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Modeling Volatility Derivatives

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MODELING VOLATILITY DERIVATIVES

A Directed Research Project
Submitted to the Faculty of the
WORCESTER POLYTECHNIC INSTITUTE
in partial fulfillment of the requirements for the
Professional Degree of Master of Science
in
Financial Mathematics
by
Justin Carr

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Approved:

Professor Marcel Blais, Advisor

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Abstract

The VIX was introduced in 1993 by the CBOE and has been commonly referred to as the fear gauge due to decreases in market sentiment leading market participants to purchase protection from declining asset prices. As market sentiment improves, declines in the VIX are generally observed. In reality the VIX measures the market's expectations about future volatility with asset prices either rising or falling in value. With the VIX gaining popularity in the marketplace a proliferation of derivative products has emerged allowing investors to trade volatility. In observance of the behavior of the VIX we attempt to model the derivative VXX as a mean reverting process via the Ornstein-Uhlenbeck stochastic differential equation. We extend this analysis by calibrating VIX options with observed market prices in order to extract the market density function. Using these parameters as the diffusion process in our Ornstein-Uhlenbeck model we derive futures prices on the VIX which serves to value our target derivative VXX.

Acknowledgements

It is with much gratitude and appreciation that I recognize my professors from the Mathematical Finance Program at Worcester Polytechnic Institute, and in particular, Professor Marcel Blais for his guidance and wisdom that he has imparted on me throughout my pursuit of this degree. I would also like to extend my appreciation to those who have supported me throughout the program.

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1. Introduction:

1.1 Background:

There are several measures of volatility. Simple historical measures of variance and ARCH models have been around for some time. The VIX, which was developed by the Chicago Board Options Exchange in 1993, is yet another approach and is measured indirectly. In earlier versions the approach calculated implied volatility via the Black-Scholes model for a range of at-the-money S&P 100 options. Over time a more robust methodology change occurred replacing the S&P 100 index with the S&P 500 index and Black-Scholes was replaced with a method employed in the variance swap market.

The volatility index is an annualized measure of the market's risk-neutral expectation of the Standard and Poor's 500 index over a 30 day period. The calculation of the index is a weighted average of both at-the-money and out-of-the money put options and call options on the S&P500 index which reflects information about the volatility smile and expected variance over all possible volatility paths. This contrasts the Black-Scholes approach which measures implied volatility over the most likely price path. When the VIX is increasing investors' expectations of a significant move to either the upside or downside is increasing. One would interpret, for example, a VIX reading of 30 to mean that the market expects a 1 sigma event to equate to an 8.66% ($30\%/\sqrt{12}$) move over the next 30 days in either direction.

1.2 Objective:

VXX is an efficient means of obtaining exposure to the CBOE volatility VIX index and it tracks the S&P 500 VIX Short-Term Futures Total Return Index. This index reflects a daily rolling long of futures positions in the first and second month VIX futures contracts maintaining a constant 30 day weighted average.¹ It is our purpose to understand and model the dynamics of the volatility derivative VXX.

Investments in equity securities tend to exhibit an asymmetrical return profile with greater downside risk than upside potential. As such, returns linked to the volatility of the underlying security spike upwards as investors' price in deteriorations in fundamentals with a more gradual return as investor sentiment improves. This suggests a positively skewed return profile associated with volatility derivatives. Nevertheless, since volatility is not a return generating asset we expect it to fluctuate around a long-term average and be mean reverting. Typical equity derivatives are modeled as a Geometric Brownian Motion whose dynamics are a function of a drift term and volatility. We seek a process that is mean reverting and thus model VXX using the Ornstein-Uhlenbeck process. We begin by confirming the mean reverting tendency of the VIX and extrapolate this out to value VIX options, which are simply forward VIX values, and VIX futures which is the basis for valuing our objective process VXX. Along the way we will implement a risk neutral density parameter estimate as the final approach to taking the risk neutral expectation of our Ornstein-Uhlenbeck process for valuing futures prices and we observe this process relative to market prices.

¹ iPath: The Basics of VIX Futures ENTs

1.3 Calculation of volatility:

$$\sigma^2 = \frac{2}{T} e^{RT} \left[\sum_i \frac{\Delta k}{K_i^2} Q(K_i) \right] - \frac{1}{T} \left[\frac{F}{K_0} - 1 \right]^2 \quad [2]$$

Where:

$$\sigma \text{ (fair value of VIX)} = 100\sqrt{\sigma^2}$$

T = time to expiration

$$F = \text{Strike Price} + e^{RT} * \min(\text{Call Price}(i) - \text{Put Price}(i))$$

K_0 = First strike below the forward index level F

K_i = Strike price of i^{th} out-of-the-money option; a call if $K_i > K_0$ and a put if $K_i < K_0$; both a put and call if $K_i = K_0$

ΔK_i = the average of the difference in the strikes

R = Risk-free interest rate

$Q(K_i)$ = The midpoint of the bid-ask spread for each option with strike K_i

Replicating Portfolio:

VIX is calculated as the square root of the forward price of a strip of SPX options. By formulating the discrete replicating portfolio we attempt to gain a better understating of this formula. The discrete replicating portfolio has the following expression:

$$\sigma^2 = 2 \left\{ \frac{(F_T - F_0)}{F_0} - \sum_0^T \frac{\Delta F_i}{F_i} + \sum_0^{K_0} (K - F_T) \frac{\Delta K}{K^2} + \sum_{K_0}^{\infty} (F_T - K) \frac{\Delta K}{K^2} \right\} - \left[\frac{F_0}{K_0} - 1 \right]^2$$

This equation states that total variance is a function of three terms. The first two terms of the equation reflects a return to a stock index futures contract held to maturity less a dynamic futures component rebalanced through time. The middle terms are the payoffs to holding $\frac{\Delta K}{K^2}$ puts and calls. The use of several options is needed to capture the volatility smile, rather than a constant volatility as implied by the Black-Scholes formula, which emerged followed the stock market crash of 1987. The skew in the implied volatilities follows from a strong demand for out-of-the money put options in order to protect against a stock market crash. The last term of this equation is an adjustment for the difference between the forward price and strike. This term is necessary to account for a strip of options that is not centered on a strike that is at-the-money. If F_0 were equal to K_0 then this term would drop out.

The value of an option is a function of several variables including interest rates, time to expiry, changes in the underlying, and changes in volatility. The above equation states that variance is replicated with a portfolio of options delta-hedged with stock index futures. By delta hedging the options, we are

² CBOE: Volatility Index - VIX

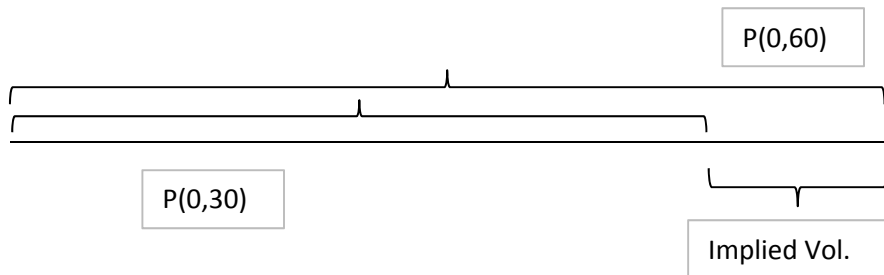
immunizing the value of our portfolio to changes in the underlying index while leaving us un-hedged to changes in volatility which is precisely what we want. Therefore, this equation attempts to isolate a constant exposure to variance.

Taking matters one step further, because 30-day options are typically not available 30, day volatility is interpolated vis-à-vis two options. The resulting VIX value is a constant 30-day weighted average of the near-term and back month contract.

$$VIX = 100 * \sqrt{\left\{ T1\sigma_1^2 \left[\frac{Nt2-N30}{Nt2-Nt1} \right] + T2\sigma_2^2 \left[\frac{N30-Nt1}{Nt2-Nt1} \right] \right\} * \frac{N365}{N30}}$$

1.4 VIX Futures:

The fair value of VIX futures is the square root of the forward price of expected 30 day variance at futures expiration minus an adjustment factor which reflects the concavity of the square root function. Thus, the estimate of the fair value of futures at T_0 reflects the markets expectation of future volatility over a 30 day period at expiry. The process of extracting the fair value of the VIX futures via a synthetic calendar spread is possible because of the linearity of variance.



$$\text{Fair value of VIX futures} = 100 * \sqrt{\left(\frac{365}{30} \right) * (e^{RT2} P(0,60) - e^{RT1} P(0,30)) - var(Ft)} \quad [3],[4]$$

Where:

$$P = 2e^{rt} \left[\sum_i \frac{\Delta k}{K_i^2} Put(k) + \sum_i \frac{\Delta k}{K_i^2} Call(k) \right] - \frac{1}{T} \left[\frac{F}{K_0} - 1 \right]^2$$

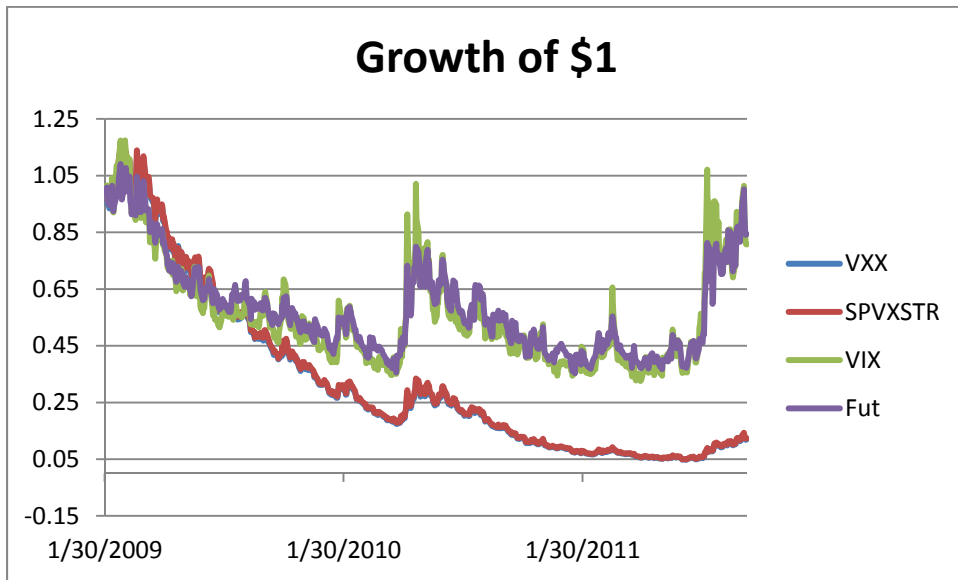
This calculation is different than the typical cost of carry model used to price index futures. The reason being volatility is not an asset, and therefore, you cannot purchase the underlying and hold it to the maturity of the futures contract. A second distinction is that while the spot VIX represents 30 day forward volatility, VIX futures represents 30 day volatility at expiry of the contracts which may be higher or lower than the spot VIX depending on market expectations.

