Hedge Funds' Performance Fees and Investments

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Hedge Funds’ Performance Fees and Investments

A Thesis

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of the

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Abstract

The high-water mark provision in hedge fund managers’ compensation raises concerns of investors, because they are worried about that fund managers would take unnecessarily high risk in the fund investment. In this paper, we theoretically analyze the optimal strategies for hedge fund managers who choose to maximize the expected power utility from fees in both discrete-time and continuous-time models. The results show that when approaching the fee payment date, hedge fund managers would take as much risk as they are allowed to in the fund investment. However, if hedge fund managers are given more time, they tend to be more conservative. In the continuous-time model, the optimal allocation of the fund in the risky asset depends on market conditions, which are measured by the state price density.
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1 Introduction

Hedge fund managers receive management fees and performance fees as the compensation for managing the funds. A manager is paid performance fees when the fund exceeds the previous highest value, which is referred to as the high-water mark. The common practice for hedge fund manager’s fee structure is 1% - 2% of the fund’s assets as management fees and 15% - 20% of the profit above the high-water mark [see Aragon and Nanda (2011)]. A criticism about the high-water mark fee schedule is that the performance fee is akin to a call option, of which the high-water mark is the strike price and the fund’s value is the underlying asset [see Carhart et al. (2002)]. If the hedge fund’s value exceeds the high-water mark, the manager earns performance fees. If not, the manager loses nothing. If the fund value follows a geometric Brownian Motion, according to Black-Scholes model, the value of this call option increases if the volatility of the fund increases, because the vega, which is the derivative of the option value with respect to the volatility of the underlying asset, is always greater than 0 [see Hull (2015)]. Therefore, the fund manager who is paid by high-water mark fees tends to increase the volatility of the fund in order to maximize the value from fees. Investors should be concerned about this moral hazard introduced by the high-water mark fee contract, which is against their interest.

Many researchers have investigated the relationship between the high-water mark fee contract and fund managers’ risk taking. In Guasoni and Oblój (2016), a fund manager with constant investment opportunity for the fund aims to maximize the expected power utility from fees in the long run. The manager’s optimal strategy is a Merton portfolio\(^1\) with an effective risk aversion \(\gamma^*_M\), where \(\gamma^*_M = \alpha + (1 - \alpha)\gamma_M\), \(\gamma_M\) is the manager’s own risk aversion, and \(0 < \alpha < 1\) is the fraction of profit that the manager receives from high-water mark provision. Thus, the risk taking of the fund is bounded. If the manager’s own risk aversion \(\gamma_M < 1\), the high-water mark fee contract will decrease the risk taking of the fund comparing to the case when the manager is managing his own money, because the manager is more risk averse (\(\gamma^*_M > \gamma_M\)). On the contrary, if the manager’s risk aversion \(\gamma_M > 1\), the risk taking is increased (\(\gamma^*_M < \gamma_M\)). Therefore, the authors conclude that the high-water mark fee contract does not necessarily lead to the moral hazard of excessively high risk.

While the research of Guasoni and Oblój (2016) is based on the assumption that the fund manager does not invest in the fund, and invests the fees in the risk-free asset, Guasoni and

---

\(^1\)Merton portfolio is the allocation that investors choose for the risky assets if they are managing their own money in order to maximize the expected power utility from personal wealth.
Wang (2015) assumes a manager who invests both the fund and the private wealth, including fees, in separate, but possibly correlated investment opportunities and wants to maximize the expected power utility from private wealth. The results show that for both hedge fund and mutual fund, the manager does not use the fund to hedge personal investment even if the investment opportunities are correlated, which is referred to as portfolio separation. The downside effect is the so called “attention separation”. The manager’s welfare is the maximum of the welfare from fees and the welfare from private wealth. If the welfare comes from fees in the long run, the manager will focus on the fund investment. If the fund can not offer sufficiently attractive welfare, then the manager will neglect the fund and focus on the private investment.

Since the previous results are all based on the assumption that the manager has a long horizon, this paper is going further to discuss how the length of the planning horizon affects the risk taking in the fund investment for a manager who is paid by the high-water mark fees. We theoretically analyze of the optimal strategies for the fund manager who aims to maximize the expected power utility from fees for discrete-time and continuous-time models.

The results show that for single period binomial tree model, the manager invests in the risky assets as much as he is allowed to. For two period binomial tree model, the manager does not necessarily invest as much as in the one period model. The manager might be conservative at the initial date, because he does not wants to lose the opportunity of earning fees in the latter period. In continuous-time model, the optimal strategy is determined by the market conditions, which are measured by the state price density. In particular, as the time approaches the fee payment date, if the investment opportunity is good, the fund manager is more likely to expect fees. Since the fund manager is risk averse, he tends to be conservative in the fund investment. However, if the investment opportunity is bad, the manager tends to perform aggressively in order to earn fees.

The rest of the paper is organized as the following: section 2 discusses the single period binomial tree model, section 3 discusses the two period binomial tree model, section 4 considers the continuous-time model and the conclusion is in section 5.
2 Single Period Binomial Tree Model

In this section, we assume there is only one period, and that the fund manager only invests the fund in one stock and the money market account. At the end of the period, the stock with initial price $S_0$ either goes up to $uS_0$ or goes down to $dS_0$ with probability $p$ and $1-p$, respectively, where the up factor $u > 1$ and the down factor $d < 1$. With initial fund value $F_0$, the fund asset at the end of the period is $F_1 = \pi F_0 \left( \frac{S_1}{S_0} \right) + (1 - \pi) F_0 (1 + r)$, where $S_1$ is the stock price at the end of the period, $\pi$ is the proportion that the fund invests in the stock and $r$ is the risk-free rate.

Given the high-water mark $H_0$ at time 0, the total fee $X$, which is paid at the end of the period, includes both the performance fee and the management fee. The performance fee is the proportion $\alpha$ of the profit of the fund above the high-water mark, which is $\alpha (F_1 - H_0)^+$. The management fee is proportion $\phi$ from the fund asset, which is $\phi F_1$. Thus, $X = \alpha (F_1 - H_0)^+ + \phi F_1$. Let $X_u$ be the fee at the end of the period if the stock price goes up, and $X_d$ be the fee if the stock price goes down. Then,

$$X_u = \alpha (F_1 - H_0)^+ + \phi F_1$$

$$= \alpha F_0 \left( \pi (u - r - 1) + (1 + r) - \frac{H_0}{F_0} \right)^+ + \phi F_0 \left( \pi (u - r - 1) + (1 + r) \right).$$

(1)

$$X_d = \alpha (F_1 - H_0)^+ + \phi F_1$$

$$= \alpha F_0 \left( \pi (d - r - 1) + (1 + r) - \frac{H_0}{F_0} \right)^+ + \phi F_0 \left( \pi (d - r - 1) + (1 + r) \right).$$

(2)

We are interested in the optimal strategy for the fund manager who wants to maximize the expected power utility from fees ($U(X) = \frac{X^{1-\gamma}}{1-\gamma}$) at the end of the period. $\gamma$ is the coefficient of the relative risk aversion.

To simplify the mathematical calculation, we assume the risk free rate $r$ is 0, the fund manager earns no management fee ($\phi = 0$) and $d = \frac{1}{u}$. Therefore,

$$X_u = \alpha (F_1 - H_0)^+ = \alpha F_0 \left( \pi (u - 1) + 1 - \frac{H_0}{F_0} \right)^+.$$  

(3)

$$X_d = \alpha (F_1 - H_0)^+ = \alpha F_0 \left( \pi \left( \frac{1}{u} - 1 \right) + 1 - \frac{H_0}{F_0} \right)^+.$$  

(4)
The maximum expected power utility from the fees is $\mathbb{E}[U(X)] = p \frac{X^{1-\gamma}}{1-\gamma} + (1-p) \frac{X^{1-\gamma}}{1-\gamma}$.

The fund manager never wants the fund to go bankrupt. Thus, the fund value at the end of the period is non-negative, which indicates $F_1 \geq 0$ holds. Therefore, both $\pi F_u + (1-\pi) F_0 \geq 0$ and $\pi F_0 + (1-\pi) F_u \geq 0$, which gives the non-bankruptcy constraint $\frac{1}{1-u} \leq \pi \leq \frac{u}{u-1}$. Letting $\pi^- = \frac{1}{1-u} < 0$ and $\pi^+ = \frac{u}{u-1} > 0$, the manager’s goal is $\max_{\pi^- \leq \pi \leq \pi^+} \mathbb{E}[U(X)]$.

**Theorem 2.1.** Let $M_d = \frac{1}{u} + 1 - \frac{H_0}{F_0}$ and $M_u = u + 1 - \frac{H_0}{F_0}$. In the one period binomial tree model, if $0 < \gamma < 1$, the optimal strategy $\pi^*$ is a function of $\frac{H_0}{F_0}$, where

$$
\pi^*(\frac{H_0}{F_0}) = \begin{cases} 
\text{Any } \pi^- \leq \pi \leq \pi^+, & \text{if } \frac{H_0}{F_0} > u + 1 \\
\pi^+, & \text{if } \frac{1}{u} + 1 \leq \frac{H_0}{F_0} \leq u + 1 \\
\pi^-, & \text{if } \frac{H_0}{F_0} < \frac{1}{u} + 1 \text{ and } (1-p)(M_d)^{1-\gamma} \geq p(M_u)^{1-\gamma} \\
\pi^+, & \text{if } \frac{H_0}{F_0} < \frac{1}{u} + 1 \text{ and } (1-p)(M_d)^{1-\gamma} < p(M_u)^{1-\gamma}.
\end{cases}
$$

The maximum expected power utility is

$$
\max_{\pi^- \leq \pi \leq \pi^+} \mathbb{E}[U(X)] = \begin{cases} 
0, & \text{if } \frac{H_0}{F_0} > u + 1 \\
\frac{(\alpha F_0 M_u)^{1-\gamma}}{1-\gamma}, & \text{if } \frac{1}{u} + 1 \leq \frac{H_0}{F_0} \leq u + 1 \\
(1-p)\frac{(\alpha F_0 M_d)^{1-\gamma}}{1-\gamma}, & \text{if } \frac{H_0}{F_0} < \frac{1}{u} + 1 \text{ and } (1-p)(M_d)^{1-\gamma} \geq p(M_u)^{1-\gamma} \\
\frac{(\alpha F_0 M_u)^{1-\gamma}}{1-\gamma}, & \text{if } \frac{H_0}{F_0} < \frac{1}{u} + 1 \text{ and } (1-p)(M_d)^{1-\gamma} < p(M_u)^{1-\gamma}.
\end{cases}
$$

**Proof.** Let $\pi_1 = \frac{H_0}{F_0} - 1 = \frac{H_0}{u-1} - 1 \leq 0$, and $\pi_2 = \frac{H_0}{F_0} - 1 \geq 0$, because $u > 1$ and $H_0 \geq F_0$.

(i) If $\pi > \pi_2$, then $X_u > 0$ and $X_d = 0$, because $X_u > \alpha F_0 \left( \frac{H_0}{F_0} - 1 \right) (u - 1) + 1 - \frac{H_0}{F_0} = 0$,

$$
X_d \leq \alpha F_0 \left( \frac{H_0}{F_0} - 1 \right) (u - 1) + 1 - \frac{H_0}{F_0} = 0,
$$

and according to (4), $X_d \geq 0$. Therefore, the expected power utility is $p \frac{X^{1-\gamma}}{1-\gamma}$.

