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Efficient Pricing of an Asian Put Option Using Stiff ODE Methods

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Efficient Pricing of an Asian Put Option Using Stiff ODE Methods

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1 Abstract

Financial mathematics is a branch of mathematics that assesses the risk and value of various financial instruments. Banks, companies, and other institutions mitigate their risk through financial instruments known as derivatives, that derive their value from some underlying asset. The equations that arise from pricing and modeling can be very complex, leading to the necessity of numerical methods.

This project studied the use of certain numerical methods for the pricing of a particular type of option called an Asian option. Asian options can provide favorable risk profiles because the payout is determined based on the average value over a time period, rather than the final value. The price of an Asian option is governed by a partial differential equation in three variables: stock price, average price over the current time interval, and time.

The solution method was first to discretize the partial differential equation into a system of ordinary differential equations. Next, the ODE system was integrated using a stiff-ODE solver available in MATLAB. Enhancements to this solution method include specifying the sparsity pattern, implementing an iterative linear solver (GMRES [2]) in place of MATLAB's built-in direct linear solver, and using preconditioning to improve the solution characteristics of that solver.

2 Introduction

Asian options are a particular class of options that calculate payout on a time-interval average as opposed to the standard final value of an underlying instrument. As a result, Asian options have less volatility than their "vanilla" European counterparts and thus, a cheaper price.

Asian options can be classified by their method of averaging, such as arithmetic or geometric. In this project, the particular averaging equation can be found as Equation 1, where A is the average, T is the time until expiration, S is the price of the underlying instrument, and t is the time until maturity.

$$A = \frac{1}{T-t}\int_0^{T-t} S(\tau)\, d\tau \tag{1}$$

Financial mathematicians have developed theory to price Asian options, resulting in Equation 2, where P is the price, σ is the volatility, S is the underlying price, r is the risk-free rate, A is the time-interval average, T is the time until expiration, and t is the time until maturity [1].

$$\frac{\partial P}{\partial t} = \frac{\sigma^2 S^2}{2}\frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} + \frac{S-A}{T-t}\frac{\partial P}{\partial A} - rP \tag{2}$$

The equation is structured so that the starting time is at expiration and the equation is solved backwards in time until $t = T$ (today). The price of the option can then be found by looking at the diagonal $S = A$ and obtaining the current underlying price S from the marketplace. Ranges of S and A depend on the price properties of the underlying instrument. In this project, we set the maximum values of S and A to two-hundred.

Asian options come in two forms, puts and calls. Both forms have a payout based on the difference between the time-interval average price and the strike price. The strike price (K) is specified in the contract, and can be any value that the buyer and seller agree upon. The payout functions for puts and calls can be found in Equations 3 and 4. Puts are favorable to the option buyer if prices go down; calls are favorable to the option buyer if prices go up. This project addressed the problem of pricing an Asian put. Since the problem is solved backward in time, the initial condition for the integration was the put payout function, found in Equation 3.

$$P(S, A, 0) = max(K - A, 0) \tag{3}$$

$$P(S, A, 0) = max(A - K, 0) \tag{4}$$

3 Solution Methodology

3.1 Method of Lines

The method of lines is a technique for solving partial differential equations where all but one dimension is discretized, leading to a set of ordinary differential equations in the non-discretized dimension [3]. After the semi-discretization, methods and software that have been developed for ordinary differential equations can be used to integrate the system, yielding a solution of the partial differential equation.

This method is only suitable for certain classes of partial differential equations, namely initial value problems. The pricing of an Asian option meets this criteria because of its structure in time. An example of an unsuitable partial differential equation would be the standard Laplace equation which does not have any initial-value type conditions.

3.2 Stiffness

Since we are dealing with numerical methods, stability problems may arise. For the Asian option, the ordinary differential equations arising from the discretization are stiff. Stiffness is a property of an equation system that leads to instability when using explicit numerical methods, unless the time-steps are kept sufficiently small. As an example, consider the one-dimensional differential equation in Equation 5 with initial condition Equation 6.

$$\frac{du}{dt} = -20u \tag{5}$$

$$u(0) = 1 \tag{6}$$

Suppose we apply an Explicit Euler Integration, as found in Equation 7. The results for a number of h values (time step) are shown in Figure 1. A very small time step is required for a reasonable solution. However, if we use an Implicit Euler Integration, as found in Equation 8, the solution is reasonable even for the relatively large value of $h = .1$. This result can be found in Figure 2. A trade-off arises for stiff numerical differential equations. Explicit methods allow for direct calculation and computational simplicity. Implicit methods allow for improved stability with more complex computa-

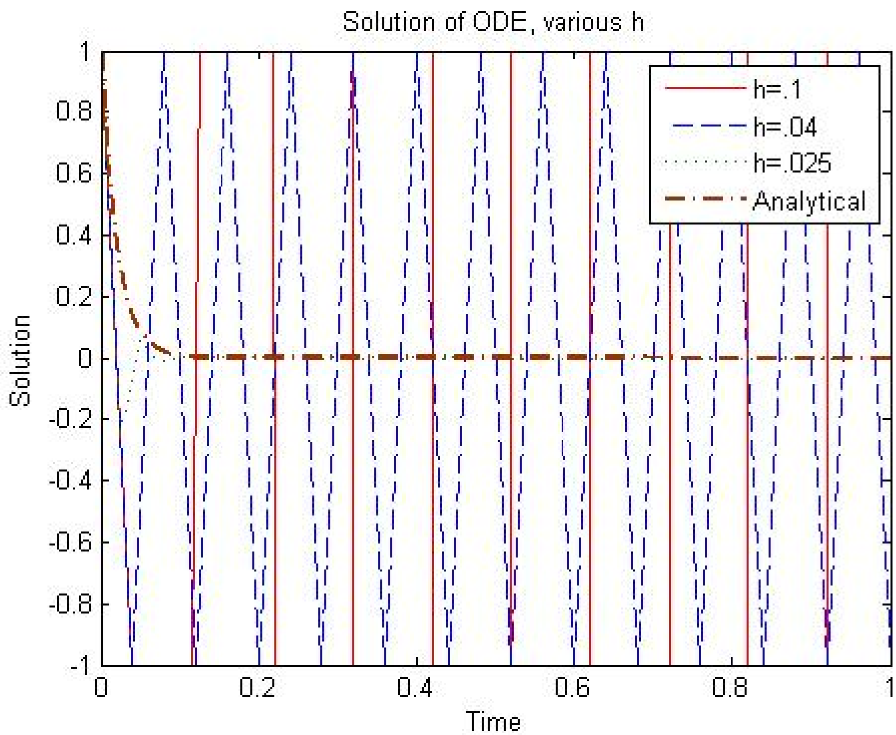

Figure 1: Explicit Euler Method

tions (solutions of linear or non-linear systems). In stiff problems, implicit methods are usually the preferred choice.

$$u_{n+1} = u_n + h(-20u_n) \tag{7}$$

$$u_{n+1} = u_n + h(-20u_{n+1}) \tag{8}$$

3.3 Linear Systems

Since we are using an implicit method in our solution, it is necessary to solve linear systems throughout the integration. There are many algorithms available. They are divided into two main types: direct and iterative solvers.

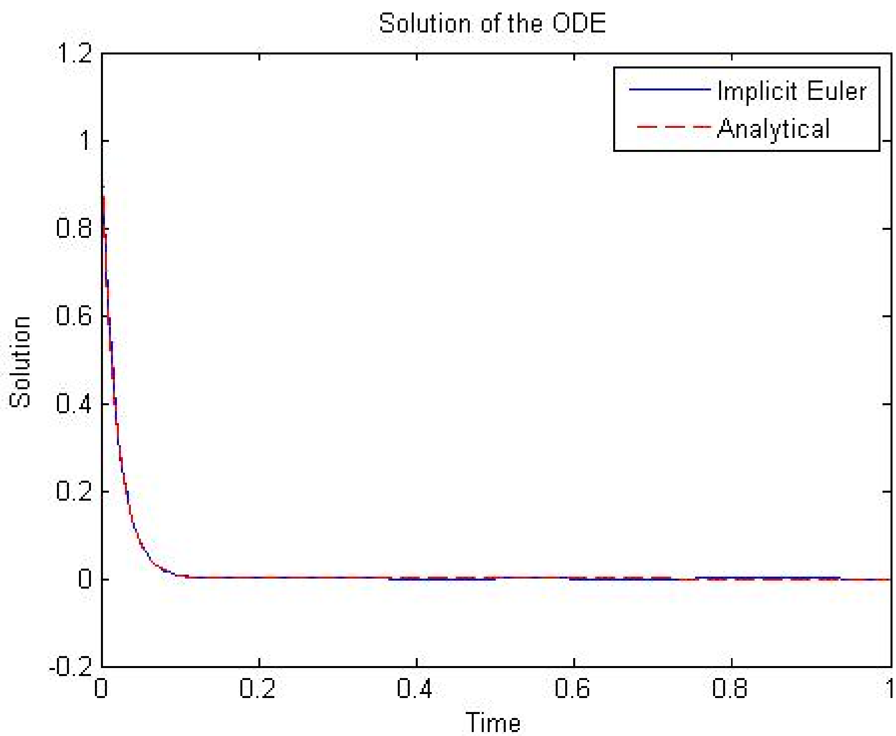

Figure 2: Implicit Euler Method, h=.1

Direct solvers attempt to solve the equation $Ax = b$ using an exact algorithm. Examples include Gaussian elimination and LU decomposition. There are many specialized algorithms within the set of direct solvers. For example, Cholesky decomposition is used for symmetric, positive-definite systems. For the Asian put pricing problem, many of the matrix elements are zero, meaning the systems that arise are extremely sparse. MATLAB's direct solvers can be chosen to exploit this sparsity.

Iterative solvers, as opposed to direct solvers, try to find successive approximations to the solution, starting from an initial guess. The stopping criteria can be a fixed number of iterations, a specified residual-norm reduction tolerance, or a combination of both. Examples include the Gauss-Seidel method and GMRES. Iterative solvers are often effective on large grid sizes where direct solvers can become inefficient or even impossible to use.

In this project, we implemented both direct and iterative solvers in order to find the best choice of method on a variety of grid sizes. The specific methods used are discussed in Chapter 4.

3.4 GMRES and Preconditioning

Other than the built-in MATLAB solvers, the linear systems solution method used was GMRES [2]. GMRES is an iterative linear systems method that is particularly suited for large, sparse systems. The method approximates the solution by the vector in a Krylov subspace with minimal residual.

