2004-04-30

Homogenization of Acoustic Wave Propagation in a Magnetorheological Fluid

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April 30, 2004

Abstract

We formulate a model for acoustic excitations in a magnetorheological fluid. Constitutive equations are derived for Navier-Stokes flow coupled with Maxwell's Equations. The viscosity of the fluid is modified to reflect the dependence of waves propagating within the fluid itself and in the case where they propagate along the network of particles.

1 Introduction

One of the defining characteristics of fluids is the inability to carry acoustical shear wave modes. To enable a fluid to carry more than one longitudinal mode one of its properties must be changed. It has been experimentally proven that with the addition of small particles two longitudinal modes may propagate through a system provided that the wavelength is comparable to the size of the particles or smaller [10, 20]. In the past decade researchers experimenting with magnetorheological fluids discovered in the large wavelength regime two acoustical longitudinal waves of differing speed and amplitude propagating simultaneously [13]. From their experimental findings Nahmad-Molinari *et al* were able to show a strong behavioral dependence upon an external magnetic field in both waves. The exact nature of these waves have been debated [2, 14, 7] and several models have been proposed [13, 3, 4] of varying degrees of computational complexity and accuracy. The motivation of this study is to produce equations using homogenization theory which correctly model the behavior of acoustical waves in a magnetorheological fluid.

Magnetorheological fluids (MR fluids) are typically classified as Bingham plastics. In the presence of an external magnetic field the effective viscosity of the fluid changes. The alteration in viscosity is sudden but not permanent and quickly returns to initial conditions once the magnetic field is removed. The variation of the viscosity is a product of the arrangement of the magnetically active particles due to dipole forces [17]. These particles arrange themselves to form web like structures [17, 21] and it is along these structures that the second longitudinal mode travels [13].

The study of the nucleation rates and responses to magnetic fields has a much larger background than the study of the acoustical wave propagation. The interest there was how to best incorporate these fluids into modern technological uses. Many of these types of simulations were based either on a modified Bingham plastic [19], on a minimum energy approach [12, 1] or incorporated static Maxwell's Equations to explain particle trajectories [11]. This study departs from these earlier works formulating our equations based upon the conservation of mass and momentum coupled with time dependent Maxwell's equations.

Homogenization techniques have been used in the past to study magnetorheological materials [8, 16] and the comparable electrorheological materials [15, 18]. Our model is based in part on prior work with waves in suspensions of particles [6] and structural deformations in MR fluids [8]. The wave motion is taken to be a plain wave eliminating the need to include time stepping algorithms into the implementation of the equations. However, due to the oscillatory nature of the acoustical excitations within the MR fluid we are not able to assume that the magnetic properties of the particles may be represented by static Maxwell's equations as was done by Levy. Furthermore, since the amplitudes of the oscillations are comparably small we are able to neglect non-linear terms in the fluid motion.

Several studies have modelled acoustical waves in MR fluids by partitioning the system into two components: the fluid itself and the rigid skeleton formed by the particles [3, 4]. In this way they obtain two waves whose characteristics approximate experimental results. However, their methodology does not reflect the dependence upon the magnetic field observed in the longitudinal mode propagating in the fluid. A more natural approach is to examine the entire system as a whole, weakening the viscosity of the fluid when modelling the waves traveling along the magnetic particles. The justification for this technique relies on the assumption that the two modes will collapse into a single mode in the absence of a magnetic field. As this paper will show not only does this happen but our equations reduce to a more familiar form of wave motion for an anisotropic viscous fluid.

2 Formalization of the Problem

We consider a suspension of magnetizable rigid spheres in a viscous incompressible fluid under the influence of a magnetic inductance, $\mathbf{B}$. For our purposes we assume that the magnetic permeability of the particles is linear and uniform, μ_S, and the electric permitivity is likewise linear and uniform, ε_s. Furthermore the particles carry no free charge and any accumulation of such charges is quickly dispersed by the surrounding fluid. We also assume that the dielectric polarization of the particles matches that of the fluid, rendering force contributions by the induced electric field negligable.

The magnitization of each particle, $\mathbf{M}$, is known and the magnetic forces on each particle are composed of the volumic density force $(\nabla \times \mathbf{M}) \times \tilde{\mathbf{B}}$ and the surface density force $(\mathbf{n} \times \mathbf{M}) \times \tilde{\mathbf{B}}$. Generally the term $\nabla \times \mathbf{M}$ is denoted $\mathbf{J}_B$, reflecting the fact that it represents the density of the bound current in each particle. Likewise, $\mathbf{n} \times \mathbf{M}$ is commonly referred to as the bound surface current density, $\mathbf{K}_B$. This nomenclature shall be used for the remainder of the study.

The oscillatory nature of the fluid's velocity, $\mathbf{V}$, is taken to follow a plane wave, $\tilde{\mathbf{V}}(\mathbf{x}, t) = \mathbf{V}(\mathbf{x})e^{i\omega t}$. Due to the fact that the currents within each particle are bound, the time dependence of the magnetic and electric fields may be expressed as $\tilde{\mathbf{B}}(\mathbf{x}, t) = \mathbf{B}(x)e^{i\omega t}$ and $\tilde{\mathbf{E}}(\mathbf{x}, t) = \mathbf{E}(x)e^{i\omega t}$ respectively. The wavelength of the acoustical excitations is assumed to be larger than the particle size. Thermal considerations are also neglected.

In the domain we have

$$\nabla \times \mathbf{E} = -i\omega \mathbf{B} \tag{1}$$

$$\nabla \cdot \mathbf{E} = 0 \tag{2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3}$$

In the fluid we have

$$\nabla \cdot \mathbf{V} = 0 \tag{4}$$

$$
\begin{aligned}
i\omega\rho_F V_i &= \frac{\partial \sigma_{ij}}{\partial x_j} && (5)\\
\nabla \times \mathbf{B} &= i\omega\mu_0\epsilon_0\mathbf{E} && (6)
\end{aligned}
$$

where

$$
\sigma_{ij} = -\delta_{ij}P + 2\mu D_{ij}(\mathbf{V})
$$
$$
D_{ij}(\mathbf{V}) = \frac{1}{2}\Big(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\Big)
$$

In each particle, S we have

$$
\begin{aligned}
D_{ij}(\mathbf{V}) &= 0 && (7)\\
\nabla \times \mathbf{B} &= \mu_0 \nabla \times \mathbf{M} + i\omega\mu_0\epsilon_S\mathbf{E} && (8)
\end{aligned}
$$