(ii) If $\pi < \pi_1$, then $X_u = 0$ and $X_d > 0$, because $X_d \geq \alpha F_0 \left( \frac{H_0}{F_0} - 1 \right) (u - 1) + 1 - \frac{H_0}{F_0} = 0$,

$$
X_u \leq \alpha F_0 \left( \frac{H_0}{F_0} - 1 \right) (u - 1) + 1 - \frac{H_0}{F_0} = 0,
$$

and according to (3), $X_u \geq 0$. Therefore, the expected power utility is $(1-p) \frac{X^{1-\gamma}}{1-\gamma}$. 4
(iii) If $\pi_1 \leq \pi \leq \pi_2$, then $X_u = 0$ and $X_d = 0$, because $X_u \leq \alpha F_0 \left( \frac{H_0}{\pi - 1} \right) = 0$, $X_d \leq \alpha F_0 \left( \frac{H_0}{\pi - 1} \right) = 0$, and according to (3) and (4), $X_u \geq 0$ and $X_d \geq 0$. Therefore, the expected power utility is 0.

We consider the following cases of $\frac{H_0}{F_0}$:

(i) If $\frac{H_0}{F_0} > u + 1$, then $\pi_1 < \pi^- < \pi^+ < \pi_2$, because $\pi_1 < \frac{u}{\pi - 1} < \frac{1}{1-u} = \pi^-$ and $\pi_2 > \frac{u}{\pi - 1} = \pi^+$. Then, for all $\pi^- \leq \pi \leq \pi^+$, $X_u = 0$ and $X_d = 0$. Therefore, any $\pi^- \leq \pi \leq \pi^+$ is the optimal strategy and $\max_{\pi^- \leq \pi \leq \pi^+} \mathbb{E}[U(X)] = 0$.

(ii) If $1 + \frac{1}{u} \leq \frac{H_0}{F_0} \leq u + 1$, then $\pi_1 < \pi^- < \pi_2 \leq \pi^+$, because $\pi_1 < \frac{u}{\pi - 1} < \frac{1}{1-u} = \pi^-$ and $\pi_2 \leq \frac{u}{\pi - 1} = \pi^+$.

(a) If $\pi^- \leq \pi < \pi_2$, then $X_u = 0$ and $X_d = 0$. Therefore, $\mathbb{E}[U(X)] = 0$ and $\frac{\partial \mathbb{E}[U(X)]}{\partial \pi} = 0$ for all $\pi^- \leq \pi < \pi_2$.

(b) If $\pi_2 \leq \pi \leq \pi^+$, then $X_u > 0$ and $X_d = 0$. Therefore,

$$
\mathbb{E}[U(X)] = p \frac{X_u^{1-\gamma}}{1-\gamma} = p \left( \frac{\alpha F_0 \left( \pi(u - 1) + 1 - \frac{H_0}{F_0} \right)}{1-\gamma} \right) > 0
$$

and

$$
\frac{\partial \mathbb{E}[U(X)]}{\partial \pi} = p \left( \frac{\alpha F_0 \left( \pi(u - 1) + 1 - \frac{H_0}{F_0} \right)}{1-\gamma} \right) > 0
$$

for all $\pi_2 \leq \pi \leq \pi^+$.

Therefore, $\pi = \pi^+$ reaches the maximum of the expected power utility, and

$$
\max_{\pi^- \leq \pi \leq \pi^+} \mathbb{E}[U(X)] = p \frac{X_u^{1-\gamma}}{1-\gamma} = p \left( \frac{\alpha F_0 \left( u + 1 - \frac{H_0}{F_0} \right)}{1-\gamma} \right) > 0.
$$

(iii) If $\frac{H_0}{F_0} < \frac{1}{u} + 1$, then $\pi^- < \pi_1 < \pi_2 < \pi^+$, because $\pi_1 > \frac{1}{u} \frac{u}{u-1} = \frac{1}{1-u} = \pi^-$ and $\pi_2 < \frac{1}{u} \frac{u}{u-1} = \pi^+$. 


(a) If \( \pi^\leq \leq \pi_1 \), then \( X_u = 0 \) and \( X_d > 0 \). Therefore,

\[
E[U(X)] = (1 - p) \frac{X_d^{1-\gamma}}{1-\gamma} = (1 - p) \frac{\left(\alpha F_0 \left( \pi \left( \frac{1}{u} - 1 \right) + 1 - \frac{H_0}{F_0} \right) \right)^{1-\gamma}}{1-\gamma},
\]

(10)

and

\[
\frac{\partial E[U(X)]}{\partial \pi} = (1 - p) \left( \alpha F_0 \left( \pi \left( \frac{1}{u} - 1 \right) + 1 - \frac{H_0}{F_0} \right) \right)^{-\gamma} \alpha F_0 \left( \frac{1}{u} - 1 \right) < 0
\]

(11)

for all \( \pi^- < \pi < \pi_1 \).

(b) If \( \pi_1 < \pi < \pi_2 \), then \( X_u = 0 \) and \( X_d = 0 \). According to the previous discussion,

\[
\max_{\pi_1 < \pi < \pi_2} E[U(X)] = 0 \quad \text{and} \quad \frac{\partial E[U(X)]}{\partial \pi} = 0
\]

for all \( \pi_1 < \pi < \pi_2 \).

(c) If \( \pi_2 < \pi < \pi^+ \), then \( X_u > 0 \) and \( X_d = 0 \). Therefore,

\[
E[U(X)] = p X_u^{1-\gamma} = p \frac{\left(\alpha F_0 \left( \pi \left( u - 1 \right) + 1 - \frac{H_0}{F_0} \right) \right)^{1-\gamma}}{1-\gamma},
\]

(12)

and

\[
\frac{\partial E[U(X)]}{\partial \pi} = p \left( \alpha F_0 \left( \pi \left( u - 1 \right) + 1 - \frac{H_0}{F_0} \right) \right)^{-\gamma} \alpha F_0 (u - 1) > 0
\]

(13)

for all \( \pi_2 < \pi < \pi^+ \).

According to the discussion above, \( E[U(X)] \) reaches the maximum at either \( \pi^- \) or \( \pi^+ \).

\[
E[U(X)]|_{\pi=\pi^+} = \left(1 - p\right) \frac{\left(\alpha F_0 \left( \frac{1}{u} + 1 - \frac{H_0}{F_0} \right) \right)^{1-\gamma}}{1-\gamma}.
\]

(14)

\[
E[U(X)]|_{\pi=\pi^-} = (1 - p) \frac{\left(\alpha F_0 \left( \frac{1}{u} + 1 - \frac{H_0}{F_0} \right) \right)^{1-\gamma}}{1-\gamma}.
\]

(15)

Thus, if \( (1 - p) \left( \frac{1}{u} + 1 - \frac{H_0}{F_0} \right)^{1-\gamma} \geq p \left( u + 1 - \frac{H_0}{F_0} \right)^{1-\gamma} \), the expected utility reaches its’ maximum at \( \pi = \pi^- \) and \( \max_{\pi^- \leq \pi \leq \pi^+} E[U(X)] = (1 - p) \frac{\left(\alpha F_0 \left( \frac{1}{u} + 1 - \frac{H_0}{F_0} \right) \right)^{1-\gamma}}{1-\gamma} \). If \( (1 - p) \left( \frac{1}{u} + 1 - \frac{H_0}{F_0} \right)^{1-\gamma} < p \left( u + 1 - \frac{H_0}{F_0} \right)^{1-\gamma} \), the expected utility reaches it maximum at
\[ \pi = \pi^+ \text{ and } \max_{\pi^- \leq \pi \leq \pi^+} \mathbb{E}[U(X)] = p \left( \frac{\alpha F_0 (u+1-H_0 F_0)}{1-\gamma} \right)^{1-\gamma}. \]

The intuition behind Theorem 2.1 is as follows. Based on the assumption of model parameters, the manager earns positive fees if he buys a sufficient amount of the stock \((\pi > \pi_2)\) and the stock price goes up. However, if the stock price goes down, the manager earns no fees. If the manager sells a sufficient amount of the stock \((\pi < \pi_1)\), then he receives positive fees if the stock price goes down. On the contrary, if the stock price goes up, the manager earns no fees. Any other strategies lead to the fees being zero. Thus, even if the marginal utility is \(+\infty\) at \(X = 0\), the manager can never avoid such a situation.

If the high-water mark is low \((\frac{H_0}{F_0} < \frac{1}{u} + 1)\), the fund manager compares the expected utility out of buying and selling and bets as much as possible in the better case. If the high-water mark is very high \((\frac{H_0}{F_0} > u + 1)\), the fund has no chance to exceed the high-water mark, which means the fund manager cannot get performance fees. Thus, any strategy works the same. If the high-water mark is \(1 + \frac{1}{u} < \frac{H_0}{F_0} < u + 1\), the fund manager only earns fees if the stock price goes up, so the manager buys in the risky asset as much as he is allowed to and the optimal strategy is \(\pi^+\).

In our model, we assume that \(0 < \gamma < 1\). Because if \(\gamma > 1\), \(U(X) = \frac{X^{1-\gamma}}{1-\gamma} \to -\infty\) if \(X \to 0^+\), while as shown in the proof of Theorem 2.1, \(X = 0\) is always of positive probability. Therefore, \(U(X) = -\infty\) and there is no optimal strategy. Hence, the optimization problem is not well defined. To analyze the problem with \(\gamma > 1\), the assumptions we made need to be modified and either the a positive management fee or a positive risk-free rate should work. If the management fee is positive, the fund manager always earns some fees regardless of the performance and the fee is not 0 even if the manager earns no performance fees. If the risk-free rate is positive, at least if the high-water mark is low, there always exists a \(\pi\) (for example, \(\pi = 0\)) such that \(X_u\) and \(X_d\) are both positive, and thus the expected power utility is not \(-\infty\).

Theorem 2.1 indicates that in the single period binomial tree model, if the fund manager receives only the performance fees at the end of the period and maximizes the expected power utility from the fees, the optimal strategy for him is always to buy or to sell the maximum amount allowed in the risky asset. However, if the high-water mark is too high, any strategy delivers zero utility.

In the next section, we are going to discuss whether the above conclusion will change if the fund manager is given a longer planning horizon.
3 Two Period Binomial Tree Model

In this section, we assume that the manager invests in the fund in a two period model, and check if the manager still chooses the most risky allocation he is allowed if he is given a longer horizon. To simplify the mathematical calculation, in addition to the assumptions in Section 2, we make the following assumptions: high water mark starts from the initial fund value \( H_0 = F_0 \), and the probability of the stock going up and down is equal \( p = \frac{1}{2} \).

Let \( \pi_0 \) and \( \pi_1 \) be the proportion of the fund invested in the stock in the first period and the second period, respectively. Because of the non-bankruptcy assumption, the constraint for \( \pi_0 \) and \( \pi_1 \) is \( \pi^- \leq \pi_0, \pi_1 \leq \pi^+ \), where \( \pi^- = \frac{1}{1-u} \) and \( \pi^+ = \frac{u}{u-1} \). The fund manager aims to maximize the expected power utility from performance fees in two periods, i.e. the maximization problem is

\[
\max_{\pi^- \leq \pi_0, \pi_1 \leq \pi^+} \mathbb{E}[U(X_1) + U(X_2)]. \tag{16}
\]

If the fund value at the end of the first period \( F_1 \) and the high-water mark at the end of the first period \( H_1 \) are given, then the optimal strategy for the second period \( \pi_1^* = \pi^* \left( \frac{H_1}{F_1} \right) \), where \( \pi^* \) is defined in Theorem 2.1. Since \( F_1, X_1 \) and \( H_1 \) are functions of \( \pi_0 \), instead of

\[
\max_{\pi^- \leq \pi_0, \pi_1 \leq \pi^+} \mathbb{E}[U(X_1) + U(X_2)],
\]

we consider the maximization problem

\[
\max_{\pi^- < \pi_0 < \pi^+} \mathbb{E}[B(\pi_0)],
\]

where

\[
B(\pi_0) = U(X_1) + \max_{\pi^- < \pi_0 < \pi^+} \mathbb{E}[U(X_2) | F_1, H_1]. \tag{17}
\]

We call \( \mathbb{E}[B(\pi_0)] \) the semi-value function.

