2007-09-12

Reflector geometry specific modeling of an annular array based ultrasound pulse-echo system

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REFLECTOR GEOMETRY SPECIFIC MODELING OF AN ANNULAR ARRAY BASED ULTRASOUND PULSE-ECHO SYSTEM

A Thesis submitted to the faculty
Of
WORCESTER POLYTECHNIC INSTITUTE
in partial fulfillment of the requirements for
The Degree of Master of Science
in
Electrical and Computer Engineering
by

__________________________
Aditya Nadkarni
May 2007

APPROVED

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Abstract

Conventional ultrasound imaging systems use array transducers for focusing and beam steering, to improve lateral resolution and permit real-time imaging. This thesis research investigates a different use of array transducers, where the acoustic field and the receiver characteristics are designed such that the energy of the output signal from targets of a specified geometry is maximized. The output signal is the sum of the received signals obtained using all the possible combinations of transducer array elements as transmitter and receiver. This work is based on annular array transducers, but is applicable for any array configuration.

The first step is the development of software for the efficient modeling of the wave interaction between transmitted field and target, and between the transducer and receiver field. Using this software, we have calculated the received signal for each combination of an array element as transmitter and the same or another array element as receiver, leading to an $N \times N$ received signal matrix for an $N$ element array transducer. A waveform optimization algorithm is then implemented for the purpose of determining the set of delays for the individual array elements, which maximizes the energy of the sum of the received signals. In one implementation of this algorithm, the received signal with the maximum energy is considered as a reference signal, and specific delays are applied to the other signals so that any two signals produce a maximum correlation. This leads to an $N \times N$ delay matrix, which, however, is not readily implemented in a practical real-time system, which uses all the elements in an array transducer simultaneously to customize
acoustic fields. Hence, the values in this delay matrix are fed into a linear programming optimizer tool to obtain a set of delay values, which makes its implementation practical.

The optimized set of delays thus obtained is used to maximize the energy of the received signal for a given transducer and target geometry and hence to enhance the reflectivity of that target. It is also important to check the robustness of the optimized set of delays obtained above, for a given target geometry. Robustness refers to the sensitivity of the optimization to variation in target geometry. This aspect is also evaluated as a part of this thesis work.
Acknowledgement

This thesis most definitely wouldn’t have been completed without the encouragement, motivation and best wishes that I have received in the past few years from several people. I would like to take this opportunity to express my gratitude to all these people. Above all, I would like to say a big big thank you to my advisor Prof. Peder.C.Pedersen.

Working as a part of the ultrasound lab has given me the opportunity to learn about some new trends in the Medical imaging industry, thanks to Prof. Pedersen. His primary objective to produce novel and good quality work taught me a lot of things in addition to the research oriented matter, which will definitely help me in the long run. I am especially grateful for the cooperation he has offered in the last two years and the time he has spent in refining my thesis write-up. I would also like to express my gratitude to Prof. Reinhold Ludwig and Prof. Sergey Makarov for readily accepting to be on my thesis committee and reviewing my work.

I would like to thank my lab mates: Dalys Sebastian, Deepti Sukhwani, Ruben Lara Montalvo and Carsten Poulsen for all their help, great memories, and a wonderful and fun work atmosphere. I have to sincerely acknowledge my dear friends: Mili, Siddhesh, Kamal-Renu and Rahul-Shivani; my roomies: Vishal, Dharmesh, Vinay and Ritesh for all the encouragement that I have received through the ups and downs I encountered while completing this thesis. Thanks to all my other friends and colleagues at EMC for the positive advice and support.

Last but not at all the least, this really wouldn’t have been possible without the blessings and best wishes from my parents, my dear brother Akshay and my family back home. They are my source of motivation and I owe this experience to them.
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Chapter 1

Introduction

1.1 Introduction to Medical Ultrasound

Medical ultrasound is a medical imaging modality that has a wide variety of clinical applications, both as a primary modality and as an adjunct to other diagnostic procedures. The basis of its operation is the transmission of high frequency sound into the body followed by the reception, processing, and parametric display of echoes returning from structures and tissues within the body. Its utility in the medical industry is largely due to the following three characteristics:

1) It is a real-time modality
2) It does not utilize ionizing radiations
3) It provides quantitative measurements and imaging of blood flow

1.2 Introduction to an ultrasound pulse-echo system

Sound is a mechanical energy transmitted by pressure waves in a medium. Sound waves, whose frequency is greater than 20 KHz, are termed as ultrasound. Medical ultrasound imaging relies solely on the fact that biological tissues scatter and tissue interfaces reflect incident sound. To be more precise, scattering refers to the interaction between sound waves and particles that are much smaller than the sound’s wavelength, while reflection refers to the interaction with particles or objects that are larger than the wavelength of sound. Scattering or reflection of acoustic waves arise from the inhomogenities in the medium’s density or compressibility. Sound is primarily scattered or reflected by a
discontinuity in the medium’s mechanical properties, to a degree proportional to the relative change in acoustic impedance. Abrupt as well as continuous changes in a medium’s material properties can cause the direction of propagation to change gradually.

The word transducer denotes any device that is used to convert signals or energy from one energy form to another. In the context of medical ultrasound and this thesis, the term transducer will refer to the ultrasonic transducer that is used to convert acoustic signals to electrical signals and electrical signals to acoustic signals. When an ultrasonic transducer transmits a short-duration acoustic pulse into a medium containing reflecting interfaces, the pulses undergo reflection at these interfaces, as discussed above due to the material properties of the object. This gives rise to echo signals returning to the receiving transducer. Such a system is called an ultrasound pulse-echo system and is illustrated in Fig 1.1, in which the same transducer acts as a transmitter and receiver.

![Figure 1.1: An ultrasound pulse-echo system](image)
The ultrasound transducer uses an array of piezoelectric elements to transmit a sound pulse into the body and to receive the echoes that return from scattering structures within. This array is often referred to as the imaging system’s aperture. The excitation signals applied to, and the received signals obtained from, these array elements can be individually delayed in time, hence the term phased array. This is done to electronically steer and focus each of a sequence of acoustic pulses through the plane or volume to be imaged in the body. This produces a 2- or 3-D map of the scattered echoes that is usually presented in the form an image to the clinician for interpretation and diagnosis. The process of steering and focusing these acoustic pulses is known as beamforming. This process is shown in the Fig 1.2 on the next page.
Figure 1.2: Phased array beamforming concept (a) Pulses delayed by some values $\tau$ are transmitted from an array of piezoelectric elements to achieve steering and focusing at the point of interest. However, only focusing delays are shown here. (b) The echoes returning are likewise delayed by $\tau$, before they are summed together to form a strong echo signal from the region of interest.

The ultrasound pulse-echo system is the basis for most practical applications of ultrasound in addition to medicine. In many situations, pulse-echo ultrasound is the only practical way that ultrasonic imaging, the most qualitative ultrasound application, can be
performed. Images of local backscatter level (B-mode images) are readily generated by using linear array transducers. Quantitative ultrasound on the other hand, often requires that the received signals in a pulse-echo are processed in specific ways. Quantitative ultrasound applications include tissue characterization, complex object recognition and identification of surface topology. Unfortunately, it is quite difficult to efficiently determine the received electrical signal in pulse-echo ultrasound systems because of the complexity of generation, propagation, backscattering and reception of the ultrasound fields in pulse-echo systems. Therefore, efficient numerical modeling tools for pulse-echo system are essential to the progress of the quantitative medical and industrial applications of ultrasound.

The relationship between the output signal from an ultrasound pulse-echo system on one hand and the excitation signal, the geometry, properties and location of the ultrasound transducers and size, geometry, location and orientation of the reflector on the other hand is very complex. Numerical modeling is the only way in which the output signal for a given measurement system can be predicted. This is in particular true when it comes to optimizing the design of ultrasound system to carry out such tasks as identifying objects of specified shapes, determining surface topology or alignment of surface, etc., numerical modeling is the only practical way. The factors that influence the received signal include: the excitation signal; the geometry, location, electro-acoustic transfer function of the transmit and receive transducers; the size, shape, surface geometry, interface orientation, location of the reflector; and the effect of attenuation, absorption, refraction and non-linearity of the coupling medium.
This thesis work primarily focuses on obtaining an efficient and effective method to numerically model (or predict) the output signal for a given pulse-echo system setup, which can further be used to optimize the design of that ultrasound system to carry out specific tasks such as identifying objects of a specified shape and enhancing the images obtained from those objects. But before we get to the outline of this thesis, let us discuss some work that has been done in the areas of modeling an ultrasound pulse-echo system and some techniques that are being applied in medical ultrasound for improving the medical image quality.

1.3 Modeling Pulse-echo Ultrasound systems

Several techniques have been applied to model pulse-echo systems. These methods include analytical approaches, a numerical method: the Finite-Element Method (FEM), the Angular Spectrum Method (ASM), and the Spatial Impulse-Response Method (SIRM). Each of these techniques has its own advantages and disadvantages, which we will discuss now.

1.3.1 Analytical approaches

In this modeling approach, an analytical solution for the received signal in a pulse-echo system is used. The analytical approaches can make the computer simulation of the system efficient and general, given the analytical solutions for the received signals are available. However, these kinds of formulations do not exist for any general transducer and reflector geometries. They are further not very implementable since they do include
attenuation, scattering and refraction effects. Thus, analytical approaches are not considered much for modeling general pulse-echo systems.

1.3.2 The Finite-Element Method (FEM)
This is a numerical method used to compute the wavefields created in a pulse-echo system. In fact, the FEM method is a well-established technique for field computations in any complex and heterogeneous media. As per this method, the field is represented by a complete set of fundamental differentiation equations with the restriction of linearity. The modeling is based on the solution of these differential equations. This FEM technique was applied to model the pulse-echo behavior of ultrasound transducers immersed in water by Lerch, Landes and Kaarmann [1]. They used the FEM to model the transducer and the reflector as well as the fluid environment to calculate the reflected acoustic pressure. The wave propagation between the transducer and reflector was calculated using the Helmholtz integral. The observation made in this case was that with FEM, it is possible to model very complex and realistic situations. However, due to the complex model, the computation time involved was long. Also, it can be interpreted that if the source and reflector are far apart, a large number of propagation steps will be required to propagate the wavefield from the transducer to the reflector and from the reflector back to the transducer. In such a case, the error introduced in the computation of the wavefield for each propagation step can accumulate to an unacceptably large error.
1.3.3 The Angular-Spectrum Method (ASM)

This modeling technique makes use of signal processing concepts. Using this method, an acoustic field can be decomposed into harmonic plane waves. Orofino and Pedersen have discussed a practical angular spectrum method based on the 2D-FFT [2]. This method was used to decompose normal velocity and pressure fields radiated by transducers of arbitrary shape into component plane waves with amplitudes and propagation directions determined by the temporal and spatial frequencies. This method was further extended to model received ultrasound signals from finite planar targets by Pedersen and Orofino [3]. Unlike, the FEM, the propagation from the transducer to reflector is directly achieved by a single phase term, thus avoiding the accumulative error problem. Thus, it is a useful technique for modeling reflections from planar reflectors of arbitrary size. However, the ASM is very computationally intensive. This is mainly because it is based on harmonic waves only, which are obtained on applying a temporal Fourier Transform to the acoustic fields before the ASM can be implemented. Also, spatial frequencies are dependent on the geometry of the transducer and reflector in this case, making their determination complicated.

1.3.4 The Spatial Impulse-Response Method (SIRM)

This is the method that is currently used for the calculation of pressure field from a variety of transducer types [3]. This method has not only been used to calculate the pulse-echo responses from a given transducer due to a point-like scatterer but has been extended to calculate the received signals due to extended reflector surfaces in a pulse-echo system. A lot of research related work has been done in applying SIRM to varying
pulse-echo systems. Weight and Hayman [4] came up with a derivation method for the received signal from a small reflector surface insonified by a transducer with short pulse excitation. This derivation is based on Rayleigh integral and the principle of acoustic reciprocity, which has been reviewed in Chapter 2, Section 2.1. McLaren and Weight made use of the results obtained in [4] to calculate the received signals from solid targets of various sizes interrogated by short pulses of ultrasound propagating in a fluid medium [5]. In addition to this, they also studied the effects of target size, field position and material on the amplitude and shape of the received signals. Lhemery then developed a model to predict the received signal from targets of complex geometry, with specific formulations for arbitrary shape targets with very high acoustic impedance, arbitrary acoustic impedance and near zero acoustic impedance [6]. Later, Li Wan and Pedersen applied the SIRM to model the pulse-echo system using annular array transducers and a flat reflector surface [7], and further calculate the received signal obtained from the reflector. The modeling technique discussed in this thesis has evolved from this work [7].

1.4 Current advances in ultrasound technology

As stated earlier, diagnostic ultrasound is becoming the preferred imaging modality in a variety of clinical situations. Also, since the equipment for ultrasound is less expensive than that used in radiographic, ionizing radiation techniques, it is becoming more widely available. In the past decades, there have been several engineering and technological innovations and breakthroughs to significantly improve the ultrasound image quality upon which the final diagnosis critically depends. Some of the techniques [8] which are
gaining attention are harmonic imaging, 3D and 4D imaging, very high frequency imaging, Doppler Ultrasound and Advanced Signal Processing.

Harmonic imaging has the capability to offer better resolution in medical imaging applications and provide more detailed and enhanced contrast images. This feature can be particularly helpful in the examining of heavy, technically challenging patients. 3D reconstruction of data obtained using an ultrasound device can provide vascular anatomy details not available using conventional gray scale, color, and power Doppler. 3D ultrasound shows great promise in improving the ability to detect and differentiate between many types of functional abnormalities in patients. 3D data provides digitally encoded images which can be manipulated by removing obscure images that may inhibit diagnosis. This feature is further providing to be of assistance to clinicians in the advanced planning of difficult surgeries. Very high frequency imaging has opened a different avenue towards the development of ultrasound images. The current imaging frequency range (1-15 MHz) can be enhanced by miniature or microsonography devices that can offer sub-millimeter resolution imaging at frequencies ranging from 20 to 100 MHz. These miniature devices operating at frequencies above 20 MHz are already available and can be placed within the blood vessels, urethras, etc., to study abnormalities from within. Also, higher frequencies can provide desirable improvement in spatial and temporal resolution, and are more quickly attenuated in the interrogated tissue than the lower frequencies. As a result, the penetration depth decreases with increasing frequencies. Doppler ultrasound is well suited for constant monitoring needed in tissue transplants. It can also be used as an early warning device for imminent rejection of a transplant. The future generation of ultrasound scanners will exhibit adequate Doppler
sensitivity to detect extremely low velocity (less than 10mm/s) which can make them useful in intra- and post micro-surgery, including implants to evaluate the flow in narrow vessels or capillaries affected by the procedure. Using Advanced Signal Processing techniques, the optimization of images can improve image quality, minimize the possible, operator dependent inconsistency in the images and, therefore, it can also contribute to the increase of diagnostic confidence. Such development can ensure that the application of ultrasound technology as the preferred imaging modality in a variety of clinical solutions will continue to grow. The optimization method for improving the quality of the received signal in an ultrasound pulse-echo system discussed in this thesis is fundamentally based on signal processing techniques.

1.5 Outline of the thesis

This section describes the overall content of the thesis. Each chapter of this thesis is individually summarized below for the convenience of the reader:

Chapter 2

This chapter primarily focuses on formulating an appropriate modeling technique for pulse-echo ultrasound systems. The former part of the chapter discusses the conventional Huygens method which is used as a numerical modeling tool for calculating the received signal in a pulse-echo ultrasound system. As we will see, when the received signal from an extended reflector is desired, the Huygens method approach is to tessellate the reflector surface into a large number of “microtiles” chosen so that all tiles are small, relative to the shortest wavelength in the frequency range of interest. The total received
signal is found as a sum of contributions from all the microtiles. Since this approach follows from the Huygens principle, it is referred to as the *Huygens Method*.

If the microtile size is chosen appropriately small, *Huygens method* is accurate, but unfortunately computationally demanding. This has motivated the development of a more effective modeling technique termed *Diffraction Response for Extended Area Method* (DREAM). Just as *Huygens Method*, DREAM determines the received electrical signal, in pulse-echo mode, from an extended reflector of arbitrary shape, location and orientation. The concept of diffraction response and an overview of the steps involved in obtaining the received signal using the DREAM modeling technique have been discussed in this chapter.

**Chapter 3**

This chapter provides a detailed study of the steps involved in obtaining the received signal from a given/arbitrary reflector geometry using the DREAM method, and explains the concept of tessellations: dividing the reflector surface into microtiles.

The former part of the chapter is mainly a discussion on what would be an ideal tessellation method to capture the geometrical properties of a given reflector surface. What is also discussed are the factors involved in selecting an accurate tessellation algorithm.
Further on, this chapter looks at various examples which help us to evaluate the DREAM method in comparison to the conventional *Huygens method* in terms of accuracy.

**Chapter 4**

In the first two chapters, we discuss the modeling of the ultrasound pulse-echo system, and the numerical modeling system, for a system with planar circular transmitter and receiver. This modeling concept can further be extended to a pulse-echo system utilizing a planar annular array transducer. The optimal design of acoustic fields and receiver characteristics using annular array transducers in addition to why we use annular array transducers for our research has been explained in this chapter.

**Chapter 5**

As we discussed earlier, this thesis has two objectives: formulating a numerical modeling tool to effectively predict the received signal obtained from a reflector and using that information to optimize the design of the pulse-echo system to enhance the image obtained from that object. The second, third and fourth chapters describe the numerical modeling method, while the sixth, seventh and eighth chapter look into the optimization concept. This chapter gives an overall perspective of the thought process behind this thesis, the conceptual approach that is followed. The chapter bridges the former part of the thesis with the latter portion.
Chapter 6

This chapter discusses in detail the optimization algorithms which have been developed to quantify specific aspects of a reflecting structure or to identify a given reflector geometry. It also states how the energy of the received signal can be used to quantify the feature of interest in the reflector, which further helps to identify the given reflector. What we will also be seeing is that the methods described in this chapter are actually not practical to execute in a real time ultrasound system, and hence they have been termed as the non-implementable optimization methods. About three such optimization methods have been presented with their advantages and drawbacks. Depending on their performance, one of them is selected as the appropriate optimization method. Although this selected method is non-implementable, we will see that it does form the basis for the selected implementable optimization method which is discussed in Chapter 7.

Chapter 7

This chapter describes some implementable optimization methods, which can be used to improve the received signal quality from a practical ultrasound pulse-echo system. These methods are discussed with their advantages and drawbacks, and their performance is evaluated. We will see that these methods have evolved from analytical and numerical approaches.

Chapter 8

As we know there are several factors that have to be accounted for, while modeling a received signal from a given reflector surface and further, to optimize the system
parameters for enhancement of the received signal from that reflector. It was important to see how robust the optimization results obtained for a given set of factors were, if these factors were to be delineated or modified by a comparable margin. This chapter discusses this robustness aspect. The selected non-implementable and implementable optimization algorithms are applied to a similar pulse-echo system like the one discussed in the previous chapters, but with physical changes in the dimensions and positioning of the reflector surface. The selected optimization method did work well and proved to be robust for the ultrasound system we defined.

