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Crack Derivatives

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Crack Derivatives

A Major Qualifying Project
Submitted to the Faculty of the
WORCESTER POLYTECHNIC INSTITUTE
in partial fulfillment of the requirements for the
Degree of Bachelor of Science
By
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Date: April 25, 2013

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Key Words: fracture mechanics, crack derivative, jump transfer, blow-up sequence.

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Abstract

In this paper, the "crack derivatives", which defined to be the limits of the blow-up sequences, are formulated and studied. Both quasi-static fracture evolution and dynamic fracture evolution are included. For quasi-static fracture evolution, the crack derivatives are proven to exist and globally minimize the energy functional on domain with arbitrary growing cracks. For dynamic fracture evolution, the crack derivatives are proven to be solutions of the wave equations on domain with arbitrary growing locally connected cracks.

Acknowledgements

The author expresses sincere appreciation to his advisors:

Prof. Christopher J. Larsen
Department of Mathematical Sciences

Prof. L. Ramdas Ram-Mohan
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1 Preliminary Knowledge and Previous Results

1.1 Griffith's criterion in Fracture Mechanics

Since the early work of Griffith, in most models for crack prediction, the propagation of a crack is considered to be driven by the energy release rate along the surface area of the crack.
Consider the propagation of a quasi-static crack in a brittle material body $\Omega \subset \mathbb{R}^N$, we define: $\Gamma \subset \Omega$ is a crack set satisfying $\mathcal{H}_{loc}^{N-1}(\Gamma) < \infty$; $u \in SBV(\Omega)$ is the displacement.
For each such pair (u, Γ), $\mathcal{E}_{el} = \frac{1}{2} \int_{\Omega \setminus \Gamma} \rho \nabla u \cdot \nabla u \, \mathrm{d}x$ is the elastic energy of the displacement field u. We will always set $\rho \equiv 1$ for simplification.
Griffith's criterion states that a crack can grow if

$$\mathcal{G}_c \leq -\frac{\delta \mathcal{E}_{el}(u, \Gamma)}{\delta \mathcal{H}(\Gamma)} \tag{1.1}$$

where $\mathcal{G}_c$ is the fracture toughness, and $-\frac{\partial}{\partial \mathcal{H}(\Gamma)} \mathcal{E}_{el}(u, \Gamma)$ is the energy release rate. We set $\mathcal{G}_c \equiv 1$ for simplification.

1.2 Global and Local

Following Griffith's work, mathematical models for quasi-static fracture evolution have been developed based on global minimization. More precisely, if we define $\mathcal{E} = \mathcal{E}_{el} + \mathcal{G}_c \mathcal{H}(\Gamma)$ to be the free energy of the elastic body, then the minimizer of $\mathcal{E}$ will satisfy Griffith's criterion. However, the reverse is not necessarily true, because Griffith's criterion is local. So a local minimizer is able to keep the crack stable. In other words, a global minimizer of the free energy function $\mathcal{E}$ is not necessarily to be a solution of quasi-static fracture evolution. Global minimization differs from local minimization for the reason that the global minimality holds for all v in the same functional space as u, and the local minimality holds for all v close enough to u. If a local minimization problem can be transformed to a global minimization problem, then the analysis developed for global minimization problems can be used for solving local minimization problems.

1.3 From Quasi-static to Dynamic

Quasi-static models, which are based on a limit case in which systems are always at equilibrium, are easier to understand and developed, but they are unphysical unless strong constraints are assumed to ensure systems to stay at equilibrium. As a result, dynamic models are needed to be introduced in order to get physical solutions.

2 Quasi-static Fracture Evolution

2.1 General Theory

The following lemmas from [3], will be used later.

Lemma 1. Let $\{u_n\}$: $\Omega \to \mathbb{R}$ be a sequence in $L^p(\Omega)$, $1 \le p < \infty$, such that $u_n \to u$ *a.e.* on Ω and $\forall n$, $\|u_n\|$ is bounded, where C is a constant. Then $u \in L^p(\Omega)$ and $u_n \rightharpoonup u$ in $L^p(\Omega)$.

Lemma 2. (products of weak-strong converging sequences)
Let $1 < p < \infty$, u_n: $\Omega \to \mathbb{R}$ be a sequence in $L^p(\Omega)$, and $u \in L^p(\Omega)$. Let v_n: $\Omega \to \mathbb{R}$ be a sequence in $L^{p^*}(\Omega)$. Suppose $u_n \rightharpoonup u$ in $L^p(\Omega)$ and $v_n \to v$ in $L^{p^*}(\Omega)$. Then $u_n v_n \rightharpoonup uv$ in $L^1(\Omega)$.

Lemma 3. Let u_n: $\Omega \to \mathbb{R}$ be a sequence in $L^1(\Omega)$. Suppose $u_n \to u$ *a.e.* on Ω and u_n is bounded in L^p for some $p > 1$. Then $u_n \to u$ in $L^1(\Omega)$.

Lemma 4. (Radon-Riesz property)
Suppose that $(X, \|\cdot\|)$ is a normed space. We say that X has the Radon-Riesz property if whenever $\{x_n\}$ is a sequence in the space and x is a member of X such that $\{x_n\}$ converges weakly to x and continuity of norm, then $\{x_n\}$ converges to x in the norm.

Lemma 5. Let $\Omega \subset \mathbb{R}^n$ and $\mathcal{U} \subset L^1(\Omega)$ be a family of integrable functions. If $|\Omega| < \infty$ and $\mathcal{U}$ is bounded in $L^1(\Omega)$, then $\mathcal{U}$ is equiintegrable if and only if

$$\mathcal{U} \subset \{u \in L^1(\Omega) : \int_\Omega \Psi(|u|)\mathrm{d}x \le 1\} \tag{2.1}$$

for some increasing function Ψ: $[0, \infty) \to [0, \infty]$ satisfying

$$\lim_{v\to\infty} \frac{\Psi(v)}{v} \to \infty. \tag{2.2}$$

Lemma 6. (Radon-Nikodỳm decomposition theorem)
Let μ be a $\sigma - finite$ positive measure and ν a real measure satisfying $\nu \ll \mu$. Then there is a unique pair of real measures ν^a, ν^s such that $\nu^a \ll \mu$, $\nu^s \perp \mu$ and $\nu = \nu^a + \nu^s$.

Lemma 7. (Calderon-Zygmund)
Any function $u \in [BV(\Omega)]^m$ is approximately differentiable at $\mathcal{H}^N$ *a.e.* of Ω. The approximate differential ∇u is the density of the absolute continuous part of D_u with respect to $\mathcal{H}^N$.

Lemma 8. (Weak lower semicontinuity of convex functions)
If Ψ: $\mathbb{R} \to \mathbb{R}$ is convex and $u_n \rightharpoonup u$ in L^1, then

$$\int \Psi(u)\mathrm{d}x \le \liminf_{n\to\infty} \int \Psi(u_n)\mathrm{d}x \tag{2.3}$$

If Ψ: $\mathbb{R} \to \mathbb{R}$ is concave and $u_n \rightharpoonup u$ in L^1, then

$$\int \Psi(u)\mathrm{d}x \ge \limsup_{n\to\infty} \int \Psi(u_n)\mathrm{d}x \tag{2.4}$$

Some definitions about BV functions from [7] are stated here.

Definition 1. Let $u \in [L^1_{loc}(\Omega)]^m$. u has an approximate limit at $x_0 \in \Omega$ if there exists $z \in \mathbb{R}^m$ such that

$$\lim_{\rho \to} \fint_{B(x_0,\rho)} |u(y) - z| \mathrm{d}y = 0. \tag{2.5}$$

Then the set S_u of points where this property does not hold is called the approximate discontinuity set. For any $x_0 \in \Omega \backslash S_u$ the vector z is called the approximate limit of u at x_0 and denoted by $\tilde{u}(x_0)$.