On the boundary ∂S the velocity, $\mathbf{V}$, is continuous. While the magnetic and electrical fields satisfy the relations

$$
\begin{aligned}
(\mathbf{B}_F - \mathbf{B}_S)\cdot\mathbf{n} &= 0\\
\mathbf{n} \times \big(\frac{1}{\mu_0}\mathbf{B}_F - (\frac{1}{\mu_0}\mathbf{B}_S - \mathbf{M})\big) &= \mathbf{n}\times\mathbf{M}\\
(\epsilon_0\mathbf{E}_F - \epsilon_S\mathbf{E}_S)\cdot\mathbf{n} &= 0\\
(\mathbf{E}_F - \mathbf{E}_S)\times\mathbf{n} &= 0
\end{aligned}
$$

The balance of forces is then expressed as

$$
\begin{aligned}
i\omega\int_S \rho_s \mathbf{V} dv &= \int_S (\nabla\times\mathbf{M})\times\mathbf{B} dv + \int_{\partial S}(\mathbf{n}\times\mathbf{M})\times\mathbf{B} d\sigma\\
&\quad - \int_{\partial S}\sigma_{ij}n_j\mathbf{e}_i d\sigma && (9)\\
i\omega\int_S \rho_s(\mathbf{x}-\mathbf{x_G})\times\mathbf{V} dv &= \int_S (\mathbf{x}-\mathbf{x_G})\times((\nabla\times\mathbf{M})\times\mathbf{B})dv\\
&\quad + \int_{\partial S}(\mathbf{x}-\mathbf{x_G})\times((\mathbf{n}\times\mathbf{M})\times\mathbf{B})d\sigma\\
&\quad - \int_{\partial S}(\mathbf{x}-\mathbf{x_G})\times(\sigma_{ij}n_j\mathbf{e}_i)d\sigma && (10)
\end{aligned}
$$

The suspension of particles is assumed to be non-dilute and of infinite dimensions. The equilibrium state shall be used as the reference configuration with the particles being distributed within a locally periodic unit cell under a uniform pressure P_0. The length of this unit cell, l, compared to the length of the macroscopic phenomena, L, denoted $\epsilon = l/L$ is small $\epsilon \ll 1$. We impose a double scale assymptotic expansion on the spacial coordinates within the unit cell. We define $\mathbf{y} = (\mathbf{x} - \mathbf{x}_G)/\epsilon$, where $\mathbf{x}_G$ is the center of mass of the solid, to be the scretched coordinates. The derivative operator takes on the form

$$\frac{\partial}{\partial x} \Rightarrow \frac{\partial}{\partial x} + \frac{1}{\epsilon}\frac{\partial}{\partial y}$$

The functions are expanded in powers of epsilon in the limit $\epsilon \to 0$,

$$\begin{aligned} \mathbf{V} = \mathbf{V}^0(\mathbf{x}) + \mathbf{V}^r(\mathbf{x},\mathbf{y}) + \epsilon\mathbf{V}^1(\mathbf{x},\mathbf{y})... \\ \mathbf{E} = \mathbf{E}^0(\mathbf{x}) + \epsilon\mathbf{E}^1(\mathbf{x},\mathbf{y}) + ... \\ \mathbf{B} = \mathbf{B}^0(\mathbf{x}) + \epsilon\mathbf{B}^1(\mathbf{x},\mathbf{y}) + ... \\ \mathbf{P} = \mathbf{P}^0(\mathbf{x},\mathbf{y}) + \epsilon\mathbf{P}^1(\mathbf{x},\mathbf{y}) + ... \end{aligned}$$

where the functions $\mathbf{V}^r$, $\mathbf{V}^1$, $\mathbf{B}^1$, and $\mathbf{E}^1$ are Y-periodic in y. Excluding pressure, the zeroth order functions are not arbitrairily chosen to be independent of y. It will be proved later that this initial assumption is valid.

The nature of the viscosity and the role that the viscosity has in the determination of the solution of the problem is of paramount concern in this study. We take the viscosity of the fluid to be a function of ϵ,

$$\mu(\eta) = \mu\epsilon^\eta$$

where η is a positive number. The physical interpritation of this mathematical definition is to say that the contribution of forces resulting from the fluid's viscosity is relatively small compared to the magnetic forces when $\eta > 0$.

3 Homogenized Maxwell Equations

3.1 Determination of $\mathbf{E}^0$ and $\mathbf{B}^0$

From eq(3) at order $O(\epsilon^{-1})$ we have

$$\nabla_{(y)} \cdot \mathbf{B}^0 = 0 \tag{11}$$

in F from eq(6) at order $O(\epsilon^{-1})$ we have

$$\nabla_{(y)} \times \mathbf{B}^0 = 0 \tag{12}$$

and in S from eq(8) at order $O(\epsilon^{-1})$ we have

$$\nabla_{(y)} \times \mathbf{B}^0 = 0 \tag{13}$$

which combined with the boundary conditions allow us to conclude that $\mathbf{B}^0(\mathbf{x}, \mathbf{y}) \equiv \mathbf{B}^0(\mathbf{x})$.

The form of the electric field may be deduced from Maxwell's equations in relation to the magnetic induction. From eq(1) at order $O(\epsilon^{-1})$ and eq(2) at order $O(\epsilon^{-1})$ we have

$$\nabla_{(y)} \times \mathbf{E}^0 = 0 \tag{14}$$

$$\nabla_{(y)} \cdot \mathbf{E}^0 = 0 \tag{15}$$

which combined with the boundary conditions allows us to conclude that $\mathbf{E}^0(\mathbf{x}, \mathbf{y}) \equiv \mathbf{E}^0(\mathbf{x})$.