Chapter 9

This is the conclusion chapter which discusses the inferences obtained from this thesis and the future work in this area.
Chapter 2

Modeling technique for a pulse-echo based ultrasound system

Modeling of pulse-echo ultrasound systems is a broad topic. There are different aspects of modeling ultrasound pulse-echo systems such as transducer modeling, acoustic field modeling etc. The specific goal of this thesis is to come up with an efficient and effective modeling technique for pulse-echo ultrasound systems with extended reflectors. There are several variables involved in obtaining the output signal from a given reflector surface in an ultrasound pulse-echo system. The excitation signal, the geometry, properties and location of the ultrasound transducers and size, geometry, location and orientation of the reflector on the other hand are some of the variables that add to the complexity of obtaining the output signal from the reflector. Numerical modeling is the only practical method that can be used to predict the output signal for a given measurement configuration, especially, when it comes to optimizing the design of an ultrasound system to carry out tasks such as identifying objects of specified shapes, determining surface topology or alignment of surface, etc.

There is one such numerical modeling tool that is used for calculating the received signal in a pulse-echo ultrasound system, which has originated from the Huygen’s principle. When the received signal from an extended reflector is desired, the approach is to tessellate the reflector surface into a large number of “microtiles” chosen so that all tiles are small, relative to the shortest wavelength in the spectrum of interest. The total
received signal is then found as a sum of contributions from all the microtiles. This approach is referred to as the Huygens Method as it follows from Huygens principle.

If the microtile size is chosen appropriately small, Huygens method is accurate, but unfortunately computationally demanding. This has motivated the development of a more effective modeling technique termed Diffraction Response for Extended Area Method (DREAM) [7]. Just like Huygens Method, DREAM determines the received electrical signal, in pulse-echo mode, from an extended reflector of arbitrary shape, location and orientation.

These numerical modeling methods will be briefly discussed in this chapter. To understand these methods better, it is important to study the concept of diffraction response. This concept has been discussed extensively in Li Wan’s thesis; in fact the former part of this chapter is a summary of the theory in Li Wan’s thesis. The following section explains the same.

2.1 Formulation of Diffraction Response

A sound field from a baffled planar piston source in a fluid can be accurately described by the Rayleigh integral. For a source with a radiating surface $S$ and a normal particle velocity function $u_n(\vec{r}, t)$, the Rayleigh integral for time-dependent velocity potential $\phi(\vec{r}, t)$ is given by (2.1) [9,13]
\[
\phi(\vec{r}, t) = \int_S \frac{u_s(\vec{r}_s, t - |\vec{r} - \vec{r}_s| / c)}{2\pi |\vec{r} - \vec{r}_s|} dS
\]

(2.1)

where \( \vec{r}_s \) represents points on the surface \( S \) and \( \vec{r} \) represents the position of the field point as shown in Figure 2.1. In the figure below, only one point has been illustrated, and its position from the center of the surface \( S \) has been represented as \( \vec{r}_s \). There can be several such points on the surface \( S \), and the velocity potential as calculated in (2.1) is a result obtained on considering all these points over the surface \( S \). The parameter \( c \) is the sound speed in the homogeneous propagation medium.

\[\text{Figure 2.1: Illustration of the simplest pulse-echo system with point scatterer as reflector.} [7]\]

If all of the points on the source vibrate with equal amplitude and in phase, i.e., the vibration of the piston is uniform, then \( u_n(\vec{r}_s, t) = u_n(t) \) on \( S \) and zero outside, and (2.1) can be expressed as (2.2).

\[
\phi(\vec{r}, t) = \int_S \frac{u_n(t - |\vec{r} - \vec{r}_s| / c)}{2\pi |\vec{r} - \vec{r}_s|} dS = u_n(t) \otimes h(\vec{r}, t)
\]

(2.2)
In (2.2), $\otimes$ is the convolution in time-domain, and $h(\vec{r}, t)$ is the spatial impulse response of the velocity potential and defined in (2.3) [13].

\[
h(\vec{r}, t) = \int \delta(t - \frac{|\vec{r} - \vec{r}_s|/c}{2\pi}) dS = \int \frac{\delta(t - r'/c)}{2\pi r'} dS
\]

(2.3)

Here $r' = |\vec{r} - \vec{r}_s|$. The pressure field at point $\vec{r}$, $p(\vec{r}, t)$, can be obtained from $\phi(\vec{r}, t)$ as [5,14]

\[
p(\vec{r}, t) = \rho_0 \frac{\partial \phi(\vec{r}, t)}{\partial t} = \rho_0 \frac{\partial [u_n(t) \otimes h(\vec{r}, t)]}{\partial t} = \rho_0 u_n(t) \otimes \frac{\partial h(\vec{r}, t)}{\partial t}
\]

(2.4)

where $\rho_0$ is the density of the medium in front of the transducer. The method discussed above is termed as the “Velocity Potential Impulse Response Method” or just the “Impulse Response Method”.

This impulse response method has been applied to calculate the received signal in a pulse-echo system. The principle of acoustic reciprocity [10, p.172] is the basis for the following derivation. One form of the acoustic reciprocity principle states that if the locations and orientations of a small source and a small receiver are interchanged, the received signal will remain the same. For pulsed radiation, the principle is stated as [2]: “For a given transducer in reception, the output voltage waveform due to a pulse emitted at a point is identical to the pressure waveform at that point resulting from transmission of the same pulse by the transducer.”

The first step towards determining the received signal in a pulse-echo system is to calculate the received signal due to small reflector surface with dimensions much smaller than a wavelength, i.e., point scatterer. This case is illustrated in Fig 2.2.
The point scatterer is at point \( \vec{r} \) and is subjected to the incident pressure field \( p_i(\vec{r},t) \).

The calculation of this pressure field is similar to the one given by (2.4). It can be assumed that the incident field is locally plane if it is observed over a very small region. It is also assumed that the impedance of the point scatterer is either zero or infinite; therefore, the reflected pressure magnitude at the surface of the point scatterer is equal to the incident pressure magnitude. According to the principle of acoustic reciprocity, the received signal for the receiving transducer can be calculated by assuming the point scatterer acts as a point source. The point source is characterized by its surface velocity \( u_s(\vec{r},t) \), which is [10, p.126]

\[
   u_s(\vec{r},t) = -\frac{p_i(\vec{r},t)}{\rho_0 c} \tag{2.5}
\]

and the surface velocity of the point source will create the reflected velocity potential over the surface of the receiver [11, p.298-303],
\[
\phi(\vec{r}, t) = \frac{u_e(\vec{r}, t - r'/c)}{4\pi r'} dA = -\frac{p_i(\vec{r}, t - r'/c)}{4\pi r' \rho_0 c} dA
\]

(2.6)

where \( r' \) is the distance from the point scatterer to the observation point on the transducer surface and \( dA \) is the small surface area of the point scatterer. By combining (2.4) and (2.6), the reflected pressure on the transducer, \( p_r(\vec{r}, t) \), can be found as:

\[
p_r(\vec{r}, t) = \rho_0 \cos[\theta(\vec{r}, t)] \frac{\partial \phi(\vec{r}, t)}{\partial t} = -\cos[\theta(\vec{r}, t)] \frac{\partial p_i(\vec{r}, t - r'/c)}{\partial t} \frac{dA}{4\pi r' c}
\]

(2.7)

In (2.7), \( \theta(\vec{r}, t) \) is the angle between the unit normal vector of the reflector surface and the particle velocity vector at \( \vec{r} \). The output voltage, \( dv_r(\vec{r}, t) \), due to the point scatterer is

\[
dv_r(\vec{r}, t) = E_r(t) \otimes p_r(\vec{r}, t)
\]

(2.8a)

For an extended reflector, with surface \( S \), and \( \vec{r} \)

\[
dv_r(\vec{r}, t) = E_r(t) \otimes \int_S p_r(\vec{r}, t) dS
\]

(2.8b)

where \( \vec{r} \) is the position vector for all points on \( S \) and in either case, \( E_r(t) \) is the acoustic-electrical impulse response of the receiving transducer. With several straightforward operations and applying (2.7), (2.8) can be rewritten as [2, 12]

\[
dv_r(\vec{r}, t) = -\frac{dA \rho_0}{2c} \cos[\theta(\vec{r}, t)]E_r(t) \otimes u_e(t) \otimes \frac{\partial^2}{\partial t^2} [h(t, \vec{r}, t) \otimes h_r(\vec{r}, t)]
\]

(2.9)

where \( u_e(t) \) is the uniform particle velocity on the surface of the transmitter and \( h(t, \vec{r}, t) \) and \( h_r(\vec{r}, t) \) are the velocity potential impulse response of the transmitter and receiver, respectively. Now, the received signal due to a small reflector surface with dimensions much smaller than a wavelength can be expressed as (2.10) by application of (2.9)
If we express \( u_n(t) \) as \( u_n(t) = v_{\text{exc}}(t) \otimes E_r(t) \) and then define \( E(t) = E_i(t) \otimes E_r(t) \), where \( E_i(t) \) is the acoustic-electrical impulse response of the transmitting transducer and \( v_{\text{exc}}(t) \) is the excitation voltage applied to the transmitting transducer, equation (2.10) can be rewritten as:

\[
dv_r(\bar{r}, t) = \frac{\rho_0}{c} \cos[\theta(\bar{r}, t)] E_r(t) \otimes u_n(t) \otimes \frac{\partial^2}{\partial t^2} [h_i(\bar{r}, t) \otimes h_r(\bar{r}, t)] dA \quad (2.11)
\]

To simplify the notation in (2.11), the Pulse-Echo Diffraction Impulse Response is defined as:

\[
D(\bar{r}, t) = \frac{\partial^2}{\partial t^2} [h_i(\bar{r}, t) \otimes h_r(\bar{r}, t)]
\]

Further simplification of (2.11) is achieved by doing the following:

i) Assuming that \( E(t) = \delta(t) \) and \( v_{\text{exc}}(t) = \delta(t) \). Although these assumptions are not realistic, they do not limit the practical value of the approach because the realistic functions for \( E(t) \) and \( v_{\text{exc}}(t) \) can be convolved onto the calculated response at any time in the process.

ii) Approximating \( \theta(\bar{r}, t) \) with \( \theta(\bar{r}) \).

iii) Defining \( A_i = \frac{\rho_0}{c} \). Applying these approximations and the definition for \( D(\bar{r}, t) \) to equation (2.11), the resulting expression becomes

\[
dv_r(\bar{r}, t) = A_i \cos[\theta(\bar{r})] D(\bar{r}, t) dA. \quad (2.13)
\]
Hence, it can be seen that it is not difficult to obtain the received signal in pulse-echo ultrasound system for a very small reflector surface as long as the diffraction response can be calculated.

2.2 Huygens Method

The Huygens method that was discussed in the previous section, to calculate the received signal for a very small reflector surface, is expanded here to calculate the received signal from an extended reflector. The most straightforward way to obtain the received signal from an extended reflector surface is to divide the reflector surface into a large number of planar small surface elements, calculate the received signals from each element and sum these received signals. This method is referred to as Huygens Method.

With the assumption of linearity, that is, considering that all effects due to multiple scattering, angle dependent reflection coefficients, etc. are excluded, the received signal from an extended reflector is just the integration or summation of the responses obtained by equation (2.13) over the reflector surface, as shown in (2.14)

\[
v_r(t) = A_t \int_A \cos[\theta(\vec{r})] D(\vec{r}, t) dA = A_t \sum \cos[\theta(\vec{r})] D(\vec{r}, t) \Delta A. \tag{2.14}
\]

As per the description here and in section 2.1, the steps for implementing Huygens method are as follows:

1) Divide (tessellate) the reflector into \( N \) microtiles, each of which is small enough to be treated as a simple source. A simple source means a reflector with
dimensions much smaller than the shortest wavelength in the frequency range of interest.

2) Calculate the velocity potential impulse responses of the transmitter transducer and receiving transducer: \( h_t(\vec{r}, t) \) and \( h_r(\vec{r}, t) \), using the multi-rate digital signal processing algorithm with final sampling rate of 400MHz, for a point on each microtile.

3) Calculate the diffraction response for each point: \( D(\vec{r}, t) = \frac{\partial^2}{\partial t^2} [h_t(\vec{r}, t) \otimes h_r(\vec{r}, t)] \).

4) Calculate received signal from the extended reflector using the following equation:

\[
v_r(t) = A_i \sum_{\text{all tiles}}^N \cos[\theta(\vec{r})] D(\vec{r}, t) \Delta A
\]

where \( A_i \) is a system related constant, \( \theta(\vec{r}) \) is the angle between the unit normal vector of the reflector surface and the particle velocity vector at the field point \( \vec{r} \); \( D(\vec{r}, t) \) is the diffraction response of one single field point and \( \Delta A \) is the area of the microtile, located at the field point, \( \vec{r} \).

2.3 DREAM method

The *Huygens method* described in the last section is an accurate and straightforward numerical modeling tool used to obtain the received signal from an extended reflector surface. However, the disadvantage is that the size of the individual surface elements must be chosen very small to satisfy the assumption of point source behavior. This further gives rise to a large number of integration points, and thus the computation time
to obtain the received signal for the whole reflector is quite long. Reducing the computation time is the motivation behind the DREAM method.

The DREAM method tessellates the surface into triangular tiles of moderate dimensions. The diffraction response, as previously defined above, is then evaluated at each corner of the tiles and the center of each tile. It is found that, for points situated not too far away from each other, the diffraction responses are quite similar. They have a similar shape or waveform and similar amplitude, but they differ in terms of their delays. Therefore, the diffraction response of the center of the triangular tile, with the propagation delay not considered, can be used to approximate the diffraction responses within the small triangular tile area. The unique aspect of DREAM is that the spatial integration of the diffraction response over surface of a given tile is replaced by an equivalent low pass filtering operation on the diffraction response at the center of the tile. Specifically, the low pass filter is defined by the relative delays of the diffraction responses from the corners of the tile, and this filter is therefore referred to as the “delay filter”, $F(t)$.

A given diffraction response consists of individual signal components separated by near zero amplitude intervals. These signal components are called segments. Each segment further exhibits unique time shifts and thus needs to be filtered by separate delay filters. Hence for a single transducer pulse-echo case, the echo signal for a given tile can be formulated as shown in (2.15), which consists of a summation over the segments in the diffraction response.
where \( A_i = \frac{A_o \rho_o}{c_o} \), \( A_o = \) reflection coeff., \( \rho_o = \) density and \( c_o = \) sound speed of medium.

The symbol \( \otimes \) denotes convolution. \( \cos[\varphi(\vec{r})] \) is a correction factor where \( \varphi(\vec{r}) \) combines the angle between tile normal and position vector \( \vec{r} \) which defines the location of tile relative to the transducer. \( M \) is the number of segments in the diffraction response, \( D_i(\vec{r}_{\text{center}}, t) \) is the diffraction response (with delay removed) of center of tile, and \( F_i(t) \) is the delay filter for the \( i^{th} \) segment.

It is also important to note that the dc component of the diffraction response is always zero and can be used as a check for a numerical method such as DREAM, to verify the accuracy of the results obtained.

Given below is a formal proof of this statement.

Diffraction response \( D(\vec{r}, t) \) is given as follows

\[
D(\vec{r}, t) = \frac{\partial^2}{\partial t^2} [h_i(\vec{r}, t) \otimes h_r(\vec{r}, t)]
\]

\[
= \frac{\partial^2}{\partial t^2} \left[ \frac{1}{2\pi} \int_S \frac{\delta(t - r/c)}{r} \, ds \otimes \frac{1}{2\pi} \int_S \frac{\delta(t - r/c)}{r} \, ds \right]
\]

Taking a Fourier transform of the above equation, we get
\[ D(\tilde{r}, \omega) = (j \omega)^2 \left[ \frac{1}{2\pi} \iint_{W} \frac{\exp(-j\omega r / c)}{r} dS \cdot \frac{1}{2\pi} \iint_{W} \frac{\exp(-j\omega r / c)}{r} dS \right] \]  

(2.16)

where (2.16) is obtained using the following Fourier transform formula:

\[
\begin{bmatrix}
\vdots \\
\frac{d^n}{dt^n} \cdot F^{-1} s(t) & \Rightarrow & (j\omega)^n S(\omega) \\
\delta(t-a) & \Rightarrow & \exp(-aj\omega)
\end{bmatrix}
\]

For dc component, \( \omega = 0 \)

\[
\therefore D(\tilde{r},0) = 0 \quad \ldots \ldots \text{from eq}(2.16)
\]

Hence the integrals are finite and evaluate to zero for \( \omega = 0 \) which proves the above statement.

The received signal from an extended, arbitrary surface is then calculated as the sum of received echo signals from the tiles which tessellate the extended surface, as shown in (2.17).

\[
v^{\text{REFL}}(t) = \sum_{\text{all tiles}} v_{\text{tile}}(t). \quad (2.17)
\]
At this stage, it is also important to explain in further detail the steps involved in applying segmentation and delay filtering, to validate the correctness of the pulse-echo system simulation based on the DREAM method.

2.3.1 Segmentation

The procedure for deciding the segments of a single diffraction response is as follows:

1) Finding the peak points of the overall diffraction response and identifying the maximal amplitude among these peak points of the response, i.e. finding the largest peak. Referring to this maximal as “the largest response amplitude”.

2) Then identifying the other extremes (or peaks). If the amplitude of one extreme is larger than a specified fraction of “the largest response amplitude”, it can be considered as peak point of the diffraction response. For our application, the specified fractional value or threshold is set to 5% of the “the largest response amplitude”.

3) Segmenting the signal with one peak point per segment, and setting the segment boundaries to occur at the time instance between the peak points where the amplitude is closest to zero.

4) Checking the boundaries between the segments in the following manner. If the amplitude at a given boundary is above a specified fraction, i.e., 20% of the smaller amplitude of peak points, the two segments separated by that boundary are merged into one segment.
2.3.2 Delay Filtering

A delay filter is used to filter the reference diffraction response for a tile i.e., the diffraction response at a reference point on the tile, which we have specified as the center of the tile and where the propagation delay can be removed, as has been discussed in the former part of this section. In the case of the triangular tile, this reference point is the center of the triangle. The delay filter function is calculated using the concept of delay linearization plane [7, p.31].

The delay filter function $F(t)$ is given by [7]:

$$
F(t) = \begin{cases} 
(t - \tau_{\text{min}}) \times 2 \times \text{area} (\Delta), & \tau_{\text{min}} \leq t < t_{\text{med}}; \\
(t_{\text{med}} - \tau_{\text{min}})(\tau_{\text{max}} - \tau_{\text{min}}), & t_{\text{med}} \leq t < \tau_{\text{max}}; \\
-(t - \tau_{\text{max}}) \times 2 \times \text{area} (\Delta), & \tau_{\text{min}} \leq t < \tau_{\text{med}}; \\
(t_{\text{max}} - \tau_{\text{med}})(\tau_{\text{max}} - \tau_{\text{min}}), & t_{\text{med}} \leq t < \tau_{\text{max}}; \\
0, & \text{otherwise}; 
\end{cases} \quad (2.18)
$$

where $\tau_{\text{min}}, \tau_{\text{med}}, \tau_{\text{max}}$ are the minimal, median and maximal of the delay values of the three corners of the triangular tile, respectively, and area($\Delta$) is the area of the triangular tile.

The filter function $F(t)$ is triangular in shape and is shown in Fig.2.3:
Figure 2.3: Delay filter function $F(t)$ for a triangular tile

However, when the duration of $F(t)$ is less than one sampling period, the original triangular filter function defaults to an impulse function. A proper weight $A$ should be assigned to this impulse function. The way to find $A$ is to make the area of the real continuous time function $F(t)$ equal to the weight of the equivalent impulse function the same, i.e. $\int_{-\infty}^{\infty} F(t) dt = \int_{-\infty}^{\infty} A \delta(t) dt$, from which it is easy to find that $A = area(\Delta)$.

