A Robust Cusum Test for SETAR-Type Nonlinearity in Time Series

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A Robust Cusum Test for SETAR-Type Nonlinearity in Time Series

by

Alina Ursan

A Thesis

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of

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APPROVED:

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Abstract

As a part of an effective SETAR (self-exciting threshold autoregressive) modeling methodology, it is important to identify processes exhibiting SETAR-type non-linearity. A number of tests of nonlinearity have been developed in the literature, including those of Keenan (1985), Petruccelli and Davies (1986), Tsay (1986, 1989), Luukkonen (1988), and Chan and Tong (1990). However, it has recently been shown that all these tests perform poorly for SETAR-type nonlinearity detection in the presence of outliers.

In this project we develop an improved test for SETAR-type nonlinearity in time series. The test is an outlier-robust variant of the Petruccelli and Davies (1986) test based on the cumulative sums of ordered weighted residuals from generalized maximum likelihood fits (which we call CUSUM-GM).

The properties of the proposed CUSUM-GM test are illustrated by means of Monte Carlo simulations. The merits, in terms of size and power, of the proposed test are evaluated relative to the test based on ordered residuals from the ordinary least squares fit (which we call CUSUM-LS) and also to that of other tests for nonlinearity developed in literature. The simulations are run for uncontaminated data and for data contaminated with additive and innovational outliers. The simulation study strongly supports the validity of the proposed robust CUSUM-GM test, particularly in situations in which outliers might be a problem.
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Finally I would like to dedicate my accomplishment to the memory of my dear father “I carry your love with me always.”
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The data is free of outliers.
Chapter 1

The Objective and Summary of this Project

Many of the time series encountered in practice exhibit characteristics not shown by linear processes, such as time irreversibility, nonnormality, asymmetric cycles, nonlinear relationship between lagged variables, and sensitivity to initial conditions. Tong (1990, 1995) and Tjosteim (1994) present many real series which do not have these characteristics. Because linear time series models cannot account for these types of nonlinear behavior, it is only with the recent development of nonlinear times series models and methodology that such features have been successfully modeled (Tong, 1990).

One useful class of nonlinear time series models are the SETAR (self-exciting threshold autoregressive) models (Tong, 1978 and 1983). Applications of SETAR models can be found in diverse fields such as: finance, economics, medicine, and
As a part of an effective SETAR modeling methodology, it is important to identify processes exhibiting SETAR-type nonlinearity. A number of tests of nonlinearity have been developed in the literature, including those of Keenan (1985), Petruccelli and Davies (1986), Tsay (1986, 1989), Luukkonen (1988), and Chan and Tong (1990). However, it has recently been shown that all these tests perform poorly for SETAR-type nonlinearity detection in the presence of outliers (Ng and Chan, 2004).

The primary goal of this project is to propose a test for SETAR-type nonlinearity in time series which is robust against outliers, and to evaluate its performance. The proposed test, which we call CUSUM-GM, is a robust version of a modified Petruccelli and Davies (1986) test based on the cumulative sums of ordered weighted residuals from generalized maximum likelihood (GM) fits. Simulations are used to obtain critical values of the test statistic. The performance, in terms of size and power, of the test on uncontaminated data and on data contaminated with additive and innovational outliers, is compared with that of a non-robust version of the tests, which we call CUSUM-LS, based on the ordered residuals from an ordinary least squares fit, and also with that of other tests for nonlinearity developed in literature.

In most of the cases considered for the Monte Carlo simulation study, the proposed CUSUM-GM test outperforms the CUSUM-LS test. Comparing the CUSUM-GM and CUSUM-LS tests to the CUSUM test of Petruccelli and Davies and other nonlinear tests on a limited range of simulated series, it appears that the CUSUM-GM test performs best in the comparisons and the CUSUM-LS does surprisingly well.
even in the presence of outliers.

The outline of this report is as follows. Chapter 2 contains the introduction and definitions of the main concepts of statistical robustness, robustness to outliers, and the two basic outlier types in time series. Chapter 3 presents two robust methods for estimation in the presence of outliers in the time series context: M-estimation and GM-estimation. In Chapter 4, the AR and SETAR time series models are introduced. Chapter 5 describes the Petruccelli and Davies CUSUM test, for detecting SETAR-type nonlinearity in time series, presents a modified version based on OLS residuals (CUSUM-LS), and introduces the robust CUSUM-GM test. In Chapter 6, the performance of the proposed CUSUM-GM test is evaluated, and its size and power compared with those of the CUSUM-LS test, the Petruccelli and Davies CUSUM test and other tests for nonlinearity proposed in literature, by means of Monte Carlo techniques. Concluding remarks are also presented in this chapter.
Chapter 2

Some Basic Concepts and Definitions

In this chapter, we present some basic concepts of statistical robustness, discuss robustness to outliers in particular, and characterize two main types of time series outliers.

2.1 Statistical Robustness

Robustness, in general, refers to the ability of a procedure or an estimator to produce results that are insensitive to departures from ideal assumptions. This definition of robustness covers all scientific research and its applications date back to the eighteenth century (Hampel et al., 1986; Barnett and Lewis, 1994). For example, early studies using astronomical observations refer to unrepresentative observations that are today known as outliers.
Robustness is a fundamental and fairly recent concept in statistics, having been first introduced as a statistical term by Box (1953). Since then, the theory of robustness has been developed as an important part of the field of statistics. For further details see the books by Huber (1981) and Hampel et al. (1986).

### 2.2 Outlier Robustness

Outliers can be thought of as observations in a data set that cause surprise in relation to the majority of the data (Ripley, 2004). For example, surprising or extreme observations might be unusually large or unusually small values compared to the remaining data. Outliers are a common occurrence in data. They may be the result of an error in measurement, of recording or transmission errors, of exceptional phenomena such as earthquakes or strikes, or they may be due to the sample not being entirely from the same population. Apparent outliers may also be due to the values being from the same, but nonnormal (in particular, heavy-tailed) distribution.

Outliers should be investigated carefully. Often they contain valuable information about the process under investigation or the data gathering and recording process. Before considering the possible elimination of these points from the data, one should try to understand why they appeared.

Many robust methods have been developed to handle data contaminated with outliers. Such methods are said to have outlier robustness. These robust methods can be used to detect outlying observations and to provide resistant results which are stable in the presence of outliers. For instance, if we are interested in estimating a
model parameter for a data contaminated with outliers from a random measurement error, it is of interest to use an estimator which is not sensitive to such outlying observations.

2.3 Outliers in Time Series

The basic theory of robustness has been developed in the i.i.d. (independent, identically distributed) context, but the study of outliers in time series is a more complicated task, due mainly to the structure of the adjacent correlated observations. To illustrate the concepts, we focus attention on the first-order autoregressive, or AR(1) model given by:

\[ x_t = \phi x_{t-1} + \epsilon_t, \quad t = 1, \ldots, T, \]  

(2.1)

where \(|\phi| < 1\) to ensure stationarity and \(\epsilon_t\) are independent and identically distributed \(N(0, \sigma^2_\epsilon)\) random variables.

Two types of outliers in time series were defined by Fox (1972): additive outliers (AO), and innovational outliers (IO). These two types of outliers and other robustness problems in time series are discussed extensively in the time series literature (Denby and Martin, 1979; Hampel et al., 1986).

2.3.1 Model AO

The first and the most often-studied type is the additive outlier (AO), also called Fox’s type I. An AO affects only a single observation. After this disturbance,
the series returns to its normal path as if nothing had happened.

We say that $y_t$ is an AR(1) time series with additive outliers if $y_t$ satisfies

$$y_t = x_t + v_t, \ 1 \leq t \leq T$$

(2.2)

where $x_t$ is as defined in (2.1) and the $v_t$ are independent random variables, independent of the sequence $x_t$ and having a Gaussian mixture density given by

$$CND(\gamma, \sigma^2) = (1 - \gamma)\delta + \gamma N(0, \sigma^2)$$

where $0 \leq \gamma \leq 1$, $\delta$ is a degenerate central component (i.e. $P(\delta = 0) = 1$), and $\sigma^2 > \sigma_c^2$.

$\gamma$ represents the proportion of contamination in the time series. In general, it is assumed that $\gamma$ is small ($\gamma \leq 0.1$) because it appears that outliers in time series are present only a small fraction of the time (Denby and Martin, 1979).

The classic reason for an AO is a recording or measurement error. Outbreaks of wars, strikes, an abrupt change in the market structure of some group of commodities, a technical change or new equipment in a communication system, or simply unexpected geophysical phenomena (e.g., earthquakes) are all possible causes of additive outliers.