3.2 Determination of $\mathbf{B}^1$

From eq(3) at order $O(1)$ we have

$$\nabla \cdot \mathbf{B}^0 = -\nabla_{(y)} \cdot \mathbf{B}^1 \tag{16}$$

in F from eq(6) at order $O(1)$ we have

$$\nabla \times \mathbf{B}^0 = i\omega\mu_0\epsilon_0\mathbf{E}^0 - \nabla_{(y)} \times \mathbf{B}^1 \tag{17}$$

and in S from eq(8) at order $O(1)$ we have

$$\nabla \times \mathbf{B}^0 = \mu_0\mathbf{J}_B + i\omega\mu_0\epsilon_0\mathbf{E}^0 - \nabla_{(y)} \times \mathbf{B}^1 \tag{18}$$

This allows us to write $\mathbf{B}^1$ as

$$\mathbf{B}^1 = -\mathbf{y}(\nabla \cdot \mathbf{B}^0) - \mathbf{y} \times (i\omega\mu_0\epsilon_0\mathbf{E}^0 - \nabla \times \mathbf{B}^0) + \mu_0[\mathbf{e}_i \cdot \mathbf{J}_B]\mathbf{\Psi}^i \tag{19}$$

with $\mathbf{\Psi}^i = -\frac{1}{3}\chi_{Y_S}(\mathbf{y} \times \mathbf{e}_i)$ where χ_{Y_S} is the charateristic function of Y_S. The strength of the first order expansion of the magnetic induction is dependent upon the frequency of the propagating wave indicating some amount of non-linearity intrinsic within the electromagnetic forces.

The integral of $\mathbf{B}^1$ over the surface of the particles Γ is computed using basic vector identities applied to eq(19) where we have grouped common terms together:

$$\int_\Gamma \mathbf{B}^1 d\sigma_y = \int_\Gamma \tau_{ij}(\mathbf{B}^0)\mathbf{e}_i d\sigma_y + (\mu_0[\mathbf{J}_B \cdot \mathbf{e}_i] + 3i\omega\mu_0\epsilon_0[\mathbf{E}^0 \cdot \mathbf{e}_i]) \int_\Gamma \mathbf{\Psi}^i d\sigma_y$$

with

$$\tau_{ij}(\mathbf{B}^0) = \Big(y_j \frac{\partial B_j^0}{\partial x_i} - y_i \frac{\partial B_j^0}{\partial x_j} - y_j \frac{\partial B_i^0}{\partial x_j}\Big)$$

4 Wave propagation in a viscous fluid

The propagation of waves within the fluid is modelled as a suspension of magnetic particles immersed in a non-magnetic fluid. The viscosity of the fluid is taken to be of the same order as the magnetic forces, $\mu = O(1)$.

4.1 Determination of $\mathbf{V}^0$

We get from eqn(4) at order $O(\epsilon^{-1})$ and eqn(5) at order $O(\epsilon^{-2})$ in F

$$\nabla_{(y)} \cdot \mathbf{V}^0 = 0 \tag{20}$$

$$\frac{\partial}{\partial y_j} D_{ij}(\mathbf{V}^0) = 0 \tag{21}$$

In S from (8) at the order $O(\epsilon^{-1})$ we have

$$D_{ij(y)}(\mathbf{V}^0) = 0 \tag{22}$$

From eqn(9) and eqn(10) at order $O(\epsilon)$ we have

$$2\mu \int_\Gamma D_{ij(y)}(\mathbf{V}^0) n_j \mathbf{e}_i d\sigma_y = 0 \tag{23}$$

$$2\mu \int_\Gamma \mathbf{y} \times D_{ij(y)}(\mathbf{V}^0) n_j \mathbf{e}_i d\sigma_y = 0 \tag{24}$$

This implies that for some function $\mathbf{\Upsilon} \in U_{ad} = \{\upsilon \in [H^1(Y)]$, Y-periodic, $\nabla \cdot \upsilon = 0$ in F, $D_{ij}(\upsilon) = 0\ \forall$ i,j in $Y_S\}$:

$$\int_{Y_S} D_{ij(y)}(\mathbf{V}^0) D_{ij(y)}(\mathbf{u}) dy = 0 \qquad \forall \mathbf{u} \in \Upsilon \tag{25}$$

with $\mathbf{u} = \alpha + \beta \times \mathbf{y}$ in S, α and β independent of $\mathbf{y}$. We therefore conclude that $\mathbf{V}^0(\mathbf{x}, \mathbf{y}) \equiv \mathbf{V}^0(\mathbf{x})$ and $\mathbf{V}^r(\mathbf{x}, \mathbf{y}) \equiv 0$

4.2 Determination of $\mathbf{V}^1$, and P^0

We may choose to normalize P^0 such that

$$\int_{Y_F} P^0 dy = 0$$

and with careful selection of normalization constants any periodic extension into the solid can produce [5]

$$\int_{Y} P^0 dy = 0 \tag{26}$$

However, we choose not to explore this option for reasons that will become obvious later on.

From eq(5) and eq(6) in F

$$\begin{aligned} \nabla_{(y)} \cdot \mathbf{V}^1 &= -\nabla_{(x)} \cdot \mathbf{V}^0 & (27) \\ \frac{\partial}{\partial y_j} \sigma_{ij}^0 &= 0 & (28) \\ \sigma_{ij}^0 &= -\delta_{ij} P^0 + 2\mu(D_{ij(x)}(\mathbf{V}^0) + D_{ij(y)}(\mathbf{V}^1)) & (29) \end{aligned}$$

and from eq(8) in S

$$D_{ij(y)}(\mathbf{V}^1) = -D_{ij(x)}(\mathbf{V}^0) \tag{30}$$

From eq(9) and eq(10) at order $O(\epsilon^2)$ combined with the conditions of eq(27) through eq(30)

$$\begin{aligned} \int_{\Gamma} \sigma_{ij}^0 n_j \mathbf{e}_i d\sigma_y &= |\Gamma|(\mathbf{K}_B \times \mathbf{B}^0) \\ \int_{\Gamma} \mathbf{y} \times (\sigma_{ij}^0 n_j \mathbf{e}_i) d\sigma_y &= -(\mathbf{K}_B \times \mathbf{B}^0) \times \int_{\Gamma} \mathbf{y} d\sigma_y \end{aligned}$$

We express the velocity of the particles as

$$\mathbf{V}^1 = \mathbf{V}'^1 + \mathbf{V}"^1$$

where $\mathbf{V}'^1$ reflects the macroscopic strains and $\mathbf{V}"^1$ reflects the magnetic forces on the particles.