Some implementation examples of the segmentation and delay filtering techniques and the actual appearance of the diffraction responses from one tile of an extended reflector surface will be illustrated in a later chapter.

On this basis, the data processing for the DREAM method can now be summarized as follows:

1) Divide (tessellate) the reflector surface into $M$ triangular tiles, which are small enough to apply DREAM. Normally, $M << N$ where $N$ is the number of microtiles tessellated by the Huygens Method.
2) Calculate the velocity potential impulse responses of the transmitter transducer and receiving transducer: $h_t(\vec{r}, t)$ and $h_r(\vec{r}, t)$, using the multi-rate digital signal processing algorithm with final sampling rate of 400MHz, for the corners and the center of the tile.

3) Calculate the diffraction response for the corners and center of the tile:

$$D(\vec{r}, t) = \frac{\partial^2}{\partial t^2} [h_t(\vec{r}, t) \otimes h_r(\vec{r}, t)].$$

4) Segment the diffraction responses from the corners and the center of the tile.

5) Determine delay filters for each segment of the responses from the corners of the tile.

6) Calculate the received signal from each tile, using segmentation and delay filtering: $v_{tile}(t)$.

7) Calculate overall received signal from the entire reflector: $v_{all}(t) = \sum_{\text{tile}} v_{tile}(t)$.

8) Calculate the spectra of the received signal: $V_{all}(\omega) = F[v_{all}(t)]$.

We have tested the DREAM method for accuracy in comparison to the standard Huygens Method, and the results for the same will be discussed in Chapter 4. As per the evaluation results, the error in the accuracy is acceptably small and hence, DREAM has been used as the numerical modeling tool for the ultrasound pulse-echo system, in this research.
Chapter 3

Evaluation of a Method for Tessellating Reflector Surfaces for DREAM

As was discussed in the previous chapter, the Huygens Method for calculating the received signal in a pulse-echo ultrasound system from an extended reflector is based on the concept of receiving a signal from a point scatterer in a pulse-echo system, using the velocity potential impulse response method. Thus, the reflector surface needs to be divided into elements, the size of which must be smaller than the shortest wavelength in the frequency range of interest, so as to satisfy the point source behavior.

The DREAM method is conceptually identical to the Huygens method, but allows for much larger tiles, and thus much less computation time. It is also based on the velocity potential impulse response and mainly comprises of the following two tasks:

1) Dividing the reflector surface into reflector elements (tiles) of moderate dimensions, such that the tessellated tiles are chosen to approximate the reflector surface well.

2) Calculating the received signal contribution from each tile and summing the received signals.

To consider the trade-off between the computation time and accuracy for DREAM method, we need to find an efficient tessellation method for arbitrary geometry reflector surfaces, which is the objective of this chapter.
3.1. Introduction to Tessellations

Before we move ahead with the factors in a pulse echo system, that may affect the type (shape, size, orientation etc.) of tiles we are using, it is important to note certain basic properties of tessellations. *The tessellation of a plane by polygons is a collection of the polygons that cover the plane without gaps or overlaps.* A regular polygon has 3, 4, 5 or more sides and angles, all equal. When a tessellation is made up of regular polygons of the same size and shape, the tessellation is a regular tessellation.

Three types of regular polygons are used for tessellations in the Euclidean plane: triangles, squares or hexagons. Given below are examples of these three tessellations:

![A tessellation of triangles](image)

![A tessellation of squares](image)

![A tessellation of hexagons](image)

*Figure 3.1: Different types of tessellations*

It can easily be seen that the squares are lined up with each other while the triangles and hexagons are not. Also, if one looks at 6 triangles at a time, they form a hexagon, so the tiling of triangles and hexagons are similar and they cannot be formed by directly lining shapes up under each other. Since the regular polygons in a tessellation must fill the plane
at each node, the interior angle must be an exact divisor of 360°. This works for the triangle, square and hexagon, and thus we can tessellate surfaces using these figures. For all other types of polygons, the interior angles are not exact divisors of 360°, and therefore those figures cannot tile or tessellate the plane. It is also very important to consider the boundary conditions of a surface with arbitrary geometry during tessellation. It can be said that the square tessellations are comparatively less accurate than triangular tessellations for filling up surfaces with curved edges or boundaries, due to the fact that triangles can be fitted in more easily as they have less number of edges than the square. Also there are software packages for surface tessellation into triangular elements available [18], which makes the practical applications of using triangular tiles much easy to implement.

3.2. The R-DREAM and T-DREAM methods

The DREAM algorithm as described in Chapter 2, has been implemented with both square (R-DREAM) and triangular (T-DREAM) tiles. From observing the diffraction responses (as discussed in section 2.1, equation (2.11)) of the individual field points, it is found that, for points situated not too far away from each other, the responses are quite similar in appearance. Specifically, they have a similar shape or waveform and similar amplitude, but different delays. Hence, in order to ignore the amplitude variation for our analysis, it is important to keep the dimensions of the tiles as small as possible.

Considering the trade-off between the computational time and accuracy, an optimal tile size was found to depend on several parameters. The typical size of the tiles used for
DREAM tessellation has been found to be small enough to tessellate square reflector surfaces in the order of 0.5 mm*0.5mm or 1mm*1mm [7]. A received signal obtained with a given tile size is dependent on the parameters of the specific simulation scenario such as the tilt angle of the reflector surface, the shape and size of reflector, the radial location of the tile on the reflector surface relative to the radius of the transducer, etc. Therefore, the optimal tile size which produces a received signal as accurate as the one obtained using the tested Huygen’s method (as described in Chapter 2), was found empirically by the “trial and error method” and was dependent on a specific simulation scenario.

A simulation of a pulse echo system with a defined transducer and reflector geometry was described in a previous thesis [7], and different tile sizes and shapes (square or triangular) were tried out. The received signal was further calculated using the DREAM method, and the accuracy of the method was verified by checking that the received signal result is within a predefined error level as compared to that obtained by the Huygens method. Considering most modeling situations involve the received signal from an extended reflector, a normalized DREAM Error (or MSE) in which the mean square error of a small tile is normalized by the energy of the received signal from a large reflector was defined [7]. This normalized DREAM Error was defined as follows:

$$\text{Normalized DREAM Error} = \frac{\int_{0}^{15\text{MHz}} \left( |V_{r_{-tile}}(f)| - |V_{ref_{-tile}}(f)| \right)^2 df}{\int_{0}^{15\text{MHz}} |V_{ref_{-reflector}}(f)|^2 df} \times 100\% \quad (3.1)$$
In the work done so far [7], the optimal tile size in the DREAM method was defined as the size which gives the shortest computation time, and which at the same time keeps the mean square error of the result obtained by DREAM less than 0.2%, compared with the result obtained by Huygens method, which is chosen as a reference.

Let us briefly see what analysis has been done so far to come up with an effective tessellation tool. The analysis not only takes into account some factors that we must consider in order to design a more reliable tessellation method, but also the drawbacks of the existing tessellation tool. Most importantly, since the optimal tile size is to be chosen so that the tessellated tiles can approximate the surface accurately, the more complicated the shape of the reflector surface is, the smaller the tile size that should be chosen. As per the research done so far [7], the largest possible side length of the square tile of R-DREAM (DREAM method using square tiles for tessellation) was limited to 1 mm and the largest possible side length of the triangular tile of T-DREAM (DREAM method using right angled triangular tiles with two 45° angles for tessellation) was limited to 1.414 mm which corresponds to the diagonal line of the tile in this case. For the ease of tessellation, the tessellated triangular tiles were all right triangles generated by splitting the square tile along the diagonal line. However, the T-DREAM is equally applicable to arbitrary shaped triangular tiles. Furthermore, it is important to note that tessellation using arbitrary triangular tiles will definitely be able to produce more accurate results because the arbitrary shaped triangular tiles can approximate the complex reflector surface more effectively. This was also one of the reasons why so far T-DREAM was preferred over R-DREAM.
There was one other reason, which helped us to decide on the use of triangular tiles as the optimal solution for tessellating the reflector surface using DREAM. When the delay of the diffraction response as a function of the position on the tile (square or triangular) area is represented as a linear function of its co-ordinates $u$ and $v$, a delay linearization plane is obtained [7]. It was observed that the delay linearization plane of T-DREAM is exactly determined by the delays of the three corners of the triangular element. For R-DREAM, the delay linearization plane is over-determined because of the availability of the delays of four corners; therefore, leading to the use of an approximated delay linearization plane. Thus although the derivation of the delay linearization plane for R-DREAM was much more straightforward, the T-DREAM results were more accurate. This was our primary motivation to consider a triangulation algorithm for mesh generation. The Delaunay algorithm, which has been discussed in the next section is an efficient triangulation algorithm. We used this algorithm to generate a mesh of equilateral triangles unlike the T-DREAM discussed above, which used the right angled isosceles triangles for tessellations.

### 3.3. Delaunay Triangulation

The Delaunay triangulation [16] is a tessellation method that has enjoyed great popularity in mesh generation ever since mesh generation was in its infancy. It is a method of constructing a surface mesh in a form, suitable for computer graphics hardware. In general, a triangulation method connects a given set of points or vertices with lines resulting into sets of triangles. There can be more than one set of triangulations possible,
given a set of points. In two dimensions, the Delaunay triangulation of a vertex set maximizes the minimum angle among all possible triangulations of that vertex set. It has been proved to be one of the most reliable and efficient tools for triangulation.

The input to the algorithm is a set of points in 2D-space (i.e. a plane) and in the case of a surface in 3D space, the height of the surface at a particular point needs to be provided as well. In the 3D case, the reference plane to calculate the height will generally be a horizontal plane passing through the center of the 3D object. The output is the connectivity information describing the surface as a series of triangles. Triangles are desirable from a computer graphics perspective because they are efficient in storage, processing and rendering. The effectiveness comes from the fact that after the first triangle has been specified, it takes only one vertex and two edges to extend this triangle by another triangle. This leads to \( k \) triangles requiring \( k+2 \) vertices and \( 2k+1 \) edges, both of which are more efficient than a triangular mesh of arbitrary connectivity.

A description of the basic concept behind the Delaunay triangulation algorithm is as follows. In two dimensions (i.e. a plane), a triangulation of a set \( V \) of vertices is a set \( T \) of triangles whose vertices collectively add to \( V \), whose interiors do not intersect each other, and whose union is the *convex hull* of \( V \), if every triangle intersects \( V \) only at the triangle’s vertices. The Delaunay triangulation of \( V \), introduced by Delaunay, is the graph defined as follows. Any circle in the plane is said to be empty if it encloses no vertex of \( V \) in its interior. (Vertices are permitted on the circle.) Let \( u \) and \( v \) be any two vertices in the set \( V \). A *circumcircle* (circumscribing circle) of the edge \( uv \) is a part of the Delaunay
triangulation, if and only if there exists an empty circumcircle of $uv$. An edge satisfying this property is said to be a Delaunay edge [16].

From the definition above, the Delaunay triangulation of a vertex set is clearly unique. Every edge that lies on the boundary of the convex hull of a vertex and has no vertex in its interior is Delaunay. For any edge $e$, lying in the convex hull, it is always possible to find an empty circumcircle of $e$ by starting with the smallest circumcircle of $e$ and “growing” it away from the triangulation as shown in the Fig 3.2.

![Diagram](image)

**Figure 3.2:** Each edge on the convex hull is Delaunay, because it is always possible to find an empty circle that passes through its endpoints. [16]

---

*The convex hull of a set of points is the smallest convex set that includes the points. In two dimensions it is a convex polygon.*

*The circumcircle is a triangle's circumscribed circle, i.e., the unique circle that passes through each of the triangles three vertices.*
Every edge connecting a vertex to its nearest neighbor has to be a Delaunay edge. For example if \( w \) is the vertex nearest \( v \), the smallest circle passing through \( v \) and \( w \) will definitely not enclose any vertices, and thus satisfy the definition of Delaunay triangulation.

Again, from the definition given above, it is important to note that the Delaunay triangulation is guaranteed to be a triangulation only if the vertices of \( V \) are in a general position, here meaning that no four vertices of \( V \) lie on a common circle. As we have seen, the circumcircle of a triangle is the unique circle that passes through all three of its vertices. A triangle is said to be Delaunay if and only if its circumcircle is empty. This defining characteristic of Delaunay triangles is illustrated in Fig 3.3 below and is called the empty circumcircle property.

![Figure 3.3: Every triangle of a Delaunay triangulation has an empty circumcircle [3]](image)

Delaunay triangulation works best when the surface has only small variations in the vertex density, that is the vertices are evenly spaced in 3D-Space. When this is the case, the algorithm will select triangles that are as close as possible to equilateral, resulting in
an efficient and attractive surface. Further, the original data points are preserved, so no data is lost or approximated through interpolation. Even if the surface has large variations in the vertex density, that is vertices which in some areas are closely grouped and in other areas are spread into vast plains, the algorithm may potentially choose long, thin triangles in the regions of highly varying vertex density.

However, there are certain limitations of Delaunay triangulations. The algorithm suffers from being computationally slow to generate an optimal tessellated surface. This can usually be avoided by allowing the surface to be close enough to the actual surface geometrically, and trying to find semi-optimal surfaces, which would be parts of the whole surface under consideration. Also data sets containing similar data points and features sometime have dissimilar surfaces generated by Delaunay triangulation. A small change in a single vertex position may alter the surrounding triangles, but this change may then have subsequent effects and repercussions throughout the rest of the surface. Thus although we can say that Delaunay meshes are good approximations to the actual surfaces, it is not necessary that the Delaunay meshes will be similar, for similar surfaces.

3.4. Desirable Properties of Meshes and Mesh Generation Tools

While considering a suitable meshing tool for software applications it is mandatory that the mesh conforms to the object or domain being modeled, and ideally should meet constraints on both the size and shape of its elements. In order to evaluate the different techniques it is useful to note the following features of automated meshing techniques [16,17]. Also, here we are talking about a general meshing tool and not just the Delaunay
triangulation method. Hence we will use general meshing terms like nodes instead of nodes.

1. **Precise modeling of surface boundaries:** Nodes at the boundary of a surface must lie precisely on the boundary of the structure. In two dimensional structures the location of interior nodes is less critical (provided the acceptable element shapes are obtained). There should be no limitations on the forms of boundary curves that can be accurately modeled. Boundaries may appear in the interior of a region as well as exterior surfaces. Exterior boundaries separate meshed and unmeshed portions of space, and are found on the outer surface and on the edges if internal holes exist in a surface. Interior boundaries appear within meshed portions of space, and enforce the constraint that elements may not pierce them. These boundaries are typically used to separate regions that have different physical properties.

2. **Good correlation between the interior mesh and information on the mesh boundary:** The curvatures and node spacings on the boundaries of the region should be well represented in the interior of the mesh. This allows the user running the triangulation algorithm to control the shape of elements in the interior of the region in a predictable fashion and thus to refine the spacing of the mesh. Unnecessary refinement of the mesh leading to wasted computations is also avoided.

3. **Minimal input information:** The amount of input data required should be reduced as much as possible. This will also reduce the chances of introducing human error into the analysis. The input information should be in a form convenient to the user that can be readily communicated to the computer.
4. **Wide range of applicability:** It is actually desirable to use a small set of mesh generation techniques that can be applied to a broad range of structural topographies, rather than to use a larger set of special purpose mesh generators. This will minimize the user learning time, the program development time and the program size. However, for our research a special mesh generation case needs to be considered which could possibly be generalized for any type of reflector surfaces in a pulse-echo ultrasound system.

5. **General topology:** The method of meshing should not impose any restrictions on the topology of a mesh within a region.

6. **Automatic topology generation:** The means of generating a mesh should create element connectivity without user intervention. Although this feature may be in conflict with the need for a general topology, this reduces the required amount of user input.

7. **Optimal numbering patterns:** The numbering of nodes and elements within the structure should be arranged such that they can be tracked after applying the meshing algorithm. For multi-region structures, interface nodes common to two or more adjacent regions should appear only once in the database.

8. **Computational efficiency:** The method of mesh generation should make efficient use of computer resources to minimize expense and to provide good response when applied in an interactive environment.

9. **Control over size of elements:** It is very important to have as much control as possible over the sizes of elements in the mesh. This control would include the ability to grade from small to large elements over a relatively short distance thus providing the option to have different local concentrations of meshes over a given surface. A mesh generator should offer rapid gradation from small to large sizes. Small, densely packed
elements offer more accuracy than larger, sparsely packed elements; but the computation time required to solve a problem is proportional to the number of elements. Hence, choosing an element size entails trading off speed and accuracy. Furthermore, the element size required to attain a given amount of accuracy depends upon the behavior of the physical phenomenon being modeled, and may vary throughout the problem domain.

3.5. Relation between System Properties and Tessellations

When coming up with an automated mesh generation technique it is important to examine the results that have been obtained in the research done so far [7] and thus generalize the properties of the system, for better efficiency and accuracy. As mentioned earlier, it is necessary to consider the trade-off between the computation time and accuracy while applying the DREAM method. We need to find some optimal tile size, which takes care of these issues. For the DREAM method to produce the received signal with a small mean square error, the tile size should be chosen so that the diffraction responses from the corners of the tile do not differ too much. The factors that may cause the change of diffraction response in both waveform (shape) and amplitude include the radial position of the field point and the radii of the transmitting and receiving transducers (planar circular piston transducers).

As seen in the previous chapter, it is actually the “change of diffraction response delay”, which is the key parameter in the calculation of the received signal for a pulse-echo system using DREAM method with the optimal tile size. However, it is hard to accurately/mathematically describe the “change of diffraction response delay” precisely.
Instead, in order to find the optimal tile size, we use an empirical method, where the mean square error of the received signal obtained by DREAM method with a given tile size is compared with the received signal, obtained by Huygens method. A large mean square error means the tile size is too large for the delay linearization of DREAM method to produce good approximation. The larger the error is when evaluated with a given tile size, the smaller is the proper tile size that must be chosen for DREAM method.

To develop rules for the optimal tile size for a range of measurement situations, the relationship between the mean square error and factors closely related to the change of diffraction response such as the reflector position, the radii of the transmitting and receiving transducers were observed. By doing this, an idea about how these factors affect the choice of tile size used by DREAM method was derived. The mean square error was named as “DREAM error” and was calculated over the frequency range from 0-15MHz as formulated in (3.1). The observations of these trial and error experiments and variations of the DREAM error over the surface of the reflector helped to define some general tile sizes over a reflector surface. These observations have been listed in table 1 and 2 in section 3.6. The radial distance in the second column specifies the distance between the point on the reflector surface and the point where the reflector intersects the transducer axis.

**3.6. Specifications for our system**

To date, many approaches of mesh generation have been studied with a view towards developing a versatile system that would require minimal user interaction. In our case, we
need a tessellation technique that can create finite elements over a domain composed of many irregular sub-regions where these regions need to be demarcated owing to the difference in the characteristic properties that are being measured along them.

Considering the above theory in devising a suitable tessellation, we decided to triangulate the reflector surface using the Delaunay triangulation method provided by MATLAB (since the existing system model is in MATLAB) and use regular equilateral triangles for tiling since they are efficient and help to provide the most accurate results using a comparatively faster computation time.

