continuous except for a at most countable subset in $[0, 1]$. Let $s \in \{ \tau \in [0, 1] : m_{\infty}(\tau) \text{ is continuous} \}$. From previous work, we see $\{ \tau \in [0, 1] : m_{\infty}(\tau) \text{ is continuous} \}^c$ is at most countable and $\forall \epsilon > 0, \exists \Delta s > 0$ s.t.

\[\|\nabla u_n(s_1) - \nabla u_n(s_2)\|_2 < \epsilon, \forall s_1, s_2 \in (s - \Delta s, s + \Delta s)\]
Let \( y, z \in (s - \Delta s, s + \Delta s) \), we see
\[
|\theta(y) - \theta(z)| \leq |\theta(y) - \theta_n(y)| + |\theta_n(y) - \theta_n(z)| + |\theta_n(z) - \theta(z)| \\
\leq O(\epsilon)
\]
Thus we have shown that \( \theta(s) \) is continuous except for a countable subset on \([0, 1]\). Let \( \epsilon > 0 \), according to lemma 23 we see
\[
\int_0^t |\theta(s) - \theta^{(p)}(s)| ds \to 0
\]
as \( p \to \infty \). Then
\[
\left| \int_0^t \int_\Omega \nabla u_n(s) \cdot \nabla \hat{g}(s) dxds - \int_0^t \int_\Omega \nabla u_n^{(p)}(s) \cdot \nabla \hat{g}(s) dxds \right|
\leq \left| \int_0^t \int_\Omega \nabla u_n(s) \cdot \nabla \hat{g}(s) dxds - \int_0^t \theta(s) ds \right| + \left| \int_0^t \theta(s) ds - \int_0^t \theta^{(p)}(s) ds \right|
+ \left| \int_0^t \theta^{(p)}(s) ds - \int_0^t \int_\Omega \nabla u_n^{(p)}(s) \cdot \nabla \hat{g}^{(p)}(s) dxds \right|
+ \left| \int_0^t \int_\Omega \nabla u_n^{(p)}(s) \cdot \nabla \hat{g}^{(p)}(s) dxds - \int_0^t \int_\Omega \nabla u_n^{(p)}(s) \cdot \nabla \hat{g}(s) dxds \right|
\]
We see
\[
\left| \int_0^t \int_\Omega \nabla u_n(s) \cdot \nabla \hat{g}(s) dxds - \int_0^t \theta(s) ds \right| + \left| \int_0^t \theta^{(p)}(s) ds - \int_0^t \int_\Omega \nabla u_n^{(p)}(s) \cdot \nabla \hat{g}^{(p)}(s) dxds \right| \to 0
\]
as \( n \to \infty \). And
\[
\left| \int_0^t \int_\Omega \nabla u_n^{(p)}(s) \cdot \nabla \hat{g}^{(p)}(s) dxds - \int_0^t \int_\Omega \nabla u_n^{(p)}(s) \cdot \nabla \hat{g}(s) dxds \right|
\leq \int_0^t \|\nabla u_n^{(p)}(s)\|_{L^2} \|\nabla \hat{g}^{(p)}(s) - \nabla \hat{g}(s)\|_{L^2} ds
\leq \sup_s \|\nabla u_n^{(p)}(s)\|_{L^2} \int_0^t \|\nabla \hat{g}^{(p)}(s) - \nabla \hat{g}(s)\|_{L^2} ds
\]
We see \( \sup_s \|\nabla u_n^{(p)}(s)\|_{L^2} < \infty \) and \( \|\nabla \hat{g}^{(p)}(s) - \nabla \hat{g}(s)\|_{L^2} \to 0 \) for all \( s \) as \( p \to \infty \). Apply D.C.T. to conclude the lemma.

Then it’s not hard to see that
\[
\lim_{p \to \infty} \lim_{n \to \infty} G_n^p(t) = 0 \quad \forall t \in I_\infty
\]
(3.9)
Next we show two applications based on above result.
3.2.2 Application of little o analysis: approximation with finite minimizers

Define the following
\[
\gamma_p^n(t) = \bigvee_{i=0}^{t_p^i \leq t} [u_n(t_p^i)]
\]
\[
\Gamma_p^n(t) = \bigcup_{i=0}^{t_p^i \leq t} S_{u_n(t_p^i)}
\]

We see \(\gamma_p^n(t)\) is defined on \(\Gamma_p^n(t)\).