2.3.2 Model IO

In contrast to the AO, an innovational outlier (IO), also known as Fox’s type II, affects several observations. We say that $y_t$ is an AR(1) time series with innovational
outliers if $y_t$ satisfies

$$y_t = x_t, \quad 1 \leq t \leq T$$

where $x_t$ is as defined in (2.1) and the $\epsilon_t$ have a heavy-tailed non-Gaussian distribution $G$. For example, $G$ can be the contaminated normal density given by

$$CN(., |\gamma, \sigma^2) = (1 - \gamma)N(., 0, 1) + \gamma N(., 0, \sigma^2)$$

with $0 \leq \gamma \leq 1$ and $\sigma^2 > 1$.

An IO is typically caused by some external shock at time $t$ that influences observations $x_t, x_{t+1}, \ldots$. For example, in a time series measuring the position of a satellite, an IO could be caused by an asteroid hitting the satellite and thereby deviating it from its track. Innovational outliers can be also found in other areas such as speech recognition (Lee and Martin, 1987).

### 2.3.3 The Difference between AOs and IOs

One way to point out the difference between additive and innovational outliers, is to see that in fact we can interpret an AO as an outlying observation added after the realization to affect a single observation and an IO as an outlying observation added during the realization with influence on all succeeding observations.

In autoregressive models, AOs are a cause of much greater concern than IOs because leverage points (outliers in the $x$-direction) pose bigger problems than outliers in the $y$-direction. For example, in the case of an AR(1) model, one IO yields one outlier in the response variable and a number of “good” leverage points (“good”
refers to the fact that the leverage points lie close to the fitted line determined by the majority of the data), which actually improve the accuracy of the parameter estimate. Therefore, one IO only affects one residual. On the other hand, one AO results in one outlier in the vertical direction and one “bad” leverage point (“bad” refers to the fact that the leverage point does not lie close to the fitted line determined by the majority of the data). Thus, AO affects the next residual as well, inflating two consecutive residuals.
Chapter 3

Robust Estimation

Two robust methods for estimation in the presence of outliers in time series context are presented in this chapter. M-estimation is described in Section 3.2 and GM-estimation in Section 3.3. For simplicity, all the results are stated for the AR(1) model defined in (2.1).

3.1 Introduction

Although least-squares estimation (LSE) has been extensively used in statistics, in particular for regression analysis, it has its shortcomings. One of the weakest points of the method is its high sensitivity to outliers: one sufficiently large outlier can ruin the estimate. One explanation for this high sensitivity of least squares to outliers is that squaring the residuals magnifies the effects of these extreme data points.

In the time series setting, least squares estimation methods also exhibit a lack of
robustness to outliers. The least squares estimator of the autoregressive parameter $\phi$ in the AR(1), IO model (2.3) is consistent, but inefficient. For the AR(1), AO model (2.2), the least squares estimator is not even consistent (see Denby and Martin, 1979).

To alleviate this sensitivity to outliers, statisticians began to develop robust estimation methods starting around 1960 (Hampel, 2001). In the regression setting, robust regression methods are expressly designed to minimize the effect of outliers while retaining much of the sensitivity and precision of LSE in the absence of outliers. Two such methods, which we present in the context of AR(1) model (2.1) are M-estimation and GM-estimation.

### 3.2 Definition of the M-estimates

One of the most popular classes of robust estimators is the maximum likelihood-type estimator, or M-estimator which was introduced by Peter Huber in 1964. It is a large class of estimators that contains maximum likelihood estimators as a subclass.

Following Denby and Martin (1979), the M-estimator of parameter $\phi$, denoted by $\hat{\phi}_M$ is defined by

$$
\hat{\phi}_M = \min_{\phi'} \sum_{t=1}^{T-1} \rho(x_{t+1} - x_t \phi'),
$$

(3.1)

where $\rho(\cdot)$ is a symmetric robustifying loss function. Equivalently, $\hat{\phi}_M$ is the solution of the following equation:

$$
\sum_{t=1}^{T-1} x_t \psi(x_{t+1} - x_t \hat{\phi}_M) = 0,
$$

(3.2)
where $\psi(u) = \rho'(u)$ is called the influence function. $\psi(\cdot)$ is chosen to be a bounded function with $u\phi(u) \geq 0$, and usually $\psi'(0) = 1$.

The most commonly used influence functions are those from the Huber family (H) and from bisquare family (B) proposed by Beaton and Tukey (1974). The Huber family is defined by

$$
\psi_{H,c}(u) = \text{sgn}(u)\min(|u|, c), \tag{3.3}
$$

where $\text{sgn}(u)$ is the sign function, and $c > 0$. The bisquare family is given by

$$
\psi_{B,c}(u) = \text{sgn}(u)\max(0, |u|(1 - u^2/c^2)^2). \tag{3.4}
$$

Beaton and Tukey (1974) expressed (3.2) in terms of the weight function

$$
w(u) = \frac{\psi(u)}{u}, \text{ as }
\sum_{t=1}^{T-1} w(e_{t+1})x_t(x_{t+1} - x_t\hat{\phi}_M) = 0. \tag{3.5}
$$

where $e_{t+1} = x_{t+1} - x_t\hat{\phi}_M$ for $t = 1, \ldots, T - 1$ are the residuals.

Equation (3.5) can be solved by the iteratively reweighted least squares (IRLS) method as follows:

1. Get an initial estimate of $\phi$, say $\hat{\phi}_0$, usually by ordinary least-squares

2. Given $\hat{\phi}_0$, compute the initial weights as

$$
w(e_{t+1}^0) = \frac{\psi(e_{t+1}^0)}{e_{t+1}^0} \tag{3.5}
$$

where $e_{t+1}^0 = x_{t+1} - x_t\hat{\phi}_0$, for $t = 1, \ldots, T - 1$
3. For $j = 0$ to convergence do

$$
\hat{\phi}_{j+1} = \left( \sum_{t=1}^{T-1} w(e^j_{t+1})x_t^2 \right)^{-1} \left( \sum_{t=1}^{T-1} w(e^j_{t+1})x_tx_{t+1} \right) 
$$

$$
e^j_{t+1} = x_{t+1} - x_t \hat{\phi}_j .
$$

Convergence can be defined in a number of ways: relative change in the estimates; relative change in the scaled residuals; relative change in weights; preselected number of steps.

An effective strategy for obtaining an M-estimate is as follows: the initial estimate is computed by using least squares estimation; then the M-estimate based on the Huber influence function (the M-H estimate) is computed by the IRLS method described above; the corresponding M-H estimate is used as a starting point for computing M-estimate based on the bisquare influence function (the M-B estimate), again using the IRLS method. The use of the Huber influence function ensures that a unique root of equation (3.5) is obtained and the choice of the bisquare influence function leads to a much more robust estimator in the case of AO model (Denby and Martin, 1979).

Denby and Martin (1979) show that the M-estimator is robust to IO outliers, but not to AO outliers. In fact, they show that M-estimators can have asymptotic bias nearly as large as least squares estimators in the AO case. Since the robustness of the M-estimator is not satisfactory, a more robust estimator, called the generalized M-estimator (GM) is proposed. This estimator is described in the following section.
3.3 Definition of the GM-estimates

GM-estimates were first used in regression analysis by Hampel (1975) and Mallows (1976). GM-estimates for autoregressive models have been proposed and studied in the literature by Denby and Martin (1979), Martin (1980) and Bustos (1982). The basic idea of the GM-estimator is to modify the minimization problem so that the summands of the estimating equation (3.2) are bounded and continuous functions of the data.

Denby and Martin (1979) defined the GM-estimator, \( \hat{\phi}_{GM} \) for an AR(1) as the solution of the minimization problem

\[
\min_{\phi'} \sum_{t=1}^{T-1} W(x_t)\rho(x_{t+1} - x_t\phi'),
\]

(3.6)

where \( \rho(\cdot) \) is the same as in (3.1) and \( W(\cdot) \) is a nonnegative symmetric weight function. If \( g(\cdot) \) is a bounded, nonnegative function such that \( g(u) = uW(u) \), then \( \hat{\phi}_{GM} \) is the solution of

\[
\sum_{t=1}^{T-1} g(x_t)\psi(x_{t+1} - x_t\hat{\phi}_M) = 0.
\]

(3.7)

GM-estimates are also computed using iteratively reweighted least squares. Denby and Martin (1979) show that GM-estimation is successful in reducing asymptotic bias when the data are contaminated with additive outliers.

3.4 GM-estimation for the Proposed Test

The proposed test, which will be fully described in the next chapter, is based on the residuals obtained from the GM fit. In order to obtain the GM-estimates, we
need to define the \( g(\cdot) \) and \( \psi(\cdot) \) functions. Following Denby and Martin (1979), the functions are given by

\[
g(u) = s_x g_0 \left( \frac{u}{s_x} \right)
\]  

(3.8)

and

\[
\psi(u) = s_r \psi_0 \left( \frac{u}{s_r} \right),
\]  

(3.9)

where \( s_x \) and \( s_r \) are robust estimates of scale for \( x_t \) and \( \epsilon_t \) respectively and \( g_0, \psi_0 \) are influence functions.