Currently, for the convenience of tessellation, if a tile size would not produce the results with required accuracy, it was tessellated into four smaller tiles with equal area. In the case of the R-DREAM, the tile side length was set to be either 1000µm, 500µm, 250µm or 125µm. For the T-DREAM, the hypotenuse was set to be either 1414µm, 707µm, 354µm or 177µm. A planar reflector surface with different tilt angles was considered to study the effects of a tilt in the surface on the optimal tile size. When the tilt angle of the reflector is small (less than 2° or 3°), a 1000µm or 500µm tile size was used for R-DREAM, and thus a 1414µm was used for T-DREAM, in most situations. The following table illustrates the optimal tile size for the R-DREAM, when the reflector is small (with the dimension of 1mm*1mm) and tilted around 6° with respect to the transducer surface [2]. Corresponding tables have been developed for smaller tilt angles.
Table 3.1: Summary of optimal tile size for R-DREAM when the reflector is small and tilted around 6º with respect to the transducer surface [15].

<table>
<thead>
<tr>
<th>Radii of the transmit and receive transducers</th>
<th>Reflector radial position $r$</th>
<th>R-DREAM optimal tile size</th>
<th>Height of the equilateral triangle tile</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3mm$  $3mm$</td>
<td>$r &lt; 3.5mm$</td>
<td>125 ${\mu}$m</td>
<td>125 ${\mu}$m</td>
</tr>
<tr>
<td></td>
<td>$3.5mm \leq r &lt; 5mm$</td>
<td>250 ${\mu}$m</td>
<td>250 ${\mu}$m</td>
</tr>
<tr>
<td></td>
<td>$5mm \leq r &lt; 11mm$</td>
<td>500 ${\mu}$m</td>
<td>500 ${\mu}$m</td>
</tr>
<tr>
<td>$3mm$  $6.3mm$</td>
<td>$r &lt; 1.5mm$</td>
<td>125 ${\mu}$m</td>
<td>125 ${\mu}$m</td>
</tr>
<tr>
<td></td>
<td>$1.5mm \leq r &lt; 5mm$</td>
<td>250 ${\mu}$m</td>
<td>250 ${\mu}$m</td>
</tr>
<tr>
<td></td>
<td>$5mm \leq r &lt; 11mm$</td>
<td>500 ${\mu}$m</td>
<td>500 ${\mu}$m</td>
</tr>
<tr>
<td>$3mm$  $9mm$</td>
<td>$r &lt; 1.5mm$</td>
<td>125 ${\mu}$m</td>
<td>125 ${\mu}$m</td>
</tr>
<tr>
<td></td>
<td>$1.5mm \leq r &lt; 5mm$</td>
<td>250 ${\mu}$m</td>
<td>250 ${\mu}$m</td>
</tr>
<tr>
<td></td>
<td>$5mm \leq r &lt; 11mm$</td>
<td>500 ${\mu}$m</td>
<td>500 ${\mu}$m</td>
</tr>
<tr>
<td>$3mm$  $12.7mm$</td>
<td>$r &lt; 1.5mm$</td>
<td>125 ${\mu}$m</td>
<td>125 ${\mu}$m</td>
</tr>
<tr>
<td></td>
<td>$1.5mm \leq r &lt; 5mm$</td>
<td>250 ${\mu}$m</td>
<td>250 ${\mu}$m</td>
</tr>
<tr>
<td></td>
<td>$5mm \leq r &lt; 11mm$</td>
<td>500 ${\mu}$m</td>
<td>500 ${\mu}$m</td>
</tr>
<tr>
<td>$6.3mm$ $6.3mm$</td>
<td>$r &lt; 1.5mm; 4mm \leq r &lt; 7mm$</td>
<td>250 ${\mu}$m</td>
<td>250 ${\mu}$m</td>
</tr>
<tr>
<td></td>
<td>$1.5mm \leq r &lt; 4mm; 7mm \leq r &lt; 9mm \leq r &lt; 11mm$</td>
<td>500 ${\mu}$m</td>
<td>500 ${\mu}$m</td>
</tr>
<tr>
<td>$6.3mm$ $9mm$</td>
<td>$r &lt; 1.5mm$</td>
<td>125 ${\mu}$m</td>
<td>125 ${\mu}$m</td>
</tr>
<tr>
<td></td>
<td>$1.5mm \leq r &lt; 9mm$</td>
<td>250 ${\mu}$m</td>
<td>250 ${\mu}$m</td>
</tr>
<tr>
<td></td>
<td>$9mm \leq r &lt; 11mm$</td>
<td>500 ${\mu}$m</td>
<td>500 ${\mu}$m</td>
</tr>
<tr>
<td>$6.3mm$ $12.7mm$</td>
<td>$r &lt; 1.5mm$</td>
<td>125 ${\mu}$m</td>
<td>125 ${\mu}$m</td>
</tr>
<tr>
<td></td>
<td>$1.5mm \leq r &lt; 9mm$</td>
<td>250 ${\mu}$m</td>
<td>250 ${\mu}$m</td>
</tr>
<tr>
<td></td>
<td>$9mm \leq r &lt; 11mm$</td>
<td>500 ${\mu}$m</td>
<td>500 ${\mu}$m</td>
</tr>
<tr>
<td>$r &lt; 2mm$</td>
<td></td>
<td>125 ${\mu}$m</td>
<td>125 ${\mu}$m</td>
</tr>
<tr>
<td>$2mm \leq r &lt; 11mm$</td>
<td></td>
<td>250 ${\mu}$m</td>
<td>250 ${\mu}$m</td>
</tr>
<tr>
<td>$r &lt; 2mm$</td>
<td></td>
<td>125 ${\mu}$m</td>
<td>125 ${\mu}$m</td>
</tr>
<tr>
<td>$2mm \leq r &lt; 11mm$</td>
<td></td>
<td>250 ${\mu}$m</td>
<td>250 ${\mu}$m</td>
</tr>
<tr>
<td>$r &lt; 2mm$</td>
<td></td>
<td>125 ${\mu}$m</td>
<td>125 ${\mu}$m</td>
</tr>
<tr>
<td>$2mm \leq r &lt; 11mm$</td>
<td></td>
<td>250 ${\mu}$m</td>
<td>250 ${\mu}$m</td>
</tr>
</tbody>
</table>

When the reflector is large and covers both regions near the transducer axis and the regions far from the transducer axis, the rules for the optimal size are a little different from those for the small reflectors. Table 3.2 summarizes the optimal tile size for R-DREAM when the reflector is large and tilted around 6º with respect to the transducer.
We can observe that the dimensions are almost the same, however the range of radii values for which they exist differs. For large reflector surfaces, we can say that the resultant tessellation effect is almost equivalent to moving a small reflector along the large reflector (along the diagonal of the large reflector surface incase of a flat planar surface).

**Table 3.2:** Summary of optimal tile size for R-DREAM when the reflector is large and tilted around 6° with respect to the transducer surface [15]

<table>
<thead>
<tr>
<th>radii of the transmit and receive transducers</th>
<th>reflector radial position r</th>
<th>R-DREAM optimal tile size</th>
<th>Height of the equilateral triangle tile</th>
</tr>
</thead>
<tbody>
<tr>
<td>3mm 3mm</td>
<td>$r &lt; 4\text{mm}$</td>
<td>125(\mu\text{m})</td>
<td>125(\mu\text{m})</td>
</tr>
<tr>
<td>3mm 6.3mm</td>
<td>$4\text{mm} \leq r &lt; 11\text{mm}$</td>
<td>500(\mu\text{m})</td>
<td>500(\mu\text{m})</td>
</tr>
<tr>
<td>3mm 9mm</td>
<td>$r &lt; 3\text{mm}$</td>
<td>125(\mu\text{m})</td>
<td>125(\mu\text{m})</td>
</tr>
<tr>
<td>3mm 12.7mm</td>
<td>$3\text{mm} \leq r &lt; 11\text{mm}$</td>
<td>500(\mu\text{m})</td>
<td>500(\mu\text{m})</td>
</tr>
<tr>
<td>6.3mm 6.3mm</td>
<td>$r &lt; 2\text{mm}$</td>
<td>250(\mu\text{m})</td>
<td>250(\mu\text{m})</td>
</tr>
<tr>
<td>6.3mm 9mm</td>
<td>$2\text{mm} \leq r &lt; 6.3\text{mm}$</td>
<td>500(\mu\text{m})</td>
<td>500(\mu\text{m})</td>
</tr>
<tr>
<td>6.3mm 12.7mm</td>
<td>$6.3\text{mm} \leq r &lt; 11\text{mm}$</td>
<td>1000(\mu\text{m})</td>
<td>1000(\mu\text{m})</td>
</tr>
<tr>
<td>9mm 6.3mm</td>
<td>$r &lt; 6.3\text{mm}$</td>
<td>500(\mu\text{m})</td>
<td>500(\mu\text{m})</td>
</tr>
<tr>
<td>9mm 12.7mm</td>
<td>$6.3\text{mm} \leq r &lt; 11\text{mm}$</td>
<td>1000(\mu\text{m})</td>
<td>1000(\mu\text{m})</td>
</tr>
<tr>
<td>12.7mm 9mm</td>
<td>$r &lt; 9\text{mm}$</td>
<td>500(\mu\text{m})</td>
<td>500(\mu\text{m})</td>
</tr>
<tr>
<td>12.7mm 12.7mm</td>
<td>$9\text{mm} \leq r &lt; 11\text{mm}$</td>
<td>1000(\mu\text{m})</td>
<td>1000(\mu\text{m})</td>
</tr>
</tbody>
</table>
Just as the dimensions of the right angled triangle for T-DREAM evolved from R-DREAM, we derived the dimensions of the equilateral triangles selected for Delaunay triangulation using the lengths of the squares’ sides using R-DREAM. We will consider the length of the square tile used in the R-DREAM tables 3.1 and 3.2 to be equal to the height of the equilateral triangle tile and thus calculate the respective equal side lengths of the equilateral triangle. *For example:* for R-DREAM optimal tile size = 500µm, height of the corresponding equilateral triangular tile = 500µm, therefore length of the side of the equilateral triangle = $0.866 \times 500µm = 433µm$.

### 3.7. Tessellation algorithm for our system

The algorithm for tessellating and obtaining the diffraction impulse response of an extended reflector surface, for our simulation, is given below. This is a generic algorithm that can be implemented in any computer language. The MATLAB version of this code which was actually executed is a part of Appendix A.

Description of the tessellation algorithm:

The algorithm has been implemented in MATLAB as a method called `tessellate(T,M)`.

**Method:** `tessellate(T,M)`

**Explanation for input parameters** $T,M$

Given: An array “A” of applicable tile sizes, used to give optimal results by the DREAM numerical modeling method as compared to the computationally demanding Huygen’s numerical modeling method, which is based on the Huygen’s Principle.
\[ M \Rightarrow \text{Dimension of the reflector surface.} \]
\[ \quad \text{e.g. for a 25 x 25 mm reflector surface, } M = 0.025 \text{ (in meters)} \]

\[ T \Rightarrow \text{Position of required tile size from the above given array.} \]
\[ \quad \text{e.g. Let the given array be } A \]
\[ \quad A=[50 \ 100 \ 250 \ 500] \text{ (all values in micrometers)} \]
\[ \quad \text{If } T=3, \text{ a tile size of 250 micrometer is selected to carry out the tessellation.} \]

The algorithm for tessellating and obtaining the diffraction impulse of an extended reflector surface comprises of the following steps:

I) **Defining the specifications**

All the required input parameters for calculating the diffraction impulse response from a tile are defined. These parameters specify the transducer geometry, location and orientation of the reflector surface. The size of the reflector surface \( (M) \) and the tile size \( (T) \) used for the tessellation are defined as a part of the method, \textit{tessellate}(T,M), as discussed above.

II) **Tessellation technique**

The tessellation technique comprises of two main tasks:

a. Laying a staggered set of vertices or a grid

b. Joining a set of three vertices with triangles, abiding by the Delaunay triangulation algorithm
c. Laying a staggered set of nodes or a grid

A staggered set of vertices is laid over the surface of the reflector. By staggered, we mean that all the vertices belonging to rows 1, 3, 5, 7, etc. are in the same position and the vertices in rows 2, 4, 6, 8, etc. are shifted ½ node spacing relative to the vertices in rows 1, 3, 5, 7, etc. A staggered set of vertices has been shown below in Fig 3.4.

![Figure 3.4: A staggered set of vertices](image)

All vertices in a row are placed equidistant from each other. Now, we are tessellating using equilateral triangular tiles with tile size as defined above by T. What this implies is that the distance between the two rows of staggered nodes or correspondingly the height of the equilateral triangle tile is T. Let S be the length of the sides of the equilateral triangle. By property of an equilateral triangle, the relation between the height and the side of the equilateral triangle is as follows: $T = 0.866S$. Since the total length of the reflector surface is $M$, the number of tiles along the length of the surface is the integer value of $M / S$.

Thus, the number of vertices along the length of a reflector surface = the integer value of

$$[(M / S) + 1]$$
Since we are aiming to obtain equilateral triangular tiles along the reflector surface, we consider the first two vertices of the row we have obtained above to be the vertices of the base of the triangle. Hence in the next (second) row, we plot the node at a height of 0.866 S from the previous row, and placing the node as a mid-point of the first two vertices of the previous node, as shown in Fig 3.5.

Figure 3.5: Staggered set of vertices with dimensions

This row is completed accordingly to obtain two rows having vertices uniformly displaced from each other. In a similar manner, these two rows are repeated until the whole reflector surface is covered with a grid of vertices as shown in Fig 3.6.

Figure 3.6: An M x M reflector surface covered with a staggered set of vertices
a. Applying the Delaunay Triangulation Algorithm

Given the set of vertices, the Delaunay triangulation is then executed. In accordance with the Delaunay triangulation algorithm (a MATLAB method), the nearest set of vertices that can be circumscribed by a circle form the three vertices of the triangle which are connected. Hence we have defined our points in such a manner that the Delaunay triangulation will produce all equilateral triangles over the reflector surface with the specified length $S$.

The Delaunay triangulation method returns an $N \times 3$ matrix where $N$ is the number of triangles or tiles and the three elements in each row specify the co-ordinates of the three vertices of the triangular tile, respectively.

![Figure 3.7: A tessellated reflector surface](image)

Figure 3.7: A tessellated reflector surface
III) Tessellations for 3D Surfaces

Once the tessellation has been obtained for 2D surfaces as described above and on
obtaining the set of vertices after using Delaunay, the x, y and z coordinates for 3D
surfaces can be defined as per the requirements.

E.g. for a 2D surface let the x-coordinate of a node be x, the y-coordinate be y and the
z-coordinate be z. Since the surface is 2D along x-y plane, z = constant.

If the 2D plane is tilted by an angle $\phi$ to the x-axis,

New x-coordinate = $x \cdot \cos \phi$

New y-coordinate = $y$

New z-coordinate = $x \cdot \sin \phi$

Similarly if a new 3D surface needs to be implemented, it can be imagined that the 2D
surface is being bent into the 3D surface. We have selected two more types of reflectors
in addition to the tilted flat reflector surface in our research, as shown in Figure 8. They
are the cylindrical reflector surface and the sinusoidal reflector surface. In all the three
types of reflectors, the vertices vary in the $x$ and $z$ coordinates, but their $y$-coordinates are
constant.

The cylindrical reflector surface is actually a curved reflector surface. It is a $16.6^\circ$ arc of
a cylinder with radius = 86mm and arc length 25mm. The sinusoidal reflector surface is
considered to be comprised of two lobes, each arcs of a cylinder of radius = 50mm and length 12.5mm, as shown in Fig 3.8(c).

All reflector surfaces are placed on the transducer axis, 50mm away from the transducer axis. Hence all the above mentioned curves are centered at a point \( z = 50\text{mm} \) away from the transducer axis. The top view of the reflector surfaces which are tilted flat, curved and sinusoidal in the x-y plane appear as shown in Fig 3.8 in the x-z plane which is the top-view of the reflector surfaces.

![Diagram](image)

**Figure 3.8:** (a) Tilted flat reflector surface (b) Curved reflector surface (c) Sinusoidal reflector surface

Hence, the new set of co-ordinates obtained from the original vertices of the tiles, using analytic formulae, form the new set of points at which the diffraction response can be calculated.
V) **Obtaining the overall diffraction impulse response**

The last step involves using the obtained vertices of the tiles and feeding them as input parameters for the calculation of the diffraction impulse response along a tile. These diffraction responses are calculated as per the DREAM implementation steps discussed in section 2.3 in Chapter 2. All such diffraction impulse responses over all the tiles are added to obtain the overall diffraction impulse response and hence the received signal from the reflector surface.

A proposed idea for obtaining tessellations for any **arbitrary non-planar geometry** on the basis of Delaunay has been explained below:

1. Plot a staggered set of vertices in the x-y plane or the projected plane of the reflector surface, stacked as shown in Fig 3.9 below. The distance between the two rows in a stack would be equal to the desired height of the equilateral triangle tile and the distance between two vertices in the same row would be equal to the desired length of the equilateral triangle tile.
2. Consider an arbitrary shaped reflector surface is introduced for tessellation in the given plane, as shown in Fig 3.10. A boundary detection software package used in other medical imaging techniques (devised in [19]), which includes a graphical interface that allows a user to digitize the region boundaries of interest from a hardcopy of the picture output from a device such as an ultrasound scanner, can then be used. Boundary point coordinates will be obtained as an output from this software tool and fed into the MATLAB system which will be able to plot the surface there.
3. After determining the boundary of the reflector surface, vertices along the boundary in line with the other collinear rows of points can be plotted. The vertices outside the determined boundary can then be eliminated. We can then start plotting points along the boundary, which are collinear with the stacked rows of points, as shown in red in Fig 3.11. Let us consider the left topmost point on the boundary of the arbitrary surface. Now, consider the next staggered column, to the right of this point by comparing their respective y-coordinates. We can then plot points collinear to the consecutive columns of staggered points to the right of the previously stated column until we reach the right topmost point. These points are plotted in green as shown below. Similarly, the above steps can be carried out for the left bottommost point.
4. The Delaunay triangulation method can be applied to the remaining vertices within the surface boundaries to obtain the required equilateral triangle tessellations. These vertices would also include the vertices plotted along the boundaries, as seen in Fig 3.12.
To obtain different tessellation densities within the area marked by the boundary shown in Figure 3.12, a similar kind of arbitrary shaped region can be considered within which the desired tessellation density is required. The staggered vertices discussed in the first step of this tessellation procedure for arbitrary geometries, would be placed closer to each other or farther from each other depending on whether the desired density needs to be more or less concentrated respectively. The other steps would remain the same and the distribution of tessellations around the boundary would remain similar to the steps discussed above. However, it is important to understand that although the distribution or density of tessellations on either side of the boundary will vary, a similar logic as provided in steps 3 and 4 for tessellations along the boundary, i.e. inner and outer, would apply.
3.8. Conclusion

Based on weighing the computational efficiency and accuracy with which the results are to be obtained, and analyzing the requirements and characteristics of an automatic tessellation tool, we would like to propose the use of the Delaunay method for generating equilateral triangular tessellations as discussed above.
Chapter 4

Modeling Technique for Annular Array transducer

In the previous chapters, we have discussed the numerical modeling of the received signal from an ultrasound pulse-echo system, especially for a system with planar circular transmitter and receiver. This modeling concept can further be extended to a pulse-echo system utilizing a planar annular array transducer. The optimal design of acoustic fields and receiver characteristics using annular array transducers has been explained in this chapter.