**Theorem 3.1.** In the two period binomial tree model,

\[
\mathbb{E}[B(\pi_0)] = \begin{cases} 
\frac{C}{2} \left( -W(\pi_0) \right)^{1-\gamma} + \frac{C}{4} (u - N(\pi_0))^{1-\gamma}, & \text{if } \frac{1}{1-u} \leq \pi_0 < \frac{u}{1-u^2} \\
\frac{C}{4} ((u+1)W(\pi_0) + u)^{1-\gamma} + \frac{C}{2} \left( \frac{W(\pi_0)}{u} \right)^{1-\gamma} + \frac{C}{4} (u - N(\pi_0))^{1-\gamma}, & \text{if } \frac{u}{1-u^2} \leq \pi_0 \leq 0 \\
\frac{C}{4} (u - \frac{1+u}{u} W(\pi_0))^{1-\gamma} + \frac{C}{2} W(\pi_0)^{1-\gamma} + \frac{C}{4} (u (N(\pi_0) + 1))^{1-\gamma}, & \text{if } 0 < \pi_0 \leq \frac{u^2}{u^2-1} \\
\frac{C}{2} W(\pi_0)^{1-\gamma} + \frac{C}{4} ((N(\pi_0) + 1)u)^{1-\gamma}, & \text{if } \frac{u^2}{u^2-1} < \pi_0 \leq \frac{u}{u-1},
\end{cases} \tag{18}
\]

where \( C = \frac{(\alpha F_0)^{1-\gamma}}{1-\gamma} \), \( W(\pi_0) = \pi_0 (u-1) \) and \( N(\pi_0) = (1-\alpha)(u-1)\pi_0 \).
Proof. In the first period, we assume \( H_0 = F_0 \), \( 0 < \alpha < 1 \), \( u > 1 \) and \( d = \frac{1}{u} < 1 \). Let \( \bar{F}_{1u} \) and \( \bar{F}_{1d} \) be the fund value at \( t = 1 \) before the manager is paid by performance fees if the stock price goes up or down in the first period respectively. Let \( F_{1u} \) and \( F_{1d} \) be the fund value at \( t = 1 \) after the manager is paid by fees, if the stock price goes up or goes down in the first period, respectively. Similarly, we define \( H_{1u}, H_{1d}, X_{1u} \) and \( X_{1d} \) for high-water mark \( H_1 \) and performance fees \( X_1 \) at the end of the first period, respectively.

Notice that since \( p = \frac{1}{2} \), the optimal \( \pi^*_1 = \pi^* \left( \frac{H_1}{F_1} \right) \) is never \( \pi^- \), because \( \frac{1}{2} \left( \frac{1}{u} + 1 - \frac{H_1}{F_1} \right)^{1-\gamma} < \frac{1}{2} \left( u + 1 - \frac{H_1}{F_1} \right)^{1-\gamma} \) always holds for \( \frac{H_1}{F_1} < \frac{1}{u} + 1 \) (see details in Theorem 2.1).

The values of \( \bar{F}_1, X_1, F_1 \) and \( H_1 \) depend on the choice of \( \pi_0 \).

(a) If \( \pi_0 \leq 0 \),

\[
\begin{align*}
\bar{F}_{1u} &= F_0 \pi_0 (u - 1) + F_0 \leq F_0 = H_0. \\
\bar{F}_{1d} &= F_0 \pi_0 (d - 1) + F_0 \geq F_0 = H_0.
\end{align*}
\]

Therefore, \( X_{1u} = 0 \), because \( \bar{F}_{1u} \leq H_0 \), and \( X_{1d} = \alpha F_0 \pi_0 (d - 1) \geq 0 \) because \( \bar{F}_{1d} \geq H_0 \). Hence, the fund value at \( t = 1 \) after the manager is paid by performance fees are

\[
\begin{align*}
F_{1u} &= F_0 \pi_0 (u - 1) + F_0 \leq F_0 = H_0. \\
F_{1d} &= \bar{F}_{1d} - X_{1d} = (1 - \alpha) F_0 \pi_0 (d - 1) + F_0 \geq F_0 = H_0.
\end{align*}
\]

The high-water mark at the end of the first period is \( H_{1u} = H_0 = F_0 \) and \( H_{1d} = F_{1d} \), respectively.

(b) If \( \pi_0 > 0 \),

\[
\begin{align*}
\bar{F}_{1u} &= F_0 \pi_0 (u - 1) + F_0 > F_0 = H_0. \\
\bar{F}_{1d} &= F_0 \pi_0 (d - 1) + F_0 < F_0 = H_0.
\end{align*}
\]

Therefore, \( X_{1u} = \alpha F_0 \pi_0 (u - 1) > 0 \) because \( \bar{F}_{1u} > H_0 \), and \( X_{1d} = 0 \) because \( \bar{F}_{1d} < H_0 \). Hence, the fund value at \( t = 1 \) after the manager is paid by performance fees are

\[
\begin{align*}
F_{1u} &= \bar{F}_{1u} - X_{1u} = (1 - \alpha) F_0 \pi_0 (u - 1) + F_0 > F_0 = H_0. \\
F_{1d} &= \bar{F}_{1d} = F_0 \pi_0 (d - 1) + F_0 < F_0 = H_0.
\end{align*}
\]
The high-water mark at the end of the first period is $H_{1u} = F_{1u}$ and $H_{1d} = H_0 = F_0$, respectively.

We discuss the following cases for $\pi_0$:

**Case 1:** $\frac{1}{1-u} \leq \pi_0 < \frac{u}{1-u^2}$.

Because $\frac{u}{1-u^2} < 0$, according to (20),

(i) $F_{1u} = F_0 \pi_0 (u - 1) + F_0 < F_0 \frac{u}{1-u^2} (u - 1) + F_0 = H_0$, $X_{1u} = 0$ and $H_{1u} = H_0 = F_0$. Therefore, $H_{1u} = \frac{F_0 \pi_0 (u - 1) + F_0}{\frac{u}{1-u^2} (u - 1) + 1} = u + 1$. According to the first case in Theorem 2.1, any $\frac{1}{1-u} < \pi_1 < \frac{u}{u-1}$ is optimal. The maximum expected power utility of $X_2$ given $H_{F_1} > u + 1$ is

\[
\max_{\pi^- \leq \pi_1 \leq \pi^+} \mathbb{E} \left[ U(X_2) \mid H_{F_1} = 1 \right] = \frac{1}{2} \left( \frac{\alpha F_{1d} u}{1-\gamma} \right)^{1-\gamma} \frac{1}{1-\gamma} = \frac{1}{2} \left( \frac{\alpha ((1-\alpha) F_0 \pi_0 (d-1) + F_0) u}{1-\gamma} \right)^{1-\gamma}. \tag{23}
\]

Notice that at the end of the first period, $\frac{H_{F_1} u}{H_{F_1} d} > u + 1$ or $\frac{H_{F_1} d}{H_{F_1} u} = 1$ if and only if $\frac{S_1}{S_0} = u$ or $\frac{S_1}{S_0} = d$, respectively. Therefore, if $\frac{1}{1-u} \leq \pi_0 < \frac{u}{1-u^2},$

\[
\mathbb{E} [B(\pi_0)] = \mathbb{E} \left[ U(X_1) + \max_{\pi^- \leq \pi_1 \leq \pi^+} \mathbb{E} \left[ U(X_2) \mid H_{F_1} = 1 \right] \right] = \left( U(X_1) + \max_{\pi^- \leq \pi_1 \leq \pi^+} \mathbb{E} \left[ U(X_2) \mid H_{F_1} > u + 1 \right] \right) P \left( \frac{S_1}{S_0} = u \right) + \left( U(X_1) + \max_{\pi^- \leq \pi_1 \leq \pi^+} \mathbb{E} \left[ U(X_2) \mid H_{F_1} = 1 \right] \right) P \left( \frac{S_1}{S_0} = d \right) = \left( \frac{1}{2} + \left( \frac{(X_{1d})^{1-\gamma}}{1-\gamma} + \frac{1}{2} \left( \frac{\alpha F_{1d} u}{1-\gamma} \right)^{1-\gamma} \right) \frac{1}{2} \right)
\]

\[
= \frac{1}{2} \left( \frac{(X_{1d})^{1-\gamma}}{1-\gamma} + \frac{1}{4} \left( \frac{\alpha F_{1d} u}{1-\gamma} \right)^{1-\gamma} \right)
\]

\[
= \frac{1}{2} \left( \frac{(\alpha \pi_0 F_0 (d-1))^{1-\gamma}}{1-\gamma} + \frac{1}{4} \left( \frac{\alpha ((1-\alpha) \pi_0 F_0 (d-1) + F_0) u}{1-\gamma} \right)^{1-\gamma} \right).
\]

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\[= \left(\frac{1}{u^2} - 1\right)^{1-\gamma} \frac{\alpha F_0(1 - (1 - \alpha)(u - 1)\pi_0)}{4(1 - \gamma)}. \] (24)

**Case 2:** \(\frac{u}{1-u^2} \leq \pi_0 \leq 0.\)

According to (20),

(i) \(F_{1u} = F_0 \pi_0(u - 1) + F_0 \leq F_0, X_{1u} = 0 \) and \(H_{1u} = H_0 = F_0.\) Therefore, \(H_{1u} = F_0 \pi_0(u - 1) + F_0 \leq u + 1,\) because \(F_0 \pi_0(u - 1) + F_0 \geq 0(u - 1) + 1 = 1\) and \(u \leq \pi_0(u - 1) + 1 = u + 1.\) \(\pi_1 = \frac{u}{u-1}\) is optimal, as is indicated in the second and fourth case of Theorem 2.1. The maximum expected power utility of \(X_2\) given \(1 < \frac{H_{1u}}{F_1} < u + 1\) is

\[
\max_{\pi^{-} \leq \pi_1 \leq \pi^{+}} \mathbb{E} \left[ U(X_2) | 1 \leq \frac{H_1}{F_1} \leq u + 1 \right] = \frac{1}{2} \left( \alpha F_{1u} \left( u + 1 - \frac{H_0}{F_{1u}} \right) \right)^{1-\gamma}
\]

\[
= \frac{1}{2} \left( \alpha (F_0 \pi_0(u - 1) + F_0)(u + 1) - \alpha H_0 \right)^{1-\gamma}.
\] (25)