We consider

\[
s_x = \text{median}(|x_i - \text{median}(x_i)|)/0.6745 \quad (3.10)
\]

and \( s_e \) is obtained from the residuals \( (e_i) \) at each step of the IRLS procedure, computing

\[
s_e = \text{median}(|e_i - \text{median}(e_i)|)/0.6745. \quad (3.11)
\]

This type of robust estimator for scale was first suggested by Hampel (1986).

We also choose \( g_0 = \psi_0 \) to be either the Huber or bisquare influence function. For the Huber case, the constant \( c \) is \( c_{H,y} = 1 \) (for \( g_0 \)) and \( c_{H,r} = 1.5 \) (for \( \psi_0 \)). For the bisquare case, \( c \) is defined as \( c_{B,y} = 3.9 \) (for \( g_0 \)) and \( c_{B,r} = 1.5 \) (for \( \psi_0 \)). These parameters are chosen to get 95% asymptotic efficiency on the standard normal and nonnormal distributions simultaneously (Andrews et al., 1972).

The GM-B estimator is preferred to the GM-H estimator because of its superiority in the AO model along with its reasonable robustness in the IO model (Denby and Martin, 1979). This leads us to choose GM-B estimation for the proposed test. Using equation (3.7), we compute the GM-B estimator as follows: we get first an initial
estimate by using ordinary least squares, then we compute the GM-estimator based
on Huber influence function (the GM-H estimate) by choosing $g_0(\cdot) = \psi_o(\cdot) = \psi_H(\cdot)$;
then the GM-H estimate is used as a starting point for GM-B estimation for which
$g_0(\cdot) = \psi_o(\cdot) = \psi_B(\cdot)$. For our computations, we consider that the convergence is
obtained when the relative change in the parameter estimates is less than $10^{-4}$. 
Chapter 4

AR and SETAR models

4.1 AR Models

Autoregressive models are frequently used and well-studied time series models and are easily estimated using regression methods. An autoregressive model of order $p$, AR($p$), can be defined as follows:

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \ldots + \phi_p x_{t-p} + \epsilon_t,$$

(4.1)

where $\phi_1, \phi_2, \ldots, \phi_p$ are constants and $\epsilon_t$ are independent and identically distributed $N(0, \sigma^2_{\epsilon})$ random variables with $\sigma^2_{\epsilon} < \infty$.

4.2 SETAR Models

A straightforward extension of AR models is the class of AR-type nonlinear models defined as

$$x_t = f(x_{t-1}, x_{t-2}, \ldots, x_{t-p}) + \epsilon_t,$$

(4.2)
where $f : \mathbf{R}^p \to \mathbf{R}$ and $\epsilon_t$ is a sequence of independent and identically distributed $N(0, \sigma^2_\epsilon)$ random variables with $\sigma^2_\epsilon < \infty$.

When $f$ is a piecewise linear function, we get the special case of threshold autoregressive or TAR models that were first proposed by Tong (1978, 1983, 1990). The main aspect of these models is to change the parameters of a linear autoregressive model according to a value of an observable variable named the threshold variable.

When the threshold variable is a lagged value of $x_t$, say $x_{t-d}$, then the model is 'self-exciting', hence the acronym SETAR (i.e., self-exciting TAR). In this case $d$ is called the delay. The SETAR model is piecewise-linear in the space of the threshold variable, rather than in time. A $k$-regime SETAR$(d; p_1, p_2, \ldots, p_k)$ model is defined by

$$x_t = \sum_{l=1}^{p_j} \phi^{(j)}_l x_{t-l} + \epsilon_t \quad \text{if} \quad x_{t-d} \in (r_{j-1}, r_j],$$

for $j = 1, 2, \ldots, k$. The thresholds, $r_i$, satisfy $-\infty = r_0 < r_1 < \cdots < r_k = \infty$, $d$, $k$ and $(p_1, p_2, \ldots, p_k)$ are positive integers, and $\epsilon_t$ are independent and identically distributed $N(0, \sigma^2_\epsilon)$ random variables with $\sigma^2_\epsilon < \infty$. We say that the process is in the $i^{th}$ regime when $r_{i-1} < x_{t-d} \leq r_i$.

SETAR models are one class of non-linear time series models that has been widely used in the literature to explain various empirical phenomena. For example, they were applied to the foreign exchange market: Kräger and Kugler (1993) modeled five currencies against the US dollar on weekly data over the last ten years, Peel and Speight (1994) modeled three weekly sterling spot market rates over the inter-war period, and Chappell et al. (1996) modeled the French franc to Deutschemark rate.
in the 1990s on daily data.

SETAR models are able to generate complex nonlinear dynamics. This and other classes of nonlinear models can produce asymmetries and jump phenomena that cannot be captured by a linear time series model. Another interesting feature of SETAR models is that the stationarity of $x_t$ does not require the model to be stationary in each regime; on the contrary the limit cycle behavior that they are able to describe arises from the alternation of regimes and thus they can be used to model periodic time series.

The definitions of AO and IO in (2.2) and (2.3) also define AO and IO for SETAR models when $x_t$ is generated by (4.3).
Chapter 5

Testing for Nonlinearity

As nonlinear time series models and methods have come into use over the past two decades, it has become important to identify nonlinear patterns in time series data. A number of tests of SETAR-type nonlinearity have been developed in the literature, including those of Keenan (1985), Petruccelli and Davies (1986), Tsay (1986, 1989), Luukkonen (1988), and Chan and Tong (1990). However, it has recently been shown that all these tests perform poorly for SETAR-type nonlinearity detection in the presence of outliers (see Ng and Chan, 2004). This is perhaps not surprising, since it appears that outliers are responsible for a considerable amount of apparent nonlinearity in the series. Balke and Fomby, 1994 give some examples in this sense using real economic data.

In this chapter we provide a review of the Petruccelli and Davies CUSUM test for detecting SETAR-type nonlinearity in time series, we briefly describe a modification of that test based on residuals from the ordinary least squares fits, which we call the
CUSUM-LS test, and finally we propose a robust version of the CUSUM-LS test, which we call the CUSUM-GM test.

5.1 The Petruccelli and Davies Test

Assume we have a time series $x_1, x_2, \ldots, x_T$ generated from the SETAR model defined in (4.3). Let $p = \max(p_1, p_2, \ldots, p_k)$, $h = \max(1, p + 1 - d)$ and $x_{(i)}$ be the $i^{th}$ smallest observation among $x_h, x_{h+1}, \ldots, x_{T-d}$, for $i = 1, 2, \ldots, T - d - h + 1$.

If $m_0 = 0$, $m_k = T$, and $m_j, j = 1, \ldots, k - 1$ are integers such that $x_{(m_j)} \leq r_j \leq x_{(m_{j+1})}$, then model (4.3) may be written in terms of ordered autoregression as

$$x_{(i)+d} = \sum_{l=1}^{p} \phi_l^{(j)} x_{(i)+d-l} + \epsilon_{(i)+d}, \quad i = m_{j-1} + 1, \ldots, m_j, \quad j = 1, \ldots, k. \quad (5.1)$$

The increasing ordered autoregression effectively divides model (4.3) into $k$ linear autoregressions, one in each of the $k$ regions defined by the thresholds. Under the null hypothesis that model (4.3) is linear, i.e. $H_0 : \phi_1^{(i)} = \cdots = \phi_k^{(i)} \quad (i = 1, \ldots, p)$

model (5.1) can be written in matrix form as follows

$$Y = X\Phi + \epsilon \quad (5.2)$$

where $Y$ is a column vector containing $x_{(i)+d}, i = 1, 2, \ldots, T - d - h + 1$, $X$ is a $(T - d - h + 1) \times p$ matrix whose columns contain appropriately lagged $x_{(i)+d}$ values, $\Phi$ is a column vector of unknown parameters $\phi_l, l = 1, \ldots, p$ and $\epsilon$ is a column vector of noise terms.
Let $m$ be the startup number of observations. For each $r \geq m$, the first $r$ rows of $Y$, denoted by $Y_{(r)}$, are regressed on the first $r$ rows of $X$, denoted by $X_{(r)}$ and the corresponding least squares estimator $\hat{\Phi}_{(r)}$ is obtained. Then the standardized one-step-ahead predictive residual $z_{(r+1)}$ is computed successively as follows

\[
    z_{(r+1)} = \frac{x_{(r+1)+d} - x_{(r+1)}\hat{\Phi}_{(r)}}{s(r)\sqrt{1 + x_{(r+1)}(X'_{(r)}X_{(r)})^{-1}x'_{(r+1)}}}, \quad r = m, m+1, \ldots, T - d - h
\]  

(5.3)

where $x_{(r+1)}$ is the row vector $(x_{(r+1)+d-1}, \ldots, x_{(r+1)+d-p})$ and $s(r)$ is the resulting residual standard deviation based on the first rows of $Y$ and $X$. The matrix $X'_{(r)}X_{(r)}$ is assumed to be non-singular. An efficient recursive updating algorithm (Brown, Durbin and Evans, 1974) is used in calculating the $z_{(r+1)}$.