³ CBOE: Additional Features of VIX Futures

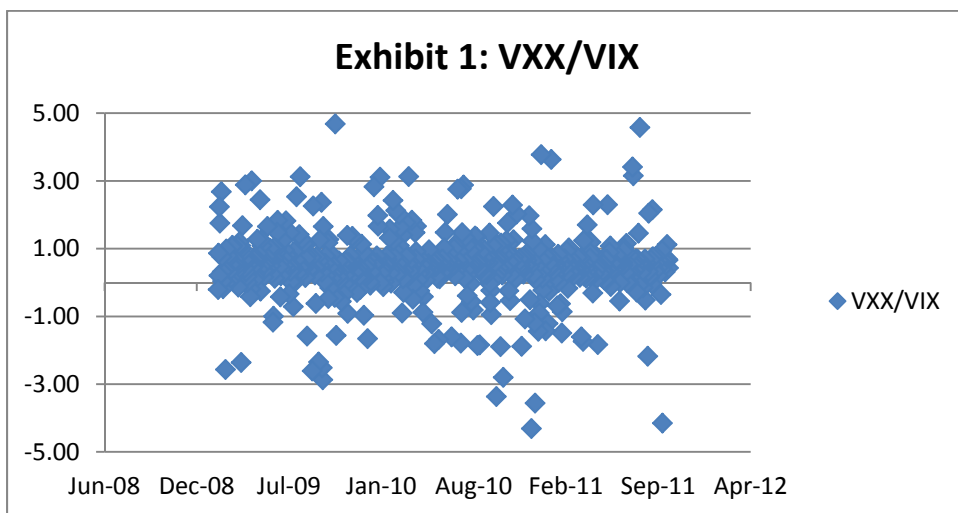
⁴ CBOE: All About VIX

2. Performance

2.1 Performance of the VXX relative to VIX:



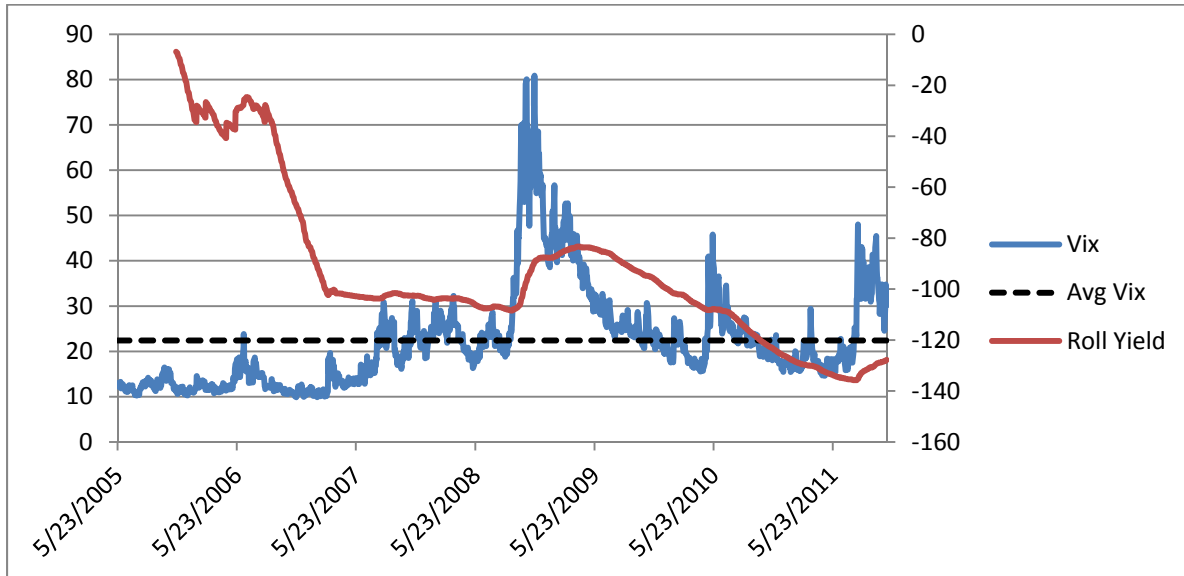
The S&P500 VIX Short-Term Futures index (SPVXSTR) is designed to capture an investment in rolling long a futures contract on the VIX index. The return to the index has two components, an implied return on a notional amount of cash as measured by a 3-month T-bill and a daily rolling of a long position in a VIX futures contract. We can see from the chart above that VXX tracks this index precisely concealing the plot of VXX but it does not track the VIX. We also observe from this chart that the continuous CBOE VIX futures track the spot VIX with a high degree of correlation.



Reviewing the daily returns of VXX we can see that the tradable derivative offers exposure to this index with a very different return profile to that of the spot VIX index. The ratio of the returns to VXX and a hypothetical investment in the spot VIX index track each other with great dispersion as shown above. As

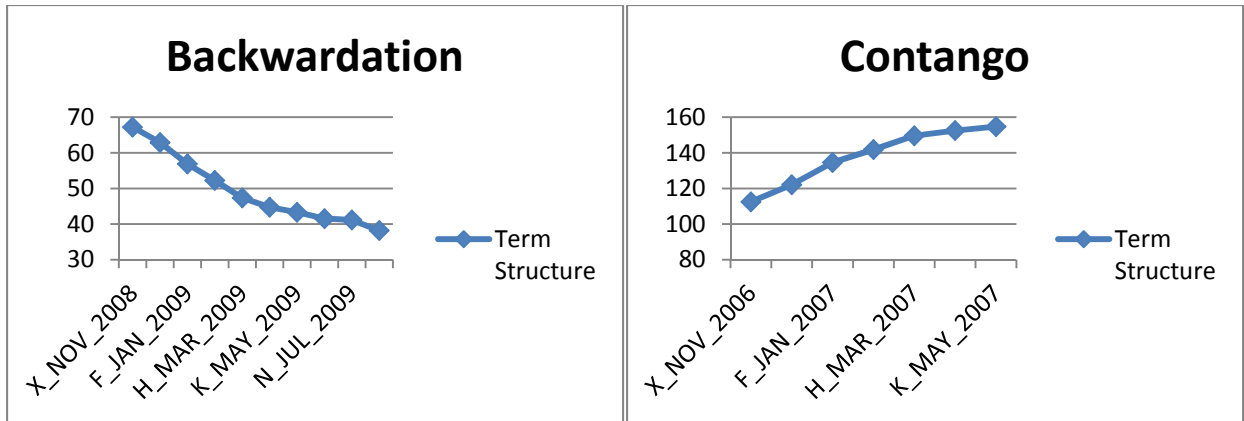
a result of this, VXX captures a fraction of the spot index return. A perfectly correlated tradable instrument with its index would have a ratio of 1. The scatter plot of the daily VXX returns relative to the daily spot VIX index returns shows that this is rarely the case.

2.2 Contango/backwardation:



Contango and backwardation refer to the term structure of the futures curve. In normal market conditions futures are said to trade in contango which refers to a state in which the futures price exceeds the spot price. In this scenario the futures prices will converge to the spot at expiry which implies futures prices are falling. In backwardation the futures price is below the expected spot price. In this situation it is suggested that the futures price will rise to the expected spot price. This is desirable for speculators who are net long the contracts.

In the context of VIX futures this has important implications. We observe that the contracts are in contango in low volatility environments and in backwardation in high volatility environments. In late 2008/2009 the VIX traded at extreme levels and well above the long-term average as the credit crisis that stemmed from the housing bubble led to significant volatility and depressed asset prices. We observed during this time that the futures contracts traded in backwardation as we would expect. Prior to the credit crisis during 2006/2007 we see that the VIX traded at levels that were in-line with the long-term average. During this volatility regime we can see that the futures contracts traded in contango.



We also observe in less extreme scenarios that it is typically the case that the term structure of volatility is steepest at the shortest maturities which the VXX is priced off of resulting in the largest differences. This lends itself to the roll yield. When the VIX futures term structure is not flat, meaning that the price difference between the front month and back month contract are different, then the daily rebalancing can generate its own profit and loss. On each business day VXX rolls a long position in the front month futures contract and purchases an equal notional amount in the back month contract maintaining a constant weighted average maturity of 30 days. Prior to expiration of the front month contract VXX will hold exclusively the back month contract. The following day 1/n (where n refers to the number of trading days) will be sold and the new back month contract will be purchased. Depending on whether the contracts are in backwardation or contango each and every day the contracts are rolled will thus have an impact and pricing of VXX. From the first chart we observe that the cumulative roll yield is declining thereby suggesting term structure decay and performance loss. This is one of the contributing factors and difficulties in replicating the spot VIX index. Other factors include continuously changing option chains and a difference in methodology. The cash index rolls forward 8 days prior to expiration to avoid anomalies in pricing whereas VXX does not.

3. Ornstein-Uhlenbeck process

3.1 Definition:

The Ornstein-Uhlenbeck process was developed in the 1930's in the context of physics as an alternative to Brownian Motion. As a continuous time stochastic process it satisfies the following properties⁵:

A stochastic process $\{X_t: t \geq 0\}$ is

- Stationary: $\forall t_1 < t_2 < \dots < t_{n-1} < t_n$ and $k > 0$ $(X_{t_1}, \dots, X_{t_k}) = (X_{t_1+k}, \dots, X_{t_n+k})$

A stationary process is a stochastic process whose joint probability distribution is identically distributed and does not change when shifted in time.

- Gaussian: $\forall t_1 < t_2 < \dots < t_{n-1} < t_n$ $X_{t_1, \dots, t_k} = (X_{t_1}, \dots, X_{t_k})$

A finite linear combination of processes is multivariate normally distributed.

- Markovian: $\forall t_1 < t_2 < \dots < t_{n-1} < t_n$ $P(X_{t_1} \leq x | X_{t_1}, \dots, X_{t_k}) = P(X_{t_n} \leq x | X_{t_{n-1}})$

A conditional probability that states our function is less than some specified value and is determined by recent events and not the past.