The equations governing the solution of $\mathbf{V}'^1$, derived from eq(27) through eq(30), are

$$\begin{aligned}
\nabla \cdot \mathbf{V}'^0 &= -\nabla_{(y)} \cdot \mathbf{V}^1 && (31)\\
\frac{\partial}{\partial y_j}\sigma^0_{ij} &= 0 && (32)\\
D_{ij}(\mathbf{V}'^0) &= -D_{ij(y)}(\mathbf{V}^1) && (33)\\
\int_{Y_S} \sigma^0_{ij} n_j \mathbf{e}_i d\sigma_y &= 0 && (34)\\
\int_{Y_S} \mathbf{y} \times (\sigma^0_{ij} n_j \mathbf{e}_i) d\sigma_y &= 0 && (35)
\end{aligned}$$

As in Levy [8] we find the local variation of $\mathbf{V}'^1$ as

$$\mathbf{V}'^1(\mathbf{x},\mathbf{y}) = -D_{ij}(\mathbf{V}^0)\mathbf{X}^{ij}(\mathbf{y})$$

where

$$\int_Y \mathbf{X}^{ij}(\mathbf{y})dy = 0$$

$\mathbf{X}^{ij} \in U_{ad}(\mathbf{P}^{ij}) = \{\mathbf{\Psi} \in [H^1(Y)]^3$, Y-periodic, $\nabla \cdot \mathbf{\Psi} = 0$ in Y_F, $D_{kl}(\mathbf{\Psi}) = D_{kl}(\mathbf{P}^{ij}) \forall$ k,l in $Y_S\}$ with $\mathbf{P}^{ij}$ such that $P^{ij}_k = y_j \delta_{ik}$, and

$$\int_{Y_F} 2\mu D_{kl}(\mathbf{X}^{ij}) D_{kl}(\mathbf{\Psi} - \mathbf{X}^{ij}) dy = 0$$

For $\mathbf{y} \in Y_s$:

$$\mathbf{X}^{ij} = \frac{1}{2}(\mathbf{P}^{ij} + \mathbf{P}^{ji}) + \alpha^{ij} + \beta^{ij} \times \mathbf{y}$$

with α^{ij} and β^{ij} independent of the local variable $\mathbf{y}$.

The equations governing the solution of $\mathbf{V}"^1$, derived from eq(27) through eq(30), are

$$\begin{aligned}
\nabla_{(y)} \cdot \mathbf{V}"^1 &= 0 && (36)\\
\frac{\partial}{\partial y_j}\sigma"^0_{ij} &= 0 && (37)\\
\sigma"^0_{ij} &= -\delta_{ij}P + 2\mu D_{ij(y)}(\mathbf{V}"^1) && (38)\\
D_{ij(y)}(\mathbf{V}"^1) &= 0 && (39)\\
2\mu \int_\Gamma \sigma"^0_{ij} n_j \mathbf{e}_i d\sigma_y &= |\Gamma|(\mathbf{K}_B \times \mathbf{B}^0)\\
2\mu \int_\Gamma \mathbf{y} \times (\sigma"^0_{ij} n_j \mathbf{e}_i) d\sigma_y &= -(\mathbf{K}_B \times \mathbf{B}^0) \times \int_\Gamma \mathbf{y} d\sigma_y
\end{aligned}$$

This leads to a solution where we are looking for $\mathbf{\Upsilon} \in U_{ad} = \{\upsilon \in [H^1(Y)],$ Y-periodic, $\nabla \cdot \upsilon = 0$ in F, $D_{ij}(\upsilon) = 0 \; \forall$ i,j in $Y_S\}$ excluding constant vectors, and

$$2\mu \int_{Y_F} D_{ij}(\mathbf{V}"^1) D_{ij}(\mathbf{u}) dy = \int_\Gamma (\mathbf{K}_B \times \mathbf{B}^0) \cdot \mathbf{u} \quad \forall \mathbf{u} \in \mathbf{\Upsilon}$$

where $\mathbf{u} = \alpha + \beta \times \mathbf{y}$ for $\mathbf{y} \in Y_S$.

The answer is

$$\mathbf{V}"^1(\mathbf{x}, \mathbf{y}, t) = [\mathbf{e}_i \cdot (\mathbf{K}_B \times \mathbf{B}^0)]\mathbf{w}^i \quad (40)$$

with $\mathbf{w}^i$ satisfying

$$\int_Y \mathbf{w}^i dy = 0$$
$$\mathbf{w}^i \in \mathbf{\Upsilon}$$

such that

$$\int_{Y_F} 2\mu D_{kj}(\mathbf{w}^i) D_{kj}(\mathbf{u}) dy = \int_\Gamma \mathbf{e}_i \cdot \mathbf{u} d\sigma_y \quad \forall \mathbf{u} \in \mathbf{\Upsilon}$$

Note that for $\mathbf{y} \in Y_S$, $\mathbf{w}^i = \alpha^i + \beta^i \times \mathbf{y}$ where α^i and β^i are independent of $\mathbf{y}$.

The asymptotic Y-periodic expansion of $\mathbf{V}$ written in terms of $\mathbf{V}^0$, $\mathbf{B}^0$ and $\mathbf{K}_B$ up to an additive constant is

$$\mathbf{V}(\mathbf{x}, t) = \mathbf{V}^0(\mathbf{x}, t) + \varepsilon \Big[- D_{ij}(\mathbf{V}^0)\mathbf{X}^{ij}(\mathbf{y}) + [\mathbf{e}_i \cdot (\mathbf{K}_B \times \mathbf{B}^0)]\mathbf{w}^i(\mathbf{y}) + \mathbf{A}(\mathbf{x}, t) \Big] + O(1)$$

4.3 Macroscopic Equations

In order to establish the correct homogenization results on the boundary we must take an additional expansion of $\mathbf{V}^k$ in the form of a Taylor series [9, 18]:

$$\mathbf{V}^k(\mathbf{x}) = \mathbf{V}^k(\epsilon \mathbf{x}_G) + \frac{\partial \mathbf{V}^k(\epsilon \mathbf{x}_G)}{\partial x_j}(x_j - \epsilon \mathbf{x}_G \cdot \mathbf{e}_j) + ...$$

We have in the fluid, Y_F,

$$\nabla \cdot \mathbf{V}^1 = -\nabla_{(y)} \cdot \mathbf{V}^2 \tag{41}$$

$$i\omega\rho_F \mathbf{V}^0 = \frac{\partial}{\partial x_j}\sigma^0_{ij} + \frac{\partial}{\partial y_j}\sigma^1_{ij} \tag{42}$$

where

$$\sigma^1_{ij} = -\delta_{ij}P^1 + 2\mu(D_{ij}(\mathbf{V}^1) + D_{ij(y)}(\mathbf{V}^2))$$