**Lemma 19.**
\[
\lim_{p \to \infty} \lim_{n \to \infty} \left( \int_{\Gamma_p^n(t)} \tilde{\varphi}(\gamma_p^n(t), \gamma_n(t)) d\mathcal{H}^{N-1} - \int_{\Gamma_p^n(t)} \varphi(\gamma_p^n(t)) d\mathcal{H}^{N-1} \right) = 0
\]

**Proof.** Due to (3.7), we have
\[
\sum_{i=0}^{t_p^{i+1} \leq t} \left( \int_{S \cap \Gamma_n(t_p^{i+1})} \tilde{\varphi}([u_n(t_p^{i+1})], \gamma_n(t_p^{i+1})) d\mathcal{H}^{N-1} - \int_{S \cap \Gamma_n(t_p^{i+1})} \tilde{\varphi}(\gamma_p^n(t), \gamma_n(t_p^{i+1})) d\mathcal{H}^{N-1} \right)
\]
\[
\leq G_p^n(t)
\]

Let \(y > z\), we see in general the value of \(\tilde{\varphi}(x, y) - \tilde{\varphi}(x, z)\) decreases as \(x\) increases. So
\[
\sum_{i=0}^{t_p^{i+1} \leq t} \left( \int_{\Gamma_n(t_p^{i+1})} \tilde{\varphi}(\gamma_p^n(t), \gamma_n(t_p^{i+1})) d\mathcal{H}^{N-1} - \int_{\Gamma_n(t_p^{i+1})} \tilde{\varphi}(\gamma_p^n(t), \gamma_n(t_p^{i+1})) d\mathcal{H}^{N-1} \right)
\]
\[
\leq \sum_{i=0}^{t_p^{i+1} \leq t} \left( \int_{S \cap \Gamma_n(t_p^{i+1})} \tilde{\varphi}([u_n(t_p^{i+1})], \gamma_n(t_p^{i+1})) d\mathcal{H}^{N-1} - \int_{S \cap \Gamma_n(t_p^{i+1})} \tilde{\varphi}(\gamma_p^n(t), \gamma_n(t_p^{i+1})) d\mathcal{H}^{N-1} \right)
\]
\[
\leq G_p^n(t)
\]

But we see the l.h.s of above inequality can be simplified to
\[
\int_{\Gamma_n(t)} \tilde{\varphi}(\gamma_p^n(t), \gamma_n(t)) d\mathcal{H}^{N-1} - \int_{\Gamma_p^n(t)} \varphi(\gamma_p^n(t)) d\mathcal{H}^{N-1}
\]

Thus we prove the lemma. \(\square\)

The result is pretty straightforward once we have (3.9). It shows that we can approximate the whole history evolution by choosing finitely many minimizers.
3.2.3 Application of little o analysis: absolute continuity

As we see in Chapter 2, absolute continuity is very important in that it allows us to use advance tools like covering theorem. But it is very hard to show the result if we take into account the history. As \( n \to \infty \), the number of minimizers in the history goes to infinity and it is hard to predict what they will behave as a whole. But little method allows us to fix finitely many times and do analysis on those finitely many chosen times. This gives us a way to prove the absolute continuity for the evolution problem. First let \( h(x) : \Gamma(t) \to (0, \infty) \) and \( h(x) \in L^1(\Gamma(t); H^{N-1}) \).

**Lemma 20.** Let \( t \in I_\infty \), define \( \mu_n := \tilde{\varphi}([u_n(t)], \gamma_n(t))H^{N-1}[S_{u_n(t)} \cup \Gamma_n(t)] \). We see there exists a Radon measure \( \mu_\infty \) and a subsequence s.t.

\[
\mu_n \overset{*}{\rightharpoonup} \mu_\infty
\]

Let \( \mu := hH^{N-1}[\Gamma(t)] \). We have \( \mu_\infty \ll \mu \).

**Proof.** The way we prove the lemma is very similar to the one we use to prove absolutely continuity in section Chapter 2. Except here we combine the little o method. Assume it is not true, we can find a set \( A \subset \Omega \) s.t. \( \mu(A) = 0 \) but \( \mu_\infty(A) = \delta > 0 \).

First we see, according to (3.9)

\[
\lim_{p \to \infty} \lim_{n \to \infty} G_n^p(t) = 0
\]

where

\[
G_n^p(t) = (\Delta t_n + \Delta t_p) \int_0^t \int_\Omega |\nabla \dot{g}(s)|^2 dxds + 2 \int_0^t \int_\Omega (\nabla u_n(s) - \nabla \gamma_n^p(s)) \cdot \nabla \dot{g}(s) dxds
\]

Fix \( p \) to be big s.t. \( \lim_{n \to \infty} G(n, p) \leq \frac{1}{8} \delta \) and let \( P \) be s.t. \( t_P = t \).