3.2 Derivation:

The Ornstein-Uhlenbeck process satisfies the following stochastic differential equation.

$$dx_t = \theta(\mu - x_t)dt + \sigma dw_t$$

An application of the Ito Lemma to $f(X_t, t) = X_t e^{\theta t}$ yields a process X that satisfies the linear stochastic differential equation. We have the following:

$$f_t = \theta X_t e^{\theta t} \quad f_x = e^{\theta t} \quad f_{xx} = 0$$

$$df(X_t, t) = f_t dt + f_x dX_t + \frac{1}{2} f_{xx} dX_t dX_t$$

$$df(X_t, t) = \theta X_t e^{\theta t} dt + e^{\theta t} dX_t + \frac{1}{2} 0 * dX_t dX_t$$

$$df(X_t, t) = \theta f(X_t, t) dt + e^{\theta t} dX_t$$

$$\int_0^t df(X_s, s) = \theta \int_0^t f(X_s, s) ds + \int_0^t e^{\theta s} dX_s$$

$$\int_0^t df(X_s, s) = \theta \int_0^t X_s e^{\theta s} ds + \int_0^t e^{\theta s} [\theta(\mu - X_s) ds + \sigma dw_s]$$

$$\int_0^t df(X_s, s) = \theta \int_0^t X_s e^{\theta s} ds - \theta \int_0^t X_s e^{\theta s} ds + \theta \int_0^t e^{\theta s} \mu ds + \int_0^t e^{\theta s} \sigma dw_s$$

⁵ S. Finch, "Ornstein-Uhlenbeck Process"

$$\int_0^t df(X_s, s) = \theta \int_0^t e^{\theta s} \mu ds + \int_0^t e^{\theta s} \sigma dw_s \text{ where } \int_0^t e^{\theta s} ds = \frac{1}{\theta} (e^{\theta t} - e^{\theta 0})$$

$$X_t e^{\theta t} - X_0 = \mu(1 - e^{-\theta t}) + \int_0^t e^{\theta s} \sigma dw_s$$

$$X_t = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma e^{-\theta t} \int_0^t e^{\theta s} dw_s$$

3.3 Estimating the parameters of our process:

Mean:

Taking an expectation of this process yields the following.

$$E[X_t] = E[X_0 e^{-\theta t}] + E[\mu(1 - e^{-\theta t})] + E[\sigma e^{-\theta t} \int_0^t e^{\theta s} dw_s]$$

Pulling out the constants and expressing the Ito stochastic integral as a Riemann-Stieltjes sum and recognizing the expectation of a Brownian Motion is zero gives us the following.

$$E[X_t] = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \sigma \int_0^t e^{\theta s} * E[\sum_{i=1}^n Bt_i - Bt_{i-1}]$$

$$E[X_t] = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t})$$

For simplicity we will assume that X_0 starts at zero.

Variance:

$$Var[X_t] = E[X^2] - [EX]^2$$

$$Var[X_t] = E[(X_0 e^{-\theta t})^2] - E\left[(X_0 e^{-\theta t} + \sigma e^{-\theta t} \int_0^t e^{\theta s} dw(s))^2\right] \text{ where } \mu \text{ is assumed to be zero.}$$

$$Var[X_t] = E[(X_0 e^{-\theta t})^2] - E\left[(X_0^2 e^{-2\theta t} + 2X_0 e^{-\theta t} \sigma e^{-\theta t} \int_0^t e^{\theta s} dw_s + \sigma^2 e^{-2\theta t} \int_0^t e^{2\theta s} d(s)\right]$$

$$Var[X_t] = E[(X_0 e^{-\theta t})^2] - E\left[(X_0^2 e^{-2\theta t} + 2X_0 e^{-\theta t} \sigma e^{-\theta t} \int_0^t e^{\theta s} dw_s + \sigma^2 e^{-2\theta t + 2\theta t} \frac{1}{2\theta} - \sigma^2 e^{-2\theta t} \frac{1}{2\theta}\right]$$

$$Var[X_t] = \frac{\sigma^2}{2\theta} (1 - e^{-2\theta t})$$

3.4 Implementation:

We implement this process regressing spot VIX values on 1-day lagged values. This regression has the linear relationship $X_{i+1} = a + X_i b + \epsilon$. Associating our linear equation with our stochastic differential equation from above along with our results of our first and second order moments provides us with the third and final parameter of our process, theta. Theta represents the decay rate or mean reversion parameter.

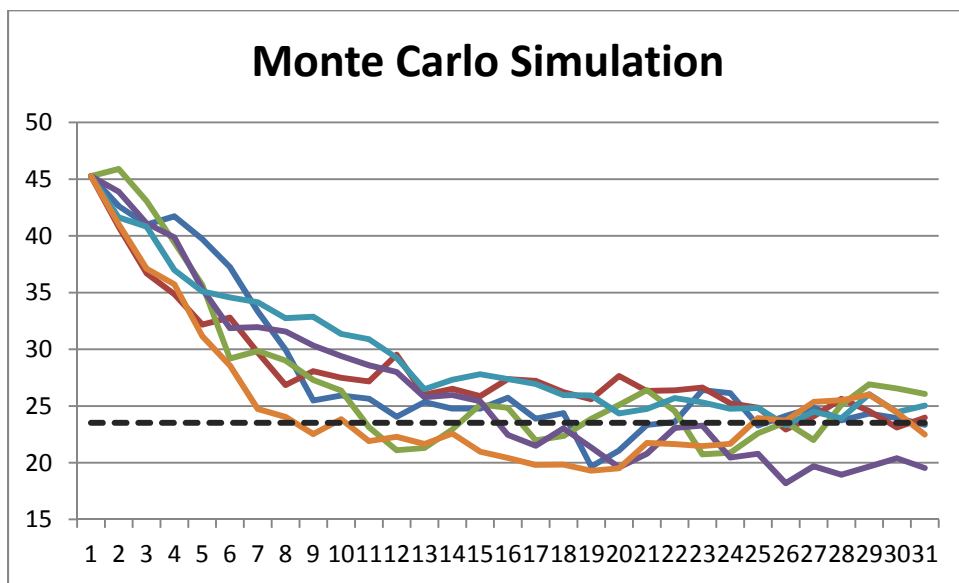
$$b = e^{-\theta t}$$

Using the slope of our regression result and rearranging terms solves for the desired parameter.

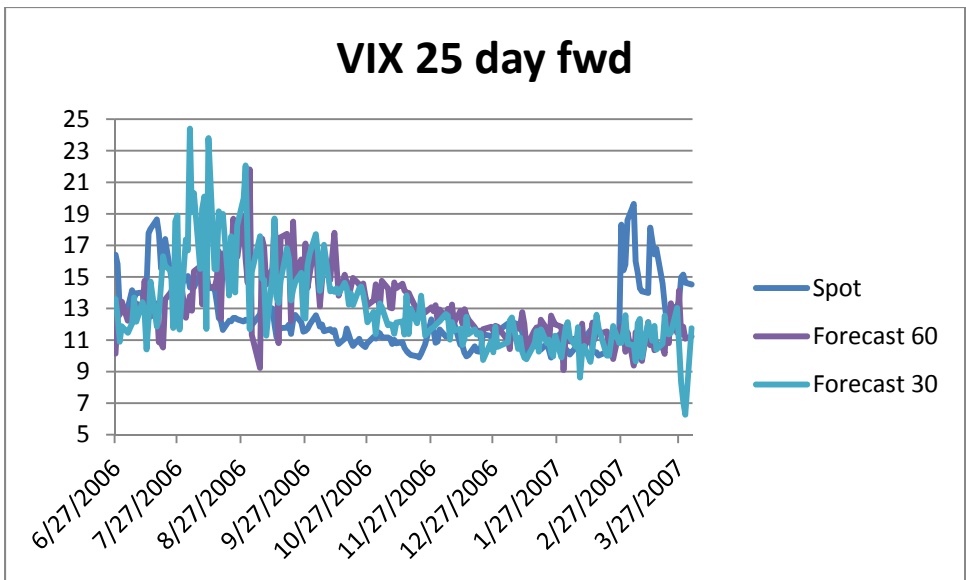
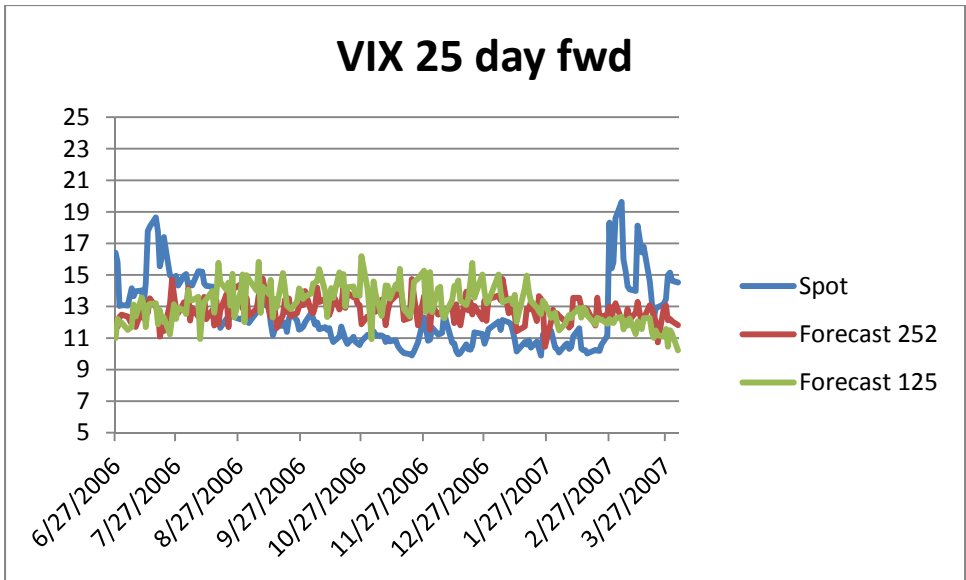
$$\theta = -\frac{\ln b}{t}$$