The CUSUM test statistic of Petruccelli and Davies is defined as

\[
    P_T = \max_{m+1 \leq r \leq T-d-h+1} |Z_r|,
\]

(5.4)

where

\[
    Z_r = \sum_{i=m+1}^{r} z_{(i)}, \quad r = m + 1, \ldots, T - d - h + 1
\]

(5.5)

Under $H_0$ and $T \to \infty$, $\frac{P_T}{\sqrt{T-p-m}} \to \sup_{0<\lambda<1} |W(\lambda)|$, where $W$ is standard Brownian motion on $[0,1]$. Therefore,

\[
    Pr \left( \frac{P_T}{\sqrt{T-p-m}} \leq t \right) \to \frac{4}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1} \exp \left[ -\frac{(2i+1)^2 \pi^2}{8t^2} \right].
\]

(5.6)

Let $1 - p^*$ denote the value computed from the right side of (5.6) with $t$ given by

\[
    t^* = \max_{m+1 \leq r \leq T-d-h+1} |Z_r|/\sqrt{T-p-m}.
\]
Then the test rejects $H_0$ at the $\alpha$ level of significance if $p^* < \alpha$.

Since we were looking for a robust test for linearity against SETAR-type non-linearity, we initially tried to directly modify the CUSUM test to obtain the recursive residuals from the GM fit instead of the LS fit. This presented two problems: (1) Because there are not recursive updating operations, the amount of computation involved in trying to assess the performance of the test through Monte Carlo simulations was excessive. (2) The large number of fits required resulted in too frequent convergence failures of the IRLS algorithm, especially for small $m$.

### 5.2 The CUSUM-LS Test

Ploberger and Krämer (1992) recently discussed the possibility of a CUSUM test in the regression setting based on the ordinary least squares (OLS) (i.e., residuals from the fit to the entire set of data) rather than recursive residuals.

Consider the linear regression model

$$y_t = x_t'\beta + u_t, \quad t = 1, \ldots, T$$

(5.7)

where $y_t$ is the dependent variable, $x_t$ is a $p$ dimensional vector of fixed regressors which includes an intercept, $\beta$ is the $p$ dimensional unknown parameter vector and $u_t$ is the error term. $u_1, \ldots, u_T$ are i.i.d. variables, not necessarily normal, with zero mean and $\sigma^2$ variance.

The null hypothesis of the test is that the unknown parameter vector $\beta$ is
constant over time

\[ H_0 : \beta^{(1)} = \ldots = \beta^{(T)}. \]

The residuals obtained from the OLS estimation are

\[ \hat{u}_t^{(T)} = y_t - x_t'\hat{\beta}^{(T)}, \]

where the superscript \( T \) emphasizes the dependence of these quantities on sample size. Then the cumulative sums of the OLS residuals are given by

\[ B^{(T)}(\lambda) = \frac{1}{\hat{\sigma}\sqrt{T}} \sum_{t=1}^{T\lambda} \hat{u}_t^{(T)}, \quad (5.8) \]

where

\[ \hat{\sigma} = \sqrt{\frac{\sum_{t=1}^{T} \hat{u}_t^2}{T}}. \]

Ploberger and Krämer (1992) show that under \( H_0 \), \( B^{(T)}(\lambda) \) converges to \( B(\lambda) \), the Brownian Bridge on \([0, 1]\) as \( T \to \infty \). Then the test statistic is

\[ \sup_{0 \leq \lambda \leq 1} |B^{(T)}(\lambda)| \quad (5.9) \]

which under \( H_0 \) converges to \( \sup_{0 \leq \lambda \leq 1} |B(\lambda)| \). Therefore, under \( H_0 \), as \( T \to \infty \)

\[ P \left( \sup_{0 \leq \lambda \leq 1} |B^{(T)}(\lambda)| > a \right) \to 2 \sum_{j=1}^{\infty} (-)^{j+1} \exp(-2j^2a^2). \quad (5.10) \]

The \( p \)-value of the test equals the right side of (5.10), where \( a \) is taken to be the observed value of the test statistic (5.9).

By applying the Ploberger and Krämer test to the residuals from the arranged autoregression (5.1), we obtain a test for SETAR-type nonlinearity that we call the CUSUM-LS test. The CUSUM-LS test is similar to the Petruccelli and Davies test,
except that it is based on the OLS residuals instead of recursive residuals. The asymptotic distribution is no longer Brownian bridge which will cause us to resort to simulation to obtain critical values for the test. The CUSUM-LS test is still not robust, but it has the benefit that it reduces computational burden, which will be important when we create a robust version.

5.3 CUSUM-GM Test for Setar-type Nonlinearity

If a time series is contaminated with additive outliers then we can obtain biased least-squares estimates. A possible solution to the outlier problem for estimation is to use robust estimation methods to guard against the harmful effects of these outlying observations. In the previous chapter we have presented a robust estimation method, GM-estimation, for first order autoregressive time series models. Chan and Cheung (1994) and Gabr (1998) discuss robust estimation in the nonlinear case of threshold autoregressive and bilinear models respectively.

Because the CUSUM-LS test is based on residuals from least squares fits, we propose to develop a robust version of the test by using attenuated residuals from GM fits. We call the test CUSUM-GM.

Let’s consider the particular first order, two-regime SETAR model or SETAR(1;1,1) model

\[
x_t = \begin{cases} 
\phi_1 x_{t-1} + \epsilon_t & \text{if } x_{t-1} \leq r \\
\phi_2 x_{t-1} + \epsilon_t & \text{if } x_{t-1} > r
\end{cases}
\]  

(5.11)
where $\epsilon_t$ are i.i.d. normal random variables with zero mean and constant variance $\sigma^2_\epsilon$.

The null hypothesis of the test is

$$H_0 : \phi_1 = \phi_2$$

and in this case model (5.11) becomes an AR(1). The alternative hypothesis is that $H_0$ does not hold.

The proposed CUSUM-GM test statistic is given by

$$\sup_{0 \leq \lambda \leq 1} \left| \frac{1}{A \sqrt{T}} Z_{CGM}^{(T)}(\lambda) \right|,$$

where $T$ is the number of observation in the time series, $A$ is defined as

$$\sqrt{\sum_{t=0}^{T-1} \psi^2(\hat{x}_{t+1} - \hat{x}_t \hat{\phi}_{GM})},$$

and $\hat{\phi}_{GM}$ is the GM-estimate as explained in Section 3.3.

To evaluate the rejection region for this test, we first thought to use the asymptotic results of Ploberger and Krämer, but we soon found these to be an unsuitable approximation. We then decided to use Monte Carlo simulation to obtain a $p$-value on each set of data separately. The algorithm is:

1. Compute $\hat{\phi}_{GM}$ and $\hat{\sigma}_\epsilon$, where $\hat{\sigma}_\epsilon$ is calculated using formula (3.11)

2. Compute the CUSUM-GM test statistic $Z^*$

3. For $j = 1$ to 1000 do:
   
   (i) Generate an AR(1) $\sim \hat{\phi}_{GM}$, $\hat{\sigma}_\epsilon$, $T$
(ii) Compute the CUSUM-GM statistic $Z_j^*$

(4) Compute the empirical $p$-value as the proportion of $Z_j^* \geq Z^*$

This method works well for individual series, but was too time-consuming when applied to the thousands of series needed to perform our simulation study. In order to overcome this problem, we created a table of critical values.

The critical values represent the 0.95 quantile of the test statistic under $H_0$. To obtain the table of critical values, we consider various sets of $\phi, \sigma_\epsilon$, and $T$, with $-0.95 \leq \phi \leq 0.95$, $0.1 \leq \sigma_\epsilon \leq 3$ and $T = 100, 200, 500$. For each of these sets, we first generated 5000 series, computed the test statistic for each of the series and then found the 0.95 quantile of the test statistic. These quantiles were then tabulated (see Tables 10 - 14 for some empirical quantiles when $T = 100$ and 500). To conduct the test for a given series, $\hat{\phi}_{GM}$ and $\hat{\sigma}_\epsilon$ are computed and the critical values obtained from the table by quadratic interpolation.