Definition 2. Let $u \in [L_{loc}(\Omega)]^m$ and let $x_0 \in \Omega \backslash S_u$; we say that u is approximately differentiable at x if there exists an linear operator L such that

$$\lim_{\rho \to 0} \fint_{B(x_0,\rho)} \frac{|u(y) - \tilde{u}(x_0) - L(y - x_0)|}{\rho} \mathrm{d}y = 0 \tag{2.6}$$

If u is approximately differentiable at x, the operator L is called the approximated differential of u at x_0 and denoted by $\nabla u(x_0)$.

Definition 3. $BV(\Omega)$ is the space of all scalar functions with bounded variation on Ω, whose weak derivative Du is a finite Radon measure.

Definition 4. $SBV(\Omega)$ is the space of special functions with bounded variation, the subspace of $BV(\Omega)$ without Cantor part, $D_c u = 0$.
Specificly,

$$Du = \nabla u \mathcal{L}_n + (u^+ - u^-) \nu_u \mathcal{H}^{n-1} \lfloor S_u \tag{2.7}$$

where S_u is Borel and ν_u is the normal to S_u.

Definition 5. (SBV convergence)
Let $\{u_n\}$ be a sequence of functions in $SBV(\Omega)$, and

1. $\nabla u_n \rightharpoonup \nabla u$ in $L^1(\Omega)$
2. $u_n \to u$ in $L^1(\Omega)$
3. $u_n \overset{*}{\rightharpoonup} u$ in $L^\infty(\Omega)$
4. $|u_n^+ - u_n^-| \nu_{u_n} \mathcal{H}^{N-1} \lfloor S_{u_n} \overset{*}{\rightharpoonup} |u^+ - u^-| \nu_u \mathcal{H}^{N-1} \lfloor S_u$ as measures.

Then up to subsequences, $\{u_n\}$ weakly* converges in $BV(\Omega)$ to a function $u \in SBV(\Omega)$.

For quasi-static fracture evolution, the total energy

$$\int_\Omega |\nabla u(t)|^2 \mathrm{d}x + \mathcal{H}^1(\Gamma(t)) \tag{2.8}$$

must be unilaterally minimized, among $\{u(t) \in SBV(\Omega) : u(t) = g(t)\}$, where $g \in L^\infty_{loc}([0,\infty); L^\infty(\partial\Omega) \cap W^{1,1}_{loc}([0,\infty); H^1(\partial\Omega))$. The unilateral minimalilty requires that when there is an energy cost for crack propagation, there is no energy reduction if at some time later, though the discontinuity of the displacement may disappear.$\Gamma(t) = \cup_{0 \leq s \leq t} S_{u(s)}$. This property requires

$$\mathcal{E}_{el\Omega}(u(t)) \leq \mathcal{E}_{el\Omega}(v) + \mathcal{H}^{N-1}(S_v \backslash \Gamma(t)) \tag{2.9}$$

where $u(t)$ and v are in $SBV(\Omega)$.

Definition 6. (Global minimizer)
A pair $(u(t), \Gamma(t))$ is a global minimizer if $u(t) \in SBV_{loc}(\mathbb{R}^2)$, $\mathcal{H}^1_{loc}(\Gamma(t)) < \infty$, and, for all $R > 0$, if $v \in SBV_{loc}(\mathbb{R}^2)$ and $\mathcal{H}^1_{loc}(C) < \infty$ with $u(t) = v$ and $\Gamma(t) = C$ outside $B(0,R)$, then $\mathcal{E}_{B(0,2R)}(u(t), \Gamma(t)) \leq \mathcal{E}_{B(0,2R)}(v, C)$.

Lemma 9. The elastic energy is lower semicontinuous with respect to SBV convergence.

$$\mathcal{E}_{el}(u) \leq \liminf_{n\to\infty} \mathcal{E}_{el}(u_n) \tag{2.10}$$

However, because of the unilateral property of the crack set,

$$\lim_{n\to\infty} \mathcal{H}^{n-1}(S_v \backslash S_{u_n}) \leq \mathcal{H}^{n-1}(S_v \backslash S_u) \tag{2.11}$$

is not necessarily true, so we need the jump transfer theorem to alter v, creating v_n such that

$$\mathcal{E}_{el}(v_n) \to \mathcal{E}_{el}(v) \tag{2.12}$$

and

$$\lim_{n\to\infty} \mathcal{H}^{n-1}(S_{v_n} \backslash S_{u_n}) \leq \mathcal{H}^{n-1}(S_v \backslash S_u). \tag{2.13}$$

Given (u, Γ), such that, $\forall t \in [0,T]$, $(u(t_0+t), \Gamma(t_0+t))$ unilaterally minimizes the given energy functional on the domain Ω, subject to its Dirichlet data on the boundary, for $x_0 \in \Gamma(t_0)$, a blow-up sequence at the point (x_0, t_0), is defined as

$$u^\epsilon(x + x_0, t + t_0) = \epsilon^{-\frac{1}{2}}[u((\epsilon x + x_0), (\epsilon t + t_0)) - \tilde{u}(x_0, t_0 + \epsilon t)] \tag{2.14}$$

$$\Gamma_\epsilon(t_0 + t) = \frac{1}{\epsilon}\Gamma(t_0 + \epsilon t). \tag{2.15}$$

More about the blow-up technique can be found in [8], [9], [2].
The definition gives that

$$\int_{B(x_0,r)} |\nabla u^\epsilon(x, t_0 + t)|^2 \,\mathrm{d}x = \frac{1}{\epsilon}\int_{B(x_0,\epsilon r)} |\nabla u(x, \epsilon t + t_0)|^2 \,\mathrm{d}x \tag{2.16}$$

$$\frac{1}{\epsilon}\mathcal{H}^1(\Gamma(\epsilon t + t_0) \bigcap B(x_0, \epsilon r)) = \mathcal{H}^1(\Gamma^\epsilon(t_0 + t) \bigcap B(x_0, r)) \tag{2.17}$$

Therefore, for all $t \in [0,T]$, there is a relation between the original functional $\mathcal{E}_{B(x_0,\epsilon r)}$ and the blow-up functional $\mathcal{E}_{B(x_0,r)}$ at time $(t + t_0)$:

$$\mathcal{H}^1(\Gamma(t_0+\epsilon t) \bigcap B(0,\epsilon r)) + \int_{B(x_0,\epsilon r)} |\nabla u(t_0+\epsilon t)|^2 \,\mathrm{d}x = \epsilon(\mathcal{H}^1(\Gamma_\epsilon(t+t_0) \bigcap B(x_0, r) + \int_{B(x_0,r)} |\nabla u_\epsilon(t_0+t)|^2 \,\mathrm{d}x) \tag{2.18}$$

In other words, $\forall t \in [0,T]$ and for any positive ϵ, whenever $(u(t_0 + \epsilon t), \Gamma(t_0 + \epsilon t))$ unilaterally minimizes $\mathcal{E}_{B(0,\epsilon r)}$, $(u_\epsilon(t + t_0), \Gamma_\epsilon(t_0 + t))$ unilaterally minimizes $\mathcal{E}_B(0, r)$ subject to its Dirichlet data on $\partial B(x_0, \epsilon r)$.
The following theorem from [6] is recalled for proving the unilateral minimality of the blow-up limit.