3.5 Results:

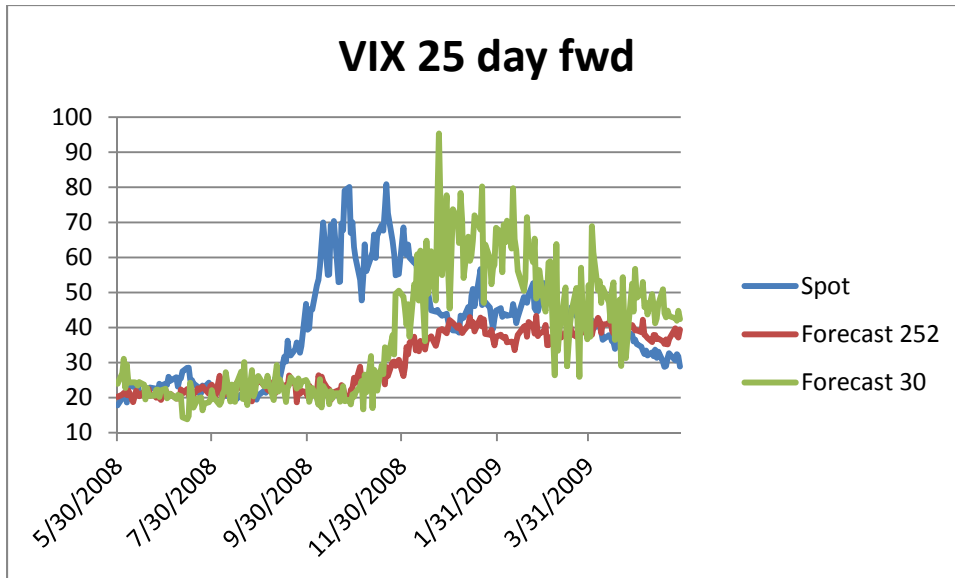
As one would expect, since volatility in and of itself is not return generating, it exhibits a mean reverting behavior. Ornstein-Uhlenbeck is such a process and is similar in ways to a Weiner process that has a central tendency. The more distant x_t is from its mean μ the larger the drift back towards its central location. The speed at which this happens is determined by the parameter θ . The chart below shows a Monte Carlo simulation of five paths forecasting out 30 days. The mean μ from this regression was 23.5058 and we can see the attraction of the simulation to tend towards its mean.



Reviewing the results of modeling the VIX in the following charts we can see that the volatility regime played an important role in determining the window of data one should use in estimating the parameters. Each simulation ran a rolling window of regressions through time using 30 days, 60 days, 125 days, and 252 days of historical data. The parameters of the regression from each rolling window of historical data served as the variables into our SDE which recursively calculated subsequent values.



We can see that the shorter window used in estimating the parameters was more responsive to increases in volatility but these estimate were also more volatile. 2005 through 2007 was a period of volatility that persisted below the long term average of 22.39, while 2008 onwards was a regime that exceeded the long term average for an extended period of time. This suggests some room for improvement and perhaps exponentially weighting the data instead of equally weighting the date points may improve our forecast.



This framework serves as the basis for valuing our options. We use the 30 day rolling window of data to forecast our VIX values using the dynamics of the Ornstein-Uhlenbeck process which we derived. Using these values we run a Monte Carlo simulation using 100,000 paths to calculate our theoretical option prices.

$$S(t_{i+1}) = S(t_i)e^{-\theta(t_{i+1}-t_i)} + \mu(1 - e^{-\theta(t_{i+1}-t_i)}) + \sigma \sqrt{\frac{1 - e^{-2\theta(t_{i+1}-t_i)}}{2\theta}} * Z_{i+1}$$

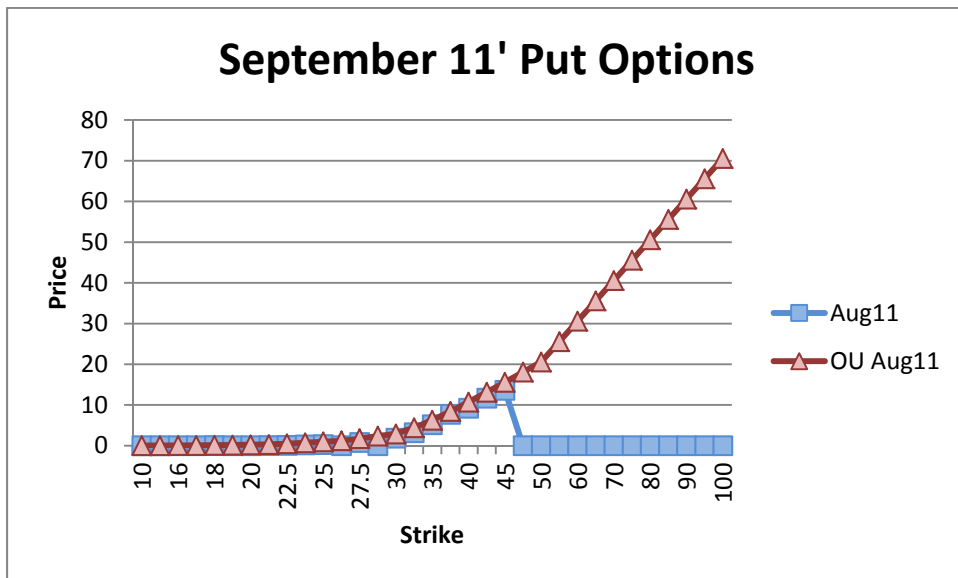
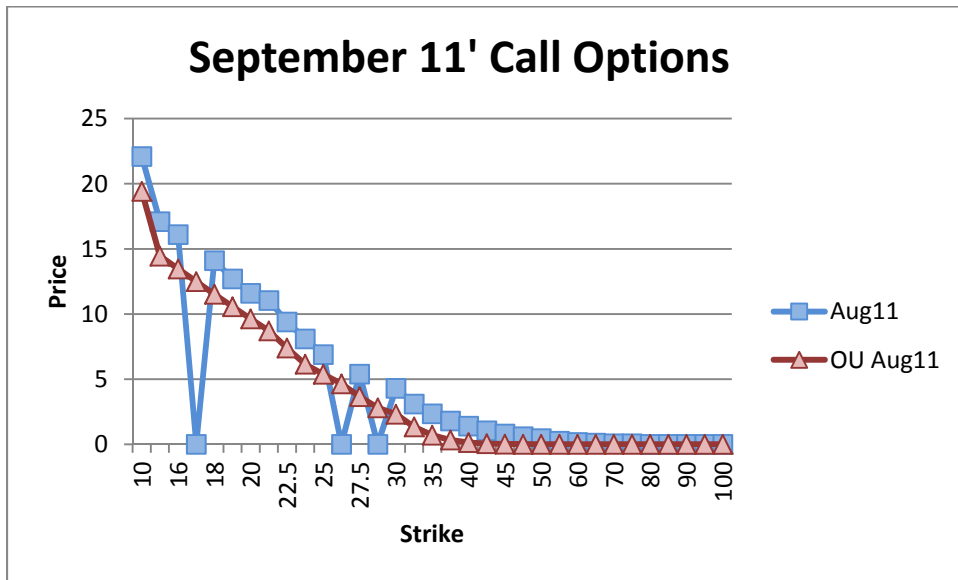
$$C(K, T) = e^{-rt} E[(S(T) - K)^+]$$

$$P(K, T) = e^{-rt} E[(K - S(T))^+] \quad [6]$$

On August 21, 2011, for example, the September puts and calls expired in 21 days. Thus, our theoretical option prices reflect a 21 day forecast of spot VIX. We achieved this by recursively calculating values for each simulation and then finally by taking an expectation of our terminal values.

⁶ P. Glasserman, "Monte Carlo Methods in Financial Engineering"

3.6 Option Estimates:



In this particular instance, the results show that the call option prices are modestly lower than the market prices while the put prices are seemingly better and slightly higher than market prices. Reviewing the results of our estimates our mean term was 29.38 as compared to the spot VIX level of 31.62. With a theta of .27 and our forecast of 100,000 simulations we generated an estimate of 25.48. Due to spot VIX starting above our mean we would expect the simulation to forecast a lower value while exhibiting some noise around our target value due to the diffusion process.

4. Estimating the Risk Neutral Density

4.1 Background:

Theoretical constructs of asset pricing provide a framework and starting point for valuing tradable instruments. Another approach, and the inverse problem, suggests a need to assess market expectations embedded in these tradable products. Extracting the implied risk-neutral density function from market option prices provides us with the forward-looking probability distribution on the underlying fundamental factors. In doing this we reconcile the differences between theoretical model prices and market expectations.

We accomplish this by employing a methodology proposed by Bahra (1997) [5]. This approach is based on a weighted average of log-normal densities.

$$L(x; \alpha, \beta) = \frac{1}{x\sqrt{2\pi\beta^2}} e^{\left(-\frac{1}{2\beta^2}(\ln x - \alpha)^2\right)} \quad , x > 0$$

$$C(K, t) = e^{-rt} \left\{ \omega \left[e^{\alpha_1 + \frac{1}{2}\beta_1^2} N(d1) - KN(d2) \right] + (1 - \omega) \left[e^{\alpha_2 + \frac{1}{2}\beta_2^2} N(d3) - KN(d4) \right] \right\}$$

$$\text{Where} \quad d_1, d_3 = \frac{-\ln K + \alpha_i + \beta_i^2}{\beta_i} \quad \text{and} \quad d_2, d_4 = d_j - \beta_i$$

Defining the parameters of our log-normal distribution, alpha and beta, in the following manner we show that the desired mixture distribution is a weighted average of Black-Scholes formula. We show this for the call option.

$$\alpha_i = \ln S_t + \left(\mu_i - \frac{1}{2}\sigma_i^2 \right) t \quad \beta_i = \sigma_i \sqrt{t}$$

$$C(K, t) = e^{-rt} \left\{ \omega \left[e^{\ln S_t + (\mu_i - \frac{1}{2}\sigma_i^2)t + \frac{1}{2}\sigma_i^2 t} N(d1) - KN(d2) \right] + (1 - \omega) \left[e^{\ln S_t + (\mu_i - \frac{1}{2}\sigma_i^2)t + \frac{1}{2}\sigma_i^2 t} N(d3) - KN(d4) \right] \right\}$$

$$C(K, t) = e^{-rt} \left\{ \omega \left[e^{\ln S_t + (\mu_i t)} N(d1) - KN(d2) \right] + (1 - \omega) \left[e^{\ln S_t + (\mu_i t)} N(d3) - KN(d4) \right] \right\}$$

$$C(K, t) = e^{-rt} \left\{ \omega \left[S_t e^{\mu_1 t} N(d1) - KN(d2) \right] + (1 - \omega) \left[S_t e^{\mu_2 t} N(d3) - KN(d4) \right] \right\}$$

$$C(K, t) = \left\{ \omega \left[S_t e^{\mu_1 t - rt} N(d1) - K e^{-rt} N(d2) \right] + (1 - \omega) \left[S_t e^{\mu_2 t - rt} N(d3) - K e^{-rt} N(d4) \right] \right\}$$

$$C(K, t) = \omega \left[S_t N(d1) - K e^{-rt} N(d2) \right] + (1 - \omega) \left[S_t N(d3) - K e^{-rt} N(d4) \right]$$