In the solid, Y_S, we have

$$D_{ij}(\mathbf{V}^1) = -D_{ij(y)}(\mathbf{V}^2) \tag{43}$$

with the balance of forces given as

$$\begin{aligned} i\omega \int_{Y_S} \rho_s \mathbf{V}^0 dy &= \int_{Y_S} \mathbf{J}_B \times \mathbf{B}^0 dy + \int_\Gamma \mathbf{K}_B \times \mathbf{B}^1 d\sigma_y \\ &\quad - \int_\Gamma (\frac{\partial \sigma^0_{ij}}{\partial x_k} y_k + \sigma^1_{ij}) n_j \mathbf{e}_i d\sigma_y \qquad (44) \\ i\omega \int_{Y_S} \rho_s \mathbf{y} \times \mathbf{V}^0 dy &= \int_{Y_S} \mathbf{y} \times (\mathbf{J}_B \times \mathbf{B}^0) dy + \int_\Gamma \mathbf{y} \times (\mathbf{K}_B \times \mathbf{B}^1) d\sigma_y \\ &\quad - \int_\Gamma \mathbf{y} \times (\frac{\partial \sigma^0_{ij}}{\partial x_k} y_k + \sigma^1_{ij}) n_j \mathbf{e}_i d\sigma_y \qquad (45) \end{aligned}$$

We integrate eq(42) on Y_F and add it to eq(44) to get

$$\begin{aligned} i\omega \int_Y \bar{\rho} \mathbf{V}^0 dy &= \frac{\partial}{\partial x_j} \int_{Y_F} \sigma^0_{ij} dy \mathbf{e}_i - \frac{\partial}{\partial x_k} \int_\Gamma \sigma^0_{ij} y_k n_j \mathbf{e}_i d\sigma_y \\ &\quad + \int_{Y_S} \mathbf{J}_B \times \mathbf{B}^0 dy + \int_\Gamma \mathbf{K}_B \times \mathbf{B}^1 d\sigma_y \qquad (46) \end{aligned}$$

where we take

$$\bar{\rho} = \frac{|Y_S|}{|Y|}\rho_S + \frac{|Y_F|}{|Y|}\rho_F$$

The average of the equations is taken

$$i\omega\bar{\rho} < \mathbf{V}^0 > -i\omega < \mathbf{E}^0 > = \frac{\partial}{\partial x_l} < \sigma^0_{kl} > + < f_l >$$

requiring a computation of an extension of the macroscopic tensor, the average velocity,

$$< \mathbf{V}^0 > = \frac{1}{|Y|} \int_Y \mathbf{V}^0 dy \tag{47}$$

the average of magnetic forces minus the frequency dependent terms,

$$
\begin{aligned}
< f_l > &= \frac{1}{|Y|}([\mathbf{K}_B \times \mathbf{e}_i] \cdot \mathbf{e}_l) \int_\Gamma \tau_{ij}(\mathbf{B}^0) d\sigma_y + \frac{|Y_S|}{|Y|}(\mathbf{K}_B \times \mathbf{B}^0) \cdot \mathbf{e}_l \\
&+ \frac{1}{|Y|}\mu_0 [\mathbf{J}_B \cdot \mathbf{e}_i] \int_\Gamma (\mathbf{K}_B \times \mathbf{\Psi}^i) \cdot \mathbf{e}_l d\sigma_y \qquad (48)
\end{aligned}
$$

and the average of the frequency dependent terms of the magnetic forces

$$
< \mathbf{E}^0 >= \frac{3}{|Y|}\mu_0 \epsilon_0 [\mathbf{E}^0 \cdot \mathbf{e}_i] \int_\Gamma (\mathbf{K}_B \times \mathbf{\Psi}^i) d\sigma_y \qquad (49)
$$

which may be reguarded as a slight pertubation due to the induced electric field by the particle oscillations.

4.4 Extension of $< \sigma^0_{ij} >$

From the previous section we must determine $< \sigma^0_{kl} >$, let us again split it into two components reflecting the contributions of the magnetic fields and those of the fluid.

$$
< \sigma^0_{kl} >=< \sigma'^0_{kl} >_Y + < \sigma"^0_{kl} >_Y + < \sigma'^0_{kl} >_\Gamma + < \sigma"^0_{kl} >_\Gamma
$$

with

$$
\sigma'^0_{kl} = -\delta_{kl} P'^0(\mathbf{x}) + 2\mu [D_{kl}(\mathbf{V}^0) + D_{kl(y)}(\mathbf{V}^1)]
$$

and

$$
\sigma"^0_{kl} = -\delta_{kl} P"^0(\mathbf{x}) + 2\mu D_{kl(y)}(\mathbf{V}^1)
$$

Therefore the sum of the fluid components is expressed up to an additive constant

$$
< \sigma'^0_{kl} >_Y + < \sigma'^0_{kl} >_\Gamma = -\delta_{kl} \pi'^0(\mathbf{x}) + a_{ijkl} D_{ij}(\mathbf{V}^0) \qquad (50)
$$

where

$$
\begin{aligned}
a_{ijkl} &= \frac{2\mu}{|Y|} \int_{Y_F} D_{pq(y)}(\mathbf{P}^{ij} - \mathbf{X}^{ij}) D_{pq(y)}(\mathbf{P}^{kl} - \mathbf{X}^{kl}) dy \\
&+ \frac{2\mu}{|\Gamma|} \int_\Gamma (\delta_{ik}\delta_{ml} + D_{im(y)}(\mathbf{X}^{kl})) y_j n_m d\sigma_y
\end{aligned}
$$

and $\pi'(\mathbf{x})$ an additive constant.

To calculate $<\sigma"^{0}_{kl}>$ we must extend $\sigma"^{0}_{kl}$ over the entire period. This extension, $\hat{\sigma}"^{0}_{kl}$, satisfies the following conditions

$$\hat{\sigma}"^{0}_{kl} = \sigma"^{0}_{kl} \quad in \quad Y_F \tag{51}$$

$$\nabla_{(y)} \cdot \hat{\sigma}"^{0}_{kl} = 0 \quad in \quad Y_S \tag{52}$$

$$\hat{\sigma}"^{0}_{kl} n_l \mathbf{e}_k = \sigma"^{0}_{kl} n_l \mathbf{e}_k - (\mathbf{K}_B \times \mathbf{B}^0) \quad on \quad \Gamma \tag{53}$$

The mean volumic value is then given as

$$\bar{\sigma}"^{0}_{kl} = \frac{1}{|Y|}\Big(\int_{Y_F} \sigma"^{0}_{kl} dy + \int_{YS} \hat{\sigma}"^{0}_{kl} dy\Big)$$