Since \( \mu(A) = 0 \), we have \( |A| = 0 \) too. Let \( U \) be open and \( A \subset U \subset \Omega \) s.t.

\[
\sup_{s} \|\nabla \dot{g}(s)\|_{L^\infty} \lim_{n \to \infty} \sup_{n \to \infty} \int_U |\nabla u_n(t^i_p)| dx \leq \frac{1}{8} \frac{\delta}{P} \quad \forall 1 \leq i \leq P
\]

(3.10)

\[
\sup_{s} \|\nabla \dot{g}(s)\|_{L^\infty} \int_U |\nabla u(t^i_p)| dx \leq \frac{1}{8} \frac{\delta}{P} \quad \forall 1 \leq i \leq P
\]

(3.11)

\[
\int_U \varphi([u(t^i_p)]) dH^{N-1} \leq \frac{1}{8} \frac{\delta}{P} \quad \forall 1 \leq i \leq P
\]

(3.12)

The last inequality comes from the fact that \( \mu(A) = 0 \) implies \( \int_A \varphi([u(t^i_p)]) = 0 \) \( \forall i \).

Then finely cover \( A \) with each ball chosen s.t.

\[
x \in A
\]

\[
B(x, r) \subset U
\]

\[
\mu_n(\partial B(x, r)) = 0 \quad \text{for} \quad \forall \ n > 0
\]

\[
\mu_\infty(\partial B(x, r)) = 0
\]

According to Besicovitch covering theorem we can find a countable disjoint family of closed balls
\{B(x_j, r_j)\}_{j=1}^{\infty} \text{ s.t.}

\mu_\infty(A \setminus \bigcup_{j}^{\infty} B(x_j, r_j)) = 0

(3.13)

Since \(\mu_\infty(\partial B(x, r)) = 0\) we can let all balls be open. Then select a finite \(N \in \mathbb{N}\) s.t. 
\(\mu_\infty(A \setminus \bigcup_{i=1}^{N} B(x_i, r_i)) < \frac{1}{8}\delta\). We see 

\[
\mu\left(\bigcup_{i=1}^{N} B(x_i, r_i)\right) < \epsilon
\]

\(\mu_\infty\left(\bigcup_{i=1}^{N} B(x_i, r_i)\right) \geq \frac{7}{8}\delta\)

Denote \(B := \bigcup_{i=1}^{N} B(x_i, r_i)\) and consider the following test functions 

\[v_n(t^i_p) = \begin{cases} 
u_n(t^i_p) & \Omega \setminus B \\ u_n(t^i_p) & B \end{cases}\]

Previously we showed that 

\[
\sum_{i=0}^{P-1} \int_0^{t} \int_{\Omega} |\nabla \tilde{g}(s)|^2 dx ds + 2 \sum_{i=0}^{P-1} \int_{t^i_p}^{t^i_{p+1}} \int_{\Omega} (\nabla u_n(s) - \nabla v_n(t^i_p)) \cdot \nabla \hat{g}(s) dx ds
\]

\[
+ \sum_{i=0}^{P-1} \int_{\Omega} |\nabla v_n(t^i_p)|^2 dx - \int_{\Omega} |\nabla u_n(t^i_p)|^2 dx
\]

\[
\leq (\Delta t_n + \Delta t_p) \int_0^{t} \int_{\Omega} |\nabla \tilde{g}(s)|^2 dx ds + 2 \sum_{i=0}^{P-1} \int_{t^i_p}^{t^i_{p+1}} \int_{\Omega} (\nabla u_n(s) - \nabla v_n(t^i_p)) \cdot \nabla \hat{g}(s) dx ds
\]

\[
+ \sum_{i=0}^{P-1} \int_{\Omega} |\nabla v_n(t^i_p)|^2 dx - \int_{\Omega} |\nabla u_n(t^i_p)|^2 dx
\]

\[
\leq G(n, p) + 2 \sum_{i=0}^{P-1} \int_{t^i_p}^{t^i_{p+1}} \int_{\Omega} (\nabla u_n(t^i_p) - \nabla v_n(t^i_p)) \cdot \nabla \hat{g}(s) dx ds
\]