Theorem 2.1. let Ω be a bounded open domain and $\{u_n\} \subset SBV(\Omega)$ be such that $|\nabla u_n|$ weakly converges in $L^1(\Omega)$ and $u_n \to u$ in $L^1(\Omega)$, there $u \in SBV(\Omega)$ with $H^{N-1}(S_u) < \infty$, there exists a sequence $\{\phi_n\} \subset SBV(\Omega)$ such that $\phi_n \to \phi$ in $L^1(\Omega)$, $\nabla\phi_n \to \nabla\phi$ in L^p, where $1 \leq p < \infty$ and $\mathcal{H}^{N-1}([S_{\phi_n} \backslash S_{u_n}] \backslash [S_\phi \backslash S_u]) \to 0$.

Theorem 2.2. The blow-up limit at (x_0, t_0) exists, and $\forall t \in [0,T]$, $(u^0(t_0+t), \Gamma^0(t_0+t))$ is a unilateral global minimizer of the functional

$$\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x + \mathcal{H}^1(\Gamma) \tag{2.19}$$

We know that

$$u^\epsilon(x+x_0, t_0+t) = \epsilon^{-\frac{1}{2}}[u(\epsilon x + x_0, \epsilon t + t_0) - \tilde{u}(x_0, t_0+\epsilon t)] \tag{2.20}$$

$$\nabla u^\epsilon(x+x_0, t_0+t) = \epsilon^{-\frac{1}{2}} \nabla u(\epsilon x + x_0, \epsilon t + t_0) \tag{2.21}$$

For $R > 0$, as $\epsilon \to 0$

$$u^\epsilon(t+t_0) \to 0 \ a.e. \text{ on } B(x_0, R) \tag{2.22}$$

$$u^\epsilon(t+t_0) \text{ is uniformly bounded in } L^\infty(B(x_0,R)) \tag{2.23}$$

$$\nabla u^\epsilon(t_0+t) \text{ is uniformly bounded in } L^1(B(x_0,R)) \tag{2.24}$$

$$\begin{aligned} &|u^{\epsilon+}(t_0+t) - u^{\epsilon-}(t_0+t)|\nu_{u^\epsilon(t_0+t)}\mathcal{H}^1_{loc}\lfloor S_{u^\epsilon(t_0+t)} \overset{*}{\rightharpoonup} \\ &|u^{0+}(t_0+t) - u^{0-}(t_0+t)|\nu_{u^0(t_0+t)}\mathcal{H}^1_{loc}\lfloor S_{u^0(t_0+t)} \text{ as measures} \end{aligned} \tag{2.25}$$

So $\{u^\epsilon(t_0+t)\}$ converges to $u^0(t_0+t)$ in $SBV_{loc}(\mathbb{R}^2)$.
In order to prove the blow-up limit is a global minimizer, we need to construct a competitor and prove by contradiction using the result $\mathcal{H}^{N-1}([S_{\phi^\epsilon} \backslash S_{u^\epsilon(t_0+\epsilon t)}] \backslash [S_{\phi^0} \backslash S_{u^0}]) \to 0$ from Theorem 2.2.
For a better geometrical understanding of this process, we will prove a special case in the following section.

2.2 A Special Case

Lemma 10. (Golab's theorem)
Let $\{\Gamma^\epsilon\}$ be a sequence of 1 dimensional connected sets in $\mathbb{R}^2$ which converges to Γ^0 in the Hausdorff metric. Then Γ^0 is connected and

$$\mathcal{H}^1(\Gamma^0 \cap O) \leq \liminf_{\epsilon \to 0} \mathcal{H}^1(\Gamma^\epsilon \cap O) \tag{2.26}$$

for every open set $O \subset \mathbb{R}^2$.

Classically, we assume the crack set is a $\mathcal{C}^{1,1}$ curve, give a special case, show how to construct a blow-up sequence and prove that a limit of the sequence exists and is a minimizer of the blow-up energy functional.
Considering the domain $(\Omega \subset \mathbb{R}^2, [0,T])$ with $\Gamma(t) \subset \Omega$, where $\Gamma(t)$ is a $\mathcal{C}^{1,1}$ curve, we have

$$\lim_{r \to 0} \frac{\mathcal{H}^1(\Gamma(T) \cap B(x_0, r))}{2r} = \frac{1}{2} \quad \text{for cracktips} \tag{2.27}$$

$$\lim_{r \to 0} \frac{\mathcal{H}^1(\Gamma(T) \cap B(x_0, r))}{2r} < \infty \quad \text{for almost every } x \in \Gamma \tag{2.28}$$

Theorem 2.3. The convergence of $\{\Gamma^\epsilon(t_0+t)\}$ to $\Gamma^0(t_0+t)$.

Proof. For all $0 < R$, the closed ball $B(x_0, R)$ is compact. The compactness of the Hausdorff metric for subsets of a closed ball gives that there exists $\Gamma^0(t_0+t)$ and a subsequence $\{\Gamma^{\epsilon_k}(t_0+t)\} \subset \{\Gamma^{\epsilon}(t_0+t)\}$ such that $\Gamma^{\epsilon_k}(t_0+t) \cap B(x_0, R)$ converges to $\Gamma^0(t_0+t) \cap B(x_0, R)$ in the Hausdorff metric. Since $\{\Gamma^{\epsilon}(t_0+t) \cap B(x_0, R)\}$ is a sequence of 1 dimensional connected sets in $\mathbb{R}^2$ which converges to $\Gamma^0 \cap B(x_0, R)$ in the Hausdorff metric, by Golab's theorem, we have

$$\mathcal{H}^1(\Gamma^0(t_0+t) \cap B(x_0, R)) \leq \liminf_{\epsilon \to 0} \mathcal{H}^1(\Gamma^{\epsilon}(t_0+t) \cap B(x_0, R)). \tag{2.29}$$

□

Lemma 11. $\mathcal{H}^1(\Gamma(t_0+t) \bigcap B(x,r)) + \int_{B(x,r)} |\nabla u(t_0+t)|^2 \, \mathrm{d}x \leq Cr.$

Theorem 2.4. The convergence of $\{u^{\epsilon}(t_0+t)\}$ to $u^0(t_0+t)$.

Proof. We notice that $\mathcal{H}^1(\Gamma(t_0+t) \bigcap B(x,r)) + \int_{B(x,r)} |\nabla u(t_0+t)|^2 \, \mathrm{d}x \leq Cr$. The SBV compactness gives that the sequence $\{u^{\epsilon}(t_0+t)\}$ converges to a function $u^0(t_0+t)$ in $H^1_{loc}(\mathbb{R}^2)$. □

Proof. It is sufficient to prove $\mathcal{H}^1(\Gamma(t_0+t) \bigcap B(x,r)) \leq C_1 r < \infty$ and $\int_{B(x,r)} |\nabla u(t_0+t)|^2 \, \mathrm{d}x \leq C_2 r$. □

The following two lemmas from [11] will be used later.

Lemma 12. (Poincaré inequality)
For each $1 \leq p < n$, there exists a constant C, depending on p and n such that

$$\left(\fint_{B(x_0,r)} |f(y) - \fint_{B(x_0,r)} f(z)\mathrm{d}z|^{p^*} \mathrm{d}y\right)^{1/p^*} \leq Cr\left(\fint_{B(x_0,r)} |Df|^p \mathrm{d}y\right)^{1/p} \tag{2.30}$$

for all $B(x_0, r) \subset \mathbb{R}^n$, $f \in W^{1,p}(B(x_0, r))$.

Lemma 13. (Urysohn's lemma)
Suppose X is a locally compact Hausdorff space, V is open in X, $K \subset V$, and K is compact. Then there exists an $f \in C_c(X), 0 \leq f < 1$ such that the support of f lies in V and $f(x) = 1$ for all $x \in K$.