4.1. Annular array transducers

Annular array transducers are comprised of individual transducer elements arranged in the form of concentric rings of different radii, as shown in Fig 4.1. As will be shown in this chapter, the received signal from any array element can be derived based on the superposition of received signal from planar circular transducers. With the annular array transducers, a large number of acoustic fields can be produced by varying the relative excitation delay and amplitude scale factor for the individual transmitting elements. Similarly, a large number of receiver characteristics can be generated by varying the relative delay and gain factor for the individual receiving element. By customizing the acoustic field and receiver characteristics of an ultrasound pulse-echo system with annular array, the system can be optimized in terms of its ability to identify a given object or reflector surface among a limited set of objects or reflector surfaces.
4.2. Analytic derivation for obtaining the received signal from an annular array transducer

4.2.1. Concept of echo signal matrix

The received signal obtained from a transducer with a flat frequency response is termed the *echo signal*. A pulse-echo system, utilizing the elements in an $N$ element annular array individually, can produce $N \times N$ echo signals for a given reflector at a given location and orientation, based on all the possible combinations of transmit and receive elements. These echo signals can be presented in a $N \times N$ echo signal matrix $V^{REFL}(t)$ of the form shown in (4.1).

$$V^{REFL}(t) = \begin{bmatrix}
    v^{REFL}_{1,1}(t) & \ldots & v^{REFL}_{1,j}(t) & \ldots & v^{REFL}_{1,N}(t) \\
    \vdots & \ddots & \vdots & \ddots & \vdots \\
    v^{REFL}_{i,1}(t) & \ldots & v^{REFL}_{i,j}(t) & \ldots & v^{REFL}_{i,N}(t) \\
    \vdots & \ddots & \vdots & \ddots & \vdots \\
    v^{REFL}_{N,1}(t) & \ldots & v^{REFL}_{N,j}(t) & \ldots & v^{REFL}_{N,N}(t)
\end{bmatrix} \quad (4.1)$$
The elements of the matrix are represented by \( v_{i,j}^{\text{REFL}}(t) \), for \( i, j \in [1, N] \), which refers to the echo signal from the entire reflector, produced with the \( i^{th} \) ring as transmitter and the \( j^{th} \) ring as receiver. Also, due to reciprocity conditions, \( v_{i,j}^{\text{REFL}}(t) = v_{j,i}^{\text{REFL}}(t) \).

The elements of the echo signal matrix in (4.1) are obtained by tessellating the entire reflector \( \text{REFL} \) into a number of triangular tiles, each of which is small enough to apply DREAM. The formulation in (4.2) is then used to calculate each element of the matrix in (4.1).

\[
v_{i,j}^{\text{REFL}}(t) = \sum_{\text{all tiles}} v_{i,j}(\vec{r},t) \tag{4.2}
\]

where \( v_{i,j}(\vec{r},t) \) is the echo signal from the tile at location \( \vec{r} \) on the reflector surface, produced with the \( i^{th} \) ring as transmitter and the \( j^{th} \) ring as receiver.

The echo signal from a given element in the array, due to transmission with any element in the same array, can be found as a superposition of the echo signals from the planar circular transducers. Formulated analytically, \( v_{i,j}(\vec{r},t) \) can be calculated similar to the formulation in (2.14), in Chapter 2, and is given in (4.3).

\[
v_{i,j}(\vec{r},t) = A_{1}\cos[\theta(\vec{r})] \int \frac{\partial^2}{\partial t^2} \left[ h_{i,j-1}(\vec{r},t) \otimes h_{j,j-1}(\vec{r},t) \right] dA \tag{4.3}
\]

\[
= A_{1}\cos[\theta(\vec{r})] \Delta A \left\{ \frac{\partial^2}{\Delta t^2} \left[ h_{i,j-1}(\vec{r},t) \otimes h_{j,j-1}(\vec{r},t) \right] \right\} \otimes F(t)
\]
where $\Delta A$ refers to the area of the small tile, $h_{i,j-1}(\vec{r},t)$ and $h_{j,j-1}(\vec{r},t)$ are the velocity potential impulse responses at the field point $\vec{r}$ for the $i^{th}$ ring as transmitter and the $j^{th}$ ring as receiver, respectively.

However, due to difficulty with the segmentation of the diffraction response, the echo signal obtained using DREAM cannot be calculated directly as in (4.3). Since there is no segmentation in the Huygens Method, the formula given by (4.3) is acceptable for calculating the received signal using Huygens Method. This is primarily because the Huygens method requires a much smaller tile size ($\Delta A$). The method for calculating the echo signal using DREAM is explained below.

Based on the assumption of linearity, the diffraction response for any combination of transmitting and receiving annulus of the transducer can be formulated as defined in (2.12) in chapter 2, and as represented below in (4.4)

$$\frac{\partial^2}{\partial t^2} [h_{i,j-1}(\vec{r},t) \otimes h_{j,j-1}(\vec{r},t)]$$

$$= \frac{\partial^2}{\partial t^2} [\{h_i(\vec{r},t) - h_{i-1}(\vec{r},t)\} \otimes [h_j(\vec{r},t) - h_{j-1}(\vec{r},t)]]$$

$$= \frac{\partial^2}{\partial t^2} [h_i(\vec{r},t) \otimes h_j(\vec{r},t) - h_i(\vec{r},t) \otimes h_{j-1}(\vec{r},t) - h_{i-1}(\vec{r},t) \otimes h_j(\vec{r},t) + h_{i-1}(\vec{r},t) \otimes h_{j-1}(\vec{r},t)]$$

$$= D_{i,j}(\vec{r},t) - D_{i,j-1}(\vec{r},t) - D_{i-1,j}(\vec{r},t) + D_{i-1,j-1}(\vec{r},t)$$

In the above formulation, $h_i(\vec{r},t)$ is the velocity potential impulse response at the field point $\vec{r}$ for the planar circular transducer with radius of $a_i$ shown in Fig. 4.1 and
$D_{i,j}(\vec{r},t)$ is the diffraction impulse response at the field point $\vec{r}$ for a pulse-echo system with a planar circular transmitter of radius $a_i$ and a planar circular receiver of radius $a_j$.

As can readily be seen, (4.3) can be expanded into four terms as given in (4.5):

\[ v_{i,j}(\vec{r},t) = A_i \cos[\theta(\vec{r})] \int_{\Delta \mathcal{A}} [D_{i,j}(\vec{r},t) - D_{i,j-1}(\vec{r},t) - D_{i-1,j}(\vec{r},t) + D_{i-1,j-1}(\vec{r},t)]dA \]

\[ = A_i \cos[\theta(\vec{r})] \int_{\Delta \mathcal{A}} D_{i,j}(\vec{r},t)dA - A_i \cos[\theta(\vec{r})] \int_{\Delta \mathcal{A}} D_{i,j-1}(\vec{r},t)dA - A_i \cos[\theta(\vec{r})] \int_{\Delta \mathcal{A}} D_{i-1,j}(\vec{r},t)dA \]

\[ + A_i \cos[\theta(\vec{r})] \int_{\Delta \mathcal{A}} D_{i-1,j-1}(\vec{r},t)dA \]

\[ = A_i \cos[\theta(\vec{r})] \Delta \mathcal{A} \left[ D_{i,j}(\vec{r},t) \otimes F_{i,j}(t) \right] - \left[ D_{i,j-1}(\vec{r},t) \otimes F_{i,j-1}(t) \right] - \left[ D_{i-1,j}(\vec{r},t) \otimes F_{i-1,j}(t) \right] \]

\[ + \left[ D_{i-1,j-1}(\vec{r},t) \otimes F_{i-1,j-1}(t) \right] \]  \hspace{1cm} (4.5)

As can be seen, we have used the concept of segmentation earlier discussed in section 2.3 of Chapter 2. The variables in the last step in (4.5) have been defined in (2.15) from the same section. Each term in (4.5) represents the echo signal from different combinations of planar circular transmitting and receiving transducers. By combining (4.2) and (4.5), the total echo signal $v_{i,j}^{REFL}(t)$ produced with the $i^{th}$ ring as transmitter and the $j^{th}$ ring as receiver can readily be calculated using the DREAM method.

### 4.2.2 Concept of delay matrix

While calculating the total echo signal produced by a given reflector surface, it is important to account for the nature of the excitation signal and receiver characteristics of the transducer. The transmitting and received signals produced by a transducer are
characterized by an amplitude scale factor and a delay value that is assigned to each of the elements in an annular array in the transmitting and receiver modes respectively. However, in formulating the transmitted field and receiver characteristics for this research, we will only take into consideration the delay values, and we assume that the excitation signals for all the elements in the array have the same amplitude and that the same gain factor is applied to the received signals from all the rings. Let us denote the transmit delays to the individual elements $\tau_1, \tau_2, \ldots, \tau_i, \ldots, \tau_N$, meaning that the excitation signals to the $i^{th}$ array element is delayed by the transmit delay $\tau_i$ relative to some time reference. The receive delays will be denoted $\tau'_1, \tau'_2, \ldots, \tau'_i, \ldots, \tau'_N$, meaning that the received signal from the $j^{th}$ element is delayed by the receive delay $\tau'_j$ relative to some time reference.

An $N \times N$ delay matrix $T^{REFL}$ can be formulated as

$$
T^{REFL} =
\begin{bmatrix}
 t_{1,1} & t_{1,2} & \cdots & t_{1,j} & \cdots & t_{1,N} \\
 t_{2,1} & t_{2,2} & \cdots & t_{2,j} & \cdots & t_{2,N} \\
 \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
 t_{i,1} & t_{i,2} & \cdots & t_{i,j} & \cdots & t_{i,N} \\
 \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
 t_{N,1} & t_{N,2} & \cdots & t_{N,j} & \cdots & t_{N,N}
\end{bmatrix}
$$

(4.6)

At this point, let us address the important question as to whether $T^{REFL}$ can be divided up into a separate transmit delay matrix and a separate receive delay matrix; in other words, whether $T^{REFL}$ can be written as $T^{REFL} = T^{REFL}_T + T^{REFL}_R$. 

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The transmit delay matrix and the receive delay matrix can be represented explicitly as

\[
T_T^{REFL} = \begin{bmatrix}
\tau_1 & \tau_1 & \cdots & \tau_1 \\
. & . & \cdots & . \\
\tau_i & \tau_i & \cdots & \tau_i \\
. & . & \cdots & . \\
\tau_N & \tau_N & \cdots & \tau_N \\
\end{bmatrix}
\] and

\[
T_R^{REFL} = \begin{bmatrix}
\tau'_1 & \tau'_2 & \cdots & \tau'_j & \cdots & \tau'_N \\
. & . & \cdots & . \\
\tau'_1 & \tau'_2 & \cdots & \tau'_j & \cdots & \tau'_N \\
. & . & \cdots & . \\
\tau'_1 & \tau'_2 & \cdots & \tau'_j & \cdots & \tau'_N \\
\end{bmatrix}. \tag{4.7}
\]

Note that \(T_T^{REFL}\) consists of a column matrix repeated \(N\) times, whereas \(T_R^{REFL}\) consists of a row matrix repeated \(N\) times. \(T^{REFL}\) is given in (4.6) and it is interesting to see whether \(T^{REFL}\) can be split up into \(T^{REFL} = T_T^{REFL} + T_R^{REFL}\). This leads to the expression in (4.8).

\[
T^{REFL} = \begin{bmatrix}
t_{1,1} & t_{1,2} & \cdots & t_{1,j} & \cdots & t_{1,N} \\
t_{2,1} & t_{2,2} & \cdots & t_{2,j} & \cdots & t_{2,N} \\
. & . & \cdots & . & \cdots & . \\
t_{i,1} & t_{i,2} & \cdots & t_{i,j} & \cdots & t_{i,N} \\
. & . & \cdots & . & \cdots & . \\
t_{N,1} & t_{N,2} & \cdots & t_{N,j} & \cdots & t_{N,N} \\
\end{bmatrix} = \begin{bmatrix}
t_1 + \tau_1 & t_1 + \tau_2 & \cdots & t_j + \tau_j & \cdots & t_N + \tau_N \\
t_1 + \tau_1 & t_1 + \tau_2 & \cdots & t_j + \tau_j & \cdots & t_N + \tau_N \\
. & . & \cdots & . & \cdots & . \\
t_1 + \tau_1 & t_1 + \tau_2 & \cdots & t_j + \tau_j & \cdots & t_N + \tau_N \\
. & . & \cdots & . & \cdots & . \\
t_1 + \tau_1 & t_1 + \tau_2 & \cdots & t_j + \tau_j & \cdots & t_N + \tau_N \\
\end{bmatrix}. \tag{4.8}
\]

Examination of \(T^{REFL}\) as shown in (4.8) reveals that the condition of symmetry cannot be imposed. Or, if symmetry is to exist in \(T^{REFL}\), then \(T^{REFL} \neq T_T^{REFL} + T_R^{REFL}\).

If the condition of symmetry is removed, then it may be possible to take the freely chosen set of delays, as formulated in (4.7), and split up the delays into the two separate matrices, as shown in \(T_T^{REFL}\) and \(T_R^{REFL}\). This possibility is presented in (4.9).
The answer to whether (4.9) is valid or not will be addressed in Sections 6.1 and 7.1 of Chapters 6 and 7 respectively. We will see that it depends on what method we are appointing out of the following two methods, to obtain the received signal from a reflector:

1) Transmitting with one transducer element at a time, and receiving with one element at a time. Although, this form of operating the array transducer consumes the greatest amount of time for carrying out the measurements, it does give the greatest degree of flexibility.

2) Transmitting with all the array elements together and receiving with all the array elements together. This form of operating the array transducer gives less flexibility but is the standard way that a pulse echo system operates.

As of now, we need to note that $t_{i,j}$, the element of the delay matrix $T^{\text{REFL}}$, is exactly the delay value that is assigned to the $(i,j)^{th}$ element of the echo signal matrix $V^{\text{REFL}}(t)$ to
obtain the summed echo signal from the entire annular array transducer. The summed echo signal from the entire annular array transducer can be calculated as shown in (4.10)

$$v_{sum}^{REFL} (t) = \sum_{i=1}^{N} \sum_{j=1}^{N} v_{i,j}^{REFL} (t) \otimes \delta(t - t_{i,j})$$  \hspace{2cm} (4.10)

### 4.2.3. Concept of received signal

The signal obtained with an ultrasound pulse-echo system, using a realistic transducer with bandlimited frequency response, is referred to as the **received signal** in contrast to the **echo signal**, previously defined. The received signal $u_{i,j}^{REFL} (t)$, for a given reflector and transmitter-receiver combination, is obtained by applying the delay values in $T^{REFL}$ to the elements in $V^{REFL}(t)$ and introducing the bandpass filtering effect of the transducer by convolving with $w(t)$, where $w(t)$ represents the combined transmit-receive impulse response of the array transducer.

Hence, the summed received signal corresponding to the summed echo signal given in (4.11) can be formulated as shown in (4.12)

$$u_{sum}^{REFL} (t) = \sum_{i=1}^{N} \sum_{j=1}^{N} v_{i,j}^{REFL} (t) \otimes \delta(t - t_{i,j}) \otimes w(t)$$  \hspace{2cm} (4.11)

After the theoretical description of the steps involved in calculating the echo signal using the DREAM method for a planar transducer, it is instructive to see some actual examples of the segmentation and delay filtering, and the actual appearance of the diffraction responses from one tile of an extended reflector surface. An illustration of the steps involved in determining the diffraction response at the center of the tile, to obtaining the
echo signal from that tile, is given in the next section for two combinations of transmitter-receiver pairs.

4.2.4 Steps involved in obtaining the echo signal from a tile on the reflector surface

This section describes the process of calculating the echo signal from a given transducer ring as transmitter and another, generally different, ring as receiver. If the $i^{th}$ is used as transmitter and the $j^{th}$ ring is used as receiver, then the received signal is $\nu_{i,j}^{\text{REFL}}(t)$.

However, the DREAM method does not perform well when applied directly to individual rings due to difficulties in segmenting the diffraction response calculated for a given ring. Therefore, the calculation of $\nu_{i,j}^{\text{REFL}}(t)$ takes place as described in (4.4) and (4.5). As can be seen from these equations, $\nu_{i,j}^{\text{REFL}}(t)$ is formed as a combination of an echo signal from a planar piston transducer with different radii.

As described earlier, $D_{i,j}(\vec{r},t)$ is the diffraction impulse response at the field point $\vec{r}$ for a pulse-echo system with a planar circular transmitter of radius $a_i$ and a planar circular receiver of radius $a_j$. The step-by-step process of calculating $D_{i,j}(\vec{r},t)$ is described in the flowchart in Figure 4.2.
Figure 4.2: Flowchart to obtain echo signal from a given tile on a reflector surface
Waveforms illustrating the steps towards calculating the echo signal from a tile (centered at (-0.007m, 0.0031mm, 0.0503mm) if center of transducer is at (0,0,0)) using planar piston transducers (transmitter radius = 9.1mm, receiver radius = 10.4mm) for a 15mm x 15mm flat reflector surface tilted at 6 degrees. The reflector is centered on the Z-axis.

Figure 4.3: Pulse echo system using 6 ring annular array transducer and a 15mm x 15mm tilted flat reflector surface
Fig 4.4 (a) Diffraction response [Y-axis in $m^4/s^3$] at center of tile

Fig 4.4 (b) Segments of the diffraction response [X-axis in sec, and Y-axis in $m^4/s^3$]

Fig 4.4 (c) Delay filters for individual segments of the diffraction response
[X-axis in sec, and Y-axis in $m^4/s^3$]
Time domain \([X\text{-axis in sec, and } Y\text{-axis in } m^4/s^3]\)

\[
\begin{align*}
\tilde{D}_1(\tilde{r},t) &= D_1(\tilde{r},t) \otimes F_1(t) \\
\tilde{D}_2(\tilde{r},t) &= D_2(\tilde{r},t) \otimes F_2(t) \quad \tilde{D}_3(\tilde{r},t) &= D_3(\tilde{r},t) \otimes F_3(t)
\end{align*}
\]

Frequency domain \([X\text{-axis in } Hz, \text{ and } Y\text{-axis in } m^4/s^3]\)

\[
\begin{align*}
\tilde{D}_1(\tilde{r},f) \\
\tilde{D}_2(\tilde{r},f) \\
\tilde{D}_3(\tilde{r},f)
\end{align*}
\]

**Fig 4.4 (d)** Filtered segments of the diffraction response

**Fig 4.4 (e)** Echo signal for tile, based on DREAM

**Figure 4.4:** Steps involved in obtaining the echo signal for a tile in a pulse-echo system described in Figure 4.3, based on DREAM
Waveforms illustrating the steps towards calculating the echo signals from a tile (centered at (-0.007m, 0.0031mm, 0.0503mm) if center of transducer is at (0,0,0)) using planar piston transducers (transmitter radius = 12.7mm, receiver = 12.7mm) for a 15mm x 15mm flat reflector surface tilted at 6 degrees.

Figure 4.5: Pulse echo system using 6 ring annular array transducer and a 15mm x 15mm tilted flat reflector surface.