In particular, if we define the nth Krylov subspace as K_n in Equation 9 and suppose we are trying to solve the matrix equation $Ax = b$, GMRES calculates $x_n = x_o + z_n, z_n \in K_n$ such that $\| Ax_n - b \|$ is minimized.

$$K_n = span(b, Ab, A^2b, ..., A^{n-1}b) \tag{9}$$

The performance of GMRES can be further enhanced through two methods. First, since GMRES is a Krylov method, matrix evaluations are not required; instead, a "matrix-vector product" routine can be supplied. Throughout the report, this method is known as the "matrix-free" method. Next, preconditioning can be implemented.

Preconditioning is the process of modifying a linear system in a specific way to improve its solvability. Suppose we are dealing with the generic linear equation $Ax = b$. The condition number, defined in Equation 10, is a crucial factor in the performance of linear solution methods. We consider left-preconditioning here, in which the linear system is transformed with a left preconditioning matrix P as found in Equation 11. Improving the condition of the system is often stated as a nominal goal of preconditioning, since ill-conditioning of A usually implies poor performance of Krylov subspace methods. A good choice of P will improve the condition number of the modified system and speed up the convergence of the method sufficiently to outweigh the cost of implementing it.

$$\kappa(A) = ||A|| \cdot ||A^{-1}|| \tag{10}$$

$$P^{-1}Ax = P^{-1}b \tag{11}$$

4 Implementation Framework

4.1 Discretization

The partial differential equation was discretized on a square grid of S and A, with both variables ranging from zero to two hundred. There were $N+1$ nodes in each direction so that $h_A = h_S = \frac{1}{N}$. Using uniform spacing, $S_i = ih_S$ and $A_j = jh_A$. The derivatives were approximated using a finite difference scheme. For the S derivatives, a second order centered-difference scheme could be used. However, for the A derivatives, it was necessary to use upwinding since there is a lack of diffusion (second order derivative). In summary, we use the discretization found in [1] as shown below in Equations 12-16.

$$\frac{\partial P}{\partial t} = \frac{\sigma^2 S_i^2}{2} D_{S,S} + rS_i D_S + \frac{S_i - A_j}{T-t} D_A - rP \tag{12}$$

$$D_{S,S} = \frac{P(S_{i+1}, A_j, t) - 2P(S_i, A_j, t) + P(S_{i-1}, A_j, t)}{h_S^2} \tag{13}$$

$$D_S = \frac{P(S_{i+1}, A_j, t) - P(S_{i-1}, A_j, t)}{2h_S} \tag{14}$$

$$D_A = \frac{P(S_i, A_{j+1}, t) - P(S_i, A_j, t)}{h_A}, S_i > A_j \tag{15}$$

$$D_A = \frac{P(S_i, A_j, t) - P(S_i, A_{j-1}, t)}{h_A}, S_i \leq A_j \tag{16}$$

Equation 12 yields an ordinary differential equation in time for each node on the S-A grid. As a result, the partial differential equation has been discretized into a system of ordinary differential equations. A numerical solution can be obtained by using an ordinary differential equation solver such as those found in MATLAB.

4.2 Boundary Conditions

We need to consider four boundary conditions for the rectangular domain, $S = 0$, $S = S_c$, $A = 0$, and $A = A_c$. Along $S = 0$, the S derivatives vanish as they are multiplied by S in the equations. For $A = 0$, there is only dependence on interior nodes since we are using upwinding. This is similar to the treatment of an outflow boundary in fluid mechanics.

For $S = S_c$, we impose the boundary condition $\frac{\partial P}{\partial S} = 0$ because the price of the option depends only on A for large values of S. For the boundary $A = A_c$, $lim_{A \to +\infty} P(t, S, A) = 0$ so we can set $P(t, S, A_c) = 0$ where $S > A_c$. For $S < A_c$, no condition has to be imposed due to upwinding and an outflow-like boundary.

4.3 MATLAB Implementation

The equation was solved using the MATLAB ODE Suite [4]. The suite has seven solvers that are used for various types of initial value problems: "ode23","ode45","ode113","ode15s","ode23s","ode23t", and "ode23tb". The solvers are distinguished from each other by their ability to solve stiff systems as well as their available order.

"ode23","ode45", and "ode113" are solvers designed for non-stiff problems. The others are used for stiff problems. The method we chose to use for this equation was "ode15s" due to its stiffness,. "ode15s" is the standard stiff solver, whereas "ode23s" and "ode23tb" are for crude error tolerances and "ode23t" is for a solution without numerical damping.

The MATLAB solvers themselves are relatively easy to implement. The minimum input includes a function handle for the time derivative, a time range, and an initial value. The solvers can accommodate systems of differential equations with ease: Rather than scalar arguments, vector arguments are used. For the implicit time-stepping solvers, performance can be enhanced by specifying a function that evaluates the Jacobian matrix, and the Jacobian's sparsity pattern.

In addition to using the standard "ode15s", several modified versions were created that implemented various versions of the GMRES algorithm for the solution of linear systems. Iterative methods will often perform better than standard direct algorithms on larger grid sizes. The standard MATLAB implementation of GMRES was used which unfortunately, is implemented as interpreted source code. This can cause the solution time to increase drastically relative to the compiled code used by MATLAB's direct solvers, as discussed in the results section.

5 Solution Methods

5.1 ode15s

In the "ode15s" case, we only provide the MATLAB stiff system solver the right-hand side of Equation 2, in addition to the time interval and initial data. The solver then proceeds to integrate the time derivatives and obtain the solution using full-matrix storage. The benefit of using this solver is that it is the most basic. However, that simplicity comes with a price because the method is very inefficient and slow. The main reason for the inefficiency is that a standard linear solution method is used that does not take advantage of the Jacobian sparsity.

5.2 ode15s with sparsity

For the "ode15s with sparsity" case, we provided MATLAB's ode15s solver with both the time derivative function and a matrix of zeros and ones indicating the sparsity pattern of the Jacobian. This allowed MATLAB's linear systems algorithms to exploit sparsity, enhancing speed and efficiency over ode15s on moderate and large grids. The sparsity pattern can be found in Figure 3. A close-up detailing the band structure can be found in Figure 4. The varying in lengths of the outer bands is a result of the upwinding scheme.

5.3 ode15s with GMRES

In the "ode15s with GMRES" case, we modified the ode15s solver code to use MATLAB's GMRES solver rather than its standard linear solver. The advantage of this is that GMRES is a solver suited especially for large sparse matrix systems. The disadvantages are that MATLAB's code had to be manually modified, and the GMRES implementation is interpreted.

5.4 ode15s with GMRES, static preconditioning

For "ode15s with GMRES, static preconditioning", a preconditioner is factored one time at the outset of the calculation and used throughout the integration. Preconditioning can have some desirable effects on the performance of GMRES, as discussed previously. Furthermore, static preconditioning has

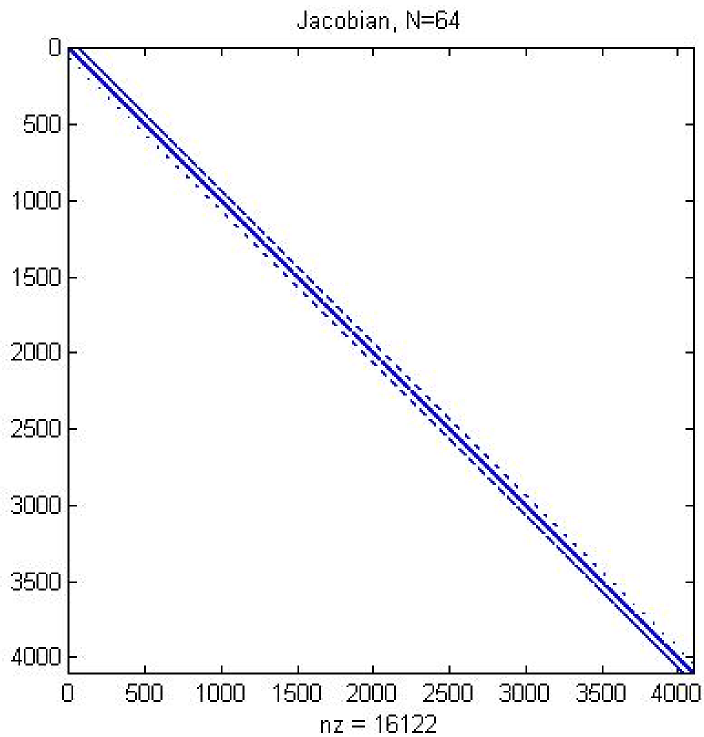

Figure 3: Jacobian Sparsity Pattern

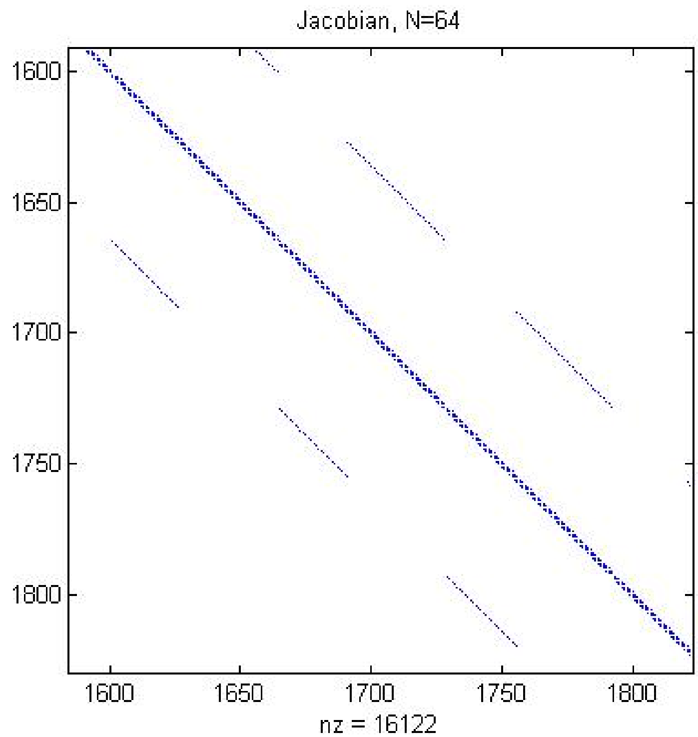

Figure 4: Jacobian Sparsity Pattern, Close-Up

a computational advantage over dynamic preconditioning as it requires only one factorization.

The static preconditioners used are based on the value of the Jacobian at the calculation outset. Various preconditioners include the tridiagonal part and main diagonal part of the Jacobian. It is difficult to find an effective static preconditioner because the implicit methods used in "ode15s" do not use the actual Jacobian matrix, but a matrix that includes method-specific constants which change at each step. An example of this "modified" matrix can be found as Equation 17 where "I" is the identity matrix of size N and α is a method constant.

$$M = I - \alpha \frac{\partial P}{\partial t} \tag{17}$$

5.5 ode15s with GMRES, dynamic preconditioning

In "ode15s with GMRES, dynamic preconditioning", we calculate a new preconditioner for each linear system to be solved. The advantage here is that a current preconditioner could be used at each time step. The disadvantage is that it requires more computation than either GMRES without preconditioning or GMRES with static preconditioning.