Lemma 14. The blow-up transformation is conformal.

Proof. Obvious. □

Theorem 2.5. The blow-up limit is a solution of the blow-up quasi-static fracture evolution.

Proof. We suppose that at some time t_0+t, $(u^0(t+t_0), \Gamma^0(t+t_0))$ is not a global minimizer of the blow-up limit energy functional. Then we choose $r > 0$, a pair (u'^0, Γ'^0) such that

1. $u'_0 = u_0(t_0+t)$ on $\mathbb{R}^2 \backslash B(x_0, r/2)$
2. $\Gamma'^0 = \Gamma^0(t_0+t)$ on $\mathbb{R}^2 \backslash B(x_0, r/2)$
3. $\mathcal{E}_{B(x_0,r)}(u'^0, \Gamma'^0) < \mathcal{E}_{B(x_0,r)}(u^0(t_0+t), \Gamma^0(t_0+t))$

which means that $(u^0(t_0+t), \Gamma^0(t_0+t))$ is not a global minimizer of the functional.
$\Gamma^0(t_0+t) \cap B(x_0,r)$ is one segment here since $\Gamma(t_0+t)$ is a $\mathcal{C}^{1,1}$ curve. We know the blow-up process is conformal. So we choose an arbitrary small angle θ at the center of $B(x_0,r)$, such that $\Gamma^0(t_0+t)$ bisects θ.
By choosing θ, we can have

$$Cr\theta < \mathcal{E}_{B(x_0,r)}(u^0(t_0+t), \Gamma^0(t_0+t)) - \mathcal{E}_{B(x_0,r)}(u'^0, \Gamma'^0), \tag{2.31}$$

where C is a constant.
Suppose that the sequence $(u^\epsilon(t_0+t), \Gamma^\epsilon(t_0+t))$ converges to $(u^0(t_0+t), \Gamma^0(t_0+t))$. We want to construct a corresponding sequence $(u'^\epsilon, \Gamma'^\epsilon)$ converging to (u'^0, Γ'^0) and show that under the assumption "$(u^0(t_0+t), \Gamma^0(t_0+t))$ is not a global minimizer of the functional", $\mathcal{E}_{B(x_0,r)}(u'^\epsilon, \Gamma'^\epsilon)$ is necessarily smaller than $\mathcal{E}_{B(x_0,r)}(u^\epsilon(t_0+t), \Gamma^\epsilon(t_0+t))$, which contradicts the minimality of $(u^\epsilon(t_0+t), \Gamma^\epsilon(t_0+t))$.
We choose

$$\Gamma'^\epsilon = (\Gamma^\epsilon(t_0+t) \backslash B(x_0, r/2)) \cup Arc(\theta r/2) \cup (\Gamma'^0 \cup B(x_0, r/2)) \tag{2.32}$$

Then we see that

$$\mathcal{H}^1(\Gamma'^\epsilon) = \mathcal{H}^1(\Gamma^\epsilon(t_0+t) \backslash B(x_0, r/2)) + r\theta/2 + \mathcal{H}^1(\Gamma^0(t_0+t) \cup B(x_0, r/2)) \tag{2.33}$$

So

$$\mathcal{H}^1(\Gamma'^\epsilon) - \mathcal{H}^1(\Gamma^\epsilon(t_0+t)) = r\sum_i \theta_i/2 + \mathcal{H}^1((\Gamma'^0 \backslash \Gamma^0(t_0+t)) \cap B(x_0, r/2)) \tag{2.34}$$

At this step we see

$$\begin{aligned}
&\mathcal{E}_{B(x_0,r)}(u^\epsilon(t_0+t), \Gamma^\epsilon(t_0+t)) - \mathcal{E}_{B(x_0,r)}(u'^\epsilon, \Gamma'^\epsilon) \\
&= \mathcal{H}^1((\Gamma^0(t_0+t) \backslash \Gamma'^0) \cap B(x_0, r/2)) - r\theta/2 + \int_{B(x_0,r)} |\nabla u^\epsilon(t_0+t)|^2 - \int_{B(x_0,r)} |\nabla u'^\epsilon|^2 \\
&> -\int_{B(x_0,r)} |\nabla u_0(t_0+t)|^2 + \int_{B(x_0,r)} |\nabla u'^0|^2 + \int_{B(x_0,r)} |\nabla u^\epsilon(t_0+t)|^2 - \int_{B(x_0,r)} |\nabla u'^\epsilon|^2 + (C - \frac{1}{2})r\theta
\end{aligned} \tag{2.35}$$

By choosing ϵ even smaller, we have

$$\mathcal{E}_{B(x_0,r)}(u^\epsilon(t_0+t), \Gamma^\epsilon(t_0+t)) - \mathcal{E}_{B_r}(u'^\epsilon, \Gamma'^\epsilon) > (C-1)r\theta + \int_{B(x_0,r)} |\nabla u_0'|^2 - \int_{B(x_0,r)} |\nabla u'^\epsilon|^2 \tag{2.36}$$

Then we choose

$$u'^\epsilon = \begin{cases} u'^0 + e & \text{in } B(x_0, r/2) \backslash \Gamma'^0 \\ u^\epsilon(t_0+t) & \text{otherwise} \end{cases} \tag{2.37}$$

where e is "ein bein" with suitable properties, such that

$$\int_{B(x_0,r)} |\nabla u'^\epsilon|^2 \to \int_{B(x_0,r)} |\nabla u'^0|^2 \tag{2.38}$$

In order to find a suitable e, we consider

$$\begin{aligned}
&\int_{B(x_0,r)} |\nabla u'^{\epsilon}|^2 \\
&= \int_{B(x_0,r/2)} |\nabla u'^{\epsilon}|^2 + \int_{(B(x_0,r)\backslash(B(x_0,r/2)} |\nabla u'^{\epsilon}|^2 \\
&\leq \int_{B(x_0,r/2)} |\nabla u'^{0}|^2 + \int_{B(x_0,r/2)} |\nabla e|^2 + \int_{B(x_0,r)\backslash B(x_0,r)} |\nabla u^{\epsilon}(t_0+t)|^2
\end{aligned} \tag{2.39}$$

Firstly, we only need to care about the continuity on $\partial B(x_0, r/2)$. By Urysohn's lemma, we can define a continuous function ϕ such that

$$\phi = \begin{cases} 1 & \text{on a compact set near the ends of } Arc \\ 1 & \text{on } \partial B(x_0, r/2)\backslash Arc \\ 0 & \text{on an open set around the middle part of } Arc \text{ crossing } \Gamma'^{0}. \end{cases} \tag{2.40}$$

Then consider the normal direction of$B(x_0, r/2)$, we define a continuous function ψ such that

$$\psi = \begin{cases} 1 & \text{on a compact set near } \partial B(x_0, r/2) \\ 0 & \text{on an open set around the center of } B(x_0, r/2) \end{cases} \tag{2.41}$$

Defining $e = \phi\psi(u^{\epsilon} - u^{0})$, we conclude that if we choose every needed small quantity to be $r \min_i\{\theta_i/2\}$, we can get

$$\mathcal{E}_{B(x_0,r)}(u^{\epsilon}(t_0+t), \Gamma^{\epsilon}(t_0+t)) - \mathcal{E}_{B(x_0,r)}(u'^{\epsilon}, \Gamma'^{\epsilon}) > (C - 3/2)r\theta \tag{2.42}$$

So as long as θ is selected such that $C \geq 3/2$, there is a contradiction. □

Therefore, $\{(u^{\epsilon}(t_0+t), \Gamma^{\epsilon}(t_0+t))\}$ converges to a blow-up limit $(u^{0}(t_0+t), \Gamma^{0}(t_0+t))$ almost everywhere on $[0, T]$. Given the boundedness properties of $\{(u^{\epsilon}(t_0+t), \Gamma^{\epsilon}(t_0+t))\}$, we can deduce that the blow-up limit is a solution of the blow-up quasi-static fracture evolution.