(ii) \(F_{1d} = (1 - \alpha)F_0 \pi_0(d - 1) + F_0 \geq F_0,\) \(X_{1d} = \alpha \pi_0 F_0(d - 1) \geq 0\) and \(H_{1d} = F_{1d}\). Therefore, \(\frac{H_{1d}}{F_{1d}} = 1.\) \(\pi_1 = \frac{u}{u-1}\) is optimal, as is indicated by the fourth case of Theorem 2.1. The maximum expected power utility of \(X_2\) given \(\frac{H_{1d}}{F_1} = 1\) is

\[
\max_{\pi^{-} \leq \pi_1 \leq \pi^{+}} \mathbb{E} \left[ U(X_2) | \frac{H_1}{F_1} = 1 \right] = \frac{1}{2} \left( \alpha F_{1d} \right)^{1-\gamma}
\]

\[
= \frac{1}{2} \left( \alpha ((1 - \alpha)F_0 \pi_0(d - 1) + F_0)u \right)^{1-\gamma}.
\] (26)

Notice that at the end of the first period, \(1 \leq \frac{H_{1u}}{F_1} \leq u + 1\) or \(\frac{H_1}{F_1} = 1\) if and only if \(\frac{S_1}{S_0} = u\) or \(\frac{S_1}{S_0} = d\), respectively. Therefore, if \(\frac{u}{1-u^2} \leq \pi_0 \leq 0,\)

\[
\mathbb{E}[B(\pi_0)] = \mathbb{E} \left[ U(X_1) + \max_{\pi^{-} \leq \pi_1 \leq \pi^{+}} \mathbb{E} \left[ U(X_2) \right] \right]
\]

\[
= \left( U(X_1) + \max_{\pi^{-} \leq \pi_1 \leq \pi^{+}} \mathbb{E} \left[ U(X_2) | 1 \leq \frac{H_{1u}}{F_{1u}} \leq u + 1 \right] \right) P \left( \frac{S_1}{S_0} = u \right)
\]

\[
+ \left( U(X_1) + \max_{\pi^{-} \leq \pi_1 \leq \pi^{+}} \mathbb{E} \left[ U(X_2) \right] \right) P \left( \frac{S_1}{S_0} = d \right)
\]

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Case 3: \(0 < \pi_0 \leq \frac{w^2}{w^2-1}\).

According to (22),

(i) \(F_{1u} = (1 - \alpha)F_0\pi_0(u-1) + F_0\), \(X_{1u} = \alpha\pi_0 F_0(u-1)\) and \(H_{1u} = F_{1u}\). Therefore, \(\frac{H_{1}}{F_{1u}} = 1\). \(\pi_1 = \frac{u}{u-1}\) is optimal, as is indicated by the fourth case of Theorem 2.1. The maximum expected power utility of \(X_2\) given \(\frac{H_{1u}}{F_{1u}} = 1\) is

\[
\max_{\pi^- \leq \pi_1 \leq \pi^+} \mathbb{E} \left[ U(X_2) \mid H_1 \right] = \frac{1}{2}\left(\frac{\alpha F_{1u}u}{1 - \gamma}\right)^{1-\gamma}
\]

(ii) \(F_{id} = F_0\pi_0(d-1) + F_0 < F_0\), \(X_{1d} = 0\) and \(H_{1d} = H_0 = F_0\). Therefore, \(\frac{H_{1d}}{F_{1d}} = \frac{F_0\pi_0(d-1)+F_0}{F_0\pi_0(d-1)+F_0} = \frac{1}{\frac{H_0}{F_0}\pi_0(d-1)+1} = u + 1\). \(\pi_1 = \frac{u}{u-1}\) is optimal, as is indicated by the second and the fourth case of Theorem 2.1. The maximum expected power utility of \(X_2\) given \(1 < \frac{H_{1d}}{F_{1d}} \leq u + 1\) is

\[
\max_{\pi^- \leq \pi_1 \leq \pi^+} \mathbb{E} \left[ U(X_2) \mid 1 < \frac{H_{1d}}{F_{1d}} \leq u + 1\right] = \frac{1}{2}\left(\frac{\alpha F_{1d} u + 1 - \frac{H_0}{F_{1d}}}{1 - \gamma}\right)^{1-\gamma}
\]
Notice that at the end of the first period, \( \frac{H_t}{F_t} = 1 \) or \( 1 < \frac{H_t}{F_t} \leq u + 1 \) if and only if \( \frac{s_1}{s_0} = u \) or \( \frac{s_1}{s_0} = d \), respectively. Therefore, if \( 0 < \pi_0 \leq \frac{u^2}{u^2 - 1} \),

\[
\mathbb{E}[B(\pi_0)] = \mathbb{E} \left[ U(X_1) + \max_{\pi^- \leq \pi_1 \leq \pi^+} \mathbb{E} \left[ U(X_2)|\frac{H_1}{F_1} \right] \right]
\]

\[
= \left( U(X_1) + \max_{\pi^- \leq \pi_1 \leq \pi^+} \mathbb{E} \left[ U(X_2)|1 < \frac{H_{id}}{F_{id}} \leq u + 1 \right] \right) P \left( \frac{S_1}{S_0} = d \right)
\]

\[
+ \left( U(X_1) + \max_{\pi^- \leq \pi_1 \leq \pi^+} \mathbb{E} \left[ U(X_2)|\frac{H_{iu}}{F_{iu}} = 1 \right] \right) P \left( \frac{S_1}{S_0} = u \right)
\]

\[
= \left( \frac{1}{2} (\frac{(x_{1u}((1-\pi_0)F_0(u-1)+F_0)\pi_0+\alpha u)}{1-\gamma}) + \frac{1}{2} (\frac{(x_{1u}((1-\pi_0)F_0(u-1)+F_0)\pi_0+\alpha u)}{1-\gamma}) \right)
\]

\[
= \frac{1}{4} \left( (\alpha u + 1) ((1-\pi_0)F_0(u-1)+F_0)\pi_0+\alpha u ) \right)
\]

\[
+ \frac{1}{4} \left( (\alpha u + 1) ((1-\pi_0)F_0(u-1)+F_0)\pi_0+\alpha u ) \right)
\]

\[
= \frac{(\alpha u + 1) ((1-\pi_0)F_0(u-1)+F_0)\pi_0+\alpha u )}{4(1-\gamma)}
\]

\[
+ \frac{(\alpha u + 1) ((1-\pi_0)F_0(u-1)+F_0)\pi_0+\alpha u )}{4(1-\gamma)}
\]

\[
= \frac{(\alpha u + 1) ((1-\pi_0)F_0(u-1)+F_0)\pi_0+\alpha u )}{4(1-\gamma)}
\]

\[
= \frac{(\alpha u + 1) ((1-\pi_0)F_0(u-1)+F_0)\pi_0+\alpha u )}{4(1-\gamma)}
\]

\[
\Rightarrow \frac{\alpha u + 1) ((1-\pi_0)F_0(u-1)+F_0)\pi_0+\alpha u )}{4(1-\gamma)}
\]

Case 4: \( \frac{u^2}{u^2 - 1} < \pi_0 \leq \frac{u}{u-1} \).

Because \( u > 1 \), \( \frac{u^2}{u^2 - 1} > 0 \). Then, according to (22),

(i) \( F_{iu} = (1-\alpha)F_0\pi_0(u-1)+F_0 > F_0 \), \( X_{iu} = \alpha\pi_0F_0(u-1) > 0 \) and \( H_{iu} = F_{iu} \).

Therefore, \( \frac{H_{iu}}{F_{iu}} = 1 \). \( \pi_1 = \frac{u}{u-1} \) is optimal, as is indicated by the fourth case of Theorem 2.1. The maximum expected power utility of \( X_2 \) given \( \frac{H_t}{F_t} = 1 \) is

\[
\max_{\pi^- \leq \pi_1 \leq \pi^+} \mathbb{E} \left[ U(X_2)|\frac{H_1}{F_1} = 1 \right] = \frac{1}{2} (\alpha F_{iu} u)^{1-\gamma}
\]

\[
\Rightarrow \frac{1}{2} (\alpha ((1-\alpha)F_0\pi_0(u-1)+F_0)\pi_0+\alpha u )^{1-\gamma}
\]

(ii) \( F_{id} = F_0\pi_0(d-1)+F_0 < F_0 \), \( X_{id} = 0 \) and \( H_{id} = H_0 = F_0 \). Therefore, \( \frac{H_{id}}{F_{id}} = \frac{H_0}{F_0\pi_0(d-1)+F_0} > \frac{\sqrt[u-1]{1-\pi_0}}{u-1} \) = \( u + 1 \). Any \( \frac{1}{u-1} < \pi_1 < \frac{u}{u-1} \) is optimal, as is indicated by the first case of Theorem 2.1. The maximum expected power utility of \( X_2 \) given
\[ \frac{H_1}{F_1} > u + 1 \text{ is } \max_{\pi^- \leq \pi_1 \leq \pi^+} \mathbb{E} \left[ U(X_2) | \frac{H_1}{F_1} > u + 1 \right] = 0. \]

Notice that at the end of the first period, \( \frac{H_1}{F_1} = 1 \) or \( \frac{H_1}{F_1} > u + 1 \) if and only if \( \frac{S_1}{S_0} = u \) or \( \frac{S_1}{S_0} = d \), respectively. Therefore, if \( \frac{u^2}{u^2 - 1} < \pi_0 \leq \frac{u}{u - 1} \),

\[
\mathbb{E}[B(\pi_0)] = \mathbb{E} \left[ U(X_1) + \max_{\pi^- \leq \pi_1 \leq \pi^+} \mathbb{E} \left[ U(X_2) | \frac{H_1}{F_1} > u + 1 \right] \right] 
= \left( U(X_1) + \max_{\pi^- \leq \pi_1 \leq \pi^+} \mathbb{E} \left[ U(X_2) | \frac{H_1}{F_1} > u + 1 \right] \right) \mathbb{P} \left( \frac{S_1}{S_0} = d \right) 
+ \left( U(X_1) + \max_{\pi^- \leq \pi_1 \leq \pi^+} \mathbb{E} \left[ U(X_2) | \frac{H_1}{F_1} = 1 \right] \right) \mathbb{P} \left( \frac{S_1}{S_0} = u \right) 
= \left( 0 \right) \frac{1}{2} + \left( \frac{(X_{1u})_{1-\gamma}}{1-\gamma} + \frac{1}{2} \left( \frac{\alpha F_{1u}}{1-\gamma} \right) \right) \frac{1}{2} 
= \frac{1}{2} \left( \frac{(X_{1u})_{1-\gamma}}{1-\gamma} \right) + \frac{1}{4} \left( \frac{\alpha F_{1u}}{1-\gamma} \right) 
= \frac{(\alpha \pi_0 F_0 (u - 1))_{1-\gamma}}{2(1-\gamma)} + \frac{(\alpha (1 - \alpha) \pi_0 F_0 (u - 1) + F_0) u}_{4(1-\gamma)}. \quad (32)
\]

Hence, the expected semi-value function \( \mathbb{E}[B(\pi_0)] \) is in (18). We are interested in the \( \pi_0^* \) such that \( \mathbb{E}[B(\pi_0)] \) reaches the maximum.

**Corollary 3.2.** In Theorem 3.1, there exists a \( \pi_i^* \) in \( (\frac{u}{u^2}, 0) \) and a \( \pi_i^* \) in \( (0, \frac{u^2}{u^2 - 1}) \) such that \( \frac{\partial \mathbb{E}[B(\pi_0)]}{\partial \pi_i} |_{\pi_i^*} = 0 \), for \( i = 2, 3 \). The optimal strategy in the first period \( \pi_0^* = \text{argmax} \{ \mathbb{E}[B(\pi_0)], \pi_0 \in \{ \pi_2^*, \pi_3^*, \pi^+ \} \} \).