The quadratic interpolation was obtained by calling the “QD2VL” routine from IMSL Fortran 90 Library. The function “QD2VL” interpolates a table of values, using quadratic polynomials, returning an approximation to the tabulated function. More information about this routine can be found online at http://www.vni.com/books/dod/pdf/MATH.pdf

In order to compute the empirical size and power of the test statistic using the table, we proceed now as follows:

(1) Specify model parameters $(\phi_1, \phi_2, \sigma_\epsilon, T)$
(2) For $i = 1$ to 1000 do

(i) Generate series $\sim (\phi_1, \phi_2, \sigma, T)$

(ii) Compute $\hat{\phi}_{GM}$ and $\hat{\sigma}$

(iii) Compute the CUSUM-GM test statistic $Z^*$

(iv) Obtain the critical value from the table by quadratic interpolation

(3) Compute the proportion of $Z^*$ greater than the corresponding critical values.

If $\phi_1 = \phi_2$ in step (1), then the above algorithm gives the empirical size of the test. If $\phi_1 \neq \phi_2$ in step (1), then it gives the empirical power of the test.
Chapter 6

Simulation Experiments

The objective of this chapter is to evaluate the performance of the proposed CUSUM-GM test, and compare its size and power with those of the CUSUM-LS test and other tests of time series nonlinearity by using Monte Carlo techniques. The simulations are run for outlier-free data and for data contaminated with additive and innovational outliers.

6.1 Design of the Monte Carlo Simulation

Two Monte Carlo simulation studies were conducted to evaluate the performance of the CUSUM-GM test. The first study compared the performance of the CUSUM-GM and CUSUM-LS tests on simulated data with corresponding results for five other nonlinearity published tests. In the second study, the CUSUM-GM test was compared to the CUSUM-LS test with respect to size and power, for a large range of SETAR(1; 1, 1) series having additive, innovational and no outliers.
For both studies, the data were generated from model (5.11) with \( r = 0 \). All of the simulations were based on 1000 replications of the AO or IO model with \( \gamma = 0.05 \), and \( \sigma_r^2 = 1 \). Sample sizes \( T = 100 \) and 200 were used for the first study, \( T = 100, 200 \) and 500 for the second. The first 500 observations in each replication were discarded to avoid dependence on the initial value which was set to zero. We assumed that \( p = 1 \) and \( d = 1 \) are known.

For the first study, comparing the CUSUM-GM and CUSUM-LS tests to previous tests for nonlinearity in literature, we considered three parameter combinations for \( \phi_1 \) and \( \phi_2 \). They are taken from Chan and Ng (2004) and are: \( \phi_1 = \phi_2 = 0.5 \), \( \phi_1 = 0.5 \), \( \phi_2 = 0.8 \) and \( \phi_1 = 0.5 \), \( \phi_2 = -0.3 \). For the second study, comparing the CUSUM-GM and CUSUM-LS tests, we considered pairwise combinations of the following values of \( \phi_1 \) and \( \phi_2 \): \( \phi_1 = -0.9, -0.5, -0.1, 0.1, 0.5, 0.9 \) and \( \phi_2 = -0.95, -0.90, -0.75, -0.5, -0.25, 0, 0.25, 0.50, 0.75, 0.90, 0.95 \).

All the computations were coded in Fortran 90 and some routines were called from the IMSL Fortran 90 Library.

Following Chan and Ng (2004), we set the outlier magnitude \( \omega = 0, 3, 6, 10 \) for the AO model and \( \omega = 1, 3, 6, 10 \) for the IO model. It should be noted that \( \omega = 0 \) under AO and \( \omega = 1 \) under IO correspond to the no-outlier case.

The null hypothesis of the tests is that the time series \( x_t \) follows the AR(1) model, \( H_0 : \phi_1 = \phi_2 \) under (5.11), while the alternative hypothesis is that \( x_t \) follows a SETAR(1; 1,1) model (\( \phi_1 \neq \phi_2 \)).
To evaluate any test, two properties are important: size and power. The size of the test is the probability of Type I error (i.e., the probability of rejecting $H_0$ when it is true). The power of the test is the probability of rejecting $H_0$ when it is false. Usually the size of a test is fixed at a pre-specified level called the significance level, so that the relative power of the tests can be compared. It is also important to compare the size of tests to the nominal level. A test with actual size less than nominal presents the possibility that it is not as powerful as it could be. A test with actual size larger than the nominal significance level presents a higher Type-I error, so that it cannot be considered reliable. For the purpose of our simulations, a size of 0.15 or more was considered unacceptable. We compared the tests both in terms of size and power when the level of significance is fixed at a nominal $\alpha = 0.05$ level.

In order to compute the size of the CUSUM-GM and CUSUM-LS tests, linear time series with and without outliers were first simulated, and then the AR(1) model was estimated, and its residuals used to compute the test statistics. The percentage of these test statistic values greater than their critical values was then computed.

Nonlinear time series with and without outliers were then generated and the same testing procedures run in order to compute the power of the tests. In the first study, these results were compared with the results from five other tests found in Chan and Ng (2004). In the second study, the simulation results were read into a SAS program, in order to plot the empirical power curves of the CUSUM-GM and CUSUM-LS tests.
6.2 Comparison of CUSUM-GM and CUSUM-LS Tests to Other Tests for SETAR-Type Nonlinearity

The proposed CUSUM-GM test is a robust version of the CUSUM-LS test which is based on the ordered residuals from the ordinary least squares fit and the CUSUM-LS test is a modified version of the Petruccelli and Davies CUSUM test which is based on the recursive residuals from the least squares fit. In this section, we compare the CUSUM-GM test (denoted here by C-GM) and the CUSUM-LS test (denoted here by C-LS) to the Petruccelli and Davies CUSUM test (denoted by C-PD) and also to other tests for SETAR-type nonlinearity that are based on the least squares approach: the reverse CUSUM (RC) test (Petruccelli, 1990), Lagrange multiplier (LM) test (Luukkonen, 1988), F-test (Tsay, 1989) and the likelihood ratio test (Chan and Tong, 1990; Chan and Ng, 2004). The simulation results of the tests being compared are taken from Chan and Ng (2004). The results displayed in Tables 6.1 - 6.4 show the following:

Outlier Free Case  There is no test unacceptable for size in this case. The power of C-GM and C-LS are greater than that of the other tests.

Additive Outlier Case

T=100:  All the tests, except C-GM, C-LS, RC and LR are unacceptable for \( \omega \geq 3 \), due to loss of size (\( > 0.15 \)). LR is unacceptable for \( \omega \geq 6 \). C-LS outperforms
RC for all $\omega$ and C-GM outperforms RC for $\omega = 3$ and $(\phi_1, \phi_2) = (0.5, 0.8)$. Where acceptable, LR outperforms C-GM.

**T=200:** All the tests, except C-GM, C-LS and RC are unacceptable for $\omega \geq 3$, due to loss of size ($> 0.15$). RC is unacceptable for $\omega \geq 6$ and C-LS is nearly so for $\omega = 6$ due to loss of size. C-GM and C-LS perform comparably, except for $\omega = 6, 10$ and $(\phi_1, \phi_2) = (0.5, -0.3)$ where C-GM has higher power than C-LS and both have substantially higher power than RC where RC is acceptable.

**Innovational Outlier Case**

**T=100:** The only unacceptable instance due to size $> 0.15$ is C-PD for $\omega = 10$. C-GM substantially outperforms all the other tests for $\omega \geq 6$. Its only competitor for $\omega \leq 3$ is C-LS which gives comparable results.

**T=200:** There are no unacceptable instances due to size $> 0.15$. C-GM outperforms (by far in most instances) all other tests for $(\phi_1, \phi_2) = (0.5, 0.8)$. For $(\phi_1, \phi_2) = (0.5, -0.3)$ and $\omega = 3$, C-GM and C-LS are comparable and outperform all other tests, though LR, LM and F are somewhat competitive. For $(\phi_1, \phi_2) = (0.5, -0.3)$ and $\omega \geq 6$, C-GM is still the best overall, but C-LS, LR, LM and F are competitive.