3 Dynamic Fracture Evolution

3.1 PDEs and Calculus of Variations

Finding the stationary points of a given action functional is an old problem in calculus of variations. The wide application of Hamilton's principle in physics, especially, in classical mechnics, since Newton and Bernoulli, abstract physics problems to variational problems.
Great achievements have been made by Euler, Lagrange and others in solving variational problems. One of the most important tool used in the classical method introduced by them, is the Euler-Lagrange equation, which reduces the problem to one involving differential equations. The idea to compute the corresponding Euler-Lagrange equations for a given variational problem by the classical techniques is:

1. Compute the differential of the given functional.
2. Find the corresponding Euler-Lagrange equations.
3. Solve the partial differential equations.

Hamilton, Jacobi and others have developed the classical methods in order to simplify the calculation, but solving complicated partial differential equations is always a big difficulty. Relying on functional analysis and measure theory, direct methods were introduced by Hilbert and others to solve a given variational problem or differential equations. The idea to solve a given variational problem by a direct method is:

1. Construct a sequence of solutions.
2. Prove the existence of a limit of the sequence.
3. Prove the limit is a solution of the given problem.

The equivalence between finding the stationary points of a functional and solving the corresponding Euler-Lagrange equations is well known in the classical sense. However, given an action functional, it is not necessary to assume the involving functions to be smooth, which means a stationary point may not be a solution of the corresponding Euler-Lagrange equations in the classical sense. That is why the theory is generalized to the distribution theory. More about the distribution theory can be found in [11].

3.2 Dynamic Fracture

From now on, we assume that crack sets are locally connected.

Lemma 15. If a set Γ is an union of finite many connected sets or local connected sets, then for every point $x \in \Gamma$, we can find $R > 0$ such that $B(x, R) \cap \Gamma$ is connected.

The following lemma is from [4]

Lemma 16. If Γ is a connected set in $\mathbb{R}^2$, then $\mathcal{H}^1(\Gamma) = \mathcal{H}^1(\bar{\Gamma})$.

Definition 7. Let X and Y be non-empty subsets of a metric space. The Hausdorff metric $d_H(X, Y)$ is defined as

$$d_H(X, Y) := \max\{\sup_{x \in X} \inf_{y \in Y} \mathrm{d}(x, y), \sup_{y \in Y} \inf_{x \in X} \mathrm{d}(x, y)\}. \tag{3.1}$$

Some definitions from [7] are stated here.

Definition 8. (Upper and lower approximate limits)
Let u: $\Omega \to \mathbb{R}$ be a Borel function and $x \in \bar{\Omega}$ a point where the lower density of Ω is strictly positive; the upper and lower approximate limits of u at x are defined by

$$u^{\vee}(x) := \inf\{t \in \bar{\mathbb{R}} : \lim_{\rho \to 0} \rho^{-N} |\{u > t\} \cap B(x, \rho)| = 0\} \tag{3.2}$$

$$u^{\wedge}(x) := \sup\{t \in \bar{\mathbb{R}} : \lim_{\rho \to 0} \rho^{-N} |\{u < t\} \cap B(x, \rho)| = 0\}. \tag{3.3}$$

If $u^{\vee}(x) = u^{\wedge}(x)$, then we call their value the weak approximate limit of u at x, denoted by $\tilde{u}_*(x)$. The set S_{u*} of points where this property does not hold is called the weak approximate discontinuity set.

Definition 9. (Weak approximate jump set)
Let u: $\Omega \to \mathbb{R}$ be a Borel function. $J_u^* \subset \Omega$ is a weak approximate jump set, if for any point $x \in J_u^*$, there exist $a, b \in \bar{\mathbb{R}}$ with $a > b$ and a unit vector $v \in \mathbb{R}^N$ such that, the weak approximate limit of u restricted to $\{y \in \Omega : \langle y - x, v \rangle > 0\}$ is a and the weak approximate limit of u restricted to $\{y \in \Omega : \langle y - x, v \rangle < 0\}$ is b.

Definition 10. (Weak approximate differentiability)
Let u: $\Omega \to \mathbb{R}$ be a Borel function. u is weakly approximately differentiable at a point x if $\tilde{u}_*(x) \in \mathbb{R}$ and there exists a linear operator L: $\mathbb{R}^N \to \mathbb{R}$ such that for any $\epsilon > 0$ the set

$$\{y \in \Omega \backslash \{x\} : \frac{|u(y) - \tilde{u}_*(x) - L(y - x)|}{|y - x|} > \epsilon\} \tag{3.4}$$

has density 0 at x; in this case we set $\nabla^* u(x) = L$.

Definition 11. $GSBV$ functions can appears as limits of sequences of SBV functions when no bound on the L^∞ norm is imposed. Let Ω be an open set of $\mathbb{R}^N$; we say that a function u: $\Omega \to \mathbb{R}^m$ is a $GSBV$ function if $\forall \phi \in \mathcal{C}^1(\mathbb{R}^m)$ with the support of $\nabla\phi$ compact, the composition $\phi \circ u$ belongs to $SBV_{loc}(\Omega)$.

Theorem 3.1. ($GSBV$ compactness)
Let ψ: $[0, \infty) \to [0, \infty]$ and θ: $(0, \infty) \to (0, \infty]$ be lower semicontinuous increasing functions satisfying

$$\lim_{t \to \infty} \frac{\psi(t)}{t} = \infty, \quad \lim_{t \to 0} \frac{\theta(t)}{t} = \infty \tag{3.5}$$

and let g: $[0, \infty) \to [0, \infty]$ be an increasing function satisfying

$$\lim_{t \to \infty} g(t) \to \infty \tag{3.6}$$

Let $\{u_n\} \subset GSBV$ be such that

$$\sup_n \{\int_\Omega [\psi(|\nabla^* u_n|) + g(|u_n|)] \mathrm{d}x + \int_{J^*_{u_n}} \theta(u_n^{\vee} - u_n^{\wedge}) \mathrm{d}\mathcal{H}^{N-1}\} < \infty \tag{3.7}$$

Then, there exist a subsequence $\{u_n\}$ and a function $u \in GSBV(\Omega)$ such that $u_n \to u$ $\mathcal{L}^N - a.e.$

in Ω and $\nabla^* u_n \rightharpoonup \nabla^* u$ in L^1 locally. Moreover, if ϕ is convex and θ is concave, we have

$$\int_\Omega \phi(|\nabla^* u|) \leq \liminf_{n\to\infty} \int_\Omega \phi(|\nabla^* u_n| \mathrm{d}x, \tag{3.8}$$

$$\int_{J_u^*} \theta(u^{vee} - u^\wedge) \mathrm{d}\mathcal{H}^{N-1} \leq \liminf_{n\to\infty} \int_{J_{u_n}^*} \theta(u_n^\vee - u_n^\wedge) \mathrm{d}\mathcal{H}^{N-1}. \tag{3.9}$$

We will work with the $GSBV_2^2$ space defined in [5].

Definition 12.

$$GSBV_2^2(\Omega, \Gamma) := \{v \in GSBV(\Omega) \cap L^2(\Omega) : \nabla \in L^2(\Omega; \mathbb{R}^N), S_v \subset \Gamma\} \tag{3.10}$$

The inner product space $GSBV_2^2(\Omega, \Gamma)$ is a Hilbert space. Moreover, if Γ is closed in Ω, then $GSBV_2^2(\Omega, \Gamma)$ coincides with $H^1(\Omega \backslash \Gamma)$.