It is the Black-Scholes formula that serves to identify the theoretical options prices. Having established these parameters we run a non-linear least squares optimization minimizing the differences between market prices and our model prices. The non-linear least squares regression has the following formulation:

$$\min \left\{ \sum_{i=1}^n [c^i - c_i(K_i, t)]^2 + \sum_{i=1}^n [p^i - p(K_i, t)]^2 + \left[\omega e^{\alpha_1 + \frac{1}{2}\beta_1^2} + (1 - \omega) e^{\alpha_2 + \frac{1}{2}\beta_2^2} - e^{rt} S_t \right]^2 \right\} \quad [7]$$

⁷ Fusai&Roncoroni: Implementing Models in Quantitative Finance

Once again substituting alpha and beta into the third term we can see more clearly that this is the weighted difference between the theoretical forward and forward spot price.

$$\min \left\{ \sum_{i=1}^n [c^i - c_i(K_i, t)]^2 + \sum_{i=1}^n [p^i - p(K_i, t)]^2 + [\omega(\ln S_t e^{\mu_1 t} + (1 - \omega)\ln S_t e^{\mu_2 t} - e^{rt} S_t)]^2 \right\}$$

We apply this approach to estimating our parameters and we recognize the differences in approach which underlie a Geometric Brownian Motion and a mean reverting process. Implementing the later, d1 should be modified to the following. We leave pursuit of this for future work.

$$d_1 = \frac{1}{\sigma_p} \log \left\{ \frac{s}{k} \right\} + \frac{1}{2} \sigma_p \quad \text{Where } \sigma_p = \frac{1}{a} \{1 - e^{-a(S-T)}\} * \sqrt{\frac{\sigma^2}{2a} \{1 - e^{-2a(T-t)}\}}$$

$$d_2 = d_1 - \sigma_p$$

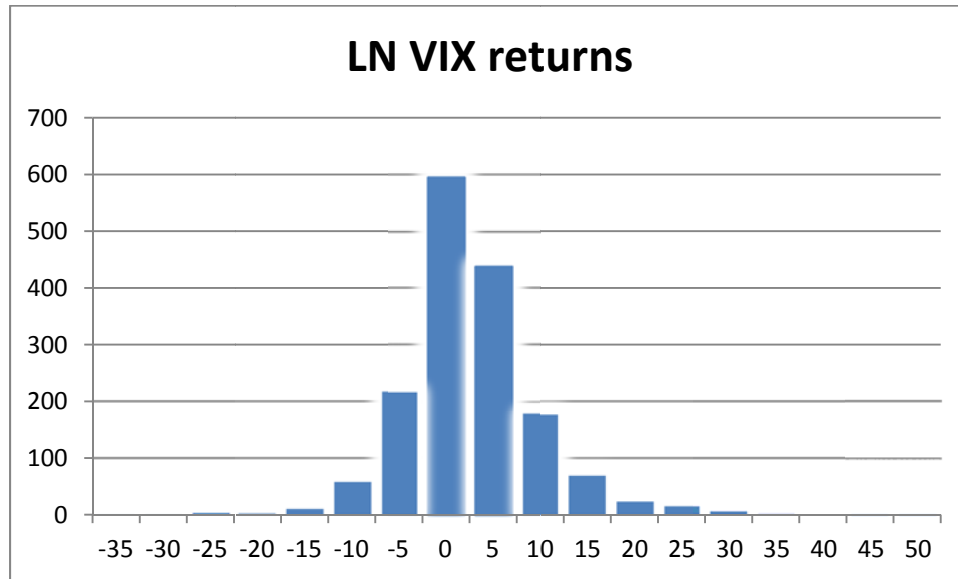
4.2 Implementation:

The calculation for VIX options and the VIX are the same; however, the S&P options used to calculate VIX options correspond to the maturity of the options and they are not the same options used to calculate the spot VIX. At the maturity of the options, VIX and VIX options should converge since the options used in both calculations are the same. Since VIX options reflect the market's expectations about the VIX level at expiry and volatility is a mean reverting process as we have established we should expect futures to be lower in elevated periods of volatility and higher in lower periods of volatility. We observe this in the market prices. On September 30, 2011 VIX futures are 38.6 while the VIX is 42.96. This suggests that the 30 day volatility of the futures at expiry in 19 days will be lower than the spot VIX 30 day forward volatility forecast. On April 29, 2011, a period of low volatility, VIX futures are 16.7 as compared to the spot VIX at 14.75. With a long-term mean near 20, this is intuitive.

Date	Fut. Mkt. Price	Vix
Nov 11'	28.3	27.8
Oct 11'	29.95	29.96
Sept 11'	38.6	42.96
Aug 11'	31.65	31.62
July 11'	21.1	22.25
Jun 11'	17.7	16.52
May 11'	16.55	15.45
Apr 11'	16.7	14.75
Mar 11'	19.3	17.74
Feb 11'	19.2	18.35
Jan 11'	19.1	19.53
Dec 10'	19.7	17.75

Option prices were determined from closing month-end values for calendar year 2011. For each month we calculated theoretical option prices C(i) and P(i) by minimizing the sum of squared differences

between our market prices and model prices. This optimization was constrained to ensure that our weights were no greater than 1 and our variances were positive. In addition to calculating our model prices we also calculated Black-Scholes prices for a base-line comparison. We observe from this scenario that using a constant volatility across all of the strikes as the Black-Scholes formula implies shows that the computed value undervalues the out-of-the money call options. Looking at the August 35 strike in section 4.3, for example, shows a computed price of \$.45 versus \$2.35 market price. If the underlying for these options were equities we would see that out-of-the money put options would be undervalued. As discussed earlier, this reflects market participants desire to protect their portfolios from a market crash. We observe this phenomenon here as well but the relationship is reversed since the VIX tends to be negatively correlated with equity prices. As such, while it is typical to treat market returns as log-normally distributed, this is shown to not be the case. Furthermore, while we would expect our calibrated lognormal mixture density to undervalue out-of-the money call options as well, the optimization results are an improvement.



4.3 Results: August 11'

S_t	r	t	σ	k	Black-Scholes		RN approach		Market Price		Theoretical Price				α_1	α_2	β_1	β_2	1-w	Min Diff
					d1	d2	d1	d3	Call	Put	Call	Put	C(i)	P(i)						
31.62	0.02	21	0.50	10.00	9.69	9.57	8.44	3.84	22.1	0	21.63	0.00	21.85	0.00	3.44	3.42	0.14	0.32	0.60	0.06
31.62	0.02	21	0.50	15.00	6.30	6.18	5.49	2.56	17.1	0	16.64	0.00	16.86	0.01	3.44	3.42	0.14	0.32	0.60	0.06
31.62	0.02	21	0.50	16.00	5.76	5.64	5.02	2.35	16.1	0	15.64	0.00	15.88	0.02	3.44	3.42	0.14	0.32	0.60	0.05
31.62	0.02	21	0.50	18.00	4.78	4.66	4.16	1.98	14.1	0.05	13.64	0.00	13.92	0.06	3.44	3.42	0.14	0.32	0.60	0.03
31.62	0.02	21	0.50	20.00	3.90	3.78	3.40	1.65	11.6	0.03	11.64	0.00	12.00	0.14	3.44	3.42	0.14	0.32	0.60	0.17
31.62	0.02	21	0.50	21.00	3.49	3.37	3.04	1.49	11.05	0.05	10.64	0.00	11.07	0.19	3.44	3.42	0.14	0.32	0.60	0.02
31.62	0.02	21	0.50	22.50	2.91	2.79	2.54	1.28	9.4	0.1	9.15	0.00	9.70	0.29	3.44	3.42	0.14	0.32	0.60	0.13
31.62	0.02	21	0.50	24.00	2.37	2.25	2.07	1.07	8.1	0.2	7.66	0.01	8.39	0.39	3.44	3.42	0.14	0.32	0.60	0.12
31.62	0.02	21	0.50	25.00	2.03	1.91	1.78	0.94	6.9	0.28	6.68	0.03	7.55	0.47	3.44	3.42	0.14	0.32	0.60	0.46
31.62	0.02	21	0.50	27.50	1.24	1.12	1.08	0.64	5.4	0.8	4.36	0.21	5.64	0.87	3.44	3.42	0.14	0.32	0.60	0.06
31.62	0.02	21	0.50	30.00	0.51	0.39	0.45	0.37	4.3	1.85	2.44	0.79	4.05	1.92	3.44	3.42	0.14	0.32	0.60	0.07
31.62	0.02	21	0.50	32.50	-0.16	-0.28	-0.13	0.11	3.1	3.2	1.14	1.99	2.84	3.61	3.44	3.42	0.14	0.32	0.60	0.24
31.62	0.02	21	0.50	35.00	-0.78	-0.90	-0.67	-0.12	2.35	5.16	0.45	3.79	1.98	5.53	3.44	3.42	0.14	0.32	0.60	0.27
31.62	0.02	21	0.50	37.50	-1.36	-1.48	-1.17	-0.34	1.8	7.7	0.15	5.98	1.40	7.43	3.44	3.42	0.14	0.32	0.60	0.24
31.62	0.02	21	0.50	40.00	-1.90	-2.02	-1.64	-0.54	1.4	9.2	0.04	8.37	1.00	9.38	3.44	3.42	0.14	0.32	0.60	0.20
31.62	0.02	21	0.50	42.50	-2.40	-2.52	-2.08	-0.73	1.05	11.7	0.01	10.84	0.72	11.46	3.44	3.42	0.14	0.32	0.60	0.17
31.62	0.02	21	0.50	45.00	-2.88	-3.00	-2.50	-0.92	0.8	13.6	0.00	13.33	0.52	13.69	3.44	3.42	0.14	0.32	0.60	0.09
sum																			2.42	

The mean and variances of our lognormal mixture density through time are shown below. These data points show an increase in our means and variances from December 2010 through November 2011. During this time the market was flat for the period but quite volatile. On August 2011, when volatility was the highest, we saw a modest increase in our estimate of volatility.