\[
+ \sum_{i=0}^{P-1} \int_{\Omega} |\nabla v_n(t^i_p)|^2 dx - \int_{\Omega} |\nabla u_n(t^i_p)|^2 dx
\]
According to the definition of $v_n(t_p^i)$, we have
\[
2 \sum_{i=0}^{P-1} \int_{t_p^i}^{t_p^{i+1}} \int_{\Omega} (\nabla u_n(t_p^{i+1}) - \nabla v_n(t_p^{i+1})) \cdot \nabla \hat{g}(s) dx ds
= 2 \sum_{i=0}^{P-1} \int_{t_p^i}^{t_p^{i+1}} \int_{B} (\nabla u_n(t_p^{i+1}) - \nabla u(t_p^{i+1})) \cdot \nabla g(s) dx ds
\leq 2 \sum_{i=0}^{P-1} \Delta t_p \sup_{s} \| \nabla \hat{g}(s) \|_{L^\infty} \int_{B} |\nabla u_n(t_p^{i+1}) - \nabla v_n(t_p^{i+1})| dx
\leq 2 \Delta t_p \sup_{s} \| \nabla \hat{g}(s) \|_{L^\infty} \sum_{i=0}^{P-1} \left[ \int_{B} |\nabla u_n(t_p^{i+1})| dx + \int_{B} |\nabla v_n(t_p^{i+1})| dx \right]
\]
Again according to the definition of $v_n(t_p^i)$
\[
\sum_{i=0}^{P-1} \int_{\Omega} |\nabla v_n(t_p^{i+1})|^2 dx - \int_{\Omega} |\nabla u_n(t_p^{i+1})|^2 dx = \sum_{i=0}^{P-1} \int_{B} |\nabla v_n(t_p^{i+1})|^2 dx - \int_{B} |\nabla u_n(t_p^{i+1})|^2 dx
\]
According to lower-semi-continuity we have
\[
\int_{B} |\nabla u(t_p^{i+1})|^2 dx \leq \liminf_{n \to \infty} \int_{B} |\nabla u_n(t_p^{i+1})|^2 dx
\]
Combine to estimates from above we can get
\[
\lim_{n \to \infty} \sum_{i=0}^{P-1} \int_{\Omega} \varphi([u_n(t_p^{i+1})], \gamma_n(t_p^{i+1})) d\mathcal{H}^{N-1} - \int_{\Omega} \varphi([v_n(t_p^{i+1})], \gamma_n(t_p^i)) d\mathcal{H}^{N-1}
\leq \frac{1}{8} \delta + 2 \Delta t_p \frac{2}{8} \delta \leq \frac{5}{8} \delta
\]
Again in light of the definition of $v_n(t_p^i)$ we have
\[
\sum_{i=0}^{P-1} \int_{\Omega \setminus B} \varphi([u_n(t_p^{i+1})], \gamma_n(t_p^{i+1})) d\mathcal{H}^{N-1} - \int_{\Omega \setminus B} \varphi([v_n(t_p^{i+1})], \gamma_n(t_p^i)) d\mathcal{H}^{N-1}
= \sum_{i=0}^{P-1} \int_{\Omega \setminus B} \varphi([u_n(t_p^{i+1})], \gamma_n(t_p^{i+1})) d\mathcal{H}^{N-1} - \int_{\Omega \setminus B} \varphi([u_n(t_p^{i+1})], \gamma_n(t_p^i)) d\mathcal{H}^{N-1}
+ \sum_{i=0}^{P-1} \int_{\partial B} \varphi([u_n(t_p^{i+1})], \gamma_n(t_p^{i+1})) d\mathcal{H}^{N-1} - \int_{\partial B} \varphi([v_n(t_p^{i+1})], \gamma_n(t_p^i)) d\mathcal{H}^{N-1}
+ \sum_{i=0}^{P-1} \int_{B} \varphi([u_n(t_p^{i+1})], \gamma_n(t_p^{i+1})) d\mathcal{H}^{N-1} - \int_{B} \varphi([v_n(t_p^{i+1})], \gamma_n(t_p^i)) d\mathcal{H}^{N-1}
\]
It is clear that

$$
\sum_{i=0}^{P-1} \int_{\Omega \setminus B} \tilde{\varphi}(u_n(t_p^{i+1}), \gamma_n(t_p^{i+1})) d\mathcal{H}^{N-1} - \int_{\Omega \setminus B} \tilde{\varphi}(u_n(t_p^{i+1}), \gamma_n(t_p^i)) d\mathcal{H}^{N-1} \geq 0
$$

And according to Lemma (6)

$$
\lim_{n \to \infty} \int_{\partial B} \tilde{\varphi}(v_n(t_p^{i+1}), \gamma_n(t_p^i)) d\mathcal{H}^{N-1} = \lim_{n \to \infty} \int_{\partial B} \tilde{\varphi}(0, \gamma_n(t_p^i)) d\mathcal{H}^{N-1}
$$

So

$$
\lim_{n \to \infty} \sum_{i=0}^{P-1} \int_{\partial B} \tilde{\varphi}(v_n(t_p^{i+1}), \gamma_n(t_p^i)) d\mathcal{H}^{N-1} - \int_{\partial B} \tilde{\varphi}(0, \gamma_n(t_p^i)) d\mathcal{H}^{N-1} \geq 0
$$