**Proof.** We can directly calculate the first order derivative and second order derivative of \( \mathbb{E}[B(\pi_0)] \) with respect to \( \pi_0 \) for each case in Theorem 3.1.

(i) For case 1,

\[
\frac{\partial \mathbb{E}[B(\pi_0)]}{\partial \pi_0} = \frac{1}{2} \left( \alpha \pi_0 F_0 (d - 1) \right)^{1-\gamma} (\alpha F_0 (d - 1)) + \frac{1}{4} (\alpha ((1 - \alpha) \pi_0 F_0 (d - 1) + F_0) u)^{1-\gamma} \alpha u (1 - \alpha) F_0 (d - 1). \quad (33)
\]

\[
\frac{\partial^2 \mathbb{E}[B(\pi_0)]}{\partial \pi_0^2} = -\frac{\gamma}{2} (\alpha \pi_0 F_0 (d - 1))^{1-\gamma-1} (\alpha F_0 (d - 1))^2
\]
\[-\frac{\gamma}{4}(\alpha((1 - \alpha)\pi_0 F_0 (d - 1) + F_0)u)^{-\gamma - 1}(\alpha u (1 - \alpha) F_0 (d - 1))^2. \tag{34}\]

Therefore, \(\frac{\partial \mathbb{E}[B(\pi_0)]}{\partial \pi_0} < 0\) because \(d < 1\). \(\frac{\partial^2 \mathbb{E}[B(\pi_0)]}{\partial \pi_0^2} < 0\) because \(\gamma > 0\).

(ii) For case 2,

\[
\frac{\partial \mathbb{E}[B(\pi_0)]}{\partial \pi_0} = \frac{1}{4} (\alpha (u + 1)(\pi_0 F_0(u - 1) + F_0) - \alpha H_0)^{-\gamma} \left[\alpha (u + 1) F_0 (u - 1)\right] \\
+ \frac{1}{2}(\alpha \pi_0 F_0(d - 1))^{-\gamma}(\alpha F_0(d - 1)) \\
+ \frac{1}{4}(\alpha((1 - \alpha)\pi_0 F_0(d - 1) + F_0)u)^{-\gamma}\alpha u (1 - \alpha) F_0 (d - 1). \tag{35}\]

\[
\frac{\partial^2 \mathbb{E}[B(\pi_0)]}{\partial \pi_0^2} = -\frac{\gamma}{4} (\alpha (u + 1)(\pi_0 F_0(u - 1) + F_0) - \alpha H_0)^{-\gamma - 1}(\alpha (u + 1) F_0 (u - 1))^2 \\
- \frac{\gamma}{2} (\alpha \pi_0 F_0(d - 1))^{-\gamma - 1}(\alpha F_0(d - 1))^2 \\
- \frac{\gamma}{4}(\alpha((1 - \alpha)\pi_0 F_0(d - 1) + F_0)u)^{-\gamma - 1}(\alpha u (1 - \alpha) F_0 (d - 1))^2. \tag{36}\]

Therefore, \(\frac{\partial^2 \mathbb{E}[B(\pi_0)]}{\partial \pi_0^2} < 0\).

(iii) For case 3,

\[
\frac{\partial \mathbb{E}[B(\pi_0)]}{\partial \pi_0} = \frac{1}{4} (\alpha (u + 1)(\pi_0 F_0(d - 1) + F_0) - \alpha H_0)^{-\gamma} \left[\alpha (u + 1) F_0 (d - 1)\right] \\
+ \frac{1}{2}(\alpha \pi_0 F_0(u - 1))^{-\gamma}(\alpha F_0(u - 1)) \\
+ \frac{1}{4}(\alpha((1 - \alpha)\pi_0 F_0(u - 1) + F_0)u)^{-\gamma}\alpha u (1 - \alpha) F_0 (u - 1). \tag{37}\]

\[
\frac{\partial^2 \mathbb{E}[B(\pi_0)]}{\partial \pi_0^2} = -\frac{\gamma}{4} (\alpha (u + 1)(\pi_0 F_0(d - 1) + F_0) - \alpha H_0)^{-\gamma - 1}(\alpha (u + 1) F_0 (d - 1))^2 \\
- \frac{\gamma}{2} (\alpha \pi_0 F_0(u - 1))^{-\gamma - 1}(\alpha F_0(u - 1))^2 \\
- \frac{\gamma}{4}(\alpha((1 - \alpha)\pi_0 F_0(u - 1) + F_0)u)^{-\gamma - 1}(\alpha u (1 - \alpha) F_0 (u - 1))^2. \tag{38}\]

Therefore, \(\frac{\partial^2 \mathbb{E}[B(\pi_0)]}{\partial \pi_0^2} < 0\).

(iv) For case 4,

\[
\frac{\partial \mathbb{E}[B(\pi_0)]}{\partial \pi_0} = \frac{1}{2}(\alpha \pi_0 F_0(u - 1))^{-\gamma}(\alpha F_0(u - 1)) \\
+ \frac{1}{4}(\alpha((1 - \alpha)\pi_0 F_0(u - 1) + F_0)u)^{-\gamma}\alpha u (1 - \alpha) F_0 (u - 1). \tag{39}\]
\[
\frac{\partial^2 \mathbb{E}[B(\pi_0)]}{\partial \pi_0^2} = -\frac{\gamma}{2}(\alpha \pi_0 F_0(u - 1))^{-\gamma-1}(\alpha F_0(u - 1))^2 \\
-\frac{\gamma}{4}(\alpha((1-\alpha)\pi_0 F_0(u - 1) + F_0)u)^{-\gamma-1}(\alpha u(1-\alpha)F_0(u - 1))^2 
\] 

(40)

Therefore, \(\frac{\partial \mathbb{E}[B(\pi_0)]}{\partial \pi_0} > 0\) and \(\frac{\partial^2 \mathbb{E}[B(\pi_0)]}{\partial \pi_0^2} < 0\).

\(\mathbb{E}[B(\pi_0)]\) is piecewise concave and continuous. Because \(\frac{\partial \mathbb{E}[B(\pi_0)]}{\partial \pi_0} < 0\) in case 1, the maximum of \(\mathbb{E}[B(\pi_0)]\) in \(\left[\frac{1}{1-u}, \frac{u}{u^2}\right]\) is achieved at \(\pi^- = \frac{1}{1-u}\). Similarly, because \(\frac{\partial \mathbb{E}[B(\pi_0)]}{\partial \pi_0} > 0\) in case 4, the maximum of \(\mathbb{E}[B(\pi_0)]\) in \(\left(\frac{u^2}{u^2-1}, \frac{u}{u-1}\right]\) is achieved at \(\pi^+ = \frac{u}{u-1}\). Next, consider the first order derivative at the endpoints in case 2 and case 3.

In case 2, if \(\pi_0 \to \left(\frac{u}{1-u^2}\right)^+\),

\[
\lim_{\pi_0 \to \left(\frac{u}{1-u^2}\right)^+} \frac{1}{4} \left(\alpha(u+1)(\pi_0 F_0(u - 1) + F_0) - \alpha H_0\right)^{-\gamma} [\alpha(u+1)F_0(u - 1)] \to +\infty. 
\] 

(41)

\[
\lim_{\pi_0 \to \left(\frac{u}{1-u^2}\right)^+} \frac{1}{2} \left(\alpha \left(\frac{u}{1-u^2}\right) F_0(d - 1)\right)^{-\gamma} (\alpha F_0(d - 1)) \\
= \frac{1}{2} \left(\alpha \left(\frac{1}{u+1}\right) F_0\right)^{-\gamma} (\alpha F_0(d - 1)). 
\] 

(42)

\[
\lim_{\pi_0 \to \left(\frac{u}{1-u^2}\right)^+} \frac{1}{4} \left(\alpha((1-\alpha)\frac{u}{1-u^2}F_0(d - 1) + F_0)u\right)^{-\gamma} \alpha u(1-\alpha)F_0(d - 1) \\
= \frac{1}{4} \left(\alpha(1-\alpha)\frac{u}{u+1}F_0 + F_0u\right)^{-\gamma} \alpha u(1-\alpha)F_0(d - 1). 
\] 

(43)

Therefore, \(\frac{\partial \mathbb{E}[B(\pi_0)]}{\partial \pi_0}\big|_{\pi_0 \to \left(\frac{u}{1-u^2}\right)^+} \to +\infty\).

If \(\pi_0 \to 0^-\),

\[
\lim_{\pi_0 \to 0^-} \frac{1}{2} \left(\alpha \pi_0 F_0(d - 1)\right)^{-\gamma} (\alpha F_0(d - 1)) \to -\infty. 
\] 

(44)

\[
\lim_{\pi_0 \to 0^-} \frac{1}{4} \left(\alpha(u+1)(\pi_0 F_0(u - 1) + F_0) - \alpha H_0\right)^{-\gamma} [\alpha(u+1)F_0(u - 1)] \\
= \frac{1}{4} \left(\alpha(u+1)(F_0 - \alpha H_0)\right)^{-\gamma} [\alpha F_0(u^2 - 1)]. 
\] 

(45)

\[
\lim_{\pi_0 \to 0^-} \frac{1}{4} \left(\alpha((1-\alpha)\pi_0 F_0(d - 1) + F_0)u\right)^{-\gamma} \alpha u(1-\alpha)F_0(d - 1) \\
= \frac{1}{4} (\alpha F_0u)^{-\gamma} \alpha u(1-\alpha)F_0(d - 1). 
\] 

(46)
Therefore, \( \frac{\partial E[B(\pi_0)]}{\partial \pi_0} \big|_{\pi_0 \to 0^-} \to -\infty \).

In case 2, the second order derivative is always negative. Therefore, the first order derivative is decreasing. Since \( \frac{\partial E[B(\pi_0)]}{\partial \pi_0} \big|_{\pi_0 \to \frac{u}{1-u^2}^+} \to +\infty \) and \( \frac{\partial E[B(\pi_0)]}{\partial \pi_0} \big|_{\pi_0 \to 0^-} \to -\infty \), by continuity, there must exist a \( \pi^*_{c2} \) in \( \left[ \frac{u}{1-u^2}, 0 \right] \) such that \( \frac{\partial E[B(\pi_0)]}{\partial \pi_0} \big|_{\pi^*_{c2}} = 0 \) and the maximum of \( E[B(\pi_0)] \) in \( \left[ \frac{u}{1-u^2}, 0 \right] \) is achieved at \( \pi^*_{c2} \).

In case 3, if \( \pi_0 \to 0^+ \),

\[
\lim_{\pi_0 \to 0^+} \frac{1}{2} (\alpha \pi_0 F_0(u - 1))^{-\gamma} (\alpha F_0(u - 1)) \to +\infty. \tag{47}
\]

\[
\lim_{\pi_0 \to 0^+} \frac{1}{4} (\alpha(u + 1)(\pi_0 F_0(d - 1) + F_0) - \alpha H_0)^{-\gamma} [\alpha(u + 1) F_0(d - 1)]
\]

\[
= \frac{1}{4} (\alpha(u + 1) F_0 - \alpha H_0)^{-\gamma} [\alpha(u + 1) F_0(d - 1)]. \tag{48}
\]

\[
\lim_{\pi_0 \to 0^+} \frac{1}{4} (\alpha((1 - \alpha) \pi_0 F_0(u - 1) + F_0) u)^{-\gamma} \alpha u(1 - \alpha) F_0(u - 1)
\]

\[
= \frac{1}{4} (\alpha F_0 u)^{-\gamma} \alpha u(1 - \alpha) F_0(u - 1). \tag{49}
\]

Therefore, \( \frac{\partial E[B(\pi_0)]}{\partial \pi_0} \big|_{\pi_0 \to 0^+} \to +\infty \).