The dynamic preconditioner used is the tridiagonal part of the matrix found in Equation 17. This is calculated at each time step and is easily factored due to its tri-diagonal structure.

5.6 ode15s with GMRES, matrix-free version

Since GMRES only requires Jacobian-vector products, a "matrix-free" method can be introduced by providing GMRES with a function routine to evaluate those Jacobian-vector products. This routine is implemented by a finite-difference approximation. The advantage is that no Jacobian evaluations are required, although there is a substantial increase in the number of function evaluations.

5.7 ode15s with GMRES, matrix-free version with static preconditioning

The final method used is a modification of "ode15s with GMRES, matrix-free version", which is to introduce static preconditioning to help improve performance. There should only be a slight increase in the computation time required over the standard matrix-free version, because the preconditioner is evaluated and factored once at the computation start.

6 Results

The first result obtained was a base-case solution using the standard MATLAB solver "ode15s" on a moderate grid size. This was to check the discretization, initial condition, and other problem parameters against known results. After it was clear the solution method was working, the other methods were implemented, including the sparsity method and the various GMRES methods.

After some initial trials, it became clear that the best GMRES variation was the dynamic preconditioning method. A parametric study was done to find a reasonable set of GMRES parameters using this preconditioner. Once the set of parameters was found, a complete numerical study on a range of grid sizes for all methods was undertaken.

6.1 Base Case

Figures 5,6,and 7 show the solution plotted on the two-dimensional grid at various times. As mentioned before, time is integrated "backwards" so $t = 0$ actually corresponds to expiration. At $t = 1$ (today), the solution depends only on the current price (S), as expected [1]. This solution had $r = .05$, $K = 100$, $S_c = A_c = 200$, $T = 1$, and $\sigma = .2$. These values were used in all tests.

6.2 GMRES Parameters

After the base case, the next step was to find a reasonable set of parameters to use within GMRES. The three parameters varied were restart value, maximum iterations, and tolerance. A restart value of n tells GMRES to restart every n inner iterations. The maximum iterations is the maximum number of outer iterations. Finally, the tolerance is the maximum allowed relative residual norm defined as in Equation 18.

The grid size was $n = 128$, a mid-level grid. The default parameters were 10 maximum iterations, a restart value of 10, and a tolerance of 10^{-6}. The number of maximum outer iterations was varied from 10 to 100 in increments of 10. The restart value was varied from 1 to 50. Finally, the tolerance was varied from 10^{-1} to 10^{-10} , increasing the negative exponent.

The study indicated that a tolerance of 10^{-6}, a restart value of 15, and a maximum number of outer iterations 10 would be a good set of parameters to

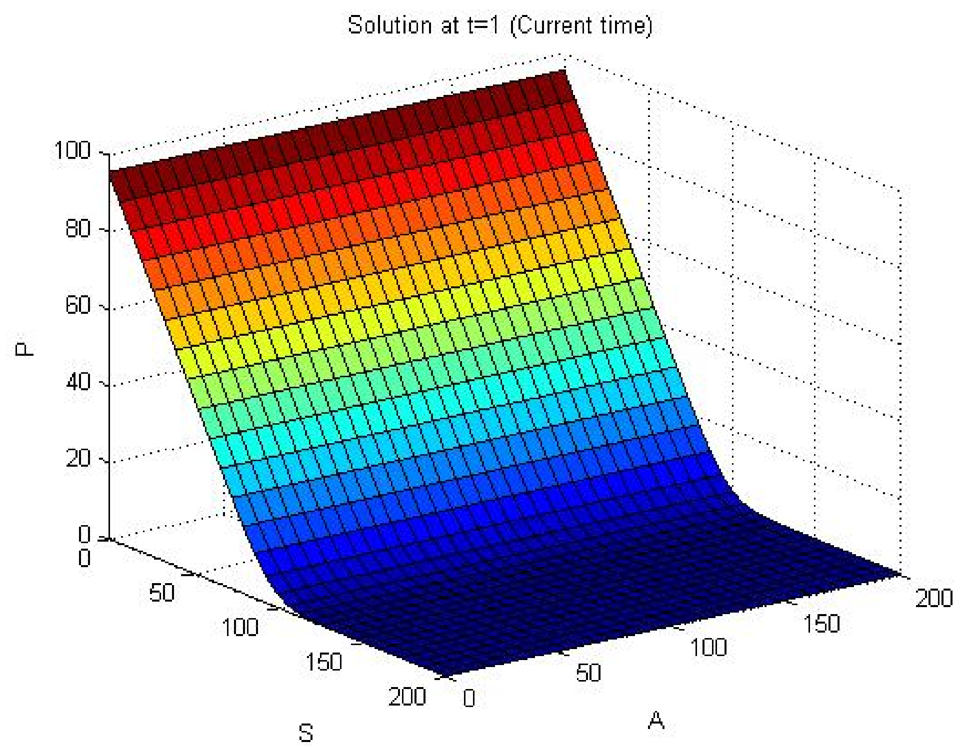

Figure 5: Solution at t=1 (Today)

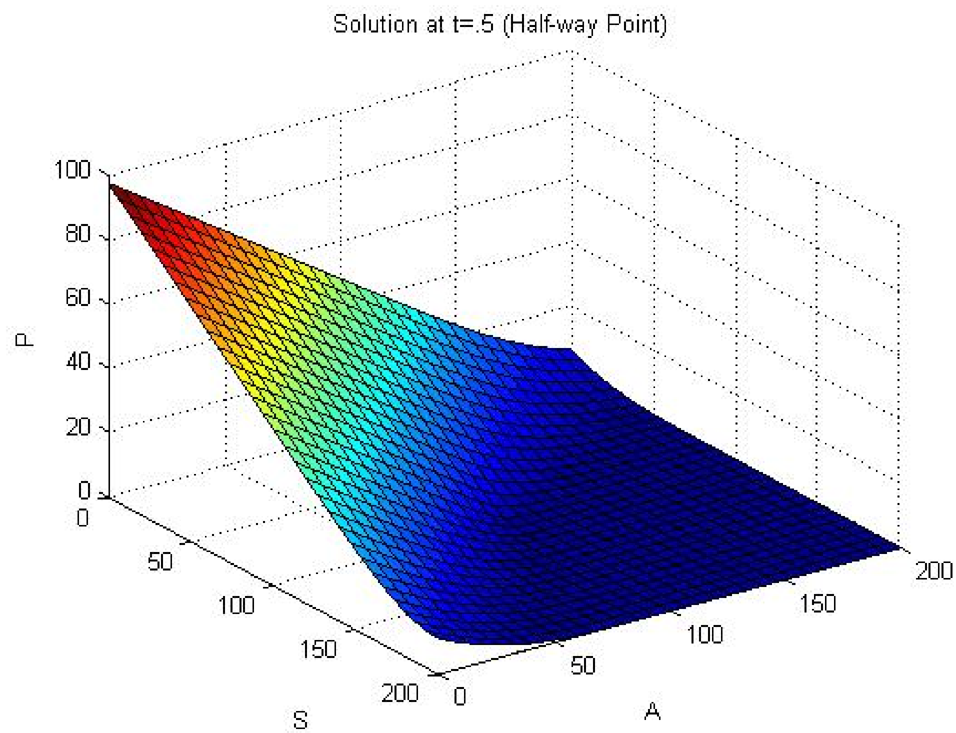

Figure 6: Solution at t=.5

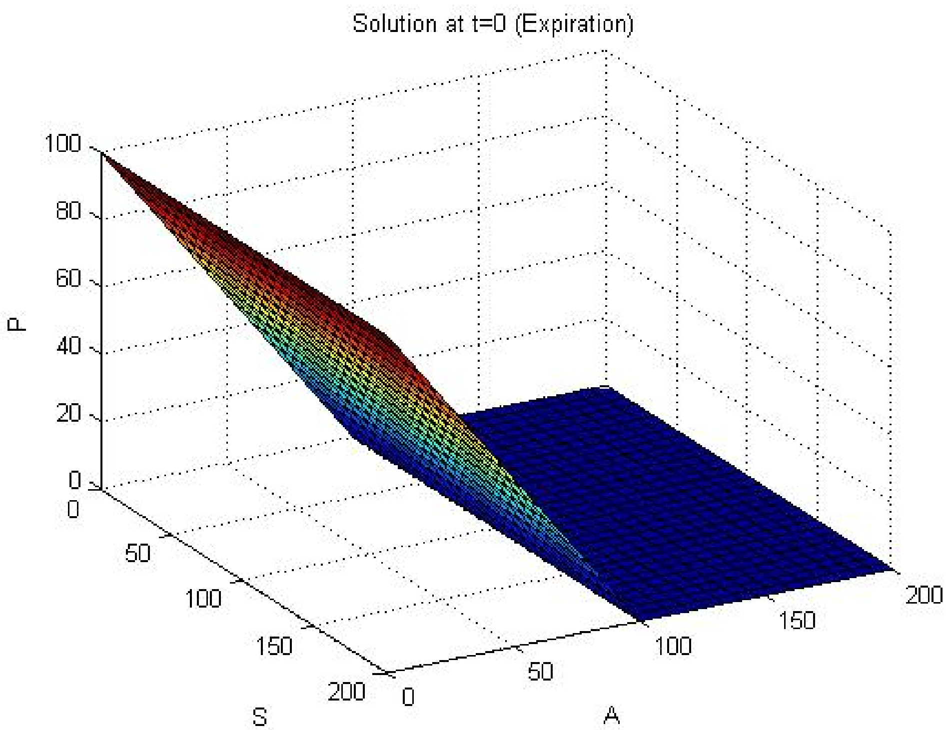

Figure 7: Solution at t=0 (Expiration)