This shows

$$
\lim_{n \to \infty} \sum_{i=0}^{P-1} \int_{B} \tilde{\varphi}(u_n(t_p^{i+1}), \gamma_n(t_p^i)) d\mathcal{H}^{N-1} - \int_{B} \tilde{\varphi}(u(t_p^{i+1}), \gamma_n(t_p^i)) d\mathcal{H}^{N-1} \leq \frac{5}{8} \delta
$$

Then we see

$$
\sum_{i=0}^{P-1} \int_{B} \tilde{\varphi}(u_n(t_p^{i+1}), \gamma_n(t_p^i)) d\mathcal{H}^{N-1} - \int_{B} \tilde{\varphi}(u(t_p^{i+1}), \gamma_n(t_p^i)) d\mathcal{H}^{N-1}
$$

$$
= \int_{B} \tilde{\varphi}(u_n(t_p^P), \gamma_n(t_p^P)) d\mathcal{H}^{N-1} + \sum_{i=1}^{P-1} \left[ \int_{B} \tilde{\varphi}(u_n(t_p^i), \gamma_n(t_p^i)) d\mathcal{H}^{N-1} - \int_{B} \tilde{\varphi}(u(t_p^i), \gamma_n(t_p^i)) d\mathcal{H}^{N-1} \right]
$$

Moreover

$$
\sum_{i=1}^{P-1} \left[ \int_{B} \tilde{\varphi}(u_n(t_p^i), \gamma_n(t_p^i)) d\mathcal{H}^{N-1} - \int_{B} \tilde{\varphi}(u(t_p^i), \gamma_n(t_p^i)) d\mathcal{H}^{N-1} \right]
$$

$$
\geq \sum_{i=1}^{P-1} \left[ \int_{B} \tilde{\varphi}(u_n(t_p^i), \gamma_n(t_p^i)) d\mathcal{H}^{N-1} - \int_{B} \tilde{\varphi}(0, \gamma_n(t_p^i)) d\mathcal{H}^{N-1} - \int_{B} \varphi(|u(t_p^i)|) d\mathcal{H}^{N-1} \right]
$$

$$
\geq \sum_{i=1}^{P-1} - \int_{B} \varphi(|u(t_p^i+1)|) d\mathcal{H}^{N-1}
$$
Thus we have

\[
\int_B \bar{\varphi}([u_n(t^p_p)], \gamma_n(t^p_p))d\mathcal{H}^{N-1} - \sum_{i=0}^{P-1} \int_B \varphi([u(t^i+1)_p)])d\mathcal{H}^{N-1}
\]

\[
\leq \sum_{i=0}^{P-1} \int_B \bar{\varphi}([u_n(t^i+1)_p)], \gamma_n(t^i+1)_p))d\mathcal{H}^{N-1} - \int_B \bar{\varphi}([u(t^i+1)_p)], \gamma_n(t^i)_p))d\mathcal{H}^{N-1}
\]

It follows

\[
\lim_{n \to \infty} \int_B \bar{\varphi}([u_n(t)], \gamma_n(t))d\mathcal{H}^{N-1} \leq \frac{5}{8} \delta + \frac{1}{8} \delta = \frac{6}{8} \delta
\]

It contradicts the fact that \( \mu_\infty(\bigcup_{i=1}^N B(x_i, r_i)) \geq \frac{7}{8} \delta \). \qed
Appendix

A. Properties of cohesive energy function $\varphi$

An example of $\varphi$ that fits the description in Chapter 1. Let

$$\varphi(x) = x + x \log \left( \frac{1}{x} \right)$$

We see $\varphi'(x) = \log \left( \frac{1}{x} \right)$ and $\varphi''(x) = -\frac{1}{x}$. So $\varphi'(0) = \infty$ and

$$\frac{\varphi'(x)x}{\varphi(x)} = \frac{\log \left( \frac{1}{x} \right)x}{x + x \log \left( \frac{1}{x} \right)} \rightarrow 1 \text{ as } x \rightarrow 0$$

Some properties of $\varphi$:

1. Let $x, \lambda > 0$, we have

   $$\varphi(\lambda x) < \lambda \varphi(x) \quad \text{if } \lambda > 1$$
   $$\varphi(\lambda x) > \lambda \varphi(x) \quad \text{if } \lambda < 1$$

2. Subadditivity

   $$\varphi \left( \sum_{i=1}^{n} x_i \right) < \sum_{i=1}^{n} \varphi(x_i) \quad \text{where } x_i > 0 \text{ for all } i$$

3. $\frac{\varphi(x)}{x} > \varphi'(x)$ for $x > 0$.