We have

$$\begin{aligned}\int_{Y_S} \hat{\sigma}"^{0}_{kl} dy &= \int_{Y_S} \frac{\partial}{\partial y_j}(\sigma"^{0}_{kj} y_l) dy - \int_{Y_S} \frac{\partial \sigma"^{0}_{kj}}{\partial y_j} y_l dy \\ &= -\delta_{kl}\pi"^0 - \int_\Gamma \hat{\sigma}"^{0}_{kj} n_j y_l dy\end{aligned}$$

Using eq(53) this becomes

$$\begin{aligned}\int_{Y_S} \hat{\sigma}"^{0}_{kl} dy &= -\int_\Gamma \sigma"^{0}_{kj} n_j y_l d\sigma_y + \int_\Gamma (\mathbf{K}_B \times \mathbf{B}^0) \cdot \mathbf{e}_k y_l d\sigma_y \\ &= -\int_{Y_F} \sigma"^{0}_{kl} dy + |Y| < \sigma"^{0}_{kl} >_Y + \int_\Gamma (\mathbf{K}_B \times \mathbf{B}^0) \cdot \mathbf{e}_k y_l d\sigma_y\end{aligned}$$

Hence

$$\bar{\sigma}"^{0}_{kl} = < \sigma"^{0}_{kl} >_Y + \frac{1}{|Y|}\int_\Gamma (\mathbf{K}_B \times \mathbf{B}^0) \cdot \mathbf{e}_k y_l d\sigma_y \tag{54}$$

Then we follow with a direct calculation of $< \sigma"^{0}_{kl} >_Y$:

$$\begin{aligned}\int_{Y_F} \sigma"^{0}_{kl} dy &= -\delta_{kl}\int_{Y_F} P"^0 dy + 2\mu \int_{Y_F} D_{kl(y)}(\mathbf{V}"^1) dy \\ &= -2\mu[(\mathbf{K}_B \times \mathbf{B}^0) \cdot \mathbf{e}_i] \int_{Y_F} D_{kl(y)}(\mathbf{w}^i) dy \\ \int_{Y_S} \hat{\sigma}"^{0}_{kl} dy &= \int_{Y_S} \hat{\sigma}"^{0}_{pq} D_{pq}(\mathbf{X}^{kl}) dy \\ &= -\int_\Gamma \hat{\sigma}"^{0}_{kl} X^{kl}_p n_q dy \\ &= -\int_{Y_F} \sigma"^{0}_{pq} D_{pq}(\mathbf{X}^{kl}) dy + [\mathbf{K}_B \times \mathbf{B}^0] \cdot \int_\Gamma \mathbf{X}^{kl} d\sigma_y\end{aligned}$$

which gives

$$\bar{\sigma}"^{0}_{kl} = -\delta_{kl}\pi"^{0} + (\mathbf{K}_B \times \mathbf{B}^0)\cdot \frac{2\mu \mathbf{e}_i}{|Y|}\int_{Y_F} D_{kl(y)}(\mathbf{w}^i)dy$$
$$+(\mathbf{K}_B \times \mathbf{B}^0)\cdot \frac{\mathbf{e}_i}{|Y|}\int_{\Gamma}(\mathbf{X}^{kl}\cdot \mathbf{e}_i)d\sigma_y \tag{55}$$

We then give a direct calculation of $< \sigma"^{0}_{kl} >_\Gamma$

$$< \sigma"^{0}_{kl} >_\Gamma = \frac{2\mu \mathbf{e}_i}{|\Gamma|}\int_{\Gamma} D_{km(y)}(\mathbf{w}^i)y_l n_m d\sigma_y \tag{56}$$

Summing our results gives

$$< \sigma^{0}_{kl} > = -\delta_{kl}\pi^0 + a_{ijkl}D_{ij}(\mathbf{V}^0) + (\mathbf{K}_B \times \mathbf{B}^0)\cdot \mathbf{b}_{kl} \tag{57}$$

with the vector $\mathbf{b}_{kl}$ defined

$$\mathbf{b}_{kl} = \frac{2\mu \mathbf{e}_i}{|Y|}\int_{Y_F} D_{kl(y)}(\mathbf{w}^i)dy + \frac{\mathbf{e}_i}{|Y|}\int_{\Gamma}(\mathbf{X}^{kl}\cdot \mathbf{e}_i)d\sigma_y$$
$$+\frac{2\mu \mathbf{e}_i}{|\Gamma|}\int_{\Gamma} D_{km(y)}(\mathbf{w}^i)y_l n_m d\sigma_y \tag{58}$$

5 Wave Propagation in a Slightly Viscous Fluid

The propagation of waves along the web like structure formed by the particles under the influence of a magnetic induction immersed in a non-magnetic fluid is modelled as a suspension of interacting particles in a slightly viscous fluid. The macroscopic stress is taken to be of a higher order, $\mu = O(\epsilon)$ resulting in the magnetic forces dominating the behavior of the acoustical waves.

5.1 Determination of V^0 and V^r

from eqn(4) at order $O(\epsilon^{-1})$ and eqn(5) at order $O(\epsilon^{-2})$ in F

$$\nabla_{(y)}\cdot(\mathbf{V}^0 + \mathbf{V}^r) = 0$$
$$\frac{\partial}{\partial y_j}\sigma^{ij} = 0$$
$$\sigma^0_{ij} = -\delta_{ij}P^0 + 2\mu D_{ij(y)}(\mathbf{V}^0 + \mathbf{V}^r)$$

In S we from (7) at the order $O(\epsilon^{-1})$

$$D_{ij(y)}(\mathbf{V}^0 + \mathbf{V}^r) = 0 \tag{59}$$

From eqn(9) and eqn(10) at order $O(\epsilon^2)$ we have

$$2\mu \int_\Gamma D_{ij(y)}(\mathbf{V}^0 + \mathbf{V}^r) d\sigma_y = \int_\Gamma \mathbf{K}_B \times \mathbf{B}^0 d\sigma_y \tag{60}$$

$$2\mu \int_\Gamma \mathbf{y} \times D_{ij(y)}(\mathbf{V}^0 + \mathbf{V}^r) d\sigma_y = -(\mathbf{K}_B \times \mathbf{B}^0) \times \int_\Gamma \mathbf{y} d\sigma_y \tag{61}$$