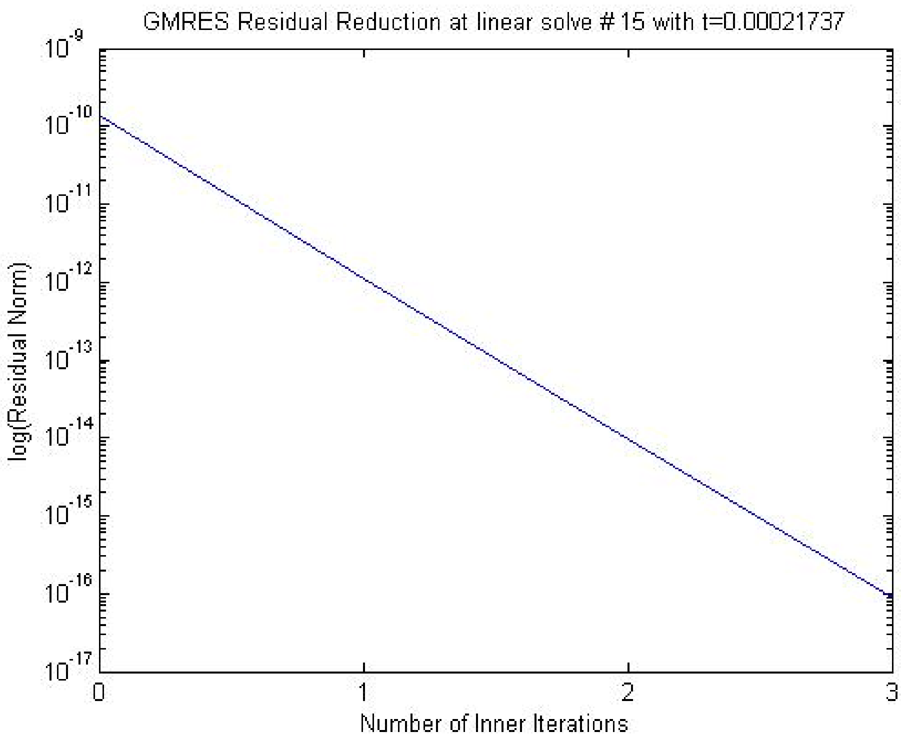

Figure 8: GMRES Performance, Preconditioned

use in the computations. Complete results for the GMRES parameter study are found in Appendix I.

$$||b - Ax||/||b|| \tag{18}$$

6.3 GMRES Performance

Residual reduction graphs of GMRES (log(residual norm) vs number of iterations) can show algorithm performance. A roughly linear behavior of residual norm reduction on a semi-log scale may show that effective preconditioning was used. The performance of GMRES was dependent on time. Figures 8 and 9 show GMRES performance at two times, a particularly easy time and one at which GMRES had trouble.

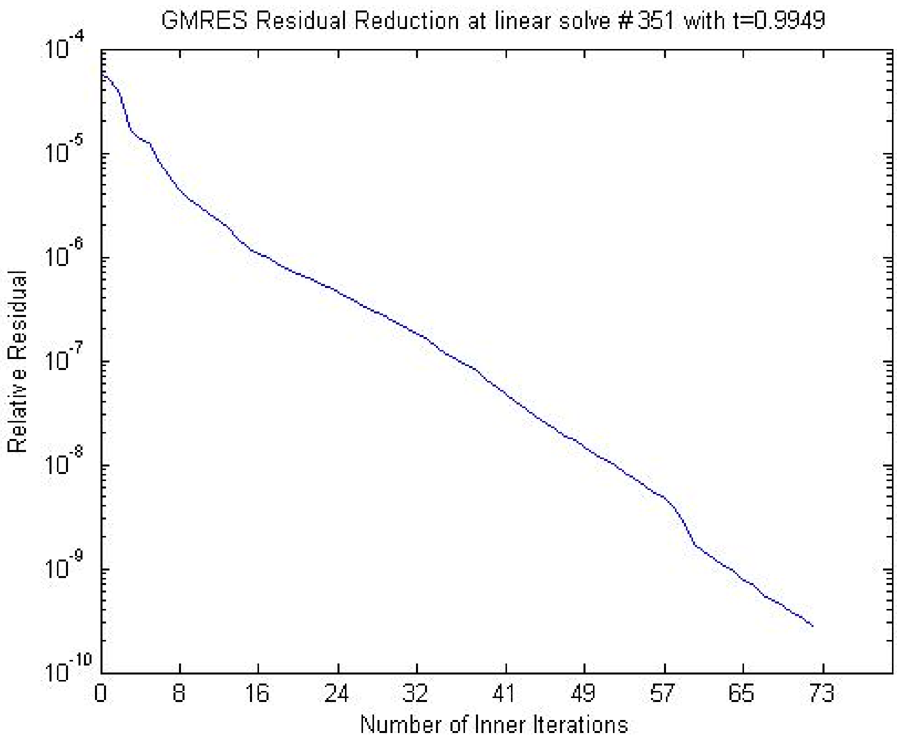

Figure 9: GMRES Performance, Preconditioned

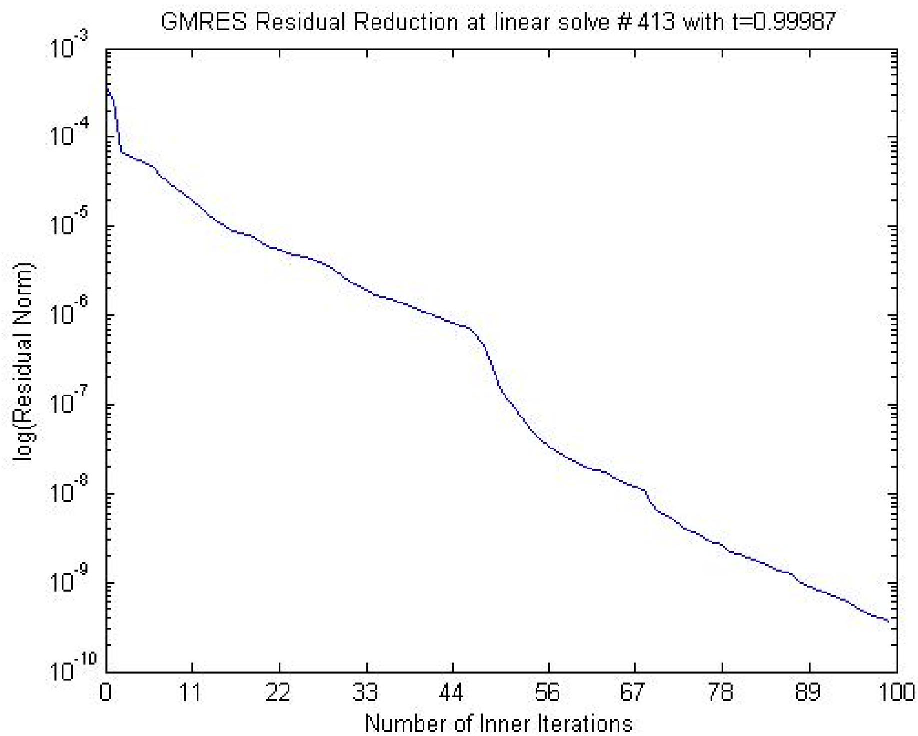

Figure 10: GMRES Performance, Not Preconditioned

Figure 10 shows performance of GMRES without preconditioning. The effect of preconditioning can be seen when comparing Figures 9 and 10.

6.4 Method Performance

Results were obtained for a variety of grid sizes ranging from $N = 16$ to $N = 512$. For the higher grid sizes, fewer methods were used because the calculation times became excessive for some methods. Tables 2-6 display the results for $N = 32, 64, 128, 256, 512$. Table 1 lists the various GMRES methods used in the cases, as well as other abbreviations used in Tables 2-5.

The "ode15s" case with full matrices was impractical to use for even moderate grid sizes, because the full matrix algorithms did not utilize the sparsity present in the jacobian. The use of a "static preconditioner" was not effective either; as the linear system changes over time, the optimal

Table 1: Abbreviations Key

Abbreviation	Full Name
GMRES-1	GMRES, No preconditioner
GMRES-2	GMRES, Updated preconditioner
GMRES-3	GMRES, Static preconditioner
GMRES-4	GMRES, Matrix-free
GMRES-5	GMRES, Matrix-free with preconditioner
NFE	Function Evaluations
NJE	Jacobian Evaluations
Jac. Fac.	Jacobian Factorizations
GMRES ITs.	GMRES Iterations
Lin. Solves	Linear System Solutions
NSS	Number of Successful Steps
NFA	Number of Failed Attempts

preconditioner to use also changes. With a static preconditioner, the user is hoping that the preconditioner calculated at the outset would demonstrate performance improvement for the method over the entire time interval, but this was not the case for moderate and large grid sizes.

The time required by each method was reported in each test case. However, on a time basis, comparing GMRES to MATLAB's default direct solver is not exactly a fair comparison. By default, within ode15s, MATLAB first calculates an L-U decomposition of the required matrix. Afterwards, it solves the system with back and forward substitution. A substantial reason for the time discrepancy is that while MATLAB's default solver is compiled, the GMRES solver used is interpreted. Compiled code will always provide a substantial time advantage. This is because compiled code is translated into machine code beforehand for fast and efficient usage, while interpreted code is translated into machine code line by line when the code is executed. However, other characteristics of the solution such as number of steps taken, number of linear solutions, and number of Jacobian evaluations can be compared fairly.

The two best methods were "ode15s with sparsity", and "GMRES with dynamic preconditioning". On a time basis, the default stiff solver had a substantial advantage. For the other performance measures, the numbers were comparable but "ode15s with sparsity" still had an advantage. This indicates that if a user were going to implement these methods within MATLAB, "ode15s with sparsity" would be the method of choice. However, if the user were going to compile his or her own code in another language (say C++), a viable alternative would be to implement "GMRES with dynamic preconditioning".

Complete results for all grid sizes can be found in Appendix II.

Table 2: $N = 32$

$N = 32$	NFE	NJE	Jac Fac.	GMRES ITs.	Lin. Solves	Time (secs.)	NSS	NFA
ode15s,Base	52799	51	106	x	515	76.21	193	80
ode15s,Sparsity	401	18	42	x	266	.75	134	24
GMRES-1	1198	77	0	5190	651	4.69	232	105
GMRES-2	1152	73	0	4532	633	7.18	230	102
GMRES-3	1216	80	0	5279	648	8.35	230	108
GMRES-4	4241	0	0	57292	442	40.08	214	55
GMRES-5	4319	0	0	58315	442	66.02	217	55

Table 3: $N = 64$

$N = 64$	NFE	NJE	Jac Fac.	GMRES ITs.	Lin. Solves	Time (secs.)	NSS	NFA
ode15s,Sparsity	1334	85	131	x	723	11.88	267	112
GMRES-1	1000	54	0	6489	607	66.46	245	86
GMRES-2	507	21	0	4186	345	26.86	174	28
GMRES-4	6498	0	0	57261	510	240.95	263	53
GMRES-5	6027	0	0	48577	462	362.83	248	45

Table 4: $N = 128$

$N = 128$	NFE	NJE	Jac Fac.	GMRES ITs.	Lin. Solves	Time (secs.)	NSS	NFA
ode15s,Sparsity	541	20	54	x	395	40.39	208	25
GMRES-1	980	47	0	17919	646	545.16	271	78
GMRES-2	601	23	0	6820	435	256.59	222	30
GMRES-4	7073	0	0	57970	527	1343.39	280	51

Table 5: $N = 256$

$N = 256$	NFE	NJE	Jac Fac.	GMRES ITs.	Lin. Solves	Time (secs.)	NSS	NFA
ode15s,Sparsity	568	15	53	x	462	313.21	254	18
GMRES-2	957	45	0	17238	642	3069.30	297	57

Table 6: $N = 512$

$N = 512$	NFE	NJE	Jac Fac.	GMRES ITs.	Lin. Solves	Time (secs.)	NSS	NFA
ode15s,Sparsity	774	23	69	x	612	3743.53	331	28
GMRES-2	1020	42	0	22247	726	15548.34	356	53

7 Conclusions and Future Work

7.1 Conclusions

An efficient solution of the pricing problem for an Asian put option was implemented on small, moderate, and large grid sizes using both MATLAB's stiff ordinary differential equations solver and several modified versions which incorporated various forms of GMRES.