4. Subadditivity of $\tilde{\varphi}(\cdot, \lambda)$ for $\lambda > 0$. i.e.

   $$\tilde{\varphi}(x + y, \lambda) \leq \tilde{\varphi}(x, \lambda) + \tilde{\varphi}(y, \lambda) \quad \forall \, x, \, y \geq 0$$

   moreover

   $$\tilde{\varphi}(x + y, \lambda) \leq \tilde{\varphi}(x, \lambda) + \varphi(y)$$

   or

   $$\tilde{\varphi}(x + y, \lambda) \leq \varphi(x) + \tilde{\varphi}(y, \lambda)$$

5. $\tilde{\varphi}(x, h) - \tilde{\varphi}(y, h) \leq \varphi(x) - \varphi(y)$
for all \( h \geq 0 \) and \( x \geq y \).

6. 
\[
\tilde{\varphi}(a + b, h) - \tilde{\varphi}(b, h) \leq \varphi(a, h) - \tilde{\varphi}(0, h)
\]
for all \( a \geq 0, b \geq 0 \) and \( h \geq 0 \).

### B. Useful lemmas

**Lemma 21.** Let \( f \in L^1(\Omega, \mu) \) where \( \Omega \) is open. Let \( \{A_n\}_{n=1}^\infty \) be a sequence of sets s.t. \( A_n \subset \Omega \) for \( \forall n \). If \( \mu(A_n) \to 0 \), then we have
\[
\int_{A_n} f \, dx \to 0
\]

*Proof.* It suffices to show \( \int_{\Omega} |f|\chi_{A_n} \, d\mu \to 0 \). Let's argue by contradiction and assume \( \lim_{n \to \infty} \int_{\Omega} |f|\chi_{A_n} \, d\mu \neq 0 \). Then we can find a subsequence \( \{A_{n_k}\} \) and \( \delta > 0 \) s.t. \( \int_{\Omega} |f|\chi_{A_{n_k}} \, d\mu > \delta \) for all \( k = 1, 2, ... \). First we see \( \int_{\Omega} |\chi_{A_{n_k}}| - 0 \, d\mu = \int_{\Omega} \chi_{A_{n_k}} \, d\mu = \mu(A_{n_k}) \to 0 \). So \( \chi_{A_{n_k}} \to 0 \) in \( L^1(\Omega, \mu) \). Thus we can extract a subsequence \( \{n_{k_i}\}_{i=1}^\infty \) s.t. \( \chi_{A_{n_{k_i}}} \to 0 \) \( \mu \text{-a.e.} \ x \in \Omega \). It follows \( |f|\chi_{A_{n_{k_i}}} \to 0 \) \( \mu \text{-a.e.} \). Then we see \( |f|\chi_{A_{n_{k_i}}} \leq |f| \in L^1(\Omega, \mu) \) for \( \forall i \), and due to D.C.T. we have \( \int_{\Omega} |f|\chi_{A_{n_{k_i}}} \, d\mu \to 0 \), which contradicts to the fact that \( \int_{\Omega} |f|\chi_{A_{n_k}} \, d\mu > \delta \) for all \( k = 1, 2, ... \). Therefore the lemma has been shown.

**Lemma 22.** Let \( f \in L^1(0,1) \), then
\[
\max_{0 \leq i < n} \int_{i/n}^{i+1/n} f \, dx \to 0 \text{ as } n \to \infty
\]

*Proof.* Assume it's not true, we can find a subsequence \( \{n_k\}_{k=1}^\infty \) and for each \( k \) an index \( i(n_k) \) that depends on \( k \) and \( \delta > 0 \) s.t.
\[
\int_{i(n_k)/n_k}^{i(n_k) + 1/n_k} f \, dx > \delta \ \forall \ k
\]
We see \( |(i(n_k)/n_k, i(n_k) + 1/n_k)| = 1/n_k \to 0 \), and according to lemma (21) we see \( \int_{i(n_k)/n_k}^{i(n_k) + 1/n_k} f \, dx \to 0 \). A contradiction.

**Lemma 23.** Let \( f \in L^\infty([0,1]) \) s.t. \( f \) is continuous a.e. Let \( f_n(x) := f(i/n) \) when \( i/n = x - \frac{i-1}{n} \), for \( \forall 1 \leq i \leq n \). Then \( \int_0^1 |f - f_n| \, dx \to 0 \).