If \( \pi_0 \to -\left( \frac{u^2}{u^2-1} \right)^- \),

\[
\lim_{\pi_0 \to -\left( \frac{u^2}{u^2-1} \right)^-} \frac{1}{4} (\alpha(u + 1)(\pi_0 F_0(d - 1) + F_0) - \alpha H_0)^{-\gamma} [\alpha(u + 1) F_0(d - 1)] \to -\infty. \tag{50}
\]

\[
\lim_{\pi_0 \to -\left( \frac{u^2}{u^2-1} \right)^-} \frac{1}{2} (\alpha \pi_0 F_0(u - 1))^{-\gamma} (\alpha F_0(u - 1)) = \frac{1}{2} \left( \frac{\alpha F_0 u}{u - 1} \right)^{-\gamma} (\alpha F_0(u - 1)). \tag{51}
\]

\[
\lim_{\pi_0 \to -\left( \frac{u^2}{u^2-1} \right)^-} \frac{1}{4} (\alpha((1 - \alpha) \pi_0 F_0(u - 1) + F_0) u)^{-\gamma} \alpha u(1 - \alpha) F_0(u - 1)
\]

\[
= \frac{1}{4} (\alpha((1 - \alpha) \pi_0 F_0(u - 1) + F_0) u)^{-\gamma} \alpha u(1 - \alpha) F_0(u - 1). \tag{52}
\]

Therefore, \( \frac{\partial E[B(\pi_0)]}{\partial \pi_0} \big|_{\pi_0 \to -\left( \frac{u^2}{u^2-1} \right)^-} \to -\infty \).

In case 3, the second order derivative is always negative. Therefore, the first order derivative is decreasing. Since \( \frac{\partial E[B(\pi_0)]}{\partial \pi_0} \big|_{\pi_0 \to 0^+} \to +\infty \) and \( \frac{\partial E[B(\pi_0)]}{\partial \pi_0} \big|_{\pi_0 \to -\left( \frac{u^2}{u^2-1} \right)^-} \to -\infty \), by continuity, there must exist a \( \pi^*_{c3} \) in \( \left( 0, \frac{u^2}{u^2-1} \right] \) such that \( \frac{\partial E[B(\pi_0)]}{\partial \pi_0} \big|_{\pi^*_{c3}} = 0 \) and the maximum of \( E[B(\pi_0)] \) in \( \left( 0, \frac{u^2}{u^2-1} \right] \) is achieved at \( \pi^*_{c3} \).
To summarize, $E[B(\pi_0)]$ must reach the maximum at one of the four points, $\pi^-$, $\pi^*_c$, $\pi^*_c$ or $\pi^+$. 

By directly plunging in $\pi^-$ and $\pi^+$,

$$E[B(\pi^-)] = \frac{(\alpha F_0(\frac{1}{u})^{1-\gamma}}{2(1-\gamma)} + \frac{(\alpha F_0(u + 1 - \alpha))^{1-\gamma}}{4(1-\gamma)}.$$

$$E[B(\pi^+)] = \frac{(\alpha F_0u)^{1-\gamma}}{2(1-\gamma)} + \frac{(\alpha F_0(u + u^2(1 - \alpha)))^{1-\gamma}}{4(1-\gamma)}. \tag{53}$$

Therefore, $E[B(\pi^-)] < E[B(\pi^+)]$ always. To maximize $E[B(\pi_0)]$ over $\pi^- \leq \pi_0 \leq \pi^+$, the optimal allocation $\pi_0^*$ is $\pi^*_c$, $\pi^*_c$ or $\pi^+$ depending on the value of $E[B(\pi_0)]$ at these points.

The $E[B(\pi_0)]$ is always positive, because $0 < \gamma < 1$. By choosing different combinations of parameters $u$, $\gamma$ and $\alpha$, the $E[B(\pi_0)]$ over $\pi^- \leq \pi_0 \leq \pi^+$ is plotted as in Figure 1. For (a), (b) and (d) in Figure 1, the fund manager invests as much as he is allowed in the risky asset.
The optimal allocation is $\pi^+$. For (c), the optimal proportion of the fund invested in the risky asset is less than $\pi^+$.

$u$, $\gamma$ and $\alpha$ are the major determinants of $\pi_0^*$. By comparing $\mathbb{E}[B(\pi_{02}^*)]$, $\mathbb{E}[B(\pi_{03}^*)]$ and $\mathbb{E}[B(\pi^+)]$, we get $\pi_0^*$ which leads to the largest $\mathbb{E}[B(\pi_0)]$. Next, we analyze the sensitivity of $\pi_0^*$ with respect to $\alpha$ and $\gamma$ as in Figure 2.

![Figure 2: Sensitivity of $\pi_0^*$ with respect to the Incentive Rate $\alpha$ and the Risk Aversion $\gamma$](image)

In Figure 2(a), $u = 2$, $\gamma = 0.7$ and $\alpha \in (0.05, 0.45)$. If the fund manager is very risk averse, for the most piratical $\alpha$, as $\alpha$ increases, the fund manager becomes more aggressive even though he does not invest in the most risky allocation $\pi^+$.

In Figure 2(b), $u = 2$, $\alpha = 0.2$, and $\gamma \in (0.2, 0.9)$. If $\gamma$ is sufficiently large, the manager is more conservative if he is more risk averse. However, if $\gamma$ is small than certain threshold, the manager invests in $\pi^+$.

In conclusion, in the two period binomial tree model, the fund manager does not simply invest as much as he is allowed in the risky asset. In order to maximize the total expected power utility from fees in two periods, the fund manager might be more conservative at the very beginning because he does not want to lose the chance to earn high-water mark fees for the latter period, in case of poor performance in the first period.
4 Continuous-time Model

In this section, we consider the model in which the stock and the money market account are traded continuously, and the fund manager aims to maximize the expected power utility from fees at the terminal date $T > 0$. Our model is a simplification of that in He and Kou (2016), and the results from our model are more explicit than theirs.

Assume a fund manager invests the fund in two assets, a risk free asset and a risky asset with price dynamics:

\[
\begin{align*}
    dS_{0,t} &= rS_{0,t}dt, \quad S_{0,0} = S_0, \\
    dS_{1,t} &= uS_{1,t}dt + \sigma S_{1,t}dW_t, \quad S_{1,0} = S_1.
\end{align*}
\]

$r \geq 0$ is the risk-free rate, $u$ is the appreciation rate and $\sigma$ is the volatility of the stock, which are all constants. $W$ is a standard Brownian motion and $\{F_t\}_{t \geq 0}$ is the filtration generated by $W$. Assume the manager invests $\pi_t$ of the total fund value in the risky asset, where $\pi$ is integrable with respect to $W$. With initial value $F_0$, the fund value follows the dynamics:

\[
dF_t = rF_t dt + \pi_tF_t[(u - r)dt + \sigma dW_t].
\]

Let $k = \frac{u - r}{\sigma}$, and the state price density in this complete market be $\xi_t = \exp \left[ -\left( r + \frac{k^2}{2} \right) t - kW_t \right]$, $0 \leq t \leq T$.

The fund manager earns $\alpha$ proportion of the fund’s profit above its initial value as performance fees. The manager’s performance fees at time $T$ is

\[
X_T = \theta(F_T) = \begin{cases} 
0, & F_T < F_0 \\
\alpha(F_T - F_0), & F_T \geq F_0.
\end{cases}
\]

The fund manager aims to maximize the expected power utility from the fees $\max_{\pi} \mathbb{E} \left[ U(\theta(F_T)) \right]$, where $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$.
Lemma 4.1. The solution to \( \max_{x \geq 0} [U(\theta(x)) - yx] \) is

\[
\arg\max_{x \geq 0} [U(\theta(x)) - yx] = \begin{cases} 
F_0 + \left(\frac{y}{\alpha^{1-\gamma}}\right)^{-\frac{1}{\gamma}}, & 0 < y < l \\
0 \text{ or } \frac{1}{\gamma}F_0, & y = l \\
0, & y > l,
\end{cases}
\]  

(57)

where \( x^* = \frac{1}{\gamma}F_0 \) and \( l = \alpha^{1-\gamma} \left( \left( \frac{1}{\gamma} - 1 \right) F_0 \right)^{-\gamma} \).

Proof. Consider the maximization problem \( \max_{x \geq 0} [U(\theta(x)) - yx] \). Let \( x^* \) be the tangent point on the curve \( U(\theta(x)) \) and the tangent line goes across \((0,0)\) and \((x^*, U(\theta(x)))\), which satisfies the following equation:

\[
U'(\theta(x^*))\theta'(x^*)(x^* - 0) = U(\theta(x^*)) - 0,
\]

i.e. \( (\alpha(x^* - F_0))^{-\gamma} \alpha(x^* - 0) = \frac{(\alpha(x^* - F_0))^{1-\gamma}}{1 - \gamma} \).

By calculation, \( x^* = \frac{1}{\gamma}F_0 \) and the slope of the tangent line is \( l = \alpha^{1-\gamma} \left( \left( \frac{1}{\gamma} - 1 \right) F_0 \right)^{-\gamma} \).

(i) If \( y > l \), \([U(\theta(x)) - yx] \leq 0 \) always. In particular, at \( x = 0 \), \( U(\theta(x)) - yx = 0 \).

(ii) If \( y = l \), \([U(\theta(x)) - yx] \leq 0 \) always. In particular, at \( x = 0 \) and \( x = x^* \), \( U(\theta(x)) - yx = 0 \).

(iii) If \( 0 < y < l \), \( \frac{\partial[U(\theta(x)) - yx]}{x} = (\alpha(x - F_0))^{-\gamma} \alpha - y \). To maximize \([U(\theta(x)) - yx]\) over \( x \), let \( (\alpha(x - F_0))^{-\gamma} \alpha - y = 0 \). Therefore, \( x = F_0 + \left(\frac{y}{\alpha^{1-\gamma}}\right)^{-\frac{1}{\gamma}} \) reaches the maximum.