Date	α_1	α_2	β_1	β_2	1-w
Nov 11'	4.04	3.42	0.95	0.20	0.99
Oct 11'	3.37	3.37	0.26	0.24	0.94
Sept 11'	3.76	3.72	0.00	0.27	0.88
Aug 11'	3.44	3.42	0.14	0.32	0.60
July 11'	2.87	3.07	0.69	0.26	0.93
Jun 11'	2.80	2.98	0.08	0.29	0.28
May 11'	2.74	2.79	0.00	0.17	0.99
Apr 11'	2.71	2.79	0.00	0.20	1.00
Mar 11'	2.91	2.93	0.07	0.20	1.00
Feb 11'	2.92	2.93	0.07	0.19	1.00
Jan 11'	2.97	2.95	0.04	0.21	1.00
Dec 10'	2.89	2.97	0.00	0.21	1.00

Having established the parameters of our lognormal distribution we are now able to replace the Brownian Motion that was driving our OU process with samples from our lognormal distribution. These samples will have the desired mean and variances we calculated above. We do this by taking our standard random normal with a mean of 0 and variance of 1 which has the form $z = \frac{x-\mu}{\sigma}$ and we express it in the form of an exponent $x = e^{\mu+\sigma z}$. Solving for our sample variable from the lognormal distribution we have the following:

$$\alpha = e^{\mu\sqrt{e^{\sigma^2}}} \quad \beta = e^{2\mu}e^{\sigma^2}(e^{\sigma^2} - 1)$$

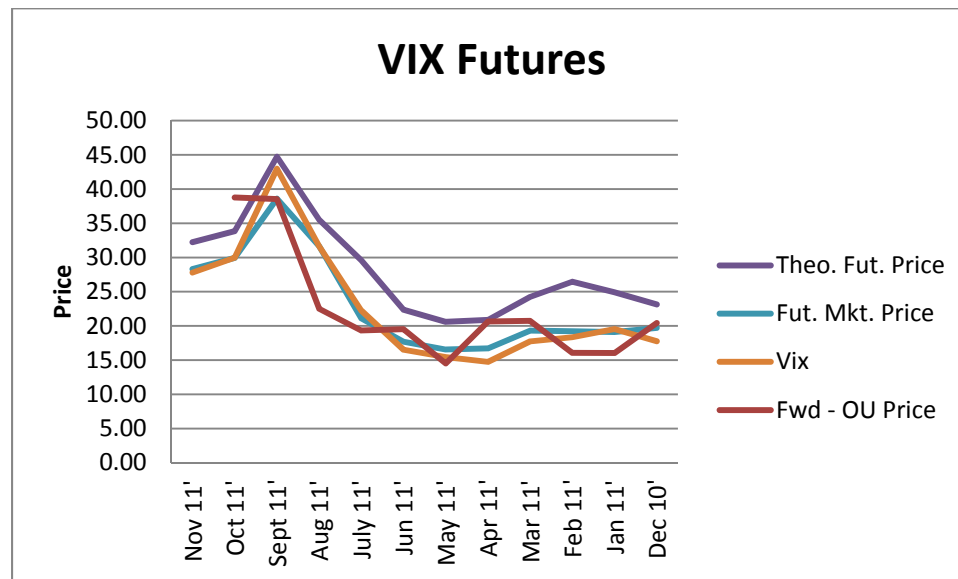
Mean:

$$\alpha^2 = e^{2\mu}e^{\sigma^2} \quad , \quad \mu = \frac{\ln(\alpha^2) - \sigma^2}{2}$$

Variance:

$$\beta = e^{2\mu+\sigma^2+\sigma^2} - e^{2\mu+\sigma^2} \quad , \quad \beta = \alpha^2 e^{\sigma^2} - \alpha^2 \quad , \quad \frac{\beta}{\alpha^2} = e^{\sigma^2} - 1 \quad , \quad \sigma = \sqrt{\ln\left(\frac{\beta}{\alpha^2} + 1\right)}$$

Having achieved this we are once again able to run simulations using the Ornstein-Uhlenbeck process with the desired lognormal distribution and calculate VIX futures. This serves as our estimate of VXX.



As we can see from above, our forecasting error when implementing our process while sampling from a lognormal distribution is lower in elevated periods of volatility with a consistently higher spread in lower

periods of volatility. Overall, modeling VXX as a mean reverting process using the Ornstein-Uhlenbeck model demonstrated better results when the diffusion term is normally distributed with a mean of zero and a variance of 1 rather than a mean of ~ 3 as we experienced using the lognormal distribution.

5. Conclusion:

The empirical investigation demonstrates the mean reverting process of volatility as measured by VIX. By extending this to model volatility derivatives we explored the use of simulating estimates of our process and calibrating these estimates using market expectations to calculate VXX. In the more general case VIX futures are calculated as we had highlighted earlier. Using S&P500 options volatility is extracted via a synthetic calendar spread by delta hedging with stock index futures thereby isolating variance. Because we were modeling VIX and its derivatives we were dealing with volatility directly. Thus, we are able to simply forecast volatility without interpolating.

Our results demonstrate the Ornstein-Uhlenbeck process to be an effective means of forecasting volatility derivatives. Despite the reasonableness of this approach, we observed an estimation error that varied over time. The reasons for the error in the model prices of VXX stems primarily from the mean parameter estimate that we obtained from the lognormal distribution. This resulted in a fairly consistent overvaluation and this was particularly noticeable during periods of lower volatility. Other contributing factors, albeit less so, included determining the proper mean for the volatility regime we were in and the decay rate, theta. We viewed these results over several different scenarios, including modeling a rolling window of data to capture parameter estimates that were reflective of that time period while also observing results over a steady state with consistent parameters that reflected a long term average.

Further study and next steps in our process would involve adapting the density function to the OU process away from a Geometric Brownian Motion to see if including market expectations in our simulations are an improvement over using a standard random normal variable in the process.

6. Appendix A

```
% Calculate the Roll Yield

% Import the file
[data3, date3]=xlsread('C:\Temp\MasterDoc Futures Prices.xlsx', 'Summary');

%Extract the data
x = datenum(date3(2:end,1));
y = data3(:,1);

%Calculate the difference in rolling the contract fwd 1 month
[M,N] = size(data3);

for i = 1:N-1
    Roll(:,i) = data3(:,i)-data3(:,i+1);
end

%Map in the locations of the roll yields
map=zeros(M,1);

for ii = 1:M
    b = find(~isnan(Roll(ii,:)));
    if b>0
        map(ii,1) = b(1,1);
    else
        map(ii,1) = 1;
    end
end

%Calculate the Roll Yield
RollYield = [];
for iii = 1:length(map)
    RollYield(iii,1) = Roll(iii,map(iii,1))*(1/(250/12)); %assuming there are
250 trading days in a year
end
```

7. Appendix B

```
%% Parameter estimation

function [theta, mu, sigma] = ParameterEstimation(Vix, deltaT)

% Create a vector of Stock price differences
Y = Vix(1:end-1);

X = ones((length(Vix)-1),2);
X(:,2)= Vix(2:end);

%Run a regression
[Coef, CoefInt, residual] = regress(Y,X);

%Extract the coefficients
Intercept=Coef(1);
Slope=Coef(2);

%Calculate the Parameters for the O-U process
theta = -log(Slope)/deltaT;
mu = (Intercept/(1-Slope));
sigma = std(residual) * sqrt(2*theta/(1-Slope^2));

end
```

8. Appendix C

```
%% Inputs
% W = Window of Vix historical data to use in the regression
% T = number of time steps
% N = # days forecasted fwd
% K = Strike price
% r = discount rate (Libor or the 90 day rate)
% M = # of Monte Carlo simulations

%% Ornstein Uhlenbeck
function [S, C, P] = OrnsteinUhlenbeckOption(VixPrices, T, N, W, K, r, M)

%Calculate deltaT
deltaT=(T/365);

%Create a Rolling window of prices to calculate the parameter estimates n day
forward estimate
if length(VixPrices)< W
    error disp('Not enough data');
else
    for i = 1:(length(VixPrices)-W+1)
        Vix = VixPrices(length(VixPrices)-W+(2-i):length(VixPrices)-i+1);

%% Estimate the parameters of our process
[theta, mu, sigma] = ParameterEstimation(Vix, deltaT);

%Pre-allocate
s=zeros(N,M);

%Form a vector of standard normal variables
z=randn(N,M);

%Generate the intial value
s(1,:) = Vix(1,1);

%Generate subsequent value of your OU process
for ii=1:M %path number
    for j=1:N %draw number
        s(j+1,ii) = s(j,ii)*exp(-theta*deltaT)+mu*(1-exp(-
theta*deltaT))+sigma*sqrt((1-exp(-2*theta*deltaT))/2*theta)*z(j,ii);
    end
end

%Average the stock price over all paths
x(:,i) = mean(s,2);
```

```
%Calculate the value of the option

    for k=1:length(K)

        Call = exp(-r*deltaT)*max(s(end,:)-K(k),0);
        CHat(k,1)=(1/(length(Call)))*sum(Call);

        Put = exp(-r*deltaT)*max(K(k)-s(end,:),0);
        PHat(k,1)=(1/(length(Put)))*sum(Put);
    end

end

S=x;
C=CHat;
P=PHat;
```

9. Appendix D

```
%% Inputs
% W = Window of Vix historical data to use in the regression
% T = number of days
% N = # time steps
% K = Strike price
% r = discount rate (Libor or the 90 day rate)
% M = # of Monte Carlo simulations

%% Ornstein Uhlenbeck
function [S] = OrnsteinUhlenbeckFutures(VixPrices, T, N, W, r, M, avg, sigma)

%Calculate deltaT
deltaT=(T/365);

%Create a Rolling window of prices to calculate the parameter estimates n day
forward estimate
if length(VixPrices)< W
    error disp('Not enough data');
else
    for i = 1:(length(VixPrices)-W+1)
        Vix = VixPrices(length(VixPrices)-W+(2-i):length(VixPrices)-i+1);

%% Estimate the parameters of our process
[theta, mu, sigma] = ParameterEstimation(Vix, deltaT);

%% Simulate the OU process

%Pre-allocate
s=zeros(N,M);

%Form a vector of standard normal variables
[Z] = RandLogNormal(avg, sigma, N, M);
z=Z;

%Generate the intial value
s(1,:) = Vix(1,1);

%Generate subsequent value of your OU process
for ii=1:M %path number
    for j=1:N %draw number
        s(j+1,ii) = s(j,ii)*exp(-theta*deltaT)+mu*(1-exp(-
theta*deltaT))+sigma*sqrt((1-exp(-2*theta*deltaT))/2*theta)*z(j,ii);
    end
end

%Average the stock price over all paths
x(:,i) = mean(s,2);

    end
end

S=x;
```