Overall, C-GM fares best in these comparisons. C-LS does surprisingly well even in the presence of outliers.
Table 6.1: The empirical frequencies of rejecting the null hypothesis of linearity under additive outlier case at the nominal 5% level; sample size $T = 100$

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Tests for SETAR-type nonlinearity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_1$  $\phi_2$ $\omega$</td>
<td>C-GM</td>
</tr>
<tr>
<td>0.5 0.5 0</td>
<td>0.052</td>
</tr>
<tr>
<td>3</td>
<td>0.057</td>
</tr>
<tr>
<td>6</td>
<td>0.050</td>
</tr>
<tr>
<td>10</td>
<td>0.058</td>
</tr>
<tr>
<td>0.5 0.8 0</td>
<td>0.159</td>
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<tr>
<td>3</td>
<td>0.223</td>
</tr>
<tr>
<td>6</td>
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</tr>
<tr>
<td>10</td>
<td>0.234</td>
</tr>
<tr>
<td>0.5 -0.3 0</td>
<td>0.774</td>
</tr>
<tr>
<td>3</td>
<td>0.709</td>
</tr>
<tr>
<td>6</td>
<td>0.652</td>
</tr>
<tr>
<td>10</td>
<td>0.654</td>
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</table>
Table 6.2: The empirical frequencies of rejecting the null hypothesis of linearity under additive outlier case at the nominal 5% level; sample size $T = 200$

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Tests for SETAR-type nonlinearity</th>
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</thead>
<tbody>
<tr>
<td>$\phi_1$</td>
<td>$\phi_2$</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>3</td>
<td>0.057</td>
</tr>
<tr>
<td>6</td>
<td>0.073</td>
</tr>
<tr>
<td>10</td>
<td>0.056</td>
</tr>
<tr>
<td>0.5</td>
<td>0.8</td>
</tr>
<tr>
<td>3</td>
<td>0.549</td>
</tr>
<tr>
<td>6</td>
<td>0.599</td>
</tr>
<tr>
<td>10</td>
<td>0.558</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.3</td>
</tr>
<tr>
<td>3</td>
<td>0.972</td>
</tr>
<tr>
<td>6</td>
<td>0.965</td>
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<tr>
<td>10</td>
<td>0.960</td>
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</table>
Table 6.3: The empirical frequencies of rejecting the null hypothesis of linearity under innovational outlier case at the nominal 5% level; sample size $T = 100$

<table>
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<tr>
<th>Parameters</th>
<th>Tests for SETAR-type nonlinearity</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
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<tbody>
<tr>
<td>$\phi_1$</td>
<td>$\phi_2$</td>
<td>$\omega$</td>
<td>C-GM</td>
<td>C-LS</td>
<td>C-PD</td>
<td>RC</td>
<td>LR</td>
<td>LM</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>1</td>
<td>0.037</td>
<td>0.040</td>
<td>0.036</td>
<td>0.014</td>
<td>0.046</td>
<td>0.032</td>
</tr>
<tr>
<td>3</td>
<td>0.044</td>
<td>0.058</td>
<td>0.034</td>
<td>0.015</td>
<td>0.035</td>
<td>0.030</td>
<td>0.035</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.061</td>
<td>0.036</td>
<td>0.059</td>
<td>0.032</td>
<td>0.038</td>
<td>0.035</td>
<td>0.052</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.056</td>
<td>0.031</td>
<td>0.153</td>
<td>0.040</td>
<td>0.037</td>
<td>0.041</td>
<td>0.073</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.8</td>
<td>1</td>
<td>0.217</td>
<td>0.205</td>
<td>0.087</td>
<td>0.033</td>
<td>0.099</td>
<td>0.102</td>
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<tr>
<td>3</td>
<td>0.261</td>
<td>0.207</td>
<td>0.089</td>
<td>0.049</td>
<td>0.107</td>
<td>0.098</td>
<td>0.113</td>
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<tr>
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<td>0.130</td>
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<td>0.133</td>
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<tr>
<td>10</td>
<td>0.520</td>
<td>0.104</td>
<td>0.162</td>
<td>0.077</td>
<td>0.099</td>
<td>0.087</td>
<td>0.146</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>-0.3</td>
<td>1</td>
<td>0.791</td>
<td>0.810</td>
<td>0.149</td>
<td>0.243</td>
<td>0.444</td>
<td>0.471</td>
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<tr>
<td>3</td>
<td>0.801</td>
<td>0.755</td>
<td>0.141</td>
<td>0.242</td>
<td>0.444</td>
<td>0.505</td>
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<tr>
<td>6</td>
<td>0.840</td>
<td>0.624</td>
<td>0.126</td>
<td>0.256</td>
<td>0.549</td>
<td>0.598</td>
<td>0.575</td>
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<tr>
<td>10</td>
<td>0.834</td>
<td>0.432</td>
<td>0.212</td>
<td>0.244</td>
<td>0.558</td>
<td>0.654</td>
<td>0.626</td>
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Table 6.4: The empirical frequencies of rejecting the null hypothesis of linearity under innovational outlier case at the nominal 5% level; sample size $T = 200$

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Tests for SETAR-type nonlinearity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_1$ $\phi_2$ $\omega$</td>
<td>C-GM</td>
</tr>
<tr>
<td>0.5 0.5 1</td>
<td>0.045</td>
</tr>
<tr>
<td>3</td>
<td>0.034</td>
</tr>
<tr>
<td>6</td>
<td>0.057</td>
</tr>
<tr>
<td>10</td>
<td>0.076</td>
</tr>
<tr>
<td>0.5 0.8 1</td>
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</tr>
<tr>
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</tr>
<tr>
<td>6</td>
<td>0.722</td>
</tr>
<tr>
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<tr>
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</tr>
<tr>
<td>3</td>
<td>0.990</td>
</tr>
<tr>
<td>6</td>
<td>0.996</td>
</tr>
<tr>
<td>10</td>
<td>0.998</td>
</tr>
</tbody>
</table>
6.3 Comparison of CUSUM-GM and CUSUM-LS Tests

**Outlier-Free Case**  The simulations carried out for the outlier-free data ($\omega = 0$ or $\omega = 1$), displayed in part (a) of Figures 6.1 - 6.36, show that, regardless of the sample size, there are no essential differences in power and size between the two tests. They have reasonable power and the empirical sizes are close to the nominal 5% level. However, there are a few exceptions. For $\phi_1 = 0.9$ and $T = 100$ (Figure 6.16), the tests perform very poorly and for $\phi_2 > 0.5$ the power of tests are quite low. When $\phi_1 = 0.9$ and $T = 200$ (Figure 6.17), the CUSUM-LS test slightly outperforms the CUSUM-GM test for $\phi_2 < 0$, but the power curves are close to each other for $\phi_2 > 0$. Nevertheless, no test would be characterized as unacceptable due to size greater than 0.15.

**Additive Outlier Case**  Simulation results for various additive outlier models (AO) are displayed in Figures 6.1 - 6.18.

**Small outlier magnitude ($\omega = 3$)**  At $\omega = 3$, for $\phi_1 = -0.9, -0.5, -0.1$ and 0.1, the robust test is slightly more powerful than the CUSUM-LS test for most $\phi_2$ values, except when $T = 100$ and $\phi_2 > 0.75$ and when $T = 200$ and $\phi_2 > 0.90$. When $\phi_1 = 0.5$ and $\phi_2 > 0$, the CUSUM-LS test performs a little bit better than the robust one in terms of power (Figures 6.13 - 6.15). When $\phi_1 = 0.9$ and $T = 100$, both tests have acceptable size ($< 0.15$) for all $\omega$. In these cases, the power of the CUSUM-LS
test is clearly larger than that of the CUSUM-GM test. At sample sizes $T = 200$ and $T = 500$, the CUSUM-LS test is unacceptable due to loss of size (> 0.15). For example, when $T = 500$, the empirical size of the CUSUM-LS test is 0.174, while that of the CUSUM-GM test is 0.103 (Table 5).

**Moderate and large outlier magnitudes ($\omega = 6, 10$).** When the data are contaminated with larger outliers ($\omega = 6$ or 10), the power curves change. We observe the same behavior in the power curves as at $\omega = 3$, but everything becomes more evident now. Where the robust test was slightly more powerful than the CUSUM-LS test, now it is much more powerful. When $\phi_1 = -0.9$, the CUSUM-LS test is unacceptable for sample size greater than 200. The CUSUM-LS test is also unacceptable when $\phi_1 = -0.5$ for $T = 500$, when $\phi_1 = 0.5$ for the case of $T = 200$ and $\omega = 10$ and for $T = 500$ and when $\phi_1 = 0.9$ for all sample sizes.

Excepting $\phi_1 = 0.5$ and 0.9, for all the other values of $\phi_1$ the power curves drop substantially when $T = 100$ and $\phi_2 > 0.75$. When $T = 200$, the power of the CUSUM-GM test still drops substantially when $\phi_2$ is close to 1. The same thing happens to the CUSUM-LS test when the data are free of outliers. A reason for this could be that the proportion of observations in the lower regime is too small to give evidence of nonlinearity. This may also be due to the fact that $\phi_2$ is near to the nonstationary boundary.

The improvement in power of the tests seems quite marked when the sample size increases.
Innovational Outlier Case  

Figures 6.19 - 6.36 show the empirical power curves in the innovational outlier case.

Both tests have acceptable size (< 0.15), for all $\omega$. The empirical size of the CUSUM-LS test is slightly deflated for $\phi_1 = \phi_1 = 0.9$, especially at $\omega = 10$. For instance, when $T = 100$ and 200, the empirical sizes of the CUSUM-LS test are 0.025 and 0.022 respectively (see Table 9).