Due to the unacceptable performance of "ode15s, base case" on even moderate grid sizes, it is clear that exploiting sparsity is necessary. The performances of the GMRES variations and "ode15s with sparsity" support this assertion.

In addition, the pricing problem showed strong sensitivity to both grid size and method selection. This is evident through the variations in performance numbers and time in all the test cases.

On most grid sizes, "ode15s with sparsity" had the best performance from both a time and efficiency perspective. However, as mentioned before, the time results are not a valid basis of comparison because the interpreted GMRES implementation is competing with the compiled internal solver. Aside from time, the performance numbers of "GMRES with dynamic preconditioning" were comparable on most grid sizes to those of "ode15s with sparsity", indicating that the GMRES method is viable although possibly inferior for these grid sizes.

7.2 Future Work

There are several improvements and new directions that were not implemented in the project. First, "ode15s" is an algorithm designed to solve non-linear differential equation systems. Thus, a Newton's Method apparatus is included in the code for new time steps. For a linear differential equation system, it is not necessary to resort to Newton's Method. As the Asian put pricing problem is linear, one modification could be to modify the "ode15s" code for all seven methods to solve for steps directly instead of using Newton's Method.

Another improvement would be to modify the GMRES code to accept right preconditioning. The MATLAB GMRES implementation only accepts left preconditioning. However, right preconditioning would be more compatible with the method by which "ode15s" evaluates tolerances and step criteria.

Finally, throughout the methods, a finite-difference approximation of the Jacobian matrix was used to obtain both Jacobians and tri-diagonal preconditioners. If an analytic Jacobian were supplied, the methods may become more accurate and efficient.

8 References

References

[1] Yves Achdou and Olivier Pironneau. *Computational Methods for Option Pricing*. SIAM Frontiers in Applied Mathematics, 2005.

[2] Youcef Saad and Martin H. Schultz. Gmres: A generalized minimal residual algorithm for solving nonsymmetric linear systems. *SIAM J. Sci. Stat. Comput.*, 1986.

[3] W. E. Schiesser. *The Numerical Method of Lines*. Academic Press, 1991.

[4] L. F. Shampine and M. W. Reichelt. The MATLAB ODE suite. *SIAM Journal on Scientific Computing*, 1997.

APPENDIX I: GMRES Parameter Study

TOLERANCE

N=128
Updated Preconditioner GMRES version of ode15s
314 successful steps
93 failed attempts
1079 function evaluations
39 partial derivatives
0 LU decompositions
801 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 117.701299 with 1110 GMRES iterates
Tolerance was 0.100000
Restart Value was 10
Max Iters Value was 10
Fail/Success Ratio of 0.296178
J-Evals per step of 0.124204
F-Evals per step of 3.436306
Linear Solves per step of 2.550955
GMRES iterates per step of 3.535032

Run of size 128 complete
****
****
N=128
Updated Preconditioner GMRES version of ode15s
251 successful steps
55 failed attempts
861 function evaluations
40 partial derivatives
0 LU decompositions
576 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 135.150369 with 2235 GMRES iterates
Tolerance was 0.010000
Restart Value was 10
Max Iters Value was 10
Fail/Success Ratio of 0.219124
J-Evals per step of 0.159363
F-Evals per step of 3.430279
Linear Solves per step of 2.294821
GMRES iterates per step of 8.904382

Run of size 128 complete
****

****
N=128
Updated Preconditioner GMRES version of ode15s
231 successful steps
42 failed attempts
719 function evaluations
32 partial derivatives
0 LU decompositions
490 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 146.521627 with 2360 GMRES iterates
Tolerance was 0.001000
Restart Value was 10
Max Iters Value was 10
Fail/Success Ratio of 0.181818
J-Evals per step of 0.138528
F-Evals per step of 3.112554
Linear Solves per step of 2.121212
GMRES iterates per step of 10.216450

Run of size 128 complete
****
****
N=128
Updated Preconditioner GMRES version of ode15s
213 successful steps
25 failed attempts
554 function evaluations
20 partial derivatives
0 LU decompositions
409 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 175.362529 with 2116 GMRES iterates
Tolerance was 0.000100
Restart Value was 10
Max Iters Value was 10
Fail/Success Ratio of 0.117371
J-Evals per step of 0.093897
F-Evals per step of 2.600939
Linear Solves per step of 1.920188
GMRES iterates per step of 9.934272

Run of size 128 complete
****
****
N=128
Updated Preconditioner GMRES version of ode15s

207 successful steps
20 failed attempts
495 function evaluations
16 partial derivatives
0 LU decompositions
378 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 227.029526 with 1970 GMRES iterates
Tolerance was 0.000010
Restart Value was 10
Max Iters Value was 10
Fail/Success Ratio of 0.096618
J-Evals per step of 0.077295
F-Evals per step of 2.391304
Linear Solves per step of 1.826087
GMRES iterates per step of 9.516908

Run of size 128 complete
****
****
N=128
Updated Preconditioner GMRES version of ode15s
212 successful steps
22 failed attempts
510 function evaluations
17 partial derivatives
0 LU decompositions
386 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 246.057617 with 2124 GMRES iterates
Tolerance was 0.000001
Restart Value was 10
Max Iters Value was 10
Fail/Success Ratio of 0.103774
J-Evals per step of 0.080189
F-Evals per step of 2.405660
Linear Solves per step of 1.820755
GMRES iterates per step of 10.018868

Run of size 128 complete
****
****
N=128
Updated Preconditioner GMRES version of ode15s
225 successful steps
36 failed attempts
671 function evaluations

29 partial derivatives
0 LU decompositions
463 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 550.033341 with 2867 GMRES iterates
Tolerance was 0.000000
Restart Value was 10
Max Iters Value was 10
Fail/Success Ratio of 0.160000
J-Evals per step of 0.128889
F-Evals per step of 2.982222
Linear Solves per step of 2.057778
GMRES iterates per step of 12.742222

Run of size 128 complete
****
****
N=128
Updated Preconditioner GMRES version of ode15s
207 successful steps
17 failed attempts
456 function evaluations
13 partial derivatives
0 LU decompositions
360 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 271.537186 with 2406 GMRES iterates
Tolerance was 0.000000
Restart Value was 10
Max Iters Value was 10
Fail/Success Ratio of 0.082126
J-Evals per step of 0.062802
F-Evals per step of 2.202899
Linear Solves per step of 1.739130
GMRES iterates per step of 11.623188

Run of size 128 complete
****
****
N=128
Updated Preconditioner GMRES version of ode15s
233 successful steps
45 failed attempts
737 function evaluations
35 partial derivatives
0 LU decompositions
487 solutions of linear systems

Updated Preconditioner GMRES finished, final time was 637.263694 with 3208 GMRES iterates
Tolerance was 0.000000
Restart Value was 10
Max Iters Value was 10
Fail/Success Ratio of 0.193133
J-Evals per step of 0.150215
F-Evals per step of 3.163090
Linear Solves per step of 2.090129
GMRES iterates per step of 13.768240

Run of size 128 complete
****
****
N=128
Updated Preconditioner GMRES version of ode15s
207 successful steps
19 failed attempts
483 function evaluations
15 partial derivatives
0 LU decompositions
373 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 449.763870 with 2253 GMRES iterates
Tolerance was 0.000000
Restart Value was 10
Max Iters Value was 10
Fail/Success Ratio of 0.091787
J-Evals per step of 0.072464
F-Evals per step of 2.333333
Linear Solves per step of 1.801932
GMRES iterates per step of 10.884058

Run of size 128 complete

RESTART

N=128
Updated Preconditioner GMRES version of ode15s
227 successful steps
37 failed attempts
657 function evaluations
28 partial derivatives
0 LU decompositions
456 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 225.396476 with 455 GMRES iterates

Tolerance was 0.000001
Restart Value was 1
Max Iters Value was 10
Fail/Success Ratio of 0.162996
J-Evals per step of 0.123348
F-Evals per step of 2.894273
Linear Solves per step of 2.008811
GMRES iterates per step of 2.004405

Run of size 128 complete
****
****
N=128
Updated Preconditioner GMRES version of ode15s
226 successful steps
38 failed attempts
668 function evaluations
29 partial derivatives
0 LU decompositions
460 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 269.004473 with 853 GMRES iterates
Tolerance was 0.000001
Restart Value was 2
Max Iters Value was 10
Fail/Success Ratio of 0.168142
J-Evals per step of 0.128319
F-Evals per step of 2.955752
Linear Solves per step of 2.035398
GMRES iterates per step of 3.774336

Run of size 128 complete
****
****
N=128
Updated Preconditioner GMRES version of ode15s
251 successful steps
65 failed attempts
907 function evaluations
48 partial derivatives
0 LU decompositions
566 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 453.095709 with 1561 GMRES iterates
Tolerance was 0.000001
Restart Value was 3
Max Iters Value was 10

Fail/Success Ratio of 0.258964
J-Evals per step of 0.191235
F-Evals per step of 3.613546
Linear Solves per step of 2.254980
GMRES iterates per step of 6.219124

Run of size 128 complete
****
****
N=128
Updated Preconditioner GMRES version of ode15s
208 successful steps
24 failed attempts
531 function evaluations
19 partial derivatives
0 LU decompositions
393 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 242.935910 with 962 GMRES iterates
Tolerance was 0.000001
Restart Value was 4
Max Iters Value was 10
Fail/Success Ratio of 0.115385
J-Evals per step of 0.091346
F-Evals per step of 2.552885
Linear Solves per step of 1.889423
GMRES iterates per step of 4.625000