10. Appendix E

```
% Generate random lognormals
function [x] = RandLogNormal(mean, sigma, N, M)

    s= sqrt(log(sigma^2/mean^2+1));
    mu = log(mean)-s^2/2;

    x = exp(mu+s*randn(N,M));

end
```

11. Appendix F

```
%% Run a quadratic least squares optimization

[x] = lsqnonlin(@QuadLstSqOpt, x0,[1e-10 1e-10 1e-10 1e-10 1e-10],[[] [] []
[] 1],options, Spot, Strike, Call, Put, deltaT, r);

%% Create a function to call the Risk Neutral Density parameters

function [X ,Y, Z] = QuadLstSqOpt(x0, Spot, Strike, Call, Put, deltaT, r)
%All parameters are scalars except x0 which is a matrix with parameters
(mean1, mean2, sigma1, sigma2, wgt)

[X] = CallDensity(x0, Spot, Strike, Call, deltaT, r);
[Y] = PutDensity(x0, Spot, Strike, Put, deltaT, r);
[Z] = SpotDensity(x0, Spot, Strike, deltaT, r);

%% Calculate the Spot density

function [x] = SpotDensity(x0, Spot, Strike, deltaT, r)
%All parameters are scalars except x0 which is a matrix with parameters
(mean1, mean2, sigma1, sigma2, wgt)

%Move the variables into model parameter names
mu1 = x0(1,1);
mu2 = x0(1,2);
sigma1 = x0(1,3);
sigma2 = x0(1,4);
wgt = x0(1,5);

%Calculate Alpha(i) and Beta(i)
alpha(:,1) = log(Spot)+ (mu1-.5*(sigma1.^2))*deltaT;
alpha(:,2) = log(Spot)+ (mu2-.5*(sigma2.^2))*deltaT;

Beta(:,1) = sigma1*sqrt(deltaT);
Beta(:,2) = sigma2*sqrt(deltaT);

%Calculate d1, d2, d3, d4
d1 = (-log(Strike)+alpha(:,1) + Beta(:,1).^2)./Beta(:,1);
d2 = d1 - Beta(:,1);
d3 = (-log(Strike) + alpha(:,2) + Beta(:,2).^2)./Beta(:,2);
d4 = d3 - Beta(:,2);

%Calculate the theoretical Put price
S = wgt.*exp(alpha(:,1)+Beta(:,1).^2./2)+(1-
wgt).*exp(alpha(:,2)+Beta(:,2).^2./2)-exp(-r*deltaT).*Spot;

%Calculate the difference between the observed market price and the
theoretical price
x=zeros(length(S),1);
x(1,1) = S(1,1);
```

```

%% Calculate the Put density

function [x] = PutDensity(x0, Spot, Strike, Put, deltaT, r)
%All parameters are scalars except x0 which is a matrix with parameters
(mean1, mean2, sigma1, sigma2, wgt)
%Put is a vector of observed market prices on put options

%Move the variables into model parameter names
mu1 = x0(1,1);
mu2 = x0(1,2);
sigma1 = x0(1,3);
sigma2 = x0(1,4);
wgt = x0(1,5);

%Calculate Alpha(i) and Beta(i)
alpha(:,1) = log(Spot)+ (mu1-.5*(sigma1.^2))*deltaT;
alpha(:,2) = log(Spot)+ (mu2-.5*(sigma2.^2))*deltaT;

Beta(:,1) = sigma1*sqrt(deltaT);
Beta(:,2) = sigma2*sqrt(deltaT);

%Calculate d1, d2, d3, d4
d1 = (-log(Strike)+alpha(:,1) + Beta(:,1).^2)./Beta(:,1);
d2 = d1 - Beta(:,1);
d3 = (-log(Strike) + alpha(:,2) + Beta(:,2).^2)./Beta(:,2);
d4 = d3 - Beta(:,2);

%Calculate the theoretical Put price
P = exp(-r*deltaT).*(wgt.*(Strike.*cdf('norm',-d2,0,1)-
exp(alpha(:,1)+Beta(:,1).^2./2).*cdf('norm',-d1,0,1)))+(1-
wgt).*(Strike.*cdf('norm',-d4,0,1)-
exp(alpha(:,2)+Beta(:,2).^2./2).*cdf('norm',-d3,0,1)));

%Calculate the difference between the observed market price and the
theoretical price
x = Put - P;

%% Calculate the Call density

function [x] = CallDensity(x0, Spot, Strike, Call, deltaT, r)
%All parameters are scalars except x0 which is a matrix with parameters
(mean1, mean2, sigma1, sigma2, wgt)
%Call is a vector of observed market prices on call options

%Move the variables into model parameter names
mu1 = x0(1,1);
mu2 = x0(1,2);
sigma1 = x0(1,3);
sigma2 = x0(1,4);
wgt = x0(1,5);

%Calculate Alpha(i) and Beta(i)
alpha(:,1) = log(Spot)+ (mu1-.5*(sigma1.^2))*deltaT;

```



```

alpha(:,2) = log(Spot)+ (mu2-.5*(sigma2.^2))*deltaT;

Beta(:,1) = sigma1*sqrt(deltaT);
Beta(:,2) = sigma2*sqrt(deltaT);

%Calculate d1, d2, d3, d4
d1 = (-log(Strike)+alpha(:,1) + Beta(:,1).^2)./Beta(:,1);
d2 = d1 - Beta(:,1);
d3 = (-log(Strike) + alpha(:,2) + Beta(:,2).^2)./Beta(:,2);
d4 = d3 - Beta(:,2);

%Calculate the theoretical Call price
C = exp(-
r*deltaT).*(wgt.*(exp(alpha(:,1)+Beta(:,1).^2./2).*cdf('norm',d1,0,1)-
Strike.*cdf('norm',d2,0,1)))+(1-wgt).*(
exp(alpha(:,2)+Beta(:,2).^2./2).*cdf('norm',d3,0,1)-
Strike.*cdf('norm',d4,0,1)));

%Calculate the difference between the observed market price and the
theoretical price
x = Call - C;

```


13. Table 2 (Forecasted option prices)

Forecasted Prices		Call												Put											
Strike	Nov/11	Oct/11	Sept/11	Aug/11	July/11	Jun/11	May/11	Apr/11	Mar/11	Feb/11	Jan/11	Dec/10	Nov/11	Oct/11	Sept/11	Aug/11	July/11	Jun/11	May/11	Apr/11	Mar/11	Feb/11	Jan/11	Dec/10	
10	21.74	25.27	26.35	19.42	8.72	8.46	6.60	10.00	10.36	7.18	6.98	6.98	0.08	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00	0.00
15	16.83	20.27	21.35	14.45	3.74	3.49	1.76	6.31	5.43	2.36	2.27	2.27	0.12	0.00	0.00	0.08	0.02	0.03	0.16	2.31	0.06	0.19	0.29	0.29	0.29
16	15.87	19.27	20.35	13.47	2.79	2.56	1.08	5.67	4.49	1.59	1.54	1.54	0.16	0.00	0.00	0.05	0.07	0.10	0.43	2.67	0.13	0.41	0.57	0.57	0.57
17	14.91	18.27	19.35	12.49	1.92	1.72	0.50	5.07	3.61	0.97	0.97	0.97	0.20	0.00	0.00	0.07	0.19	0.27	0.90	3.07	0.24	0.79	0.99	0.99	0.99
18	13.97	17.27	18.35	11.52	1.18	1.04	0.20	4.51	2.79	0.53	0.55	0.55	0.26	0.00	0.00	0.10	0.45	0.58	1.60	3.51	0.43	1.35	1.58	1.58	1.58
19	13.04	16.27	17.35	10.57	0.68	0.55	0.06	3.99	2.07	0.25	0.28	0.28	0.33	0.00	0.00	0.15	0.91	1.09	2.46	3.99	0.71	2.08	2.31	2.31	2.31
20	12.13	15.27	16.35	9.63	0.29	0.25	0.02	3.52	1.47	0.10	0.13	0.13	0.42	0.00	0.00	0.21	1.56	1.79	3.42	4.52	1.10	2.93	3.15	3.15	3.15
21	11.24	14.27	15.35	8.72	0.11	0.09	0.00	3.08	0.98	0.04	0.05	0.05	0.53	0.00	0.00	0.30	2.38	2.64	4.40	5.08	1.62	3.86	4.08	4.08	4.08
22.5	9.95	12.77	13.85	7.40	0.02	0.01	0.00	2.49	0.48	0.01	0.01	0.01	0.74	0.00	0.00	0.48	3.79	4.06	5.90	5.99	2.62	5.33	5.54	5.54	5.54
24	8.71	11.28	12.36	6.16	0.00	0.00	0.00	1.99	0.21	0.00	0.00	0.00	1.01	0.00	0.00	0.73	5.28	5.54	7.40	6.99	3.84	6.82	7.08	7.08	7.08
25	7.98	10.28	11.37	5.38	0.00	0.00	0.00	1.69	0.11	0.00	0.00	0.00	1.22	0.00	0.02	0.96	6.28	6.54	8.40	7.69	4.74	7.82	8.08	8.08	8.08
26	7.18	9.28	10.39	4.65	0.00	0.00	0.00	1.44	0.05	0.00	0.00	0.00	1.47	0.01	0.04	1.23	7.28	7.54	9.40	8.44	5.69	8.82	9.08	9.08	9.08
27.5	6.12	7.79	8.93	3.66	0.00	0.00	0.00	1.11	0.01	0.00	0.00	0.00	1.91	0.02	0.08	1.74	8.78	9.04	10.90	9.61	7.15	10.32	10.53	10.53	10.53
29	5.15	6.33	7.51	2.81	0.00	0.00	0.00	0.84	0.00	0.00	0.00	0.00	2.44	0.06	0.16	2.39	10.28	10.54	12.40	10.84	8.64	11.82	12.08	12.08	12.08
30	4.55	5.39	6.59	2.31	0.00	0.00	0.00	0.69	0.00	0.00	0.00	0.00	2.84	0.11	0.24	2.89	11.28	11.54	13.40	11.69	9.64	12.82	13.08	13.08	13.08
32.5	3.25	3.23	4.48	1.34	0.00	0.00	0.00	0.41	0.00	0.00	0.00	0.00	4.04	0.46	0.68	4.42	13.77	14.04	15.90	13.91	12.14	15.32	15.53	15.53	15.53
35	2.22	1.98	2.74	0.70	0.00	0.00	0.00	0.23	0.00	0.00	0.00	0.00	5.51	1.30	1.38	6.28	16.27	16.54	18.40	16.23	14.64	17.82	18.02	18.02	18.02
37.5	1.45	0.99	1.47	0.33	0.00	0.00	0.00	0.13	0.00	0.00	0.00	0.00	7.24	2.82	2.62	8.41	18.77	19.04	20.90	18.63	17.14	20.32	20.52	20.52	20.52
40	0.89	0.16	0.68	0.14	0.00	0.00	0.00	0.07	0.00	0.00	0.00	0.00	9.18	4.89	4.32	10.72	21.27	21.54	23.40	21.06	19.64	22.82	23.02	23.02	23.02
42.5	0.52	0.08	0.26	0.05	0.00	0.00	0.00	0.08	0.00	0.00	0.00	0.00	11.31	7.25	6.41	13.13	23.77	24.04	25.90	23.53	22.14	25.32	25.52	25.52	25.52
45	0.29	0.00	0.08	0.02	0.00	0.00	0.00	0.02	0.00	0.00	0.00	0.00	13.58	9.73	8.73	15.60	26.27	26.54	28.40	26.01	24.64	27.82	28.02	28.02	28.02
47.5	0.15	0.00	0.02	0.01	0.00	0.00	0.00	0.01	0.00	0.00	0.00	0.00	15.94	12.22	11.17	18.08	28.77	29.04	30.90	28.50	27.14	30.32	30.52	30.52	30.52
50	0.07	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	18.36	14.72	13.65	20.58	31.27	31.54	33.40	31.00	29.64	32.82	33.02	33.02	33.02
55	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	23.30	19.72	18.65	25.58	36.27	36.54	38.40	36.00	34.64	37.82	38.02	38.02	38.02
60	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	28.29	24.72	23.65	30.58	41.27	41.54	43.40	41.00	39.63	42.82	43.02	43.02	43.02
65	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	33.29	29.72	28.65	35.58	46.27	46.54	48.40	46.00	44.63	47.82	48.02	48.02	48.02
70	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	38.29	34.72	33.65	40.58	51.27	51.54	53.40	51.00	49.63	52.82	53.02	53.02	53.02
75	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	43.29	39.72	38.65	45.58	56.27	56.54	58.40	56.00	54.63	57.82	58.02	58.02	58.02
80	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	48.29	44.72	43.65	50.58	61.27	61.54	63.40	61.00	59.63	62.82	63.02	63.02	63.02
85	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	53.29	49.72	48.65	55.58	66.27	66.54	68.40	66.00	64.63	67.82	68.02	68.02	68.02
90	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	58.29	54.72	53.65	60.58	71.27	71.54	73.40	71.00	69.63	72.82	73.02	73.02	73.02
95	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	63.29	59.72	58.65	65.57	76.27	76.54	78.40	75.99	74.63	77.82	78.02	78.02	78.02
100	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	68.29	64.72	63.65	70.57	81.27	81.54	83.40	80.99	79.63	82.82	83.02	83.02	83.02