For $\phi_1 = -0.9$ and $-0.5$, the CUSUM-GM test outperforms the CUSUM-LS test for most $\phi_2$ values, but not for large $\phi_2$ values (e.g. $\phi_2 > 0.50$ for $T = 100$) and this is more evident with increasing the magnitude of the outliers. For $\phi_1 = -0.1, 0.1$ and 0.5, the CUSUM-GM test is much more powerful than the CUSUM-LS test excepting the case with $\phi_1 = -0.1$, very large values of $\phi_2$, $T = 500$, and $\omega \geq 6$ (Figure 6.27, (c) and (d)). This difference is greater for larger $T$. When $\phi_1 = 0.9$, the CUSUM-LS test performs better than the CUSUM-GM test for small $\phi_2$ values and worse for large $\phi_2$ values. However, as in the AO case, both tests perform poorly for sample size 100.

We have seen that, when the data are contaminated with additive outliers, the CUSUM-LS test shows severe size distortions, while in the case of data contaminated with innovational outliers, it keeps close to the nominal 5% level. This is somehow understandable, since it appears that the least-squares estimates of an AR with innovational outliers are consistent for any $\omega < \infty$ (Mann and Wald, 1943).

It seems therefore, that innovational outlier contamination is not as harmful to the tests as additive outlier contamination, in terms of its effects on size.
6.4 Concluding Remarks

Comparison of CUSUM-GM, CUSUM-LS and Other Nonlinear Tests

In most cases considered in Section 6.2, it appears that the CUSUM-LS test is more powerful than the CUSUM test of Petruccelli and Davies and also than the RC, LR LM and F tests. We also observed that the proposed CUSUM-GM is even more powerful than the CUSUM-LS test which confirms our conclusions from the previous section. But we still cannot draw a clear conclusion about the performance of the CUSUM-GM or CUSUM-LS test with respect to the other tests considered by Chan and Ng (2004) because of the small number of parameter combinations considered for $\phi_1$ and $\phi_2$.

Comparison of CUSUM-GM and CUSUM-LS

In most of the cases considered for this Monte Carlo study, the proposed CUSUM-GM test outperforms in terms of power and size the CUSUM-LS test which is based on the ordered residuals from the least squares fit.

In outlier-free time series, no test would be characterized as unacceptable due to size greater than 0.15. The CUSUM-LS and CUSUM-GM tests perform almost equally well, excepting the case of $\phi_1 = 0.9$ with $T = 100$ and 200 where the CUSUM-LS is slightly more powerful than the CUSUM-GM test for negative values of $\phi_2$ and the reverse is true for positive values of $\phi_2$.

The CUSUM-GM test is acceptable due to size smaller than 0.15 for all para-
meter combinations of $\phi_1$ and $\phi_2$ in both AO and IO cases. For most of the cases the empirical size of the CUSUM-GM test keeps very close to the nominal 5% level. On the other hand, in the case of additive outliers, the CUSUM-LS test is unacceptable when: (1) $\phi_1 = -0.9$, for sample size greater than 200; (2) $\phi_1 = -0.5$ for $T = 500$; (3) $\phi_1 = 0.5$ for $T = 200$ and $\omega = 10$, and for $T = 500$; (4) $\phi_1 = 0.9$ for all sample sizes. The CUSUM-LS test is acceptable for all parameter combinations of $\phi_1$ and $\phi_2$ in the IO case.

For $\phi_1 = -0.9$ and $-0.5$, the robust test outperforms the CUSUM-LS test for large $\phi_2$ values in both AO and IO cases. When the data are contaminated with additive outliers, the CUSUM-GM test is still more powerful than the CUSUM-LS test for $\phi_1 = -0.1, 0.1$ with large $\phi_2$ values, but the CUSUM-LS test is more powerful than the CUSUM-GM test for $\phi_1 = 0.5$, when $\phi_2$ is large. For innovational outlier contamination, the CUSUM-GM test performs quite successfully, independently of $\phi_2$ for $\phi_1 = -0.1, 0.1$ and 0.5. When $\phi_1 = 0.9$, the CUSUM-LS test has higher power than the CUSUM-GM test for small $\phi_2$ values and smaller power for large $\phi_2$ values in the IO case.

As a final conclusion, we may say that the simulation study strongly supports the validity of the proposed robust CUSUM-GM test, particularly in situations in which outliers might be a problem.
AO, $\phi_1 = -0.9$, $T = 100$

(a) $\omega = 0$

(b) $\omega = 3$

(c) $\omega = 6$

(d) $\omega = 10$

Figure 6.1: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with additive outliers; $T=100$; $\phi_1 = -0.9$; $-1 < \phi_2 < 1$ is shown on the horizontal axis.
AO, $\phi_1 = -0.9$, $T = 200$

(a) $\omega = 0$

(b) $\omega = 3$

(c) $\omega = 6$

(d) $\omega = 10$

Figure 6.2: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with additive outliers; $T=200$; $\phi_1 = -0.9$; $-1 < \phi_2 < 1$ is shown on the horizontal axis.
AO, $\phi_1 = -0.9$, $T = 500$

(a) $\omega = 0$

(b) $\omega = 3$

(c) $\omega = 6$

(d) $\omega = 10$

Figure 6.3: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with additive outliers; $T=500$; $\phi_1 = -0.9$; $-1 < \phi_2 < 1$ is shown on the horizontal axis.
AO, $\phi_1 = -0.5$, $T = 100$

(a) $\omega = 0$  
(b) $\omega = 3$

(c) $\omega = 6$  
(d) $\omega = 10$

Figure 6.4: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with additive outliers; $T=100$; $\phi_1 = -0.5$; $-1 < \phi_2 < 1$ is shown on the horizontal axis.
AO, $\phi_1 = -0.5$, $T = 200$

(a) $\omega = 0$

(b) $\omega = 3$

(c) $\omega = 6$

(d) $\omega = 10$

Figure 6.5: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with additive outliers; $T=200$; $\phi_1 = -0.5$; $-1 < \phi_2 < 1$ is shown on the horizontal axis.
Figure 6.6: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with additive outliers; T=500; $\phi_1 = -0.5$; $-1 < \phi_2 < 1$ is shown on the horizontal axis.
Figure 6.7: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with additive outliers; \( T=100; \phi_1 = -0.1; -1 < \phi_2 < 1 \) is shown on the horizontal axis.
$AO, \phi_1 = -0.1, T = 200$

(a) $\omega = 0$  
(b) $\omega = 3$  
(c) $\omega = 6$  
(d) $\omega = 10$

Figure 6.8: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with additive outliers; $T=200; \phi_1 = -0.1; -1 < \phi_2 < 1$ is shown on the horizontal axis.
AO, , \( \phi_1 = -0.1, T = 500 \)

(a) \( \omega = 0 \)

(b) \( \omega = 3 \)

(c) \( \omega = 6 \)

(d) \( \omega = 10 \)

Figure 6.9: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with additive outliers; \( T=500; \phi_1 = -0.1; -1 < \phi_2 < 1 \) is shown on the horizontal axis.
Figure 6.10: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with additive outliers; $T=100$; $\phi_1 = 0.1$; $-1 < \phi_2 < 1$ is shown on the horizontal axis.
Figure 6.11: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with additive outliers; $T=200$; $\phi_1 = 0.1$; $-1 < \phi_2 < 1$ is shown on the horizontal axis.
Figure 6.12: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with additive outliers; $T=500; \phi_1 = 0.1; -1 < \phi_2 < 1$ is shown on the horizontal axis.
AO, $\phi_1 = 0.5$, $T = 100$

(a) $\omega = 0$

(b) $\omega = 3$

(c) $\omega = 6$

(d) $\omega = 10$

Figure 6.13: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with additive outliers; $T=100$; $\phi_1 = 0.5$; $-1 < \phi_2 < 1$ is shown on the horizontal axis.
Figure 6.14: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with additive outliers; T=200; $\phi_1 = 0.5; -1 < \phi_2 < 1$ is shown on the horizontal axis.
AO, $\phi_1 = 0.5, T = 500$

(a) $\omega = 0$

(b) $\omega = 3$

(c) $\omega = 6$

(d) $\omega = 10$

Figure 6.15: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with additive outliers; T=500; $\phi_1 = 0.5; -1 < \phi_2 < 1$ is shown on the horizontal axis.
Figure 6.16: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with additive outliers; T=100; φ₁ = 0.9; −1 < φ₂ < 1 is shown on the horizontal axis.
Figure 6.17: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with additive outliers; T=200; \( \phi_1 = 0.9 \); \(-1 < \phi_2 < 1\) is shown on the horizontal axis.
AO, $\phi_1 = 0.9$, $T = 500$