Run of size 128 complete
****
****
N=128
Updated Preconditioner GMRES version of ode15s
233 successful steps
39 failed attempts
684 function evaluations
30 partial derivatives
0 LU decompositions
469 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 309.912998 with 1087 GMRES iterates
Tolerance was 0.000001
Restart Value was 5
Max Iters Value was 10
Fail/Success Ratio of 0.167382
J-Evals per step of 0.128755
F-Evals per step of 2.935622

Linear Solves per step of 2.012876
GMRES iterates per step of 4.665236

Run of size 128 complete
****
****
N=128
Updated Preconditioner GMRES version of ode15s
246 successful steps
51 failed attempts
811 function evaluations
38 partial derivatives
0 LU decompositions
540 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 403.633216 with 2259 GMRES iterates
Tolerance was 0.000001
Restart Value was 6
Max Iters Value was 10
Fail/Success Ratio of 0.207317
J-Evals per step of 0.154472
F-Evals per step of 3.296748
Linear Solves per step of 2.195122
GMRES iterates per step of 9.182927

Run of size 128 complete
****
****
N=128
Updated Preconditioner GMRES version of ode15s
212 successful steps
22 failed attempts
510 function evaluations
17 partial derivatives
0 LU decompositions
386 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 281.307864 with 1818 GMRES iterates
Tolerance was 0.000001
Restart Value was 7
Max Iters Value was 10
Fail/Success Ratio of 0.103774
J-Evals per step of 0.080189
F-Evals per step of 2.405660
Linear Solves per step of 1.820755
GMRES iterates per step of 8.575472

Run of size 128 complete
****
****
N=128
Updated Preconditioner GMRES version of ode15s
212 successful steps
22 failed attempts
510 function evaluations
17 partial derivatives
0 LU decompositions
386 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 241.135959 with 1946 GMRES iterates
Tolerance was 0.000001
Restart Value was 8
Max Iters Value was 10
Fail/Success Ratio of 0.103774
J-Evals per step of 0.080189
F-Evals per step of 2.405660
Linear Solves per step of 1.820755
GMRES iterates per step of 9.179245

Run of size 128 complete
****
****
N=128
Updated Preconditioner GMRES version of ode15s
212 successful steps
22 failed attempts
510 function evaluations
17 partial derivatives
0 LU decompositions
386 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 236.981114 with 2082 GMRES iterates
Tolerance was 0.000001
Restart Value was 9
Max Iters Value was 10
Fail/Success Ratio of 0.103774
J-Evals per step of 0.080189
F-Evals per step of 2.405660
Linear Solves per step of 1.820755
GMRES iterates per step of 9.820755

Run of size 128 complete
****
****

N=128
Updated Preconditioner GMRES version of ode15s
212 successful steps
22 failed attempts
510 function evaluations
17 partial derivatives
0 LU decompositions
386 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 244.331942 with 2124 GMRES iterates
Tolerance was 0.000001
Restart Value was 10
Max Iters Value was 10
Fail/Success Ratio of 0.103774
J-Evals per step of 0.080189
F-Evals per step of 2.405660
Linear Solves per step of 1.820755
GMRES iterates per step of 10.018868

Run of size 128 complete
N=128
Updated Preconditioner GMRES version of ode15s
212 successful steps
22 failed attempts
510 function evaluations
17 partial derivatives
0 LU decompositions
386 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 193.821828 with 5262 GMRES iterates
Fail/Success Ratio of 0.094017
J-Evals per step of 0.072650
F-Evals per step of 2.179487
Linear Solves per step of 1.649573
GMRES iterates per step of 22.487179
GMRES iterates per solve of 13.632124
Tolerance was 0.000001
Restart Value was 15
Max Iters Value was 10
Proportion of Failures was 0.000000

Run of size 128 complete
****
****
N=128
Updated Preconditioner GMRES version of ode15s
212 successful steps

22 failed attempts
510 function evaluations
17 partial derivatives
0 LU decompositions
386 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 220.961693 with 4946 GMRES iterates
Fail/Success Ratio of 0.094017
J-Evals per step of 0.072650
F-Evals per step of 2.179487
Linear Solves per step of 1.649573
GMRES iterates per step of 21.136752
GMRES iterates per solve of 12.813472
Tolerance was 0.000001
Restart Value was 30
Max Iters Value was 10
Proportion of Failures was 0.000000

Run of size 128 complete
****
****
N=128
Updated Preconditioner GMRES version of ode15s
212 successful steps
22 failed attempts
510 function evaluations
17 partial derivatives
0 LU decompositions
386 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 236.547107 with 4788 GMRES iterates
Fail/Success Ratio of 0.094017
J-Evals per step of 0.072650
F-Evals per step of 2.179487
Linear Solves per step of 1.649573
GMRES iterates per step of 20.461538
GMRES iterates per solve of 12.404145
Tolerance was 0.000001
Restart Value was 45
Max Iters Value was 10
Proportion of Failures was 0.000000

Run of size 128 complete
****
****
N=128
Updated Preconditioner GMRES version of ode15s

212 successful steps
22 failed attempts
510 function evaluations
17 partial derivatives
0 LU decompositions
386 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 366.141697 with 4770 GMRES iterates
Fail/Success Ratio of 0.094017
J-Evals per step of 0.072650
F-Evals per step of 2.179487
Linear Solves per step of 1.649573
GMRES iterates per step of 20.384615
GMRES iterates per solve of 12.357513
Tolerance was 0.000001
Restart Value was 60
Max Iters Value was 10
Proportion of Failures was 0.000000

Run of size 128 complete
****
****
N=128
Updated Preconditioner GMRES version of ode15s
212 successful steps
22 failed attempts
510 function evaluations
17 partial derivatives
0 LU decompositions
386 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 240.151578 with 4755 GMRES iterates
Fail/Success Ratio of 0.094017
J-Evals per step of 0.072650
F-Evals per step of 2.179487
Linear Solves per step of 1.649573
GMRES iterates per step of 20.320513
GMRES iterates per solve of 12.318653
Tolerance was 0.000001
Restart Value was 75
Max Iters Value was 10
Proportion of Failures was 0.000000

Run of size 128 complete
****

MAXIMUM ITERATIONS

N=128
Updated Preconditioner GMRES version of ode15s
212 successful steps
22 failed attempts
510 function evaluations
17 partial derivatives
0 LU decompositions
386 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 241.339120 with 2124 GMRES iterates
Tolerance was 0.000001
Restart Value was 10
Max Iters Value was 10
Fail/Success Ratio of 0.103774
J-Evals per step of 0.080189
F-Evals per step of 2.405660
Linear Solves per step of 1.820755
GMRES iterates per step of 10.018868

Run of size 128 complete
****
****
N=128
Updated Preconditioner GMRES version of ode15s
212 successful steps
22 failed attempts
510 function evaluations
17 partial derivatives
0 LU decompositions
386 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 240.551120 with 2124 GMRES iterates
Tolerance was 0.000001
Restart Value was 10
Max Iters Value was 20
Fail/Success Ratio of 0.103774
J-Evals per step of 0.080189
F-Evals per step of 2.405660
Linear Solves per step of 1.820755
GMRES iterates per step of 10.018868

Run of size 128 complete
****
****
N=128
Updated Preconditioner GMRES version of ode15s

212 successful steps
22 failed attempts
510 function evaluations
17 partial derivatives
0 LU decompositions
386 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 238.050222 with 2124 GMRES iterates
Tolerance was 0.000001
Restart Value was 10
Max Iters Value was 30
Fail/Success Ratio of 0.103774
J-Evals per step of 0.080189
F-Evals per step of 2.405660
Linear Solves per step of 1.820755
GMRES iterates per step of 10.018868

Run of size 128 complete
****
****
N=128
Updated Preconditioner GMRES version of ode15s
212 successful steps
22 failed attempts
510 function evaluations
17 partial derivatives
0 LU decompositions
386 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 240.912704 with 2124 GMRES iterates
Tolerance was 0.000001
Restart Value was 10
Max Iters Value was 40
Fail/Success Ratio of 0.103774
J-Evals per step of 0.080189
F-Evals per step of 2.405660
Linear Solves per step of 1.820755
GMRES iterates per step of 10.018868

Run of size 128 complete
****
****
N=128
Updated Preconditioner GMRES version of ode15s
212 successful steps
22 failed attempts
510 function evaluations

17 partial derivatives
0 LU decompositions
386 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 239.437258 with 2124 GMRES iterates
Tolerance was 0.000001
Restart Value was 10
Max Iters Value was 50
Fail/Success Ratio of 0.103774
J-Evals per step of 0.080189
F-Evals per step of 2.405660
Linear Solves per step of 1.820755
GMRES iterates per step of 10.018868

Run of size 128 complete
****
****
N=128
Updated Preconditioner GMRES version of ode15s
212 successful steps
22 failed attempts
510 function evaluations
17 partial derivatives
0 LU decompositions
386 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 243.650618 with 2124 GMRES iterates
Tolerance was 0.000001
Restart Value was 10
Max Iters Value was 60
Fail/Success Ratio of 0.103774
J-Evals per step of 0.080189
F-Evals per step of 2.405660
Linear Solves per step of 1.820755
GMRES iterates per step of 10.018868

Run of size 128 complete
****
****
N=128
Updated Preconditioner GMRES version of ode15s
212 successful steps
22 failed attempts
510 function evaluations
17 partial derivatives
0 LU decompositions
386 solutions of linear systems

Updated Preconditioner GMRES finished, final time was 238.925048 with 2124 GMRES iterates
Tolerance was 0.000001
Restart Value was 10
Max Iters Value was 70
Fail/Success Ratio of 0.103774
J-Evals per step of 0.080189
F-Evals per step of 2.405660
Linear Solves per step of 1.820755
GMRES iterates per step of 10.018868

Run of size 128 complete
****
****
N=128
Updated Preconditioner GMRES version of ode15s
212 successful steps
22 failed attempts
510 function evaluations
17 partial derivatives
0 LU decompositions
386 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 237.655957 with 2124 GMRES iterates
Tolerance was 0.000001
Restart Value was 10
Max Iters Value was 80
Fail/Success Ratio of 0.103774
J-Evals per step of 0.080189
F-Evals per step of 2.405660
Linear Solves per step of 1.820755
GMRES iterates per step of 10.018868

Run of size 128 complete
****
****
N=128
Updated Preconditioner GMRES version of ode15s
212 successful steps
22 failed attempts
510 function evaluations
17 partial derivatives
0 LU decompositions
386 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 245.551476 with 2124 GMRES iterates
Tolerance was 0.000001