Therefore, the solution to \( \arg\max_{x \geq 0} [U(\theta(x)) - yx] \) is

\[
\arg\max_{x \geq 0} [U(\theta(x)) - yx] = \begin{cases} 
F_0 + \left(\frac{y}{\alpha^{1-\gamma}}\right)^{-\frac{1}{\gamma}}, & 0 < y < l \\
0 \text{ or } \frac{1}{\gamma}F_0, & y = l \\
0, & y > l,
\end{cases}
\]  

(59)

Theorem 4.2. If the fund manager aims to maximize the expected power utility from fees in
the continuous-time model, at time $t$, the optimal value of the fund $F_t^*$ is

$$F_t^* = e^{-r(T-t)} F_0 \left[ \Phi(d_{1,t}) + \left( \frac{1}{\gamma} - 1 \right) e^{-\frac{1}{2} d_{1,t}^2 + \frac{1}{2} d_{2, t}^2} \Phi(d_{2, t}) \right]. \quad (60)$$

The optimal allocation of the fund in the risky asset $\pi_t^*$ is

$$\pi_t^* = \frac{k}{\sigma} \left[ \frac{1}{\gamma} + \frac{F_0}{e^{r(T-t)} F_t^*} \left( \frac{1}{\gamma k \sqrt{T-t}} \Phi'(d_{1,t}) - \frac{1}{\gamma} \Phi(d_{1,t}) \right) \right], \quad (61)$$

where $\Phi$ is the cumulative distribution function of a standard normal random variable.

$$d_{1,t} = \ln \frac{v^*}{\xi_t} + \left( r - \frac{k^2}{2} \right) (T-t), \quad d_{2, t} = d_{1, t} + \frac{k \sqrt{T-t}}{\gamma}, \quad (62)$$

and $v^*$ is the unique solution to the equation:

$$e^{-rT} \Phi \left( \frac{\ln v^* + \left( r - \frac{3}{2} k^2 \right) T}{k \sqrt{T}} \right) + \left( \frac{1}{\gamma} - 1 \right) v^* e^{-\frac{1}{2} \left( \frac{k^2}{2\gamma} + r \right) T} \Phi \left( \frac{\ln v^* + \left( r - \frac{3\gamma - 4}{2\gamma} k^2 \right) T}{k \sqrt{T}} \right) = 1. \quad (63)$$

**Proof.** First consider the problem of $\max_{F_T \geq 0} \mathbb{E}[U(\theta(F_T))]$, where $F_T$ is $\mathcal{F}_T$-measurable and $\mathbb{E}[\xi_T F_T] \leq F_0$. We solve the maximization problem by Lagrange Multiplier:

$$\max_{F_T \geq 0} \mathbb{E} \left[ U(\theta(F_T)) - l \frac{\xi_T F_T}{v} \right] + \frac{l}{v} F_0, \quad (64)$$

where $v$ is the Lagrange multiplier and $l$ is defined in Lemma 4.1. By maximizing the above problem, $\mathbb{E}[U(\theta(F_T))]$ is also maximized if $\mathbb{E}[\xi_T F_T(v)] = F_0$ holds.

According to Lemma 4.1, the optimal solution $F_T^*$ to maximization problem of

$$\max_{F_T \geq 0} \mathbb{E} \left[ U(\theta(F_T)) - l \frac{\xi_T F_T}{v} \right]$$

is

$$F_T^*(v) = \left[ 1 + \left( \frac{1}{\gamma} - 1 \right) \left( \frac{\xi_T}{v} \right)^{-\frac{1}{2}} \right] 1_{\{\xi_T < v\}} F_0 + \left[ \left( \frac{1}{\gamma} \right) 1_{\{\xi_T = v\}} \right] F_0 + (0) 1_{\{\xi_T > v\}}, \quad (65)$$

given that $\mathbb{E}[\xi_T F_T^*(v)] = F_0$, which is an equation in $v$. 

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Notice that $\mathbb{E}[\xi_T F_T^*(v)]$ is continuous and non-decreasing in $v$. Since $0 < \gamma < 1$, for $v < 1$,

$$
\xi_T F_T^*(v) = \xi_T \left(1 + \left(\frac{1}{\gamma} - 1\right) \left(\xi_T \frac{1}{v}\right)^{-\frac{1}{\gamma}}\right) \mathbb{1}_{\{\xi_T < v\}} F_0 \leq \xi_T \left(1 + \left(\frac{1}{\gamma} - 1\right) \left(\xi_T^{-\frac{1}{\gamma}}\right)\right) F_0. \quad (66)
$$

Therefore, if $v \to 0$, $|\xi_T F_T^*(v)|$ is bounded by $\xi_T \left(1 + \left(\frac{1}{\gamma} - 1\right) \left(\xi_T^{-\frac{1}{\gamma}}\right)\right) F_0$.

Since $\mathbb{E}\left[\xi_T \left(1 + \left(\frac{1}{\gamma} - 1\right) \left(\xi_T^{-\frac{1}{\gamma}}\right)\right) F_0\right] < +\infty$, by the dominated convergence theorem,

$$
\lim_{v \to 0} \mathbb{E}[\xi_T F_T^*(v)] = \mathbb{E}\left[\lim_{v \to 0} \xi_T F_T^*(v)\right] \to 0. \quad (67)
$$

Since $\xi_T F_T^*(v)$ is non-negative, by the monotone convergence theorem,

$$
\lim_{v \to \infty} \mathbb{E}[\xi_T F_T^*(v)] = \mathbb{E}\left[\lim_{v \to \infty} \xi_T F_T^*(v)\right] \to +\infty. \quad (68)
$$

Thus, there is a $v^*$ such that $\mathbb{E}[\xi_T F_T^*(v^*)] = F_0$ holds. Since

$$
F_0 = \mathbb{E}[\xi_T F_T^*(v^*)]
= \mathbb{E}\left[\xi_T \left(1 + \left(\frac{1}{\gamma} - 1\right) \left(\xi_T \frac{1}{v^*}\right)^{-\frac{1}{\gamma}}\right) \mathbb{1}_{\{\xi_T \leq v^*\}} F_0\right]
= \mathbb{E}\left[\xi_T \mathbb{1}_{\{\xi_T \leq v^*\}} F_0\right] + \left(\frac{1}{\gamma} - 1\right) \left(v^*\right)^{\frac{1}{\gamma}} F_0 \mathbb{E}\left[\xi_T^{-\frac{1}{\gamma}} \mathbb{1}_{\{\xi_T \leq v^*\}}\right]
= \left(\frac{1}{\gamma} - 1\right) \left(v^*\right)^{\frac{1}{\gamma}} e^{-\frac{1}{\gamma} \left(\frac{k^2}{2\gamma} + r\right) T} F_0 \Phi \left(\frac{\ln v^* + \left(r - \frac{3\gamma - 4}{2\gamma} k^2\right) T}{k \sqrt{T}}\right)
+ e^{-r T} F_0 \Phi \left(\frac{\ln v^* + \left(r - \frac{3}{2} k^2\right) T}{k \sqrt{T}}\right),
$$

$v^*$ satisfies the equation

$$
e^{-r T} \Phi \left(\frac{\ln v^* + \left(r - \frac{3}{2} k^2\right) T}{k \sqrt{T}}\right) + \left(\frac{1}{\gamma} - 1\right) (v^*)^{\frac{1}{\gamma}} e^{-\frac{1}{\gamma} \left(\frac{k^2}{2\gamma} + r\right) T} \Phi \left(\frac{\ln v^* + \left(r - \frac{3\gamma - 4}{2\gamma} k^2\right) T}{k \sqrt{T}}\right) = 1. \quad (70)
$$

Notice that in the general model in He and Kou (2016), they cannot find similar explicit equation for $v^*$. 

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\( F_T^* \) is the theoretical value of the fund at time \( T \) that the fund manager needs to achieve if he aims to maximize the expected power utility from fees. Let \( F_t^* = \xi_t^{-1} \mathbb{E}[\xi_t F_T^* | \mathcal{F}_t] \). Therefore,

\[
F_t^* = \xi_t^{-1} \mathbb{E} \left[ \xi_T \left( 1 + \left( \frac{1}{\gamma} - 1 \right) \left( \frac{\xi_T}{v^*} \right)^{-\frac{1}{\gamma}} \right) \mathbb{1}_{\{\xi_T \leq v^*\}} F_0 | \mathcal{F}_t \right] = \mathbb{E} \left[ \frac{\xi_T}{\xi_t} \left( 1 + \left( \frac{1}{\gamma} - 1 \right) \left( \frac{\xi_T}{v^*} \right)^{-\frac{1}{\gamma}} \right) \mathbb{1}_{\{\xi_t \xi_T \leq v^*\}} F_0 | \mathcal{F}_t \right].
\tag{71}
\]

Since \( Z = \frac{\xi_T}{\xi_t} = e^{-\left( r + \frac{k^2}{2}\right)(T-t)} \) is independent of \( \mathcal{F}_t \), and \( \xi_t \) is known at time \( t \), \( F_t^* = f(t, \xi_t) \), where

\[
f(t, \xi) = \mathbb{E} \left[ Z \left( 1 + \left( \frac{1}{\gamma} - 1 \right) \left( \frac{\xi}{v^*} \right)^{-\frac{1}{\gamma}} \right) \mathbb{1}_{\{\xi \leq v^*\}} F_0 \right] = \mathbb{E} \left[ Z \left( 1 + \left( \frac{1}{\gamma} - 1 \right) \left( \frac{\xi}{v^*} \right)^{-\frac{1}{\gamma}} (Z)^{-\frac{1}{\gamma}} \right) \mathbb{1}_{\{Z \leq v^*\}} F_0 \right] = F_0 \mathbb{E} \left[ Z \mathbb{1}_{\{Z \leq v^*\}} \right] + F_0 \left( \frac{1}{\gamma} - 1 \right) \left( \frac{\xi}{v^*} \right)^{-\frac{1}{\gamma}} \mathbb{E} \left[ Z^{1-\frac{1}{\gamma}} \mathbb{1}_{\{Z \leq v^*\}} \right],
\tag{72}
\]

and \( \ln Z \sim \mathcal{N} \left( -(r + \frac{k^2}{2})(T-t), T-t \right) \). Let \( y = -\frac{1}{k} \left( r + \frac{k^2}{2} \right) (T-t) - \frac{1}{k} \ln Z \), which follows \( \mathcal{N}(0, T-t) \).

If \( 0 < Z < \frac{v^*}{\xi} \), then \(-\frac{1}{k} \left( r + \frac{k^2}{2} \right) (T-t) - \frac{1}{k} \ln \frac{v^*}{\xi} < y < +\infty \). Thus,

\[
\mathbb{E} \left[ Z \mathbb{1}_{\{Z \leq v^*\}} \right] = \int_{-\frac{1}{k} \left( r + \frac{k^2}{2} \right) (T-t) - \frac{1}{k} \ln \frac{v^*}{\xi}}^{+\infty} \frac{1}{\sqrt{2\pi (T-t)}} e^{-\frac{(y-\frac{1}{k} \left( r + \frac{k^2}{2} \right) (T-t) - \frac{1}{k} \ln \frac{v^*}{\xi})^2}{2(T-t)}} e^{-r(T-t)} dy
\]

\[
= \int_{-\frac{1}{k} \left( r + \frac{k^2}{2} \right) (T-t) - \frac{1}{k} \ln \frac{v^*}{\xi}}^{+\infty} \frac{1}{\sqrt{2\pi (T-t)}} e^{-\frac{\left( y - r(T-t) - \frac{1}{k} \ln \frac{v^*}{\xi} \right)^2}{2(T-t)}} e^{-r(T-t)} dy
\tag{73}
\]

\[
=r^{T-t} \left[ 1 - \Phi \left( -\frac{1}{k} \left( r + \frac{k^2}{2} \right) (T-t) - \frac{1}{k} \ln \frac{v^*}{\xi} + k(T-t) \right) \right]
\tag{74}
\]

\[
=r^{T-t} \Phi \left( \frac{\ln \frac{v^*}{\xi} + \left( r + \frac{k^2}{2} \right) (T-t)}{k \sqrt{T-t}} \right),
\tag{75}
\]
and

\[
\mathbb{E}
\left[
Z^{1-\frac{1}{2} \mathbb{1}_{\{Z \leq \frac{\gamma}{T}\}}}
\right] = \int_{-\frac{1}{k}(r + \frac{k^2}{2})/(T-t)-\frac{1}{k}ln \frac{\gamma}{T}}^{+\infty} e^{\frac{1-\gamma}{\gamma}(r + \frac{k^2}{2})(T-t) + \frac{1-\gamma}{\gamma}ky - \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{y^2}{2(T-t)}} dy
\]

\[
= \int_{-\frac{1}{k}(r + \frac{k^2}{2})/(T-t)-\frac{1}{k}ln \frac{\gamma}{T}}^{+\infty} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(y + \frac{(\gamma-1)(T-t)k}{\gamma})^2}{2(T-t)}} e^{-\frac{2-1}{\gamma}(r + \frac{k^2}{2})(T-t)} dy
\]

\[
= e^{-\frac{2-1}{\gamma}(r + \frac{k^2}{2})(T-t)} \left(1 - \Phi\left(-\frac{1}{k} \frac{r + \frac{k^2}{2}}{(T-t) - \frac{1}{k} ln \frac{\gamma}{T} + 2-1 (T-t)k}{\sqrt{T-t}}\right)\right)
\]

\[
= e^{-\frac{2-1}{\gamma}(r + \frac{k^2}{2})(T-t)} \Phi\left(\frac{ln \frac{\gamma}{T} + (r - \frac{k^2}{2})(T-t)}{k\sqrt{T-t}} + \frac{k\sqrt{T-t}}{\gamma}\right). \quad (76)
\]