*Proof.* Let \( E = \{x \in [0,1] : f(x) \text{ is not continuous}\} \). Let \( \epsilon > 0 \) and \( U_E \subset (0,1) \) be open s.t. \( E \subset U_E \) and \( |U_E| \leq \frac{\epsilon}{4|\|f\|_{L^\infty}} \).
First we claim \( \exists N \in \mathbb{N} \) s.t. \( \forall n \geq N \)
\[
\text{if } \left| \sup_{i-1/n \leq x \leq i/n} f(x) - \inf_{i-1/n \leq x \leq i/n} f(x) \right| > \frac{\epsilon}{2}, \text{ then } \left[ \frac{i-1}{n}, \frac{i}{n} \right] \subset U_E
\]
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We argue by contradiction, assume the claim is not true, we can find a subsequence \( \{n_j\}_{j=1}^\infty \) s.t. for each \( n_j \), we can find an index \( i(n_j) \) that depends on \( n_j \) s.t.

\[
\left| \sup_{\frac{i(n_j)-1}{n_j} \leq x < \frac{i(n_j)}{n_j}} f(x) - \inf_{\frac{i(n_j)-1}{n_j} \leq x \leq \frac{i(n_j)}{n_j}} f(x) \right| > \frac{\epsilon}{2}
\]

and \( \left[ \frac{i(n_j)-1}{n_j}, \frac{i(n_j)}{n_j} \right] \cap U_E^c \neq \emptyset \). Meanwhile we can find \( x_{n_j}, y_{n_j}, z_{n_j} \in \left[ \frac{i(n_j)-1}{n_j}, \frac{i(n_j)}{n_j} \right] \) s.t. \( |f(x_{n_j}) - f(y_{n_j})| > \frac{\epsilon}{4} \) and \( z_{n_j} \in U_E^c \). We see there’s a further subsequence and \( x \in [0, 1] \) s.t. \( x = \lim_{j \to \infty} x_{n_j} = \lim_{j \to \infty} y_{n_j} = \lim_{j \to \infty} z_{n_j} \). It follows that \( f \) is not continuous at \( x \) since \( |f(x_{n_j}) - f(y_{n_j})| > \frac{\epsilon}{4} \) for \( \forall j \), so \( x \in U_E \). But since \( U_E^c \) is closed we also deduce \( x \in U_E \), a contradiction. Thus the claim has been shown.

Next we see

\[
\int_0^1 |f - f_n| \, dx = \sum_{|\sup f(x) - \inf f(x)| \leq \frac{\epsilon}{4}} \int_{i=1}^{\frac{n}{n}} |f - f_n| \, dx + \sum_{|\sup f(x) - \inf f(x)| > \frac{\epsilon}{4}} \int_{i=1}^{\frac{n}{n}} |f - f_n| \, dx
\]

\[
\leq \sum_{|\sup f(x) - \inf f(x)| \leq \frac{\epsilon}{4}} \frac{1}{n} \frac{\epsilon}{4} n + \frac{\epsilon}{4} \cdot 2 \cdot \|f\|_{L^\infty}
\]

\[
\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]

\[\square\]

Lemma 24. Let \( f_n : [0, 1] \to \mathbb{R} \) be a sequence of non-decreasing functions defined on \([0, 1] \). Assume \( \sup_{n, x} |f_n(x)| < \infty \), then there’s a subsequence \( \{n_j\}_{j=1}^\infty \) and a non-decreasing bounded \( f \) defined on \([0, 1] \) s.t.

\[
f_{n_j}(x) \to f(x) \text{ for } \forall x \in [0, 1]
\]

Proof. Let \( D \subset [0, 1] \) be dense and countable. Then by compactness in \( \mathbb{R} \) and diagonal argument we can extract a subsequence \( \{n_j\}_{j=1}^\infty \) and \( f \) defined on \( D \) s.t. \( f_{n_j}(x) \to f(x) \) for \( \forall x \in D \). We see \( f(x) \) is non-decreasing and bounded on \( D \), i.e. \( f(x) \leq f(y) \) whenever \( x, y \in D \) and \( x \leq y \). Next let \( x \in [0, 1] \setminus D \), define

\[
f_+(x) = \inf_{y>x, y \in D} f(y) \quad f_-(x) = \sup_{y<x, y \in D} f(y)
\]

We see \( f_-(x) \leq f_+(x) \) and define \( f(x) = \frac{f_+(x) - f_-(x)}{2} \) for \( x \in [0, 1] \setminus D \). We see \( f(x) \) is non-decreasing and bounded on \([0, 1] \). Then we see \( f(x) \) is continuous except on a set \( E \) that is at most countable.