Fig 4.6 (a) Diffraction response [Y-axis in $m^4/s^3$] at center of tile
**Fig 4.6 (b)** Segments of the diffraction response [X-axis in sec, and Y-axis in $m^4 / s^3$]

\[ D_1(\tilde{r}, t) \]

\[ D_2(\tilde{r}, t) \]

**Fig 4.6 (c)** Delay filters for individual segments of the diffraction response
[X-axis in sec, and Y-axis in $m^4 / s^3$]

**Fig 4.6 (c)** Delay filters for individual segments of the diffraction response
[X-axis in sec, and Y-axis in $m^4 / s^3$]
**Time domain** [X-axis in sec, and Y-axis in m$^4$/s$^3$]

\[ \tilde{D}_1(\tilde{r}, t) = D_1(\tilde{r}, t) \otimes F_1(t) \]

\[ \tilde{D}_2(\tilde{r}, t) = D_2(\tilde{r}, t) \otimes F_2(t) \]

**Frequency domain** [X-axis in Hz, and Y-axis in m$^4$/s$^3$]

\[ \tilde{D}_1(\tilde{r}, f) \]

\[ \tilde{D}_2(\tilde{r}, f) \]

**Fig 4.6 (d) Filtered segments of the diffraction response**
Fig 4.6 (e) Echo signal for tile, based on DREAM

**Figure 4.6:** Steps involved in obtaining the echo signal for a tile in a pulse-echo system described in Figure 4.5, based on DREAM

It is evident from the results above that the echo signal for a tile is more a function of the delay filter response than the actual diffraction response.

### 4.3. Comparison of echo signals obtained using DREAM and Huygens method

Since we have selected the DREAM method as a numerical modeling tool for the pulse-echo system, it is important to determine the accuracy of the echo signals for annular array transducers based on the DREAM method by using the Huygens method as a reference. While calculating these echo signals we will also be determining the accuracy of its components, which are obtained using planar piston transducers. These components have been stated in (4.4). In the following pages are two illustrations of the components involved to calculate the received signal; the first illustration is for annular rings with \( i = 3, j = 4 \) and the second illustration is for \( i = 6, j = 6 \), in a 6 ring annular array.
transducer with rings of outer radii of 5.2mm, 7.3mm, 9mm, 10.4mm, 11.6mm and 12.7mm.

The blue curve is obtained using the Huygens method while the red curve is obtained using the DREAM method. We can quantify the accuracy of the DREAM method by calculating the “DREAM error”, discussed in (3.1) in the frequency domain (so that we can concentrate on a specific range of frequency). The DREAM error formulation has been repeated in (4.12) for the convenience of the reader. The reference signal \( V_{\text{ref}}(f) \) is obtained using Huygens method for the same transducer and reflector geometry.

\[
\text{DREAM error} = \text{MSE} = \frac{\int_{0}^{15\text{MHz}} \left| V_r(f) - V_{\text{ref}}(f) \right|^2 df}{\int_{0}^{15\text{MHz}} \left| V_{\text{ref}}(f) \right|^2 df} \times 100 \% \quad (4.12)
\]

In Figure 4.7, components used to obtain received signal for \( i = 3, j = 4 \) in a 6-ring annular array transducer are placed in a set-up as shown in Figure 4.3. The blue curve is obtained using the Huygens method while the red curve is obtained using the DREAM method. Both results are obtained for a 15mm x 15mm flat reflector surface tilted at 6 degrees. The DREAM method is evaluated using triangular tiles of side 500\( \mu \text{m} \) while the Huygens method is evaluated for triangular tiles of side 50\( \mu \text{m} \).

From (4.4)

\[
D_{34}^{\text{annular}} = D_{34}^{\text{planar}} - D_{33}^{\text{planar}} - D_{24}^{\text{planar}} + D_{23}^{\text{planar}} \quad (4.13)
\]
**Time domain**  
[X-axis in sec, and Y-axis in $m^4/s^3$]

**Frequency domain**  
[X-axis in Hz, and Y-axis in $m^4/s^3$]

DREAM error = 0.0672%

**Fig 4.7(a)** Received signals obtained with $D_{ld}^\text{planar}$

DREAM error = 0.0516%

**Fig 4.7(b)** Received signals obtained with $D_{lz}^\text{planar}$
Fig 4.7(c) Received signals obtained with $D^\text{planar}_{14} - D^\text{planar}_{33}$

**Time domain**
[X-axis in sec, and Y-axis in $m^4/s^3$]

**Frequency domain**
[X-axis in Hz, and Y-axis in $m^4/s^3$]

DREAM error = 0.1938%

Fig 4.7(d) Received signals obtained with $D^\text{planar}_{24}$

DREAM error = 0.0880%
**Fig 4.7(e)** Received signals obtained with $D_{34}^{planar} - D_{33}^{planar} - D_{24}^{planar}$

**Time domain**
[X-axis in sec, and Y-axis in m$^4$/s$^3$]

**Frequency domain**
[X-axis in Hz, and Y-axis in m$^4$/s$^3$]

DREAM error = 0.0930%

**Fig 4.7(f)** Received signals obtained with $D_{33}^{planar}$

DREAM error = 0.1326%
DREAM error = 1.0485%

**Fig 4.7(g)** Received signals obtained for $D_{34}^{\text{annular}} = D_{34}^{\text{planar}} - D_{33}^{\text{planar}} - D_{24}^{\text{planar}} + D_{23}^{\text{planar}}$

**Figure 4.7**: Components used to obtain received signal for $i = 3, j = 4$ in a 6-ring annular array transducer placed in a set-up as shown in Figure 4.3. The blue curve is obtained using the Huygens method while the red curve is obtained using the DREAM method.

In Figure 4.8, components used to obtain received signal for $i = 6, j = 6$ in a 6-ring annular array transducer are placed in a set-up as shown in Figure 4.4.

From eq. (4.4)

$$D_{66}^{\text{annular}} = D_{66}^{\text{planar}} - 2D_{56}^{\text{planar}} + D_{55}^{\text{planar}} \quad (4.14)$$
**Time domain**
[X-axis in sec, and Y-axis in m$^4$/s$^3$]

![Time domain graph]

\[ x \times 10^8 \]

**Frequency domain**
[X-axis in Hz, and Y-axis in m$^4$/s$^3$]

![Frequency domain graph]

\[ x \times 10^5 \]

DREAM error = 0.0399%

**Fig 4.8(a)** Received signals obtained with \( D_{56}^{planar} \)

DREAM error = 0.0410%

**Fig 4.8(b)** Received signals obtained with \( D_{56}^{planar} \)
Time domain
[X-axis in sec, and Y-axis in $m^4/s^3$]

Frequency domain
[X-axis in Hz, and Y-axis in $m^4/s^3$]

DREAM error = 0.0410%

Fig 4.8(c) Received signals obtained with $2D_{56}^{planar}$

DREAM error = 0.1003%

Fig 4.8(d) Received signals obtained with $D_{56}^{planar} - 2D_{56}^{planar}$
**Time domain**

[X-axis in sec, and Y-axis in $m^4 / s^3$]

\[x \times 10^8\]

**Frequency domain**

[X-axis in Hz, and Y-axis in $m^4 / s^3$]

\[x \times 10^{10}\]

DREAM error = 0.0619%

**Fig 4.8(e)** Received signals obtained with $D_{35}^{\text{planar}}$

\[x \times 10^8\]

\[x \times 10^{-5}\]

DREAM error = 12.0456%

**Fig 4.8(f)** Received signals obtained for $D_{66}^{\text{annular}} = D_{66}^{\text{planar}} - 2D_{56}^{\text{planar}} + D_{55}^{\text{planar}}$

\[x \times 10^8\]

\[x \times 10^8\]

**Figure 4.8**: Components used to obtain received signal for $i = 6, j = 6$ in a 6-ring annular array transducer placed in a set-up as shown in Figure 4.4.
As can be seen from the above results, the small errors observed in each component used to obtain the total diffraction response increase cumulatively the error of the total diffraction response. Also the error increases significantly as the ring number in the annular array transducer increases. However, the contribution of the diffraction response from the outermost rings towards the total received signal energy is less compared to the diffraction response from the inner rings, for a reflector surface with such dimensions, placed on axis. Hence this type of an error is tolerable.
Chapter 5

The Energy Optimization Method

5.1. Introduction

In the previous chapters we have demonstrated how numerical modeling, based on the diffraction response, can be used to efficiently model an ultrasound pulse-echo system. Our approach has been to develop a software tool based on the DREAM method, which can be used to obtain the wave interaction between transmitted field and target, and between the receiving transducer and backscattered field.

In the remaining part of the thesis, we will propose and evaluate several methods to optimally design an ultrasound system, in order to identify a given object or interface among a limited set of objects or interfaces. These methods may also be used to selectively enhance the received signals from anatomical structures of a specified geometry. Hence, this method can be termed as a reflector-geometry specific ultrasound object recognition or feature extraction method.

With ultrasound array transducers, a large number of different transmitting and receiving ultrasound fields can be produced, including ultrasound fields which can maximize the energy of the received signal for a given array transducer and a given reflector geometry at a given location and orientation. We are interested in the calculation of such fields.

In this chapter, we will lay the foundation for the following chapters and discuss the basic building blocks of this research.
5.2. General overview of the thesis approach

We had seen in previous chapters how a received signal matrix [Chapter 4, section 4.2.3] is obtained as an output from the numerical modeling software (DREAM). The delay matrix is based on the transducer geometry and reflector geometry, orientation and location for the ultrasound pulse-echo system. The received signals, obtained with a single element transducer, from a given reflector at a given location, provide only a static (non-optimizable) ability to identify specific features of the reflecting object.

In the previous chapter, we saw that with ultrasound array transducers, such as linear and annular arrays, a large number of different insonifying fields can be produced by varying the relative excitation delay and amplitude applied to the individual transducer elements. In a similar fashion, a large number of different receiver characteristics can be specified. Also, for a given array transducer and reflector geometry, the energy of the received signal is the specific parameter which can be used to quantify and hence identify the reflector misalignment or reflector topology. Considering these underlying concepts, the objective of this thesis work is to come up with sets of excitation and receiving delay values which can be applied to the transducer elements to produce customized ultrasound fields. Using these delays, we can customize the transmitted acoustic field and receiver characteristics in order to maximize the energy of the received signal and hence improve the identification of structures, over conventional ultrasound systems. The method we have proposed to identify a given reflector geometry is called as the energy optimization method.
We developed two approaches to calculate these delay values:

1) An analytical approach, where we calculate delay values that can be applied if we transmit with one element at a time and receive with one element at a time. This form of operating the array transducer does give the greatest degree of flexibility, but is not how the practical ultrasound pulse-echo system operates. Hence, we termed the delay matrix calculated using this method as the non-implementable method.

2) A numerical approach, where we calculate delay values that can be applied if we transmit with all the array elements together and receive with all the array elements together. This form of operating the array transducer does give less flexibility relative to the operation discussed in the previous method, however, it is the practical way in which an ultrasound system operates. Thus, we termed the delay matrix calculated using this method as the implementable method.

Below is a flowchart which captures all this information and gives an overview of the thesis approach at a glance.
5.2.1 Energy Optimization Method

The specific optimization approach, which has been used in this research, utilizes the energy of the (normalized) received signal. It is important to note that we operate on the received signal, which is the signal obtained from a realistic transducer with bandlimited frequency response. This is in contrast to the echo signals, which are signals obtained from an ideal transducer with flat frequency response. Hence, the optimization
incorporates the frequency response of the array transducer as a parameter. This strategy is both practical and readily implemented into the optimization algorithm.

Although we have explained the details of the numerical modeling technique in the previous chapters, given below is a short description of the received signal matrix, which is required to be processed for energy optimization. Basically, the echo signal matrix, $V_{REFL}^R(t)$ [Chapter 4, section 4.2.1], is an $N \times N$ symmetric matrix of signals, based on a specified $N$ element annular array transducer and a specified reflector $REFL$, and is obtained using the DREAM method. Each signal in $V_{REFL}^R(t)$ is calculated by tessellating the reflector surface into triangular ‘tiles’ and summing the received signal contribution from each tile. An $N \times N$ received signal matrix, $U_{REFL}^R(t)$, given in (5.1) is calculated by convolving each element in the echo signal matrix with $w(t)$, the transmit-receive impulse response of the transducer, where $w(t)$ is modeled as a bandpass filter function.

$$U_{REFL}^R(t) = \begin{bmatrix}
  u_{1,1}^R(t) & \cdots & u_{1,j}^R(t) & \cdots & u_{1,N}^R(t) \\
  \vdots & \ddots & \vdots & & \vdots \\
  u_{i,1}^R(t) & \cdots & u_{i,j}^R(t) & \cdots & u_{i,N}^R(t) \\
  \vdots & \ddots & \vdots & & \vdots \\
  u_{N,1}^R(t) & \cdots & u_{N,j}^R(t) & \cdots & u_{N,N}^R(t)
\end{bmatrix}$$

The $N^2$ signals in the received signal matrix $U_{REFL}^R(t)$ generally have a good deal of similarity in terms of shape, but the elements differ in terms of their delay with respect to a common time reference. Different reflector geometries would result in unique
distributions of relative delays among the signal elements, as well as in unique patterns of energy distribution among the signals in $U_{REFL}(t)$.

We have tested several algorithms for the purpose of obtaining the delays between the signal elements which optimally align them without the constraint of the implementable format, as previously defined. The *Adaptive Waveform Alignment Algorithm*, which will be discussed in the next chapter, best aligns the signals in the received signal matrix. This algorithm operates on $U_{REFL}(t)$, given in (1), for the purpose of determining the time shifts, $t_{i,j}$, which best align the signals in $U_{REFL}(t)$. In general, the received signal is calculated as

$$u_{REFL}(t) = \sum_{i=1}^{N} \sum_{j=1}^{N} u_{i,j}(t - t_{i,j})$$  \hspace{1cm} (5.2)

where $u_{i,j}(t)$ are the elements in $U_{REFL}(t)$ and $t_{i,j}$ combines both the transmit and the receive delays. Let the delay values $\tau_{i,j}$ represent the delay values that are obtained in an attempt to optimally align the received signals in the signal matrix $U_{REFL}(t)$, using the *Adaptive Waveform Alignment Algorithm*. As a result, the new received signal with the maximized energy can be calculated as formulated in (5.3)

$$u_{max}^{REFL}(t) = \sum_{i=1}^{N} \sum_{j=1}^{N} u_{i,j}(t - \tau_{i,j})$$ \hspace{1cm} (5.3)

These delay values are selected to produce a received signal, $u_{max}^{REFL}(t)$, with a maximized energy, given by (5.4)

$$E_{REFL,max} = \int_{-\infty}^{\infty} (u_{max}^{REFL}(t))^2 \, dt$$ \hspace{1cm} (5.4)
This operation leads to an optimal delay matrix for the specified reflector, \( REFL \), of the form given in (5.5).

\[
T^{REFL, \text{OPT}} = \begin{bmatrix}
\tau_{1,1}^{\text{opt}} & \ldots & \tau_{1,j}^{\text{opt}} & \ldots & \tau_{1,N}^{\text{opt}} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\tau_{i,1}^{\text{opt}} & \ldots & \tau_{i,j}^{\text{opt}} & \ldots & \tau_{i,N}^{\text{opt}} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\tau_{N,1}^{\text{opt}} & \ldots & \tau_{N,j}^{\text{opt}} & \ldots & \tau_{N,N}^{\text{opt}}
\end{bmatrix}
\]  

(5.5)

The delay values in \( T^{REFL, \text{Opt}} \) represent our best approximation to the calculation of \( u_{\text{max}}^{REFL}(t) \) and can be used to determine a signal with the maximum energy. However, as we will see in the latter part of the thesis, the \emph{Adaptive Waveform Correlation Method} and hence the optimal delay matrix \( T^{REFL, \text{Opt}} \) has been developed such that the delay values in this matrix can be applied if we transmit with one element at a time and receive with one element at a time. This method is a time consuming and non-practical method and is hence termed as the \emph{non-implementable energy optimization method}.

In actuality, we transmit with all the array elements together and receive with all the array elements together. Also, it will be shown in Chapter 7, Section 7.1, that for an implementable system or for a conventional pulse-echo ultrasound system, the time values in a delay matrix fulfill the following constraint:

\[
\tau_{i,j} + \tau_{i+1,j+1} = \tau_{i+1,j} + \tau_{i,j+1}; \text{ i.e. } \tau_{i+1,j} = \tau_{i,j} + \tau_{i+1,j+1} - \tau_{i,j+1}.
\]  

(5.6)

Thus, the next processing step is to modify the delay values in the optimal delay matrix \( T^{REFL, \text{Opt}} \) in such a way that it fulfills (5.6), yet maintains the energy of \( u_{\text{max}}^{REFL}(t) \) as well
as possible. Different implementations of this operation will be investigated. The most
effective of these utilizes the mathematical optimization software CPLEX (ILOG Inc.,
Mountain View, CA), which permits solutions to be obtained for tasks that can be
modeled as linear programming problems. The modification of $T^{REFL, Opt}$ has been carried
out subject to a cost function, so that delay values in $T^{REFL, Opt}$ associated with large
energy signals will be shifted less than delay values, associated with low energy signals.
The result of these implementations, including operating with CPLEX, is a modified
delay matrix, $\Gamma^{REFL, OPT}$ with the delay elements $\gamma_{i,j}$ and with the same dimensions as
$T^{REFL, Opt}$, but fulfilling the conditions in (5.6). This leads to a new optimal energy

$$E^{REFL, OPT} = \int_{-\infty}^{\infty} (u_{opt}^{REFL}(t))^2 \, dt \text{, where } u_{opt}^{REFL}(t) = \sum_{i=1}^{N} \sum_{j=1}^{N} u_{i,j} (t - \gamma_{i,j})$$  (5.7)

This concept about the non-implementable and implementable delay matrices have been
represented by the last two building blocks of Fig 5.1.

Let us now illustrate the thesis approach discussed above with a more detailed block
diagram, indicating the actual terminology and mathematical representation of the
variables involved.

The more detailed block diagrams of the process that we finally appointed to obtain a
delay matrix that can be used to maximize the energy of the received signal are as shown
in Fig 5.2. They are in line with Fig 5.1, however, it can be observed that the numerical approach is dependent on the analytical approach in this case.

5.3. Relation between energy optimization and object recognition

Now, let us see how the above designed energy optimization method can be used in the task of identifying which one among a specified set of reflectors is placed in front of the array transducer. Consider a limited set of reflectors, consisting of A, B and C, where a given reflector is defined by its geometry, location and orientation. Using the DREAM method, three echo signal matrices, $V^A(t)$, $V^B(t)$ and $V^C(t)$, can be calculated. The

![Diagram](image-url)
corresponding received signal matrices are obtained by introducing the bandpass filtering effect of the transducer by convolving each element of the respective echo signal matrix with \( w(t) \), where \( w(t) \) is the combined transmit-receive impulse response of the array transducer. This convolution operation results in the received signal matrices \( U^A(t) \), \( U^B(t) \) and \( U^C(t) \).