14. Table3 (Forecasted VIX)

	Spot	1 day Fwd	2 day fwd	3 day fwd	4 day Fwd	5 day Fwd	6 day Fwd	7 day Fwd	8 day Fwd	9 day Fwd	48 day Fwd	49 day Fwd	50 day Fwd
11/4/2011	30.16	30.26	32.92	31.92	29.61	27.28	29.34	30.34	27.09	25.12	32.48	31.18	30.30
11/3/2011	30.50	28.07	27.11	26.98	24.20	23.43	25.26	26.89	30.06	30.24	38.93	38.41	37.55
11/2/2011	32.74	33.73	34.43	35.86	35.21	36.73	34.66	33.68	34.60	35.95	40.81	39.37	38.36
11/1/2011	34.77	34.89	35.23	36.39	36.31	33.19	35.88	33.74	31.90	31.85	40.41	36.83	33.06
10/31/2011	29.96	27.88	31.18	36.66	37.62	36.31	36.22	32.71	32.62	32.88	34.69	31.26	34.52
10/28/2011	24.53	28.66	32.78	28.78	32.76	33.27	30.35	28.58	33.00	35.72	30.59	31.20	34.14
10/27/2011	25.46	26.03	29.20	27.78	28.65	28.52	24.25	28.25	30.33	35.30	39.21	37.49	34.89
10/26/2011	29.86	31.82	32.25	33.93	35.27	35.58	38.25	40.69	38.88	38.75	35.09	36.79	36.11
10/25/2011	32.22	34.74	39.84	36.33	33.37	30.19	32.46	32.94	34.40	33.85	38.74	41.06	37.99
10/24/2011	29.26	28.91	26.88	32.56	34.32	35.44	39.85	39.47	39.61	40.01	39.37	38.58	35.30
10/21/2011	31.32	34.85	35.04	37.82	35.81	37.08	36.36	35.32	34.79	34.81	33.68	33.31	33.03
10/20/2011	34.78	29.65	32.32	35.68	36.64	37.10	38.03	36.92	37.08	35.64	39.20	39.93	38.48
...
8/2/2005	11.75	10.25	11.04	10.49	11.47	11.47	11.70	11.62	10.97	11.81	12.20	12.55	11.52
8/1/2005	12.08	10.90	12.03	12.17	12.91	12.63	12.30	11.58	10.38	10.72	12.06	11.44	12.11
7/29/2005	11.57	10.95	9.79	10.65	10.56	10.13	11.12	11.24	11.46	12.45	10.89	10.25	10.31
7/28/2005	10.52	10.81	10.97	11.42	11.48	11.88	10.84	10.54	10.33	11.35	11.56	12.40	12.55
7/27/2005	10.36	10.62	10.08	10.72	11.12	11.68	11.20	10.98	10.80	11.77	11.73	11.01	10.62
7/26/2005	10.99	11.17	11.10	11.36	11.57	10.75	11.10	11.16	11.25	11.61	10.71	12.06	11.43
7/25/2005	11.10	11.48	11.26	11.84	11.04	11.56	11.22	11.49	11.44	11.06	11.38	11.73	11.69
7/22/2005	10.52	10.58	10.38	10.68	11.00	11.11	11.28	11.49	10.76	11.90	11.41	11.90	12.23
7/21/2005	10.97	11.20	11.41	11.00	11.69	10.89	10.78	10.48	12.01	11.84	10.79	11.32	10.95
7/20/2005	10.23	11.28	10.34	9.64	10.54	11.08	11.51	11.88	12.33	11.77	12.40	12.44	12.61
7/19/2005	10.45	10.34	10.52	10.83	10.64	10.40	10.51	9.80	10.84	11.32	11.19	10.95	10.71
7/18/2005	10.77	11.14	10.71	10.89	9.98	10.51	11.42	11.67	11.16	10.85	12.06	11.72	10.70
7/15/2005	10.33	10.34	11.12	12.59	11.14	11.55	10.54	11.03	10.93	10.97	12.02	11.12	11.55
7/14/2005	10.81	11.29	11.99	10.64	10.52	10.69	10.36	9.78	9.56	10.67	12.74	12.64	12.11
7/13/2005	10.84	9.54	10.20	11.55	10.95	11.16	11.48	12.93	11.43	12.17	10.59	10.18	12.51
7/12/2005	10.95	11.13	11.75	11.04	11.07	12.03	11.13	11.92	12.29	11.72	10.54	11.05	10.62
7/11/2005	11.28	12.76	12.64	11.98	11.38	10.71	11.26	12.49	11.70	11.88	10.00	13.88	12.85
7/8/2005	11.45	11.95	13.57	12.24	10.95	13.37	12.59	10.42	11.88	10.37	14.00	11.93	9.60
7/7/2005	12.49	12.41	12.12	12.33	11.09	10.93	12.86	11.80	10.68	10.65	10.34	12.98	12.31
7/6/2005	12.27	11.92	11.84	12.03	11.66	11.28	11.56	11.37	10.77	11.17	11.29	11.77	12.28
7/5/2005	11.68	11.68	12.57	14.41	13.79	11.65	11.48	11.46	11.91	12.41	11.14	12.07	11.87

15. References:

- [1] Chicago Board Options Exchange, 2009 “The CBOE Volatility Index – VIX.”
<<http://www.cboe.com/micro/vix/vixwhite.pdf>>
- [2] Chicago Board Options Exchange, 2009 “Additional Features of VIX Futures,” VIX Primer
<<http://cfe.cboe.com/education/vixprimer/Features.aspx>>
- [3] Chicago Board Options Exchange, 2009 “All About VIX,” VIX Primer
<<http://cfe.cboe.com/education/vixprimer/About.aspx>>
- [4] S. Finch, May 2004, “Ornstein-Uhlenbeck Process”
- [5] P Glasserman, 2004. Monte Carlo Methods in Financial Engineering. Springer, New York, NY.
- [6] T. Berg, May 2011 “Calibrating the Ornstein-Uhlenbeck (Vasicek) model”
- [7] T. Mikosch, 2008. Elementary Stochastic Calculus. World Scientific Publishing. Hackensack, NJ.
- [8] Fusai & Roncoroni, 2008. “Estimating the Risk-Neutral Density,” Implementing Models in Quantitative Finance. Springer, New York.
- [9] G. Parker. January 2009. “Directional Exposure to Volatility via Listed Futures.” Standard & Poor’s.
<http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1359135>
- [10] T. Bjork, 2004. Arbitrage Theory in Continuous Time. Oxford University Press Inc., New York, NY.
- [11] “The Basics of VIX Futures ETNs.” IPath Exchange-Traded Notes (ETN’s). Barclays Bank PLC.
<<http://www.ipathetn.com/static/pdf/VIX-basics.pdf>>
- [12] Bloomberg L.P. (2011) Market data for VIX, VXX .
- [13] Chicago Board Options Exchange, 2011. Market data for VXX futures and options.