Definition 13. Let Ω be a bounded open set in $\mathbb{R}^2$. The capacity of a set O of Ω is defined as

$$\mathrm{cap}(O, \Omega) = \inf_{u \in \mathcal{U}_O^\Omega} \int_\Omega |\nabla u|^2 \mathrm{d}x, \tag{3.11}$$

where $\mathcal{U}_O^\Omega$ is the set of all functions $u \in H_0^1(\Omega)$ such that $u \geq 1$ *a.e.* in a neighborhood of O. And a property is true $quasi - everywhere$ on a set if it holds on it except on a set of capacity 0.

A solution of the dynamic crack growth problem is a weak solution of

$$\ddot{u}(t) - \Delta u(t) = 0 \tag{3.12}$$

in the form

$$\langle \ddot{u}(t), \phi \rangle_{GSBV_2^2(\Omega, \Gamma(t))} + \langle \nabla u(t), \nabla \phi \rangle = 0 \tag{3.13}$$

for all $t \in [0, T]$ and all $\phi \in GSBV_2^2(\Omega, \Gamma(t))$ with $\mathrm{supp}(\phi) \subset\subset \Omega$.

Theorem 3.2. The differential operator δ and the weak Laplace operator acting on the classical elastic energy functional $\mathcal{E}_{el\Omega} = \int \frac{1}{2} \nabla u \cdot \nabla u \, \mathrm{d}x$, with $u, v \in H^1(\Omega)$, are equivalent. Moreover, if u is twice differentiable, then the differential operator δ and the Laplacian operator Δ are equivalent.

Proof.

$$\delta \mathcal{E}_{el\Omega}(u, v) = \lim_{t\to 0} \frac{1}{2t} \int_\Omega (\nabla(u + tv) \cdot \nabla(u + v) - \nabla u \cdot \nabla u) \, \mathrm{d}x \tag{3.14}$$

Apply Lebesgue's dominated convergence theorem to the equation.

$$\delta \mathcal{E}_{el\Omega}(u, v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x \tag{3.15}$$

We conclude that u is the solution of the weak Laplace equation $\nabla u \nabla \phi = 0 \ \forall \phi$ on Ω if and only if the differential of the given elastic energy functional $\delta \mathcal{E}_{el} = \int \frac{1}{2} \nabla u \cdot \nabla u \, \mathrm{d}x$ is zero for $u \in H^1(\Omega)$. If u is twice differential, then integrating by parts, we have $\Delta u \phi = 0 \ \forall \phi$ on Ω, which means $\Delta u = 0$ on Ω. □

Given a weak solution of the given wave equation, (u, Γ), with the domain $(\Omega, [0, T])$, we can choose a blow-up sequence at (x_0, t_0)

$$\begin{cases} u^\epsilon(x_0+x, t_0+t) := \epsilon^{-\frac{1}{2}}[u(x_0+\epsilon x, t_0+\epsilon t) - \tilde{u}(x_0+\epsilon x, t_0+\epsilon t)] \\ \Gamma^\epsilon(t_0+t) = \epsilon^{-1}\Gamma(t_0+\epsilon t) \end{cases} \tag{3.16}$$

Considering the domain $(\Omega \subset \mathbb{R}^2, [0,T])$ with $\Gamma \subset \Omega$, where Γ is of finite many connected components, we have

$$\limsup_{r\to 0} \frac{\mathcal{H}^1(\Gamma(T)\cap B(x_0,r))}{2r} = \frac{1}{2} \quad \text{for cracktips} \tag{3.17}$$

$$\limsup_{r\to 0} \frac{\mathcal{H}^1(\Gamma(T)\cap B(x_0,r))}{2r} < \infty \quad \text{for almost every } x \in \Gamma \tag{3.18}$$

Theorem 3.3. The convergence of $\{\Gamma^\epsilon(t_0+t)\}$ to $\Gamma^0(t_0+t)$.

Proof. For all $0 < R$, the closed ball $B(x_0, R)$ is compact. The compactness of the Hausdorff metric for subsets of a closed ball gives that there exists $\Gamma^0(t_0+t) \cap B(x_0,R)$ and a subsequence $\{\Gamma^{\epsilon_k}(t_0+t)\} \subset \{\Gamma^\epsilon(t_0+t)\}$ such that $\Gamma^{\epsilon_k} \cap B(x_0,R)$ converges to $\Gamma^0 \cap B(x_0,R)$ in the Hausdorff metric. By the Golab's theorem, we conclude that

$$\mathcal{H}^1(\Gamma^0(t_0+t)\cap B(x_0,R)) \le \liminf_{\epsilon\to 0} \mathcal{H}^1(\Gamma^\epsilon(t_0+t)\cap B(x_0,R)). \tag{3.19}$$

□

To prove the strong convergences of the test functions, firstly we will follow the argument in [4] to construct a sequence of harmonic conjugates and prove the convergence of the sequence.

Definition 14. (Harmonic conjugate)
Let $\mathcal{R}$ be to the operator which transfer (x, y) to $(-y, x)$. For a function $u \in GSBV_2^2(\Omega,\Gamma)$, $v \in H^1(\Omega)$ which satisfies $\nabla v = \mathcal{R}(\nabla^* u)$ is called a harmonic conjugate of $u \in GSBV_2^2(\Omega,\Gamma)$.

Theorem 3.4. Let Γ be a connected set in $\Omega \subset \mathbb{R}^2$ and let $u \in GSBV_2^2(\Omega,\Gamma)$ be a solution of

$$\int_{\Omega\backslash\Gamma} \nabla u \nabla z \, \mathrm{d}x = 0, \quad \forall z \in GSBV_2^2(\Omega,\Gamma) \text{ with } \mathrm{supp}(z) \subset\subset \Omega \tag{3.20}$$

Then there exists a function $v \in H^1(\Omega)$ which is a harmonic conjugate of u. v is constant *q.e.* on Γ.

Proof. If $\phi \in \mathcal{C}_c^\infty(\Omega)$ with $\mathrm{supp}(\phi) \subset\subset \Omega$, we have

$$\int_\Omega \nabla u \nabla \phi \, \mathrm{d}x = \int_{\Omega\backslash\Gamma} \nabla u \nabla \phi \mathrm{d}x = 0 \tag{3.21}$$

which implies that

$$\mathrm{div}(\nabla u) = 0 \text{ on } D'(\Omega) \tag{3.22}$$

Consider the function u' which satisfies that $u' = u$ on $\Gamma \cap (\Omega\backslash\bar{\Gamma})$, there exists $v \in H^1(\Omega)$ such that $\nabla v = \mathcal{R}\nabla u'$ *a.e.* on Ω.
Because $\mathcal{H}^1(\bar{\Gamma}\backslash\Gamma) = 0$, so $\nabla v = \mathcal{R}\nabla^* u$ *a.e.* on Ω with $\nabla v = 0$ *a.e.* on Γ. v is constant *q.e.* on Γ. □

Theorem 3.5. Let Γ be connected in Ω, and let $u \in GSBV_2^2(\Omega, \Gamma)$. Assume that there exists $v \in H^1(\Omega)$ with $\nabla u = \mathcal{R}(\nabla v)$ *a.e.* on Ω such that v is constant *q.e.* on Γ. Then u is a solution of

$$\int_{\Omega\setminus\Gamma} \nabla u \nabla \phi = 0, \quad \forall z \in GSBV_2^2(\Omega, \Gamma) \text{ with } \mathrm{supp}(\phi) \subset\subset \Omega \tag{3.23}$$

Proof. We can construct a sequence of functions $\{v_n\}$ in $\mathcal{C}^\infty(\mathbb{R}^2)$ which converges to the extension of v in $H^1(\mathbb{R}^2)$ such that v_n is constant in a neighborhood $\mathcal{U}_n$ of Γ. For every ϕ, we have

$$\int_{\Omega\setminus\Gamma} \mathcal{R}\nabla v_n \nabla \phi \mathrm{d}x = -\int_{\Omega\setminus\Gamma} \mathrm{div}(\mathcal{R}\nabla v_n)\phi \mathrm{d}x + \int_{\partial\Omega} \mathcal{R}\nabla v_n \phi = 0 \tag{3.24}$$

Let $n \to \infty$,

$$\int_{\Omega\setminus\Gamma} \mathcal{R}\nabla v \nabla \phi \mathrm{d}x = \int_{\Omega\setminus\Gamma} \nabla u \nabla \phi = 0 \tag{3.25}$$

□

The following lemma from [1] will be used.