By means of the energy optimization algorithm, an optimal delay matrix, \( T^{A,\text{Opt}} \), can be determined such that the energy of the received signal from the reflector A is maximized. The received signal, \( u^A_{\max}(t) \), is obtained by applying the delay values in \( T^{A,\text{Opt}} \) which is in the form of (5.3), to the elements in \( U^A(t) \).

\[
\begin{align*}
u^A_{\max}(t) &= \sum_{i=1}^{N} \sum_{j=1}^{N} u^A_{i,j}(t) \otimes \delta(t - \tau^{A,\text{Opt}}_{i,j}) \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} u^A_{i,j}(t) \otimes \delta(t - \tau^{A,\text{Opt}}_{i,j}) \tag{5.8}
\end{align*}
\]

The corresponding maximum energy obtainable from reflector A is then given by (5.9).

\[
E_{\max}^A = \int_{-\infty}^{\infty} (u^A_{\max}(t))^2 dt. \tag{5.9}
\]

In a similar manner, the optimal delay matrices \( T^{B,\text{Opt}} \) and \( T^{C,\text{Opt}} \) can be formulated so that they will maximize the energy of the received signal when reflectors B and C, respectively, are present.

Considering a case where the delay values are chosen to optimize the received signal energy from, say, reflector A, while in fact one of the other reflectors, say reflector C, is
present, the delay values in $\mathbf{T}^{A,Opt}$ are applied to the elements in $\mathbf{U}^C(t)$, giving the received signal $u_{\text{sum}}^{C,A}(t)$ as shown in (5.10)

$$u_{\text{sum}}^{C,A}(t) = \sum_{i=1}^{N} \sum_{j=1}^{N} u_{i,j}^C(t) \otimes \delta(t - \tau_{i,j}^{A,Opt})$$

(5.10)

This received signal is characterized by the energy given by

$$E^{C,A} = \int_{-\infty}^{\infty} (u_{\text{sum}}^{C,A}(t))^2 dt.$$  

(5.11)

Similarly, a complete set of energies for all combinations of delay matrices and reflectors can be calculated and described in what is called as an energy table, as shown below:

**Table 5.1: Format of a standard energy optimization table**

<table>
<thead>
<tr>
<th>Transmit/Receive with delay matrix $T^{A,Opt}$</th>
<th>Reflector A present</th>
<th>Reflector B present</th>
<th>Reflector C present</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E^A_{\text{max}}$</td>
<td>$E^{A,B}$</td>
<td>$E^{A,C}$</td>
<td></td>
</tr>
<tr>
<td>Transmit/Receive with delay matrix $T^{B,Opt}$</td>
<td>$E^{B,A}$</td>
<td>$E^B_{\text{max}}$</td>
<td>$E^{B,C}$</td>
</tr>
<tr>
<td>Transmit/Receive with delay matrix $T^{C,Opt}$</td>
<td>$E^{C,A}$</td>
<td>$E^{C,B}$</td>
<td>$E^C_{\text{max}}$</td>
</tr>
</tbody>
</table>

These optimal delay matrices can further be processed using one of the implementable methods, such as CPLEX, to obtain an implementable set of delay matrices, for which the above table would remain in the same format, with $\mathbf{T}^{A,Opt}, \mathbf{T}^{B,Opt}$ and $\mathbf{T}^{C,Opt}$ being replaced by $\mathbf{\Gamma}^{A,OPT}, \mathbf{\Gamma}^{B,OPT}$ and $\mathbf{\Gamma}^{C,OPT}$ respectively.
Also, it is observed that when the optimal delay matrix, obtained for a specific reflector is used to customize the transmitting and receiving fields for that reflector, the energy of the received signal thus obtained, has been maximized. This concept can be used to identify the type of reflector and extract a particular feature of that reflector, considering we know what kind of reflector we are looking at. Also, if we are given three reflectors, and we know their optimal delay matrices before hand, we can apply the delay values in each of these matrices, one at a time, to identify which reflector geometry is in fact in front of the transducer. Considering the above energy table explanation, for each optimal delay matrix, one of the three reflectors will produce a received signal with maximized energy. This property can help us to identify which reflector surface has been placed in front of the transducer provided we know what type of reflectors the optimal delay matrices were designed for.

Thus the energy optimization method aids in identifying the geometry of a reflector among a given set of reflectors and enhancing the received signal obtained from the reflector.
Chapter 6

The non-implementable energy optimization algorithm

This chapter gives a detailed description of the optimization algorithm that has been developed to improve the ability to quantify specific aspects of a reflecting structure or to identify a given reflector geometry. As was discussed in the previous chapter, the energy of the signal received by the transducer, due to a given reflector, may be used to quantify the feature of interest in that reflector. In our case, this feature of interest would be the reflector misalignment or reflector topology, which would further help us in identifying the reflector under consideration. On this basis, the specific optimization approach, which has been incorporated in this research, utilizes the energy parameter of the (normalized) received signal. We might as well say that the acoustic field producing the ‘strongest’ received signal from a given reflector can be used to identify the features of interest for that reflector, where ‘strongest’ implies a received signal with maximum energy.

Thus, as we move further, it is important to keep in mind that our main objective is to determine the set of delay values to the array elements, which will maximize the energy of the received signal which in turn a combination of the signals received from the reflector by the individual elements of the array transducer. In the previous chapter, we have discussed how a given energy optimization algorithm defines a method to calculate the time shifts or delay values that are used while firing the transmitting signals from the individual elements of the array transducer and to do the same when receiving the
reflected signals. The set of such delays, which results in a received signal with maximized energy, is the optimal delay matrix. This chapter deals with the development of several non-implementable energy optimization algorithms, which are used to obtain a non-implementable optimal delay matrix. Recall that implementation of a system based on the delay values in a non-implementable optimal delay set requires that one element of an array transducer be fired at a time, rather than firing all array elements simultaneously with appropriate time shifts applied, as is done conventionally. Similarly, the array transducer receives with one element at a time. This condition makes the system impractical to implement. Hence, this delay set is termed as the non-implementable optimal delay set and thus we felt the need to come up with an implementable optimal delay set for practical purposes. In Chapter 7, implementable energy optimization algorithms are developed, where one of these algorithms will utilize the most efficient of the non-implementable algorithms as its basis.

In the course of this thesis, we have formulated several different ways to obtain these delay sets, however, based on the energy-maximizing ability of each of these methods, we selected one method to obtain the non-implementable delay values and one method to obtain the implementable optimal delay values. In this chapter we will present the principles behind and the performance of the non-implementable energy optimization algorithms, while in the next chapter we will present a similar kind of analysis for the implementable energy optimization algorithm.
It is instructive to examine the series of methods that led to an effective and efficient optimization algorithm. However, before we proceed it is important to analyze the formulation of the delay matrices produced as a result of the non-implementable energy optimization method.

### 6.1 Formulation of delay matrices for the non-implementable optimization method

As has been mentioned earlier, in the non-implementable optimization formulation we transmit with one element at a time and receive with one element at a time. This mode of operating the array transducer gives the greatest degree of flexibility, but also consumes the greatest amount of measurement time (by a factor of \( N \) for an \( N \) element array). This operating mode is also likely to produce a poor signal-to-noise ratio. Assume that the measurements with an \( N \) element transducer on a given reflector, \( \text{REFL} \), results in a square received signal matrix, \( U_{\text{REFL}}^{(t)} \), which contains the \( N^2 \) signals.

\[
U_{\text{REFL}}^{(t)} = \begin{bmatrix}
    u_{1,1}^{\text{REFL}}(t) & \ldots & u_{1,j}^{\text{REFL}}(t) & \ldots & u_{1,N}^{\text{REFL}}(t) \\
    \vdots & \ddots & \vdots & \ddots & \vdots \\
    u_{i,1}^{\text{REFL}}(t) & \ldots & u_{i,j}^{\text{REFL}}(t) & \ldots & u_{i,N}^{\text{REFL}}(t) \\
    \vdots & \ddots & \vdots & \ddots & \vdots \\
    u_{N,1}^{\text{REFL}}(t) & \ldots & u_{N,j}^{\text{REFL}}(t) & \ldots & u_{N,N}^{\text{REFL}}(t)
\end{bmatrix}
\]  

(6.1)

The matrix is symmetrical, in that \( u_{i,j}(t) = u_{j,i}(t) \), due to reciprocity considerations. Here, \( u_{i,j}(t) \) is the signal obtained with element \( i \) as the transmitter and element \( j \) as the receiver. The fact that \( U_{\text{REFL}}^{(t)} \) is symmetrical does not specifically require that the corresponding
delay matrix be symmetrical as well. However, if one wants to optimize the beamformed received signal so that the energy is maximized, then two identical received signals, such as \( u_{i,j}(t) \) and \( u_{j,i}(t) \), indeed need to undergo the same delay or time shift. This will in fact double the energy relative to what would be the case if the two signals are not aligned.

Let \( t_{i,j} \) be the delay, associated with the signal \( u_{i,j}(t) \). In the most general formulation, the received beamformed signal from the reflector \( \text{REFL} \) is given as

\[
u_{\text{SUM}}(t) = \sum_{i=1}^{N} \sum_{j=1}^{N} u_{i,j}(t - t_{i,j}).
\]

(6.2)

Note that no weighing factor (apodization) is applied here. The delays associated with the summation in (6.2) can be expressed in a square delay matrix \( \mathbf{T}^{\text{REFL}} \), given in (6.3). The matrix \( \mathbf{T}^{\text{REFL}} \) contains \( N^2 \) delay values, selected without any \textit{a priori} constraints. As a consequence, the symmetry requirement for \( \mathbf{T}^{\text{REFL}} \) has not been applied in (6.3).

\[
\mathbf{T}^{\text{REFL}} = \begin{bmatrix}
t_{1,1} & t_{1,2} & \cdots & t_{1,j} & \cdots & t_{1,N} \\
t_{2,1} & t_{2,2} & \cdots & t_{2,j} & \cdots & t_{2,N} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
t_{i,1} & t_{i,2} & \cdots & t_{i,j} & \cdots & t_{i,N} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
t_{N,1} & t_{N,2} & \cdots & t_{N,j} & \cdots & t_{N,N}
\end{bmatrix}
\]

(6.3)

We will now address the important question as to the circumstances under which \( \mathbf{T}^{\text{REFL}} \) can be divided up into a separate transmit delay matrix and a separate receive delay matrix; in other words, when \( \mathbf{T}^{\text{REFL}} \) can be written as \( \mathbf{T}^{\text{REFL}} = \mathbf{T}_T^{\text{REFL}} + \mathbf{T}_R^{\text{REFL}} \).
The transmit delay matrix and the receive delay matrix are given explicitly as

\[
\mathbf{T}^{REFL}_T = \begin{bmatrix}
\tau_1 & \tau_1 \cdots \tau_1 & \cdots & \\
\vdots & \vdots & \ddots & \\
\tau_i & \tau_i \cdots \tau_i & \cdots & \\
\vdots & \vdots & \ddots & \\
\tau_N & \tau_N \cdots \tau_N & \cdots & \\
\end{bmatrix}
\quad \text{and} \quad
\mathbf{T}^{REFL}_R = \begin{bmatrix}
\tau_1' & \tau_2' \cdots \tau_j' & \cdots & \\
\vdots & \vdots & \ddots & \\
\tau_1' & \tau_2' \cdots \tau_j' & \cdots & \\
\vdots & \vdots & \ddots & \\
\tau_1' & \tau_2' \cdots \tau_j' & \cdots & \\
\end{bmatrix}
\]

Note that \(\mathbf{T}^{REFL}_T\) consists of a column matrix repeated \(N\) times, whereas \(\mathbf{T}^{REFL}_R\) consists of a row matrix repeated \(N\) times.

\(\mathbf{T}^{REFL}\) is given in (6.3) and is assumed to contain the delay values, which will optimize the energy of \(u^{SUM}(t)\) as formulated in (6.2); at the same time, we wish to explore whether \(\mathbf{T}^{REFL}\) can be formulated as a sum of the transmit and receive delay matrices, i.e, \(\mathbf{T}^{REFL} = \mathbf{T}^{REFL}_T + \mathbf{T}^{REFL}_R\). Formulating \(\mathbf{T}^{REFL}\) this way leads to the expression in (6.4).

\[
\mathbf{T}^{REFL} = \begin{bmatrix}
t_{1,1} & t_{1,2} \cdots t_{1,j} \cdots t_{1,N} \\
t_{2,1} & t_{2,2} \cdots t_{2,j} \cdots t_{2,N} \\
\vdots & \vdots & \ddots & \\
t_{i,1} & t_{i,2} \cdots t_{i,j} \cdots t_{i,N} \\
t_{N,1} & t_{N,2} \cdots t_{N,j} \cdots t_{N,N} \\
\end{bmatrix}
= \begin{bmatrix}
\tau_1 + \tau_1' & \tau_1 + \tau_2' \cdots \tau_1 + \tau_j' \cdots \tau_1 + \tau_N' \\
\tau_2 + \tau_1' & \tau_2 + \tau_2' \cdots \tau_2 + \tau_j' \cdots \tau_2 + \tau_N' \\
\vdots & \vdots & \ddots & \\
\tau_i + \tau_1' & \tau_i + \tau_2' \cdots \tau_i + \tau_j' \cdots \tau_i + \tau_N' \\
\tau_N + \tau_1' & \tau_N + \tau_2' \cdots \tau_N + \tau_j' \cdots \tau_N + \tau_N' \\
\end{bmatrix}
\]

(6.4)
Examination of $T_{REFL}$ as shown in (6.4) reveals that the condition of symmetry cannot in general be imposed. Specifically, symmetry requires that $\tau_i = \tau_j$, $\tau_j = \tau_j'$, or $T_{TREFL} = T_{RREFL}$. Or, if symmetry is to exist in $T_{REFL}$, then $T_{REFL} \neq T_{TREFL} + T_{RREFL}$.

If the condition of symmetry were removed, we can examine whether it may then be possible to take the freely chosen set of delays, as formulated in (6.3), and split up the delays into the two separate matrices, as shown in $T_{TREFL}$ and $T_{RREFL}$. This possibility is presented in (6.5).

$$
T_{REFL} = \begin{bmatrix}
  t_{1,1} & t_{1,2} & \cdots & t_{1,j} & \cdots & t_{1,N} \\
  t_{2,1} & t_{2,2} & \cdots & t_{2,j} & \cdots & t_{2,N} \\
  \vdots & \vdots & & \vdots & & \vdots \\
  t_{i,1} & t_{i,2} & \cdots & t_{i,j} & \cdots & t_{i,N} \\
  \vdots & \vdots & & \vdots & & \vdots \\
  t_{N,1} & t_{N,2} & \cdots & t_{N,j} & \cdots & t_{N,N}
\end{bmatrix}
$$

$$
\tau = \begin{bmatrix}
  \tau_1 & \tau_1 & \cdots & \tau_1 & \cdots & \tau_1 \\
  \vdots & \vdots & & \vdots & & \vdots \\
  \tau_i & \tau_i & \cdots & \tau_i & \cdots & \tau_i \\
  \vdots & \vdots & & \vdots & & \vdots \\
  \tau_N & \tau_N & \cdots & \tau_N & \cdots & \tau_N
\end{bmatrix}

\begin{bmatrix}
  \tau_1' & \tau_2' & \cdots & \tau_j' & \cdots & \tau_N' \\
  \vdots & \vdots & & \vdots & & \vdots \\
  \tau_1' & \tau_2' & \cdots & \tau_j' & \cdots & \tau_N'
\end{bmatrix}

= \begin{bmatrix}
  \tau_1 & \tau_1 & \cdots & \tau_1 & \cdots & \tau_1 \\
  \vdots & \vdots & & \vdots & & \vdots \\
  \tau_i & \tau_i & \cdots & \tau_i & \cdots & \tau_i \\
  \vdots & \vdots & & \vdots & & \vdots \\
  \tau_N & \tau_N & \cdots & \tau_N & \cdots & \tau_N
\end{bmatrix}

+ \begin{bmatrix}
  \tau_1' & \tau_2' & \cdots & \tau_j' & \cdots & \tau_N' \\
  \vdots & \vdots & & \vdots & & \vdots \\
  \tau_1' & \tau_2' & \cdots & \tau_j' & \cdots & \tau_N'
\end{bmatrix}

$$

The number of independent terms in $T_{REFL}$ is $N^2$. The number of unknowns to solve for, if indeed the delay matrix can be written as $T_{REFL} = T_{TREFL} + T_{RREFL}$, are $2N$. These unknowns comprise $\tau_1, \tau_2 \ldots \tau_j \ldots \tau_N$ and $\tau_1', \tau_2' \ldots \tau_j' \ldots \tau_N'$. In general,
\( N^2 > 2N; \) (only for \( N = 2 \), these two terms are equal). Thus, in the general case, attempting to divide \( T^{REFL} \) into \( T_T^{REFL} \) and \( T_R^{REFL} \) results in an overdetermined case for which there generally is no solution. Hence, we must conclude that in the case of delay matrices for the non-implementable optimization, the delay matrix cannot be divided up into a separate transmit delay matrix and a separate receive delay matrix for an arbitrary set of delay values in \( T^{REFL} \).

The practical measurements for the non-implementable solution can be done in two ways for an \( N \) element array:

1. One element transmits at a time, and the received signals for all \( N \) elements are stored (as a row in the received signal matrix, as shown in (6.1)). This continues with transmission with subsequent elements until transmission has been carried out with all \( N \) elements, and the received signal matrix has been completely filled. Then the overall beamformed signal is generated as described in equation (6.2).

2. One element transmits at a time, and the received signals from all \( N \) receive elements are beamformed into a single signal by applying a set of receive delays, such as \( \tau_{1,1}, \tau_{1,2}, \ldots, \tau_{1,j}, \ldots, \tau_{1,N} \) for transmit element 1; the beamformed signal is then stored. This is repeated with the next following transmit elements until transmission has been carried out with all \( N \) elements, and beamformed signals have been produced for each transmit element. These individual
beamformed signals are then added to form the overall beamformed signal, using a set of transmit delay values.

The next derivation aims at showing that there is no difference between implementing either of these two approaches. Furthermore, when we assume that the delay matrix is symmetrical, we will show that the second implementation can be carried out solely with receive delay values, that is, all transmit delay values are set to zero. Alternatively, this implementation can be carried out solely with transmit delay values.

We will start with developing the formulation for the second implementation. In order to simplify this derivation, consider a 4 element array transducer where the transmission (obviously) will occur with one element at a time, while the reception will occur with all 4 elements, but with an individual delay applied to each element.

Let $\tau_{1,1}', \tau_{1,2}', \tau_{1,3}'$ and $\tau_{1,4}'$ be the receive delays for all 4 elements, respectively, when transmitting with element 1, as illustrated in Figure 6.1. Now, let the corresponding received signal, due to the reflecting structure, be $u_{REC,1}^{REC}(t)$.

\[
u_{REC,1}^{REC}(t) = \sum_{j=1}^{4} u_{1,j}(t - \tau_{1,j}') \quad (6.6)\]

This corresponds to summing over the elements in the first row in received signal matrix, $U_{REFL}^{REFL}(t)$. Figure 6.2 shows the first row of $U_{REFL}^{REFL}(t)$ highlighted.
Figure 6.1: Transmitting with element 1 and receiving with elements 1 – 4.

\[
\begin{bmatrix}
  u_{1,1}(t) & u_{1,2}(t) & u_{1,3}(t) & u_{1,4}(t) \\
  u_{2,1}(t) & u_{2,2}(t) & u_{2,3}(t) & u_{2,4}(t) \\
  u_{3,1}(t) & u_{3,2}(t) & u_{3,3}(t) & u_{3,4}(t) \\
  u_{4,1}(t) & u_{4,2}(t) & u_{4,3}(t) & u_{4,4}(t)
\end{bmatrix}
\]

\[U^{\text{REFL}}(t) = \]

Figure 6.2: Elements in the received signal matrix involved in determining the received signal component when transmitting with element 1.