Therefore,

\[
F^*_t = f(t, \xi_t) = e^{-r(T-t)}F_0 \left[\Phi(d_{1,t}) + \left(\frac{1}{\gamma} - 1\right) e^{-\frac{k^2}{2}d_{1,t}^2 + \frac{1}{2}d_{2,t}^2} \Phi(d_{2,t})\right], \quad (77)
\]

where

\[
d_{1,t} = \frac{ln \frac{\gamma}{\xi_t} + (r - \frac{k^2}{2})(T-t)}{k\sqrt{T-t}}, \quad \text{and} \quad d_{2,t} = d_{1,t} + \frac{k\sqrt{T-t}}{\gamma}. \quad (78)
\]

Next, the goal is to construct a strategy in the fund such that the fund replicates \(F^*_T\) at time \(T\). The dynamics of \(d(\xi_t, f(t, \xi_t))\) is

\[
d(\xi_t, f(t, \xi_t)) = \xi_t d(f(t, \xi_t)) + f(t, \xi_t) d\xi_t + d\xi_t d(f(t, \xi_t))
\]

\[
= \xi_t \left(\frac{\partial f(t, \xi_t)}{\partial t} dt + \frac{\partial f(t, \xi_t)}{\partial \xi} d\xi_t + \frac{1}{2} \frac{\partial^2 f(t, \xi_t)}{\partial \xi^2} (d\xi_t)^2\right) + f(t, \xi_t) d\xi_t
\]

\[
+ \left(\frac{\partial f(t, \xi_t)}{\partial t} dt + \frac{\partial f(t, \xi_t)}{\partial \xi} d\xi_t + \frac{1}{2} \frac{\partial^2 f(t, \xi_t)}{\partial \xi^2} (d\xi_t)^2\right) d\xi_t
\]

\[
= \xi_t G_t dt - \left(\xi_t \frac{\partial f(t, \xi_t)}{\partial \xi} + f(t, \xi_t)\right) k\xi_t dW_t
\]

\[
= \xi_t G_t dt - \left(k\xi_t \frac{\partial f(t, \xi_t)}{\partial \xi} + f(t, \xi_t) + k\right) \xi_t f(t, \xi_t) dW_t \quad (79)
\]

where \(d\xi_t = -r\xi_t dt - k\xi_t dW_t\). Notice that \(\xi_t f(t, \xi_t)\) is a martingale, because \(\mathbb{E}[\xi_t f(t, \xi_t) | \mathcal{F}_s] = \mathbb{E}[\mathbb{E}[\xi_T F^*_T | \mathcal{F}_t] | \mathcal{F}_s] = \xi_s f(s, \xi_s)\) for \(s < t\). Therefore, \(G_t = 0\).
On the other hand, for every strategy \( \pi \) on the fund, the fund follows the dynamics

\[
dF_t = rF_t dt + \pi_t F_t ((u - r) dt + \sigma dW_t) = rF_t dt + \pi_t F_t (k dt + dW_t),
\]

\[
d(\xi_t F_t) = \xi_t dF_t + F_t d\xi_t + d\xi_t dF_t = r\xi_t F_t dt + \pi_t \xi_t F_t (k dt + dW_t) - r\xi_t F_t dt - k\xi_t F_t dW_t - k\xi_t \pi_t F_t \sigma dt = (\pi_t \sigma - k)\xi_t F_t dW_t.
\]

We choose the \( \pi^*_t \) such that \( \pi^*_t \sigma - k = -k\xi_t \frac{\partial f(t, \xi_t)}{\partial \xi} - \frac{1}{f(t, \xi_t)} \) - \( \frac{1}{k} \). For the corresponding \( \xi_t f(t, \xi_t) \) and \( \xi_t F_t \), the dynamics of \( \xi_t f(t, \xi_t) \) is the same as \( \xi_t F_t \). Because \( \xi_t f(t, \xi_t) \) and \( \xi_t F_t \) have the same initial value \( \xi_0 f(0, \xi_0) = E[\xi_T F^*_T] = F_0, F_T = F^*_T \). Thus, the optimal investment strategy for the fund is

\[
\pi^*_t = -\frac{k\xi_t}{\sigma F^*_t} \frac{\partial f(t, \xi_t)}{\partial \xi}
\]

\[
= \frac{k\xi_t}{\sigma F^*_t} e^{-r(T-t)} F_0 \left[ \Phi'(d_{1,t}) \frac{\xi_t}{v^*} \frac{1}{k\sqrt{T-t}} \left( \frac{1}{\gamma} - 1 \right) \Phi'(d_{1,t}) \frac{\xi_t}{v^*} + \frac{1}{\gamma^2} \frac{\xi_t}{\Phi'(d_{2,t})} \right]
\]

\[
= \frac{k\xi_t}{\sigma F^*_t} e^{-r(T-t)} F_0 \left[ \frac{1}{\xi_t k\sqrt{T-t}} \Phi'(d_{1,t}) + \frac{1}{\gamma^2} \frac{1}{\xi_t} \Phi'(d_{2,t}) \right]
\]

\[
= \frac{k}{\sigma} \frac{F_0}{e^{r(T-t)} F^*_t} \left[ \frac{1}{\gamma k\sqrt{T-t}} \Phi'(d_{1,t}) + \frac{1}{\gamma} \left( \frac{f(t, \xi_t)e^{-r(T-t)}}{F_0} - \Phi(d_{1,t}) \right) \right]
\]

\[
= \frac{k}{\sigma} \left[ \frac{1}{\gamma} + \frac{F_0}{e^{r(T-t)} F^*_t} \left( \frac{1}{\gamma k\sqrt{T-t}} \Phi'(d_{1,t}) - \frac{1}{\gamma \Phi(d_{1,t})} \right) \right]
\]

\[
= \frac{k}{\sigma} \left[ \frac{1}{\gamma} + \frac{F_0}{e^{r(T-t)} F^*_t} \left( \frac{1}{\gamma k\sqrt{T-t}} \Phi'(d_{1,t}) - \frac{1}{\gamma \Phi(d_{1,t})} \right) \right].
\]

\( \square \)

In continuous-time model, the optimal allocation in the risk asset \( \pi^*_t \) and the optimal fund value \( F^*_t \) are determined by market conditions, which is measured by the state price density \( \xi_t \). Intuitively, if \( \xi_t \) is large, in order to earn the same amount money at time \( t \), it requires
more money at the initial time. Therefore, the market condition is bad if $\xi_t$ is large. On the contrary, the market condition is good if $\xi_t$ is small. From Theorem 4.2, the optimal fund value $F_t^*$ increases if the market condition becomes better. Next, we analyze the relationship of $\pi_t^*$ and $\xi_t$ with a graph. In Figure 3, let $T = 1$, $r = 0.03$, $u = 0.08$, $\gamma = 0.8$, $\sigma = 0.3$, $t = 0.5$

![Figure 3: $\pi_t^*$ with respect to the Price Density $\xi_t$](image)

and $\xi_t \in (0, 0.9)$. $v^*$ can be calculated numerically according to (63). The optimal allocation $\pi_t^*$ is decreasing with $\xi_t$ until $\xi_t$ reaches 0.67, and increasing with $\xi_t$ for $\xi_t$ greater than 0.67. If the market condition is bad ($\xi_t > 0.67$), the fund manager tends to be aggressive in the fund investment in order to get fees. If the market condition is good, the manager does not invest much in the risky asset, because he is risk averse.

Finally we discuss the property of $\pi_t^*$ as $t \to T$. If $\xi_t < v^*$, $d_{1,t} \to +\infty$ and $d_{2,t} \to +\infty$, because $\frac{\ln v^*}{k\sqrt{T-t}} \to +\infty$, $\frac{(r-k^2)\sqrt{T-t}}{k} \to 0$ and $\frac{k\sqrt{T-t}}{\gamma} \to 0$. Therefore, $\Phi'(d_{1,t}) \to 0$, $\Phi(d_{1,t}) \to 1$ and $\pi_t^* \to \frac{k}{\sigma} \left[ \frac{1}{\gamma} + \frac{F_0}{e^{r(T-t)\gamma}} \left( -\frac{1}{\gamma} \right) \right]$.

Similarly, if $\xi_t > v^*$, $d_{1,t} \to -\infty$ and $d_{2,t} \to -\infty$. Thus, $\Phi'(d_{1,t}) \to 0$, $\Phi(d_{1,t}) \to 0$ and $\pi_t^* \to \frac{k}{\sigma\gamma}$. Therefore, $\pi_t^*$ is smaller if $\xi_t < v^*$ comparing with the $\pi_t^*$ if $\xi_t > v^*$.

If the market condition is relatively bad, the fund manager tends to take more risk comparing with the case that the market condition is good.

To summarize, in the continuous-time model, the fund manager does not take extreme risk as in the one period binomial tree model. The allocation in the risky asset dynamically changes and depends on the market condition. If the market condition is good, the fund manager tends to take less risk, and vice versa. Thus, the moral hazard of unnecessarily high risk does not always arise.
5 Conclusion

The high-water mark fee is an important component of the hedge fund’s managerial contract. There are concerns that the high-water mark fee structure might lead to moral hazard: the fund manager takes unnecessarily high risks in order to earn more fees, which is against the interest of the investors’. Many researchers show that fund managers do not necessarily increase risk under a high-water mark contract both theoretically and empirically. However, most of the previous research is based on the assumption that the hedge fund managers who earn cumulative fees have a long planning horizon, while in reality, the hedge fund managers’ horizon is rarely infinite.

In this paper, the theoretical optimal strategies for the fund manager who chooses to maximize the expected power utility from fees are given in both discrete-time and continuous-time models. In the single period binomial tree model, the fund manager always invests as much as he is allowed in the risky assets, unless the high-water mark is too high so that the manager does not receive performance fees by any investment strategy. In the two period binomial tree model, even though the optimal strategy for the second period is the same as in the one period model, the fund manager does not necessarily invest in the most in the first period. The manager tends to be conservative at the initial date because he does not want to lose the opportunity to earn performance fee in the latter period. In general, the optimal allocation in the first period $\pi_0^*$ is deceasing with $\gamma$ and increasing with $\alpha$, and it is more sensitive to $\gamma$.

In the continuous-time model, the optimal investment strategy and the optimal fund value are determined by market conditions, which are measured by the state price density $\xi_t$. If the market condition is good, the fund manager tends to be more conservative because he is more likely to expect fees. On the contrary, if the market condition is bad, the fund manager tends to preform aggressively in order to earn fees. Thus, the moral hazard of unnecessarily high risk does not always arise.
References