Restart Value was 10
Max Iters Value was 90
Fail/Success Ratio of 0.103774
J-Evals per step of 0.080189
F-Evals per step of 2.405660
Linear Solves per step of 1.820755
GMRES iterates per step of 10.018868

Run of size 128 complete
****
****
N=128
Updated Preconditioner GMRES version of ode15s
212 successful steps
22 failed attempts
510 function evaluations
17 partial derivatives
0 LU decompositions
386 solutions of linear systems
Updated Preconditioner GMRES finished, final time was 242.660955 with 2124 GMRES iterates
Tolerance was 0.000001
Restart Value was 10
Max Iters Value was 100
Fail/Success Ratio of 0.103774
J-Evals per step of 0.080189
F-Evals per step of 2.405660
Linear Solves per step of 1.820755
GMRES iterates per step of 10.018868

Run of size 128 complete

APPENDIX II: Complete Performance Results

Solution of Asian Put Pricing Problem with various methods on various grid sizes
Method List:
Case #1: GMRES
Case #2: GMRES, with Update at Each Step Preconditioning
Case #3: GMRES, with Once and Forget Preconditioning
Case #4: GMRES, Matrix Free Mode
Case #5: GMRES, Matrix Free Mode with Once and Forget Preconditioning
Case #6: ode15S with Sparsity Pattern
Case #7: unmodified ode15s

N=8

GMRES version of ode15s
GMRES finished, final time was 4.411854 with 497 GMRES iterates
117 Successful Steps
48 Failed Attempts
505 Function Evaluations
27 Partial Derivatives
0 LU Decompositions
316 Solution of Linear Systems
Fail/Success Ratio of 0.290909
J-Evals per step of 0.163636
F-Evals per step of 3.060606
Linear Solves per step of 1.915152
GMRES iterates per step of -3.012121
GMRES iterates per solve of -1.572785
Proportion of failed GMRES solves was 0.000000

Updated Preconditioner GMRES version of ode15s
Updated Preconditioner GMRES finished, final time was 0.602076 with 615 GMRES iterates
75 Successful Steps
15 Failed Attempts
214 Function Evaluations
10 Partial Derivatives
0 LU Decompositions
144 Solution of Linear Systems
Fail/Success Ratio of 0.166667
J-Evals per step of 0.111111
F-Evals per step of 2.377778
Linear Solves per step of 1.600000
GMRES iterates per step of 6.833333
GMRES iterates per solve of 4.270833
Proportion of failed GMRES solves was 0.000000

Static Preconditioner GMRES version of ode15s
Static Preconditioner GMRES finished, final time was 0.569827 with 639 GMRES iterates
75 Successful Steps
15 Failed Attempts
214 Function Evaluations
10 Partial Derivatives
0 LU Decompositions
144 Solution of Linear Systems
Fail/Success Ratio of 0.166667
J-Evals per step of 0.111111
F-Evals per step of 2.377778
Linear Solves per step of 1.600000
GMRES iterates per step of 7.100000
GMRES iterates per solve of 4.437500
Proportion of failed GMRES solves was 0.000000

GMRES Matrix Free version of ode15s
GMRES Matrix Free finished, time was 7.545923 with 33559 GMRES iterates
134 Successful Steps
57 Failed Attempts
1314 Function Evaluations
0 Partial Derivatives
0 LU Decompositions
337 Solution of Linear Systems
Fail/Success Ratio of 0.298429
J-Evals per step of 0.000000
F-Evals per step of 6.879581
Linear Solves per step of 1.764398
GMRES iterates per step of 175.701571
GMRES iterates per solve of 99.581602
Proportion of failed GMRES solves was 0.718101

GMRES Matrix Free Preconditioned version of ode15s
GMRES Matrix Free Preconditioned finished, time was 1.902139 with 11496 GMRES iterates
67 Successful Steps
9 Failed Attempts
422 Function Evaluations
0 Partial Derivatives
0 LU Decompositions
112 Solution of Linear Systems
Fail/Success Ratio of 0.118421
J-Evals per step of 0.000000
F-Evals per step of 5.552632
Linear Solves per step of 1.473684
GMRES iterates per step of 151.263158
GMRES iterates per solve of 102.642857
Proportion of failed GMRES solves was 0.660714

ode15s sparsity specified
ode15S with Sparsity finished, time was 0.470627
65 Successful Steps
19 Failed Attempts
227 Function Evaluations
12 Partial Derivatives
28 LU Decompositions
142 Solution of Linear Systems
Fail/Success Ratio of 0.226190
J-Evals per step of 0.142857
F-Evals per step of 2.702381
Linear Solves per step of 1.690476

ode15s, base case
ode15s base-case finished, time was 0.290125
113 Successful Steps
58 Failed Attempts
2480 Function Evaluations
33 Partial Derivatives
76 LU Decompositions
334 Solution of Linear Systems
Fail/Success Ratio of 0.339181
J-Evals per step of 0.192982
F-Evals per step of 14.502924
Linear Solves per step of 1.953216

Time Summary
GMRES 4.411854 seconds
GMRES-Preconditioner 0.602076 seconds
GMRES-One Time Preconditioner 0.569827 seconds
GMRES-Matrix Free 7.545923 seconds
GMRES-Matrix Free Preconditioner 1.902139 seconds
ode15s with sparsity 0.470627 seconds
ode15s base-case 0.290125 seconds

N=16

GMRES version of ode15s
GMRES finished, final time was 5.200158 with 832 GMRES iterates
159 Successful Steps
63 Failed Attempts
669 Function Evaluations
35 Partial Derivatives
0 LU Decompositions
421 Solution of Linear Systems
Fail/Success Ratio of 0.283784
J-Evals per step of 0.157658
F-Evals per step of 3.013514
Linear Solves per step of 1.896396
GMRES iterates per step of 3.747748
GMRES iterates per solve of 1.976247
Proportion of failed GMRES solves was 0.000000

Updated Preconditioner GMRES version of ode15s
Updated Preconditioner GMRES finished, final time was 0.859521 with 1082 GMRES iterates
105 Successful Steps
16 Failed Attempts
268 Function Evaluations
9 Partial Derivatives
0 LU Decompositions
202 Solution of Linear Systems
Fail/Success Ratio of 0.132231
J-Evals per step of 0.074380
F-Evals per step of 2.214876
Linear Solves per step of 1.669421
GMRES iterates per step of 8.942149
GMRES iterates per solve of 5.356436
Proportion of failed GMRES solves was 0.000000

One-Time Preconditioner GMRES version of ode15s
One-Time Preconditioner GMRES finished, final time was 0.879426 with 1165 GMRES iterates
105 Successful Steps
16 Failed Attempts
268 Function Evaluations
9 Partial Derivatives
0 LU Decompositions
202 Solution of Linear Systems
Fail/Success Ratio of 0.132231
J-Evals per step of 0.074380
F-Evals per step of 2.214876
Linear Solves per step of 1.669421
GMRES iterates per step of 9.628099
GMRES iterates per solve of 5.767327
Proportion of failed GMRES solves was 0.000000

GMRES Matrix Free version of ode15s
GMRES Matrix Free finished, time was 13.807347 with 47150 GMRES iterates
174 Successful Steps
56 Failed Attempts
2873 Function Evaluations
0 Partial Derivatives
0 LU Decompositions
382 Solution of Linear Systems
Fail/Success Ratio of 0.243478
J-Evals per step of 0.000000
F-Evals per step of 12.491304
Linear Solves per step of 1.660870
GMRES iterates per step of 205.000000
GMRES iterates per solve of 123.429319

Proportion of failed GMRES solves was 0.853403

GMRES Matrix Free Preconditioned version of ode15s
GMRES Matrix Free Preconditioned finished, time was 5.549520 with 17709 GMRES iterates
99 Successful Steps
9 Failed Attempts
900 Function Evaluations
0 Partial Derivatives
0 LU Decompositions
153 Solution of Linear Systems
Fail/Success Ratio of 0.083333
J-Evals per step of 0.000000
F-Evals per step of 8.333333
Linear Solves per step of 1.416667
GMRES iterates per step of 163.972222
GMRES iterates per solve of 115.745098
Proportion of failed GMRES solves was 0.758170

ode15s sparsity specified
ode15S with Sparsity finished, time was 0.186403
97 Successful Steps
25 Failed Attempts
319 Function Evaluations
15 Partial Derivatives
38 LU Decompositions
210 Solution of Linear Systems
Fail/Success Ratio of 0.204918
J-Evals per step of 0.122951
F-Evals per step of 2.614754
Linear Solves per step of 1.721311

ode15s, base case
ode15s base-case finished, time was 3.756385
170 Successful Steps
87 Failed Attempts
14880 Function Evaluations
56 Partial Derivatives
102 LU Decompositions
484 Solution of Linear Systems
Fail/Success Ratio of 0.338521
J-Evals per step of 0.217899
F-Evals per step of 57.898833
Linear Solves per step of 1.883268

Time Summary
GMRES 5.200158 seconds
GMRES-Preconditioner 0.859521 seconds
GMRES-One Time Preconditioner 0.879426 seconds
GMRES-Matrix Free 13.807347 seconds
GMRES-Matrix Free Preconditioner 5.549520 seconds
ode15s with sparsity 0.186403 seconds
ode15s base-case 3.756385 seconds

N=32

GMRES version of ode15s
GMRES finished, final time was 4.688987 with 5190 GMRES iterates
232 Successful Steps
105 Failed Attempts
1198 Function Evaluations
77 Partial Derivatives
0 LU Decompositions
651 Solution of Linear Systems

Fail/Success Ratio of 0.311573
J-Evals per step of 0.228487
F-Evals per step of 3.554896
Linear Solves per step of 1.931751
GMRES iterates per step of 15.400593
GMRES iterates per solve of 7.972350
Proportion of failed GMRES solves was 0.000000

Updated Preconditioner GMRES version of ode15s
Updated Preconditioner GMRES finished, final time was 7.183947 with 4532 GMRES iterates
230 Successful Steps
102 Failed Attempts
1152 Function Evaluations
73 Partial Derivatives
0 LU Decompositions
633 Solution of Linear Systems
Fail/Success Ratio of 0.307229
J-Evals per step of 0.219880
F-Evals per step of 3.469880
Linear Solves per step of 1.906627
GMRES iterates per step of 13.650602
GMRES iterates per solve of 7.159558
Proportion of failed GMRES solves was 0.000000

One-Time Preconditioner GMRES version of ode15s
One-Time Preconditioner GMRES finished, final time was 8.353660 with 5279 GMRES iterates
230 Successful Steps
108 Failed Attempts
1216 Function Evaluations
80 Partial Derivatives
0 LU Decompositions
648 Solution of Linear Systems
Fail/Success Ratio of 0.319527
J-Evals per step of 0.236686
F-Evals per step of 3.597633
Linear Solves per step of 1.917160
GMRES iterates per step of 15.618343
GMRES iterates per solve of 8.146605
Proportion of failed GMRES solves was 0.000000