Next let \( x \in [0, 1] \setminus E \), let \( \epsilon > 0 \) we can find \( \Delta x > 0 \) s.t. \( |f(y) - f(x)| < \epsilon \) whenever \( |y - x| < \Delta x \).
Now restrict \(y, z \in D\), s.t. \(x - \Delta x < y < x < z < x + \Delta x\). Then we have

\[
|f_{n_j}(x) - f(x)| \leq |f_{n_j}(x) - f_{n_j}(y)| + |f(y) - f(x)|
\]

\[
\leq |f_{n_j}(z) - f_{n_j}(y)| + |f_{n_j}(y) - f(y)| + |f(y) - f(x)|
\]

\[
\leq |f_{n_j}(z) - f(z)| + |f(z) - f(y)| + |f(y) - f_{n_j}(y)| + |f_{n_j}(y) - f(y)| + |f(y) - f(x)|
\]

\[
\leq O(\epsilon)
\]

Thus we see \(f_{n_j}(x) \to f(x)\) for \(x \in [0, 1] \setminus E\). But since \(E\) is at most countable, we can use compactness and diagonal argument to extract a further subsequence s.t. \(f_{n_j}\) (not relabeled) converges to \(f(x)\) on \([0, 1]\). And it’s not hard to show \(f\) is non-decreasing. \(\square\)

Let \(\Omega \subset \mathbb{R}^n\) be open. Let \(\mu\) be Radon on \(\Omega\) and \(\{f_n\}_{n=1}^\infty \subset L^1(\Omega; \mu)\) s.t. \(f_n \overset{L^1}{\to} 0\). Then

**Lemma 25.** \(\exists\) a subsequence \(\{n_j\}_{j=1}^\infty\) s.t.

\[
\lim_{j \to \infty} \lim_{r \to 0} \frac{\int_{B(x,r)} f_{n_j} \, d\mu}{\mu(B(x,r))} = 0
\]

for \(\mu\)-a.e. \(x \in \Omega\).

**Proof.** First we see \(\exists\) a subsequence s.t. \(f_{n_j} \to 0\) for \(\mu\)-a.e. Let \(D_0 \subset \Omega\) be the set s.t. \(f_{n_j} \to 0\) for \(\forall x \in D_0\). We see \(\mu(\Omega \setminus D_0) = 0\).

Then for each \(j\), according to Lebesgue-Besicovitch differentiation theorem,

\[
\lim_{r \to 0} \frac{\int_{B(x,r)} f_{n_j} \, d\mu}{\mu(B(x,r))} = f_{n_j}(x)
\]

for \(\mu\)-a.e. \(x \in \Omega\). Let \(D_j \subset \Omega\) be the set s.t. the above is true for \(x \in D_j\). Then consider the set \(D := \cap_{j=0}^\infty D_j\). We see

\[
\lim_{j \to \infty} \lim_{r \to 0} \frac{\int_{B(x,r)} f_{n_j} \, d\mu}{\mu(B(x,r))} = \lim_{j \to \infty} f_{n_j}(x) = 0
\]

for all \(x \in D\). And \(\mu(\Omega \setminus D) \leq \sum_{j=0}^\infty \mu(\Omega \setminus D_j) = 0\). Thus the lemma has been proved. \(\square\)
Notation

\(a \lor b, a \land b\)  maximum and minimum of \(a\) and \(b\)

\(B(x, r)\)  open ball centered at \(x\) with radius \(r\)

\(Q(x, r)\)  open cube centered at \(x\) with side \(r\)

\(\chi_E\)  indicator function of set \(E\)

\(\overline{E}\)  closure of set \(E\)

\(\partial E\)  topological boundary of set \(E\)

\(V \subseteq U\)  \(V\) is compactly contained in \(U\)

\(\mathcal{L}^N\)  Lebesgue measure on \(\mathbb{R}^N\)

\(\mathcal{H}^{N-1}\)  \(N - 1\) dimensional Hausdorff measure

\(\mu|A\)  \(\mu\) restricted to the set \(A\)

\(D_\nu \mu\)  derivative of \(\mu\) with respect to \(\nu\)

\(\mu \ll \nu\)  \(\mu\) is absolutely continuous with respect to \(\nu\)

\(BV\)  functions of bounded variation

\(SBV\)  special functions of bounded variation

\(S_u\)  jump set of \(u\)

\([u]\)  size of the jump on \(S_u\)

\(\nu\)  unit normal to the jump set

\(D_u\)  distributional derivative of \(u\)

\(D^a u\)  absolutely continuous part of \(D_u\) with respect to \(\mathcal{L}^N\)

\(\partial^* E\)  reduced boundary of \(E\)

\(E_t\)  \(\{x \in \Omega : u > t\}\)

\(u_n \overset{SBV}{\rightarrow} u\)  \(u_n\) converges to \(u\) in the sense of \(SBV\)

\(\mu_n \rightharpoonup^* \mu\)  \(\mu_n\) converges weak* to \(\mu\)
Bibliography