(a) $\omega = 0$  

(b) $\omega = 3$  

(c) $\omega = 6$  

(d) $\omega = 10$

Figure 6.18: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with additive outliers; T=500; $\phi_1 = 0.9$; $-1 < \phi_2 < 1$ is shown on the horizontal axis.
IO, $\phi_1 = -0.9$, $T = 100$

(a) $\omega = 1$

(b) $\omega = 3$

(c) $\omega = 6$

(d) $\omega = 10$

Figure 6.19: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with innovational outliers; $T=100$; $\phi_1 = -0.9$; $-1 < \phi_2 < 1$ is shown on the horizontal axis.
IO, $\phi_1 = -0.9, T = 200$

(a) $\omega = 1$

(b) $\omega = 3$

(c) $\omega = 6$

(d) $\omega = 10$

Figure 6.20: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with innovational outliers; $T=200$; $\phi_1 = -0.9; -1 < \phi_2 < 1$ is shown on the horizontal axis.
IO, $\phi_1 = -0.9$, $T = 500$

(a) $\omega = 1$

(b) $\omega = 3$

(c) $\omega = 6$

(d) $\omega = 10$

Figure 6.21: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with innovational outliers; $T=500$; $\phi_1 = -0.9$; $-1 < \phi_2 < 1$ is shown on the horizontal axis.
IO, \( \phi_1 = -0.5, \, T = 100 \)

(a) \( \omega = 1 \)  
(b) \( \omega = 3 \)  
(c) \( \omega = 6 \)  
(d) \( \omega = 10 \)

Figure 6.22: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with innovational outliers; \( T=100; \, \phi_1 = -0.5; \, -1 < \phi_2 < 1 \) is shown on the horizontal axis.
IO, $\phi_1 = -0.5$, $T = 200$

(a) $\omega = 1$

(b) $\omega = 3$

(c) $\omega = 6$

(d) $\omega = 10$

Figure 6.23: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with innovational outliers; $T=200$; $\phi_1 = -0.5$; $-1 < \phi_2 < 1$ is shown on the horizontal axis.
IO, $\phi_1 = -0.5$, $T = 500$

(a) $\omega = 1$

(b) $\omega = 3$

(c) $\omega = 6$

(d) $\omega = 10$

Figure 6.24: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with innovational outliers; $T=500; \phi_1 = -0.5; -1 < \phi_2 < 1$ is shown on the horizontal axis.
IO, $\phi_1 = -0.1$, $T = 100$

(a) $\omega = 1$

(b) $\omega = 3$

(c) $\omega = 6$

(d) $\omega = 10$

Figure 6.25: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with innovational outliers; $T=100$; $\phi_1 = -0.1$; $-1 < \phi_2 < 1$ is shown on the horizontal axis.
IO, \( \phi_1 = -0.1 \), \( T = 200 \)

(a) \( \omega = 1 \)  

(b) \( \omega = 3 \)  

(c) \( \omega = 6 \)  

(d) \( \omega = 10 \)

Figure 6.26: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with innovational outliers; \( T=200; \phi_1 = -0.1; -1 < \phi_2 < 1 \) is shown on the horizontal axis.
IO, $\phi_1 = -0.1$, $T = 500$

(a) $\omega = 1$  
(b) $\omega = 3$

(c) $\omega = 6$  
(d) $\omega = 10$

Figure 6.27: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with innovational outliers; T=500; $\phi_1 = -0.1$; $-1 < \phi_2 < 1$ is shown on the horizontal axis.
IO, $\phi_1 = 0.1$, $T = 100$

(a) $\omega = 1$

(b) $\omega = 3$

(c) $\omega = 6$

(d) $\omega = 10$

Figure 6.28: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with innovational outliers; $T=100$; $\phi_1 = 0.1$; $-1 < \phi_2 < 1$ is shown on the horizontal axis.
IO, $\phi_1 = 0.1$, $T = 200$

(a) $\omega = 1$  

(b) $\omega = 3$

(c) $\omega = 6$  

(d) $\omega = 10$

Figure 6.29: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with innovational outliers; T=200; $\phi_1 = 0.1$; $-1 < \phi_2 < 1$ is shown on the horizontal axis.
IO, $\phi_1 = 0.1, T = 500$

(a) $\omega = 1$

(b) $\omega = 3$

(c) $\omega = 6$

(d) $\omega = 10$

Figure 6.30: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with innovational outliers; $T=500; \phi_1 = 0.1; -1 < \phi_2 < 1$ is shown on the horizontal axis.
Figure 6.31: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with innovational outliers; $T=100; \phi_1 = 0.5; -1 < \phi_2 < 1$ is shown on the horizontal axis.
IO, $\phi_1 = 0.5$, $T = 200$

(a) $\omega = 1$

(b) $\omega = 3$

(c) $\omega = 6$

(d) $\omega = 10$

Figure 6.32: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5\% level based on 1000 replications from the SETAR(1; 1,1) model with innovational outliers; $T=200$; $\phi_1 = 0.5$; $-1 < \phi_2 < 1$ is shown on the horizontal axis.
Figure 6.33: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with innovational outliers; T=500; $\phi_1 = 0.5$; $-1 < \phi_2 < 1$ is shown on the horizontal axis.
Figure 6.34: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with innovational outliers; \( T=100; \phi_1 = 0.9; \phi_2 < 1 \) is shown on the horizontal axis.
Figure 6.35: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with innovational outliers; T=200; \( \phi_1 = 0.9; -1 < \phi_2 < 1 \) is shown on the horizontal axis.
Figure 6.36: Empirical power curves of the CUSUM-GM and CUSUM-LS tests at nominal 5% level based on 1000 replications from the SETAR(1; 1,1) model with innovational outliers; $T=500$; $\phi_1 = 0.9$; $-1 < \phi_2 < 1$ is shown on the horizontal axis.
References


McCabe, B. P. M., Harrison, M.J. (1980) Testing the Constancy of Regression


University Press, Oxford.


Appendix

Table 5: The empirical size of the CUSUM-LS test (denoted in the table by LS) and CUSUM-GM test (denoted in the table by GM) under additive outlier case when $\phi_1 = \phi_2 = -0.9$ and $\alpha = 0.05$

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<td>LS</td>
<td>GM</td>
<td>LS</td>
<td>GM</td>
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<td>0.05</td>
<td>0.082</td>
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<td>0.064</td>
<td>0.088</td>
<td>0.064</td>
</tr>
<tr>
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Table 6: The empirical size of the CUSUM-LS test (denoted in the table by LS) and CUSUM-GM test (denoted in the table by GM) under additive outlier case when $\phi_1 = \phi_2 = -0.5$ and $\alpha = 0.05$

<table>
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Table 7: The empirical size of the CUSUM-LS test (denoted in the table by LS) and CUSUM-GM test (denoted in the table by GM) under additive outlier case when $\phi_1 = \phi_2 = 0.5$ and $\alpha = 0.05$

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Table 8: The empirical size of the CUSUM-LS test (denoted in the table by LS) and CUSUM-GM test (denoted in the table by GM) under additive outlier case when \( \phi_1 = \phi_2 = 0.9 \) and \( \alpha = 0.05 \)

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<th>( \omega = 6 )</th>
<th>( \omega = 10 )</th>
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Table 9: The empirical size of the CUSUM-LS test (denoted in the table by LS) and CUSUM-GM test (denoted in the table by GM) under innovational outlier case when \( \phi_1 = \phi_2 = 0.9 \) and \( \alpha = 0.05 \)

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Table 10: Empirical quantiles of the CUSUM-LS and CUSUM-GM test statistics under the null hypothesis of linearity, $\phi_1 = \phi_2 = -0.9$ and $\alpha = 0.05$. The data is free of outliers.

<table>
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Table 11: Empirical quantiles of the CUSUM-LS and CUSUM-GM test statistics under the null hypothesis of linearity, $\phi_1 = \phi_2 = -0.5$ and $\alpha = 0.05$. The data is free of outliers.

<table>
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Table 12: Empirical quantiles of the CUSUM-LS and CUSUM-GM test statistics under the null hypothesis of linearity, $\phi_1 = \phi_2 = 0$ and $\alpha = 0.05$. The data is free of outliers.

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<td>1.82 1.942 2.113 2.325 2.596</td>
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Table 13: Empirical quantiles of the CUSUM-LS and CUSUM-GM test statistics under the null hypothesis of linearity, $\phi_1 = \phi_2 = 0.5$ and $\alpha = 0.05$. The data is free of outliers.

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Table 14: Empirical quantiles of the CUSUM-LS and CUSUM-GM test statistics under the null hypothesis of linearity, $\phi_1 = \phi_2 = 0.9$ and $\alpha = 0.05$. The data is free of outliers.

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