Lemma 17. Let $\{K_n\}$ be a sequence of non empty closed convex sets and K a non empty closed convex set in a reflexive Banach space X, then the following statement are equivalent:

1. $\{K_n\}$ Mosco-converges to K.
2. $\forall x \in X$, $\mathrm{proj}_{K_n} x \to \mathrm{proj}_K x$ in X.
3. $\forall x \in X$, $\mathrm{dist}(x, K_n) \to \mathrm{dist}(x, K)$.

Theorem 3.6. Let $\{\Gamma_n\}$ be a sequence of connected sets in Ω which converges to Γ_∞ in the Hausdorff metric, and let $\{v_n\}$ be a sequence in $H^1(\Omega)$ which converges to v_∞ weakly in $H^1(\Omega)$. Assume that $v_n = 0$ *q.e.* on Γ_n. Then $v_\infty = 0$ *q.e.* on Γ_∞.

Proof. We construct extensions of v_n and v_∞ such that $v_n, v_\infty \in H_0^1(\Omega')$ and $v_n \rightharpoonup v_\infty$ weakly in $H^1(\Omega')$.
For every $f \in L^2(\Omega')$, the solutions ϕ_n of the Dirichlet problems

$$\phi_n \in H_0^1(\Omega'\setminus\bar{\Gamma}_n), \quad \Delta\phi_n = f \text{ on } \Omega'\setminus\bar{\Gamma}_n \tag{3.26}$$

converge strongly in $H_0^1(\Omega')$ to the solution ϕ_∞ of the Dirichlet problem

$$\phi_\infty \in H_0^1(\Omega'\setminus\bar{\Gamma}_n), \quad \Delta z_\infty = f \text{ on } \Omega'\setminus\bar{\Gamma}_\infty \tag{3.27}$$

So we have $H_0^1(\Omega'\setminus\bar{\Gamma}_n) \to H_0^1(\Omega'\setminus\bar{\Gamma}_\infty)$ in the sense of Mosco. We conclude that $v_\infty \in H_0^1(\Omega'\setminus\bar{\Gamma}_\infty)$ and $v_\infty = 0$ *q.e.* on Γ_∞. □

Theorem 3.7. Let $\{\Gamma_n\}$ be a sequence of connected sets which converges to Γ_∞ in the Hausdorff metric, and let u_n be a solution of the minimum problem

$$\min_{v\in\mathcal{V}(\phi,\Gamma_n)} \int_{\Omega\setminus\Gamma_n} |\nabla v|^2 \mathrm{d}x \tag{3.28}$$

and let u_∞ be a solution of the minimum problem

$$\min_{v \in \mathcal{V}(\phi, \Gamma_\infty)} \int_{\Omega \setminus \Gamma_\infty} |\nabla v|^2 \mathrm{d}x \tag{3.29}$$

where $\mathcal{V}(\phi, \Gamma) := \{v \in GSBV_2^2(\Omega, \Gamma) : v = \phi \ q.e. \text{ on } \partial\Omega \setminus \Gamma\}$.

Proof. Because $\{u_n\}$ is bounded in $GSBV_2^2(\Omega, \Gamma_n)$, there exists subsequence of $\{\nabla u_n\}$ and a function $u^* \in GSBV_2^2(\Omega, \Gamma_\infty)$ such that $\nabla u_n \rightharpoonup \nabla u^*$ weakly in $L^2(\Omega)$.
Let v_n be a harmonic conjugate of u_n on Ω, then $\nabla v_n = \mathcal{R} \nabla u_n$ $a.e.$ on Ω. We assume that $\int_\Omega v_n \mathrm{d}x = 0$.
$\{\nabla v_n\}$ converges to $\mathcal{R} \nabla u^*$ weakly in $L^2(\Omega, \mathbb{R}^2)$ and by the Poincare inequality $\{v_n\}$ converges to a function v_∞ in $H^1(\Omega)$ which satisfies $\nabla v_\infty = \mathcal{R} \nabla u^*$ $a.e.$ on Ω.
As $v_n = c_n$ $q.e.$ on Γ_n for constants c_n, using the Poincare inequality again, it follows that $v_n - c_n$ is bounded in $H^1(\Omega)$, hence the sequence $\{c_n\}$ is bounded and therefore, passing to a subsequence, we have $c_n \to c_\infty$ for a constant c_∞ with $\{v_n - c_n\}$ converges to $v_\infty - c_\infty$ weakly in $H^1(\Omega)$ and then we have $v_\infty = c_\infty$ $q.e.$ on Γ_∞. So u^* is a solution of the given minimum problem.
By the uniqueness of the gradients of the solutions, we conclude that $\nabla u^* = \nabla u_\infty$. As $\nabla u_n \rightharpoonup \nabla u_\infty$ weakly in $L^2(\Omega; \mathbb{R}^2)$ and $\|\nabla u_n\|_{L^2}$ converges to $\|\nabla u_\infty\|_{L^2}$, we deduce that

$$\nabla u_n \to \nabla u_\infty \text{ strongly in } L^2(\Omega; \mathbb{R}^2) \tag{3.30}$$

From the $GSBV_2^2$ compactness, we have u_n converges to u_∞ in measure. By the Poincare inequality, we have $\|u_n\|_{L^2} \to \|u_\infty\|_{L^2}$.
So we also have a subsequence $\{u_n\}$ such that

$$u_n \to u_\infty \text{ strongly in } L^2(\Omega) \tag{3.31}$$

. □

Theorem 3.8. The convergence of $\{\phi^\epsilon\}$ to ϕ^0

Proof. In a ball $B(x_0, R)$, containing the compact support of ϕ^0,we want to construct a collection, $\mathcal{P} := \{\{v^\epsilon\} : v^\epsilon \in GSBV_2^2(B(x_0, R), \Gamma^\epsilon(t)) \text{ and } \operatorname{supp}(v^\epsilon) \in \epsilon^{-1}\Omega\}$, such that, $\forall \phi^0 \in GSBV_2^2(B(x_0, R), \Gamma^0)$ with compact support, there exists a sequence, $\{\phi^\epsilon\} \in \mathcal{P}$ satisfying

$$\nabla \phi^\epsilon \to \nabla \phi^0 \quad \text{strongly in } L^2(B(x_0, R)) \tag{3.32}$$