Then for some function $\boldsymbol{\Upsilon} \in U_{ad} = \{\nu \in [H^1(Y)]$, Y-periodic, $\nabla \cdot \nu = 0$ in F, $D_{ij}(\boldsymbol{\Upsilon}) = 0 \; \forall$ i,j in Y_S $\}$ we have

$$2\mu \int_{Y_F} D_{ij(y)}(\mathbf{V}^0 + \mathbf{V}^r) D_{ij(y)}(\mathbf{u}) dy = \int_\Gamma (\mathbf{K}_B \times \mathbf{B}^0) \cdot \mathbf{u} d\sigma_y \qquad \forall \mathbf{u} \in \boldsymbol{\Upsilon} \tag{62}$$

with $\mathbf{u} = \alpha + \beta \times \mathbf{y}$ in S, α and β independent of $\mathbf{y}$. We choose $\mathbf{V}^0$ to be the terms which are independent of $\mathbf{y}$. Hence eq(62) becomes by the homogeneous nature of the suspension and the condition that $\mathbf{V}^0$ be independent of the local variable $\mathbf{y}$:

$$2\mu \int_{Y_F} D_{ij(y)}(\mathbf{V}^r) D_{ij(y)}(\boldsymbol{\Upsilon}) dy = \int_\Gamma (\mathbf{K}_B \times \mathbf{B}^0) \cdot \mathbf{u} d\sigma_y$$

This implies that $\mathbf{V}^r$ can be expressed as

$$\mathbf{V}^r(\mathbf{x}, \mathbf{y}) = [\mathbf{e}_i \cdot (\mathbf{K}_B \times \mathbf{B}^0)] \mathbf{w}^i(\mathbf{y}) \tag{63}$$

with $\mathbf{w}^i$ satisfying

$$\int_Y \mathbf{w}^i dy = 0$$

$\mathbf{w}^i \in \boldsymbol{\Upsilon}$ such that

$$\int_{Y_F} 2\mu D_{kj}(\mathbf{w}^i) D_{kj}(\mathbf{u}) dy = \mathbf{e}_i \cdot \int_\Gamma \mathbf{u} dy \qquad \forall \mathbf{u} \in \boldsymbol{\Upsilon}$$

Note that for $\mathbf{y} \in Y_S$, $\mathbf{w}^i = \alpha^i + \beta^i \times \mathbf{y}$ where α^i and β^i are independent of $\mathbf{y}$.

5.2 Macroscopic Equations

We have in the fluid, Y_F,

$$\begin{aligned} \nabla \cdot \mathbf{V}^1 &= -\nabla_{(y)} \cdot \mathbf{V}^2 && (64) \\ i\omega\rho_F(\mathbf{V}^0 + \mathbf{V}^r) &= \frac{\partial}{\partial x_j}\sigma^0_{ij} + \frac{\partial}{\partial y_j}\sigma^1_{ij} && (65) \end{aligned}$$

where

$$\sigma^1_{ij} = -\delta_{ij}P^1 + 2\mu(D_{ij}(\mathbf{V}^0 + \mathbf{V}^r) + D_{ij(y)}(\mathbf{V}^1))$$

In the solid, Y_S, we have

$$D_{ij}(\mathbf{V}^1) = -D_{ij(y)}(\mathbf{V}^2) \quad (66)$$

with the balance of forces given as

$$\begin{aligned} i\omega\int_{Y_S}\rho_s(\mathbf{V}^0 + \mathbf{V}^r)dy &= \int_{Y_S}\mathbf{J}_B \times \mathbf{B}^0 dy + \int_\Gamma \mathbf{K}_B \times \mathbf{B}^1 d\sigma_y \\ &\quad - \int_\Gamma (\frac{\partial \sigma^0_{ij}}{\partial x_k}y_k + \sigma^1_{ij})n_j\mathbf{e}_i d\sigma_y && (67) \\ i\omega\int_{Y_S}\rho_s\mathbf{y} \times (\mathbf{V}^0 + \mathbf{V}^r)dy &= \int_{Y_S}\mathbf{y} \times (\mathbf{J}_B \times \mathbf{B}^0)dy + \int_\Gamma \mathbf{y} \times (\mathbf{K}_B \times \mathbf{B}^1)d\sigma_y \\ &\quad - \int_\Gamma \mathbf{y} \times (\frac{\partial \sigma^0_{ij}}{\partial x_k}y_k + \sigma^1_{ij})n_j\mathbf{e}_i d\sigma_y && (68) \end{aligned}$$

We integrate eq(65) on Y_F and add it to eq(67) to get

$$\begin{aligned} i\omega\int_Y \bar{\rho}(\mathbf{V}^0 + \mathbf{V}^r)dy &= \frac{\partial}{\partial x_j}\int_{Y_F}\sigma^0_{ij}dy + \frac{\partial}{\partial x_k}\int_\Gamma \sigma^0_{ij}y_k n_j \mathbf{e}_i d\sigma_y \\ &\quad + \int_{Y_S}\mathbf{J}_B \times \mathbf{B}^0 dy + \int_\Gamma \mathbf{K}_B \times \mathbf{B}^1 d\sigma_y && (69) \end{aligned}$$

where $\bar{\rho}$ is previously defined.

The average of the relative velocity is given by

$$\begin{aligned} < \mathbf{V}^r > &= \frac{1}{|Y|}\int_Y \mathbf{V}^r dy \\ &= 0 && (70) \end{aligned}$$

so that the macroscopic equation can be expressed as

$$i\omega\bar{\rho} < \mathbf{V}^0 > - i\omega < \mathbf{E}^0 > = \frac{\partial}{\partial x_l} < \sigma^0_{kl} > + < f_l > \quad (71)$$

where $< f_l >$ is the average frequency independent magnetic force contributions as defined in eq(48) and $< \mathbf{E}^0 >$ is the average frequency dependent magnetic force as define in eq(49).