GMRES Matrix Free version of ode15s
GMRES Matrix Free finished, time was 40.075402 with 57292 GMRES iterates
214 Successful Steps
55 Failed Attempts
4241 Function Evaluations
0 Partial Derivatives
0 LU Decompositions
442 Solution of Linear Systems
Fail/Success Ratio of 0.204461
J-Evals per step of 0.000000
F-Evals per step of 15.765799
Linear Solves per step of 1.643123
GMRES iterates per step of 212.981413
GMRES iterates per solve of 129.619910
Proportion of failed GMRES solves was 0.954751

GMRES Matrix Free Preconditioned version of ode15s
GMRES Matrix Free Preconditioned finished, time was 66.018611 with 58315 GMRES iterates
217 Successful Steps
55 Failed Attempts
4319 Function Evaluations
0 Partial Derivatives
0 LU Decompositions
442 Solution of Linear Systems
Fail/Success Ratio of 0.202206
J-Evals per step of 0.000000
F-Evals per step of 15.878676
Linear Solves per step of 1.625000

GMRES iterates per step of 214.393382
GMRES iterates per solve of 131.934389
Proportion of failed GMRES solves was 0.950226

ode15s sparsity specified
ode15S with Sparsity finished, time was 0.757616
134 Successful Steps
24 Failed Attempts
401 Function Evaluations
18 Partial Derivatives
42 LU Decompositions
266 Solution of Linear Systems
Fail/Success Ratio of 0.151899
J-Evals per step of 0.113924
F-Evals per step of 2.537975
Linear Solves per step of 1.683544

ode15s, base case
ode15s base-case finished, time was 76.211247
193 Successful Steps
80 Failed Attempts
52799 Function Evaluations
51 Partial Derivatives
106 LU Decompositions
515 Solution of Linear Systems
Fail/Success Ratio of 0.293040
J-Evals per step of 0.186813
F-Evals per step of 193.402930
Linear Solves per step of 1.886447

Time Summary
GMRES 4.688987 seconds
GMRES-Preconditioner 7.183947 seconds
GMRES-One Time Preconditioner 8.353660 seconds
GMRES-Matrix Free 40.075402 seconds
GMRES-Matrix Free Preconditioner 66.018611 seconds
ode15s with sparsity 0.757616 seconds
ode15s base-case 76.211247 seconds

N=64

GMRES version of ode15s
GMRES finished, final time was 66.456174 with 6489 GMRES iterates
245 Successful Steps
86 Failed Attempts
1000 Function Evaluations
54 Partial Derivatives
0 LU Decompositions
607 Solution of Linear Systems
Fail/Success Ratio of 0.259819
J-Evals per step of 0.163142
F-Evals per step of 3.021148
Linear Solves per step of 1.833837
GMRES iterates per step of 19.604230
GMRES iterates per solve of 10.690280
Proportion of failed GMRES solves was 0.000000

Updated Preconditioner GMRES version of ode15s
Updated Preconditioner GMRES finished, final time was 26.862827 with 4186 GMRES iterates
174 Successful Steps

28 Failed Attempts
507 Function Evaluations
21 Partial Derivatives
0 LU Decompositions
345 Solution of Linear Systems
Fail/Success Ratio of 0.138614
J-Evals per step of 0.103960
F-Evals per step of 2.509901
Linear Solves per step of 1.707921
GMRES iterates per step of 20.722772
GMRES iterates per solve of 12.133333
Proportion of failed GMRES solves was 0.000000

GMRES Matrix Free version of ode15s
GMRES Matrix Free finished, time was 240.947143 with 57261 GMRES iterates
263 Successful Steps
53 Failed Attempts
6498 Function Evaluations
0 Partial Derivatives
0 LU Decompositions
510 Solution of Linear Systems
Fail/Success Ratio of 0.167722
J-Evals per step of 0.000000
F-Evals per step of 20.563291
Linear Solves per step of 1.613924
GMRES iterates per step of 181.205696
GMRES iterates per solve of 112.276471
Proportion of failed GMRES solves was 0.974510

GMRES Matrix Free Preconditioned version of ode15s
GMRES Matrix Free Preconditioned finished, time was 362.827408 with 48577 GMRES iterates
248 Successful Steps
45 Failed Attempts
6027 Function Evaluations
0 Partial Derivatives
0 LU Decompositions
462 Solution of Linear Systems
Fail/Success Ratio of 0.153584
J-Evals per step of 0.000000
F-Evals per step of 20.569966
Linear Solves per step of 1.576792
GMRES iterates per step of 165.791809
GMRES iterates per solve of 105.145022
Proportion of failed GMRES solves was 0.967532

ode15s sparsity specified
ode15S with Sparsity finished, time was 11.875045
267 Successful Steps
112 Failed Attempts
1334 Function Evaluations
85 Partial Derivatives
131 LU Decompositions
723 Solution of Linear Systems
Fail/Success Ratio of 0.295515
J-Evals per step of 0.224274
F-Evals per step of 3.519789
Linear Solves per step of 1.907652

Time Summary
GMRES 66.456174 seconds
GMRES-Preconditioner 26.862827 seconds
GMRES-One Time Preconditioner 0.000000 seconds
GMRES-Matrix Free 240.947143 seconds
GMRES-Matrix Free Preconditioner 362.827408 seconds
ode15s with sparsity 11.875045 seconds
ode15s base-case 0.000000 seconds

N=128

GMRES version of ode15s
GMRES finished, final time was 545.157086 with 17919 GMRES iterates
271 Successful Steps
78 Failed Attempts
980 Function Evaluations
47 Partial Derivatives
0 LU Decompositions
646 Solution of Linear Systems
Fail/Success Ratio of 0.223496
J-Evals per step of 0.134670
F-Evals per step of 2.808023
Linear Solves per step of 1.851003
GMRES iterates per step of 51.343840
GMRES iterates per solve of 27.738390
Proportion of failed GMRES solves was 0.006192

Updated Preconditioner GMRES version of ode15s
Updated Preconditioner GMRES finished, final time was 256.593042 with 6820 GMRES iterates
222 Successful Steps
30 Failed Attempts
601 Function Evaluations
23 Partial Derivatives
0 LU Decompositions
435 Solution of Linear Systems
Fail/Success Ratio of 0.119048
J-Evals per step of 0.091270
F-Evals per step of 2.384921
Linear Solves per step of 1.726190
GMRES iterates per step of 27.063492
GMRES iterates per solve of 15.678161
Proportion of failed GMRES solves was 0.000000

GMRES Matrix Free version of ode15s
GMRES Matrix Free finished, time was 1343.387580 with 57970 GMRES iterates
280 Successful Steps
51 Failed Attempts
7073 Function Evaluations
0 Partial Derivatives
0 LU Decompositions
527 Solution of Linear Systems
Fail/Success Ratio of 0.154079
J-Evals per step of 0.000000
F-Evals per step of 21.368580
Linear Solves per step of 1.592145
GMRES iterates per step of 175.135952
GMRES iterates per solve of 110.000000
Proportion of failed GMRES solves was 0.907021

ode15s sparsity specified
ode15S with Sparsity finished, time was 40.389768
208 Successful Steps
25 Failed Attempts
541 Function Evaluations
20 Partial Derivatives
54 LU Decompositions
395 Solution of Linear Systems
Fail/Success Ratio of 0.107296
J-Evals per step of 0.085837
F-Evals per step of 2.321888
Linear Solves per step of 1.695279

Time Summary
GMRES 545.157086 seconds
GMRES-Preconditioner 256.593042 seconds
GMRES-One Time Preconditioner 0.000000 seconds
GMRES-Matrix Free 1343.387580 seconds
GMRES-Matrix Free Preconditioner 0.000000 seconds
ode15s with sparsity 40.389768 seconds

ode15s base-case 0.000000 seconds

N=256

Updated Preconditioner GMRES version of ode15s
Updated Preconditioner GMRES finished, final time was 3069.300131 with 17238 GMRES iterates
297 Successful Steps
57 Failed Attempts
957 Function Evaluations
45 Partial Derivatives
0 LU Decompositions
642 Solution of Linear Systems
Fail/Success Ratio of 0.161017
J-Evals per step of 0.127119
F-Evals per step of 2.703390
Linear Solves per step of 1.813559
GMRES iterates per step of 48.694915
GMRES iterates per solve of 26.850467
Proportion of failed GMRES solves was 0.000000

ode15s sparsity specified
ode15S with Sparsity finished, time was 313.210936
254 Successful Steps
18 Failed Attempts
568 Function Evaluations
15 Partial Derivatives
53 LU Decompositions
462 Solution of Linear Systems
Fail/Success Ratio of 0.066176
J-Evals per step of 0.055147
F-Evals per step of 2.088235
Linear Solves per step of 1.698529

Time Summary
GMRES 0.000000 seconds
GMRES-Preconditioner 3069.300131 seconds
GMRES-One Time Preconditioner 0.000000 seconds
GMRES-Matrix Free 0.000000 seconds
GMRES-Matrix Free Preconditioner 0.000000 seconds
ode15s with sparsity 313.210936 seconds
ode15s base-case 0.000000 seconds

N=512

Updated Preconditioner GMRES version of ode15s
Updated Preconditioner GMRES finished, final time was 15548.336204 with 22247 GMRES iterates
356 Successful Steps
53 Failed Attempts
1020 Function Evaluations
42 Partial Derivatives
0 LU Decompositions
726 Solution of Linear Systems
Fail/Success Ratio of 0.129584
J-Evals per step of 0.102689
F-Evals per step of 2.493888
Linear Solves per step of 1.775061
GMRES iterates per step of 54.393643
GMRES iterates per solve of 30.643251
Proportion of failed GMRES solves was 0.004132

ode15s sparsity specified
ode15S with Sparsity finished, time was 3743.534588
331 Successful Steps
28 Failed Attempts
774 Function Evaluations
23 Partial Derivatives
69 LU Decompositions

612 Solution of Linear Systems
Fail/Success Ratio of 0.077994
J-Evals per step of 0.064067
F-Evals per step of 2.155989
Linear Solves per step of 1.704735

Time Summary
GMRES 0.000000 seconds
GMRES-Preconditioner 15548.336204 seconds
GMRES-One Time Preconditioner 0.000000 seconds
GMRES-Matrix Free 0.000000 seconds
GMRES-Matrix Free Preconditioner 0.000000 seconds
ode15s with sparsity 3743.534588 seconds
ode15s base-case 0.000000 seconds