Since Γ is locally connected, for small enough ϵ, $\Gamma^\epsilon \cap B(x_0, R)$ is connected. Let $\{C_i : C_i \cap C_j = \phi, \text{ if } i \neq j\}$ be the collection of the connected components of $B(x_0, R) \setminus \Gamma^0$.
Given $\delta > 0$, we define $\phi^\epsilon(x) := \phi^0(x) \forall x \in C_i \setminus N_\delta$, where $N_\delta := \{x : d(x, \Gamma) < \delta\}$.
For the given δ, we define v_δ^ϵ is a minimizer of $\int_{N_\delta} |\nabla v|^2$ with $S_{v_\delta^\epsilon} \subset \Gamma^\epsilon$, then by $GSBV_2^2$ compactness, we have v_δ^0 as a minimizer of $\int_{N_\delta} |\nabla v|^2$ with $S_{v_\delta^0} \subset \Gamma^0$.
δ can be defined as a function of ϵ, and $\delta \to 0$ as $\epsilon \to 0$. Then we have

$$\nabla v_\delta^\epsilon - \nabla \phi^0 \to \nabla v_\delta^0 - \nabla \phi^0 \to 0 \quad \text{ in } L^2(N_\delta) \tag{3.33}$$

By defining $\phi^\epsilon := v_\delta^\epsilon$, and adding all the parts in $\{C_i\}$ and N_δ together, we conclude that for all ϕ^0,there exists a sequance$\{\nabla \phi^\epsilon\}$ converging to $\nabla \phi^0$ strongly in $L^2(B(x_0, R))$.

□

Theorem 3.9. The convergence of $\{u^\epsilon(t)\}$ to $u^0(t)$

Proof. Fo every t, we have that

$$\sup_{\epsilon\to 0}\{\int_{B(x_0,R)} |\nabla^* u^\epsilon(t)|^2 + \int_{\Gamma^\epsilon\cap B(x_0,R)} (u^{\epsilon\vee}(t) - u^{\epsilon\wedge}(t))\} < \infty \tag{3.34}$$

So by $GSBV$ compactness and L^2 compactness, there exists a subsequence $u^\epsilon(t)$ converging to $u^0(t) \in GSBV_2^2(B(x_0,R),\Gamma^0)$.
We have $\nabla u^\epsilon(t)$ is uniformly bounded in L^2.
Then, we have $\ddot{u}^\epsilon(t)$ is uniformly bounded in $GSBV_2^{2*}$.
Hence we can choose a subsequence, also denoted by $\{u^\epsilon\}$ such that

$$\nabla u^\epsilon(t) \rightharpoonup \nabla u^0(t) \text{ in } L^2 \tag{3.35}$$

and

$$\ddot{u}^\epsilon(t) \rightharpoonup \ddot{u}^0(t) \text{ in } GSBV_2^{2*} \tag{3.36}$$

□

Theorem 3.10. The limit u is a solution of the equation

$$\ddot{u}(t) - \Delta u(t) = 0 \tag{3.37}$$

satisfying

$$\langle \ddot{u}^0(t), \phi^0\rangle + \langle \nabla u^0(t)\cdot\nabla\phi^0\rangle = 0 \tag{3.38}$$

$\forall \phi^0 \in GSBV_2^2(B(x_0,R))$ with $\mathrm{supp}(\phi^0) \subset\subset B(x_0,R)$

Proof. Given u^0 and ϕ^0, we choose sequences $\{u^\epsilon\}$ such that

$$\nabla u^\epsilon(t) \rightharpoonup \nabla u^0(t) \tag{3.39}$$

$$\ddot{u}^\epsilon(t) \rightharpoonup \ddot{u}^0(t) \tag{3.40}$$

and $\{\phi^\epsilon\}$ is defined above.
By the strong convergence of $\nabla\phi^\epsilon$, we can choose σ such that

$$\|\nabla\phi^\epsilon - \nabla\phi^0\|_{L^2} \le \sigma \tag{3.41}$$

and $\phi^\epsilon = \phi^0$ outside a small neighborhood $N_{\delta(\epsilon)}$ of $\Gamma^0(t)$.
So we have

$$\begin{aligned}
&\langle \ddot{u}^0(t), \phi^0\rangle_{GSBV_2^2(B(x_0,R),\Gamma^0(t))} + \langle \nabla u^0(t), \nabla\phi^0\rangle_{L^2} \\
&= \langle \ddot{u}^0(t), \phi^0\rangle_{GSBV_2^2(N_{\delta(\epsilon)},\Gamma^0(t))} + \langle \nabla u^0(t), \nabla\phi^0 - \nabla\phi^\epsilon\rangle_{L^2} + \langle \nabla u^0(t), \nabla\phi^\epsilon\rangle_{L^2} \\
&\le \langle \ddot{u}^0(t), \phi^0\rangle_{GSBV_2^2(N_{\delta(\epsilon)},\Gamma^0(t))} + \sigma\|\nabla u^0(t)\|_{L^2} + \ \langle \nabla u^\epsilon(t), \nabla\phi^\epsilon\rangle_{L^2} + \langle \nabla u^0(t) - \nabla u^\epsilon(t), \nabla\phi^\epsilon\rangle_{L^2}
\end{aligned} \tag{3.42}$$

By the weak convergence of $\nabla u^\epsilon(t)$,

$$\int_{B((x_0,t_0),R)} [\nabla u^0(t) - \nabla u^\epsilon(t)]\nabla\phi^\epsilon \to 0 \text{ as } \epsilon \to 0 \tag{3.43}$$

and choosing $\delta \to 0$ as $\epsilon \to 0$, we have

$$\langle \ddot{u}^0(t), \phi^0 \rangle_{GSBV_2^2(N_{\delta(\epsilon)}, \Gamma^0(t))} \to 0 \tag{3.44}$$

So we conclude that

$$\langle \ddot{u}^0(t), \phi^0 \rangle_{GSBV_2^2(B(x_0,R), \Gamma^0(t))} + \langle \nabla u^0(t), \nabla \phi^0 \rangle_{L^2} = 0 \tag{3.45}$$

So the blow-up limit is a weak solution of the wave equation at almost every time t. Given the boundedness properties of $\{(u^\epsilon, \Gamma^\epsilon)\}$, we can deduce that the blow-up limit is a weak solution of the wave equation. □

3.3 The Eshelby-Kostrov Property

Given (u, Γ) a solution of the wave equation on Ω, with Dirichlet boundary condition, we can construct a blow-up sequence of it, at (x_0, t_0), converging to a limit (u^0, Γ^0) which is a solution of the blow-up problem. If $\Gamma(t)$ is assumed to be a $\mathcal{C}^{1,1}$ curve, then $\Gamma^0(t)$ is a segment. Then the Eshelby-Kostrov property for the wave equation on the plane can give us more information about the blow-up limits. Detailed results can be found in [10].
Assume the given solution is (u, Γ), where the crack set $\Gamma(t)$ is a $C^{1,1}$ curve in the domain Ω. Then, the blow-up limit $\Gamma^0(t)$ is a straight line, and the velocity of $\Gamma^0(t)$ is a constant V.
The Eshelby-Kostrov property is an important result showing that there exist functions K_1 and K_2 such that

$$K(t) = K_1(\lambda(t)) K_2(\dot{\lambda}(t)) \tag{3.46}$$

for the function

$$u^0(x,t) \sim K(t)|x - \lambda(t)|^{1/2} B(\theta, \dot{\lambda}(t)) \tag{3.47}$$

as $x \to \lambda(t)$, satisfying

$$\ddot{u}^0 - \Delta u^0 = 0 \text{ in } \mathbb{R}^2 \backslash \Gamma^0(t),\ t > 0 \tag{3.48}$$

where

$$\Gamma^0(t) = \{(x_1, 0) | -\infty < x_1 < \lambda(t)\} \tag{3.49}$$

For cracks growing at a constant speed, $K_2(\dot{\lambda}(t))$ is a constant.

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