5.3 Extention of $< \sigma^0_{ij} >$

To calculate $< \sigma^0_{kl} >$ we must extend σ^0_{kl} over the entire period. This extension, $\hat{\sigma}^0_{kl}$, satisfies the following conditions

$$\hat{\sigma}^0_{kl} = \sigma^0_{kl} \quad in \quad Y_F \tag{72}$$

$$\nabla_{(y)} \cdot \hat{\sigma}^0_{kl} = 0 \quad in \quad Y_S \tag{73}$$

$$\hat{\sigma}^0_{kl} n_l \mathbf{e}_k = \sigma^0_{kl} n_l \mathbf{e}_k - (\mathbf{K}_B \times \mathbf{B}^0) \quad on \quad \Gamma \tag{74}$$

The mean volumic value is then given as

$$\bar{\sigma}^0_{kl} = \frac{1}{|Y|}\Big(\int_{Y_F} \sigma^0_{kl} dy + \int_{YS} \hat{\sigma}^0_{kl} dy \Big)$$

We have

$$\begin{aligned} \int_{Y_S} \hat{\sigma}^0_{kl} dy &= \int_{Y_S} \frac{\partial}{\partial y_j}(\sigma^0_{kj} y_l) dy - \int_{Y_S} \frac{\partial \sigma^0_{kj}}{\partial y_j} y_l dy \\ &= -\int_\Gamma \hat{\sigma}^0_{kj} n_j y_l dy \end{aligned}$$

Using eq(74) this becomes

$$\begin{aligned} \int_{Y_S} \hat{\sigma}^0_{kl} dy &= -\int_\Gamma \sigma^0_{kj} n_j y_l d\sigma_y + \int_\Gamma (\mathbf{K}_B \times \mathbf{B}^0) \cdot \mathbf{e}_k y_l d\sigma_y \\ &= -\int_{Y_F} \sigma^0_{kl} dy + |Y| < \sigma^0_{kl} > + \int_\Gamma (\mathbf{K}_B \times \mathbf{B}^0) \cdot \mathbf{e}_k y_l d\sigma_y \end{aligned}$$

Hence

$$\bar{\sigma}^0_{kl} =< \sigma^0_{kl} > + \frac{1}{|Y|} \int_\Gamma (\mathbf{K}_B \times \mathbf{B}^0) \cdot \mathbf{e}_k y_l d\sigma_y \tag{75}$$

Then we follow with a direct calculation of σ^0_{kl}:

$$\begin{aligned} \int_{Y_F} \sigma^0_{kl} dy &= -\delta_{kl} \int_{Y_F} P^0 dy + 2\mu \int_{Y_F} D_{kl(y)}(\mathbf{V}^r) dy \\ &= -\delta_{kl} p i^0 - 2\mu [(\mathbf{K}_B \times \mathbf{B}^0) \cdot \mathbf{e}_i] \int_{Y_F} D_{kl(y)}(\mathbf{w}^i) dy \end{aligned}$$

$$\int_{Y_S} \hat{\sigma}^0_{kl} dy = \int_{Y_S} \hat{\sigma}^0_{pq} D_{pq}(\mathbf{X}^{kl}) dy$$
$$= -\int_{\Gamma} \hat{\sigma}^0_{kl} X^{kl}_p n_q dy$$
$$= -\int_{Y_F} \sigma^0_{pq} D_{pq}(\mathbf{X}^{kl}) dy + [\mathbf{K}_B \times \mathbf{B}^0] \cdot \int_{\Gamma} \mathbf{X}^{kl} d\sigma_y$$

From this we say

$$\bar{\sigma}"^0_{kl} = -\delta_{kl}\pi^0 + (\mathbf{K}_B \times \mathbf{B}^0) \cdot \frac{2\mu \mathbf{e}_i}{|Y|} \int_{Y_F} D_{kl(y)}(\mathbf{w}^i) dy$$
$$+(\mathbf{K}_B \times \mathbf{B}^0) \cdot \frac{\mathbf{e}_i}{|Y|} \int_{\Gamma} (\mathbf{X}^{kl} \cdot \mathbf{e}_i) d\sigma_y \Big) \quad (76)$$

Using the result of $< \sigma^0_{kl} >_\Gamma$ in the previous section we may express the macroscopic stress tensor as

$$< \sigma^0_{kl} >= -\delta_{kl}\pi^0 + (\mathbf{K}_B \times \mathbf{B}^0) \cdot \mathbf{b}_{kl} \quad (77)$$

with the vector $\mathbf{b}_{kl}$ defined

$$\mathbf{b}_{kl} = \frac{2\mu \mathbf{e}_i}{|Y|} \int_{Y_F} D_{kl(y)}(\mathbf{w}^i) dy + \frac{\mathbf{e}_i}{|Y|} \int_{\Gamma} (\mathbf{X}^{kl} \cdot \mathbf{e}_i) d\sigma_y$$
$$+\frac{2\mu \mathbf{e}_i}{|\Gamma|} \int_{\Gamma} D_{km(y)}(\mathbf{w}^i) y_l n_m d\sigma_y$$

6 Discussion of Results

The propagation of acoustical waves in a magnetorheological fluid is characterized by the macroscopic stress tensor, a symmetric tensor, plus some additional force terms resulting from the electromagnetic fields. The volume fraction of the particles, $|Y_S|/|Y|$, and the geometry of the system plays an important role in determining the constants of the constitutive equations. In the case of a viscous fluid the equations are eq(46), eq(57), eq(48), and eq(49)

$$i\omega\bar{\rho} < \mathbf{V}^0 > -i\omega < \mathbf{E}^0 >= \frac{\partial}{\partial x_l} < \sigma^0_{kl} > + < f_l >$$

where

$$< \sigma^0_{kl} >= -\delta_{kl}\pi^0 + a_{ijkl} D_{ij}(\mathbf{V}^0) + (\mathbf{K}_B \times \mathbf{B}^0) \cdot \mathbf{b}_{kl}$$

We find that there is one dispersive, disipative wave within the fluid, magnetic forces working against the waves propagation. In the abscence of a magnetic field the average electromagnetic force terms and the magnetic contribution of the macroscopic stress tensor disappear leaving an equation for acoustic waves propagating in a viscous anisotropic fluid.

In the case of a slightly viscous fluid the equations are eq(71) and eq(77), eq(48), and eq(49).

$$i\omega\bar{\rho} < \mathbf{V}^0 > -i\omega < \mathbf{E}^0 >= \frac{\partial}{\partial x_l} < \sigma^0_{kl} > + < f_l >$$

where

$$< \sigma^0_{kl} >= -\delta_{kl}\pi^0 + (\mathbf{K}_B \times \mathbf{B}^0) \cdot \mathbf{b}_{kl}$$

We find that there is one dispersive, disipative wave present which is predominately affected by the electromagnetic forces and lacks the standard viscous tensor. In the abscence of a magnetic field the right side of the equation reduces to a pressure term which may be selected so that it too disapears. Therefore, we conclude that this precludes a second longitudinal mode travelling along the network of particles unless an external magnetic induction is applied to the system.

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