Similarly, \(\tau_{2,1}^i, \tau_{2,2}^i, \tau_{2,3}^i, \text{ and } \tau_{2,4}^i\) are the receive delays for all 4 elements, respectively, when transmitting with element 2. The received signal is here \(u^{\text{REC,2}}(t)\).

\[u^{\text{REC,2}}(t) = \sum_{j=1}^{4} u_{2,j}(t - \tau_{2,j}^i) \] (6.7)
Calculation of $u^{REC,2}(t)$ corresponds to summing over the elements in the second row in received signal matrix. When transmitting with element 3 and with element 4, we obtain the following received signals:

$$u^{REC,3}(t) = \sum_{j=1}^{4} u_{3,j}(t - \tau'_{3,j})$$  \hspace{1cm} (6.8)

$$u^{REC,4}(t) = \sum_{j=1}^{4} u_{4,j}(t - \tau'_{4,j})$$  \hspace{1cm} (6.9)

The overall received signal is a summation of $u^{REC,i}(t), i \in [1,4]$, with the transmit delays $\tau_1, \tau_2, \tau_3$ and $\tau_4$, yielding $u^{SUM}(t)$:

$$u^{SUM}(t) = \sum_{i=1}^{4} u^{REC,i}(t - \tau_i)$$

$$= \sum_{i=1}^{4} u^{REC,i}(t) \otimes \delta(t - \tau_i).$$  \hspace{1cm} (6.10)

Based on the formulation in (6) - (9), $u^{REC,i}(t)$ can be written as

$$u^{REC,i}(t) = \sum_{j=1}^{4} u_{i,j}(t - \tau'_{i,j})$$

$$= \sum_{j=1}^{4} u_{i,j}(t) \otimes \delta(t - \tau'_{i,j}).$$  \hspace{1cm} (6.11)

Applying (11) to (10) gives the overall received signal, $u^{SUM}(t)$:

$$u^{SUM}(t) = \sum_{i=1}^{4} \left[ \sum_{j=1}^{4} u_{i,j}(t) \otimes \delta(t - \tau'_{i,j}) \right] \otimes \delta(t - \tau_i)$$

$$= \sum_{i=1}^{4} \sum_{j=1}^{4} u_{i,j}(t) \otimes \delta(t - \tau'_{i,j}) \otimes \delta(t - \tau_i)$$  \hspace{1cm} (6.12)

$$= \sum_{i=1}^{4} \sum_{j=1}^{4} u_{i,j}(t) \otimes \delta(t - (\tau_{i,j} + \tau_i)).$$
Define $T^{NI}$ as a square delay matrix, consisting of the delay elements $\tau_{i,j}^\prime + \tau_i$; here, ‘$NI$’ refers to non-implementable.

$$T^{NI} = \begin{bmatrix}
\tau_{1,1} & \tau_{1,2} & \tau_{1,3} & \tau_{1,4} \\
\tau_{2,1} & \tau_{2,2} & \tau_{2,3} & \tau_{2,4} \\
\tau_{3,1} & \tau_{3,2} & \tau_{3,3} & \tau_{3,4} \\
\tau_{4,1} & \tau_{4,2} & \tau_{4,3} & \tau_{4,4}
\end{bmatrix} + \begin{bmatrix}
\tau_1 & \tau_1 & \tau_1 \\
\tau_2 & \tau_2 & \tau_2 \\
\tau_3 & \tau_3 & \tau_3 \\
\tau_4 & \tau_4 & \tau_4
\end{bmatrix} \quad (6.13)$$

Remember that this matrix cannot in general be separated into a transmit matrix and a receive matrix, as previous proven. This also means that the relationship between adjacent delay elements, referred in (5.6), $t_{i+1,j} = t_{i,j} + t_{i+1,j+1} - t_{i,j+1}$, is neither fulfilled.

Yet, as discussed earlier, for energy optimization the $T^{NI}$ matrix needs to be symmetrical. If we add the two terms of the $T^{NI}$ matrix in (6.13) together, and then try to apply the symmetry requirement, we will end up with a complex and generally unsolvable problem. Instead, we will impose symmetry to the first matrix term in (6.13). This leads to the set of requirements, given in (6.14).

$$\tau_{1,2}^\prime = \tau_{2,1}^\prime; \ \tau_{1,3}^\prime = \tau_{3,1}^\prime; \ \tau_{1,4}^\prime = \tau_{4,1}^\prime; \ \tau_{2,3}^\prime = \tau_{3,2}^\prime; \ \tau_{2,4}^\prime = \tau_{4,2}^\prime; \ \tau_{3,4}^\prime = \tau_{4,3}^\prime \quad (6.14)$$

Applying these six equalities to (6.13) gives
Next, we will consider the fact that the beamforming is determined by the relative delays among the transmit delays and the receive delays. Hence, we can arbitrarily set one of the delay values for each row of the receive delay matrix equal to zero. If we allow positive as well as negative delay values, then we can choose $\tau_{1,1}^{\prime}, \tau_{1,2}^{\prime}, \tau_{1,3}^{\prime}$ and $\tau_{1,4}^{\prime}$ to be zero (or equivalent the first column AND the first row in the receive delay matrix to zero). Likewise, we can arbitrarily set one of the four transmit delay values (such as $\tau_1$) equal to zero. (To be technically correct, all the non-zero delay values should be adjusted correspondingly, such that $\tau_{2,2}$ should be replaced with $\tau_{2,2} - \tau_{1,2}$, $\tau_{2,3}$ should be replaced with $\tau_{2,3} - \tau_{1,2}$ etc.)

This gives

$$
\mathbf{T}^{NI} = \begin{bmatrix}
\tau_{1,1} & \tau_{1,2} & \tau_{1,3} & \tau_{1,4} \\
\tau_{1,2} & \tau_{2,2} & \tau_{2,3} & \tau_{2,4} \\
\tau_{1,3} & \tau_{2,3} & \tau_{3,3} & \tau_{3,4} \\
\tau_{1,4} & \tau_{2,4} & \tau_{3,4} & \tau_{4,4}
\end{bmatrix} + \begin{bmatrix}
\tau_1 & \tau_1 & \tau_1 & \tau_1 \\
\tau_2 & \tau_2 & \tau_2 & \tau_2 \\
\tau_3 & \tau_3 & \tau_3 & \tau_3 \\
\tau_4 & \tau_4 & \tau_4 & \tau_4
\end{bmatrix} \quad (6.15)
$$

Next, we have

$$
\mathbf{T}^{NI} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \tau_{2,2} & \tau_{2,3} & \tau_{2,4} \\
0 & \tau_{2,3} & \tau_{3,3} & \tau_{3,4} \\
0 & \tau_{2,4} & \tau_{3,4} & \tau_{4,4}
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 & 0 \\
\tau_2 & \tau_2 & \tau_2 & \tau_2 \\
\tau_3 & \tau_3 & \tau_3 & \tau_3 \\
\tau_4 & \tau_4 & \tau_4 & \tau_4
\end{bmatrix} \quad (6.16)
$$
The transmit and receive delays in (6.16) can now be combined into a single matrix, as given in (6.17).

\[
T^{NI} = \begin{bmatrix}
  t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} \\
  t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} \\
  t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} \\
  t_{4,1} & t_{4,2} & t_{4,3} & t_{4,4}
\end{bmatrix}
= \begin{bmatrix}
  0 & 0 & 0 & 0 \\
  \tau_2 & \tau_{2,2} + \tau_2 & \tau_{2,3} + \tau_2 & \tau_{2,4} + \tau_2 \\
  \tau_3 & \tau_{2,3} + \tau_3 & \tau_{3,3} + \tau_3 & \tau_{3,4} + \tau_3 \\
  \tau_4 & \tau_{2,4} + \tau_4 & \tau_{3,4} + \tau_4 & \tau_{4,4} + \tau_4
\end{bmatrix}
\]

(6.17)

It is at this point easy to observe that in order for (6.17) to become a symmetric matrix, \( \tau_2 = \tau_3 = \tau_4 \) must be set equal to zero. In other words, we have shown that in the case of the non-implementable optimization, the values in the transmit matrix are all zero, leading to:

\[
T^{NI} = \begin{bmatrix}
  0 & 0 & 0 & 0 \\
  0 & \tau_{2,2} & \tau_{2,3} & \tau_{2,4} \\
  0 & \tau_{2,3} & \tau_{3,3} & \tau_{3,4} \\
  0 & \tau_{2,4} & \tau_{3,4} & \tau_{4,4}
\end{bmatrix}
\]

(6.18)

As mentioned, we may alternatively set all the receive delay values equal to zero. We further note that there are only 6 unique delay values for the 4 element array transducer.

We can easily calculate that the number of unique delay values is given as \( \frac{N^2 - N}{2} \).
Keeping this in mind, let us move on to evaluating the different approaches for obtaining non-implementable delay matrices. To start off with, we will discuss a brute force method for calculating the optimal delay sets, called the Global Search Algorithm.

### 6.2 The Global Search Method

It is easy to explain this method with the help of an example. Let us assume that a given reflector (denoted by the superscript \( REFL \)) is specified, together with its location and orientation with respect to a given annular array transducer with \( N \) elements. Referring to eq.11 section 4.2.3, the total received signal for a reflector is given by

\[
\sum_{i=1}^{N} \sum_{j=1}^{N} u_{i,j}^{REFL} (t - t_{i,j}) = \sum_{i=1}^{N} \sum_{j=1}^{N} v_{i,j}^{REFL} (t) \otimes \delta(t - t_{i,j}) \otimes w(t) \tag{6.19}
\]

where \( v_{i,j}^{REFL} (t) \) is the echo signal received by the \( j^{th} \) annular ring while transmitting with the \( i^{th} \) annular ring of the transducer, \( t_{i,j} \) is the delay or time shift applied to the echo signal, and \( w(t) \) is a function which emulates the combined transmit-receive bandpass filtering effect of the transducer. The most straightforward way to find the optimal set of delay values would be to search through all the possible delay combinations for the elements of the annular array transducer in either transmitting or receiving mode. Such an approach is referred to as the *Global Search Method*. All possible combinations of \( t_{i,j} \) values are applied to the above equation and the corresponding energy of the total received signal is calculated for each combination as follows:
\[ E^{\text{REFL}} = \int_{-\infty}^{\infty} (u_{\text{sum}}^{\text{REFL}}(t))^2 \, dt. \] (6.20)

The set of \( t_{i,j} \) values, which produces the largest energy \( E^{\text{REFL}} \) in \( u_{\text{sum}}^{\text{REFL}}(t) \) is then given in the optimal delay matrix \( T^{\text{REFL},\text{Opt}} \). An arbitrary element in \( T^{\text{REFL},\text{Opt}} \) is \( t_{i,j}^{\text{opt}} \), which represents the delay value that is applied to the signal obtained with the \( i^{th} \) ring of the annular array transducer as transmitter and the \( j^{th} \) ring as receiver. A set of such optimal delay values for different possible combinations of transmitting and receiving array elements forms the optimal delay matrix \( T^{\text{REFL},\text{Opt}} \) for that reflector.

By the theory of formulation for non-implementable delay matrices, developed in the previous section, the Global Search Method, requires that \( \frac{N^2 - N}{2} \) independent delay values have to be chosen for an \( N \) element array transducer. Let us assume that the search range is divided up into \( m \) delay steps for each delay value (i.e. \( t_{i,j}^{\text{opt}} \) can take up \( m \) different values from a given range of delay values). Then \( \frac{N^2 - N}{2} \) calculations are needed to find the optimal delay set for an \( N \)-ring array with the Global Search Method. Thus, we can see that the time required to obtain the optimal delay values with this method, is computationally infeasible when the number of elements \( N \) in the array is large. For example, if we have a transducer with \( N = 8 \) array elements, and a delay range of 2 \( \mu s \) with delay steps of 20ns, \( m = 2 \, \mu s / 20ns = 100. \frac{N^2 - N}{2} = 28 \), for \( N = 8 \). Thus in this
case, \((100)^{28} = 10^{56}\) calculations are needed to find the optimal delay set, which as we can see is infeasible.

### 6.3 The Waveform Alignment Method

The previous section has demonstrated that the brute force optimization using the Global Search Method is not in general feasible. It is important to note that the total received signal for a given reflector will have the maximum energy if its composite received signals obtained using different transmitting and receiving annular array elements are aligned. This can be inferred based on the notion that all the signals \(u_{i,j}^{REFL}(t)\) in a received signal matrix can \(approximately\) be represented by a time-shifted and amplitude scaled version of some prototype received signal, \(u^{REFL}(t)\). Thus, we can write

\[
 u_{i,j}^{REFL}(t) \approx A_{i,j} u^{REFL}(t - t_{i,j}) \tag{6.20a}
\]

where \(A_{i,j}\) is the scale factor and \(t_{i,j}\) is the time shift.

Hence, an alternative strategy to the energy based optimization is to determine the transmit and receive delay values or rather the time shifts that need to be applied while transmitting or receiving these signals using the individual array elements, which at least approximately, align these signals so that they add constructively. This resulting method is termed as the \(Waveform Alignment Method.\)
Further, we are already aware that $u^{\text{REFL}}(t)$ can be considered as the convolution of a prototype echo signal, $v^{\text{REFL}}(t)$ with $w(t)$, where $w(t)$ models the combined transmit-receive impulse response of the transducer. Therefore, we now have this relation

$$u^{\text{REFL}}_{i,j}(t) \cong A_{i,j} u^{\text{REFL}}(t-t_{i,j}) \cong A_{i,j} v^{\text{REFL}}(t-t_{i,j}) \otimes w(t)$$  

(6.20b)

where $u^{\text{REFL}}_{i,j}(t)$ is the received signal obtained by using the $i^{th}$ ring and $j^{th}$ ring of the annular array transducer as transmitter and receiver respectively.

To estimate $t_{i,j}$ from (6.20b), $u^{\text{REFL}}_{i,j}(t)$ is cross-correlated with $u^{\text{REFL}}(t)$ or equivalently convolved with $u^{\text{REFL}}(-t)$ and the location of the peak of the cross-correlation function determines $t_{i,j}$. But it is not practical to make the waveform alignment algorithm specifically dependent on the knowledge of the reflector and the transducer array in the given pulse-echo system, so the estimation of $t_{i,j}$ can instead be based on cross correlation with the known function $w(t)$ or convolution with $w(-t)$. In this way, we use the assumption that the pulse-echo response is mainly a function of the pulse-echo response of the transducer and less a function of the reflector geometry. This holds in particular true if the transducer is rather narrowband. The band limiting effect of transducers improves the performance of the waveform alignment algorithm.

Let $r(t) = w(t) \otimes w(-t)$ be an autocorrelation function of the combined transmit-receive impulse response of the array transducer. We can obtain a correlation matrix, $R^{\text{REFL}}(t)$, by a correlation operation on the elements in $V^{\text{REFL}}(t)$ (which is an echo signal matrix of
the form in eq.1 section 4.2.1) with \( r(t) \), or a correlation operation on the elements in \( U^{REFL}(t) \) with \( w(t) \) as shown in (6.21).

Let an arbitrary element in \( R^{REFL}(t) \) be \( r_{i,j}^{REFL}(t) \). The time shift, \( t_{i,j}^{REFL,shift} \), may thus be found by locating the time of the peak in \( r_{i,j}^{REFL}(t) = u_{i,j}^{REFL}(t) \otimes w(t) \), \( i, j \in [1, N] \):

\[
t_{i,j}^{REFL,shift} = \text{peak}\{ u_{i,j}^{REFL}(t) \otimes w(-t) \} = \text{peak}\{ u_{i,j}^{REFL}(t) \otimes w(t) \} = \text{peak}\{ v_{i,j}^{REFL}(t) \otimes r(t) \}
\]

(6.21)

From the calculation of time shifts, \( t_{i,j}^{REFL,shift} \), a time shift matrix for the reflector \( REF\), \( T^{REFL,shift} \), can be created as shown in (6.22).

\[
T^{REFL,shift} = \begin{bmatrix}
  t_{1,1}^{REFL,shift} & \cdots & t_{1,j}^{REFL,shift} & \cdots & t_{1,N}^{REFL,shift} \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  t_{i,1}^{REFL,shift} & \cdots & t_{i,j}^{REFL,shift} & \cdots & t_{i,N}^{REFL,shift} \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  t_{N,1}^{REFL,shift} & \cdots & t_{N,j}^{REFL,shift} & \cdots & t_{N,N}^{REFL,shift}
\end{bmatrix}
\]

(6.22)

In order to maximize the energy of the received signal from the array transducer, an additional delay must be applied to each individual delay element in the delay matrix \( T^{REFL,shift} \) so that the sum of the two is the same for all elements in \( T^{REFL,shift} \). With the practical implementation of the waveform alignment algorithm in mind, only positive time shifts will be considered. For this purpose, we identify the maximum delay values among the elements in \( T^{REFL,shift} \) as \( t_{i,j}^{REFL,Opt,max} \). Therefore, the additional delay value, \( t_{i,j}^{REFL,Opt} \), necessary for the waveforms in \( U^{REFL}(t) \) to be aligned is found as
When $t_{i,j}^{\text{REFL,Opt}}$ is determined for $i, j \in [1, N]$, an optimal delay matrix, $T^{\text{REFL,Opt}}$, can be defined as in (6.24).

$$T^{\text{REFL,Opt}} = \begin{bmatrix}
    t_{1,1}^{\text{REFL,Opt}} & \cdots & t_{1,j}^{\text{REFL,Opt}} & \cdots & t_{1,N}^{\text{REFL,Opt}} \\
    \vdots & \ddots & \vdots & \ddots & \vdots \\
    t_{i,1}^{\text{REFL,Opt}} & \cdots & t_{i,j}^{\text{REFL,Opt}} & \cdots & t_{i,N}^{\text{REFL,Opt}} \\
    \vdots & \ddots & \vdots & \ddots & \vdots \\
    t_{N,1}^{\text{REFL,Opt}} & \cdots & t_{N,j}^{\text{REFL,Opt}} & \cdots & t_{N,N}^{\text{REFL,Opt}}
\end{bmatrix}$$

(6.24)

These values in $T^{\text{REFL,Opt}}$ from (6.24), are used in (6.19) as the respective time shift values $t_{i,j}$. Under the assumption that these are the optimal delay values, we can expect the energy $E^{\text{REFL}}$ of the total received signal $u_{\text{sum}}^{\text{REFL}}(t)$ obtained by using (6.19), to be the maximum energy from this total received signal. This maximum energy is calculated as follows:

$$E^{\text{REFL}} = \int_{-\infty}^{\infty} (u_{\text{sum}}^{\text{REFL}}(t))^2 \, dt = E_{\text{max}}^{\text{REFL}}$$

(6.25)

Let us look at an illustrative representation of the above algorithm for some reflector with a specific geometry, location and orientation and a 3 elements annular array transducer.
Filter

\[ w(t) \]

Echo signal matrix

\[ V(t) \]

Received signal matrix

\[ U(t) \]

Correlation signal matrix

\[ R(t) \]
Time at which peak (point of highest magnitude) of signal $R_{i,j}(t)$ in $R(t)$ is reached = $t_{i,j}$

Thus, from the above correlation matrix $R(t)$, we get the following $t_{i,j}$ values:

$$\begin{bmatrix}
    t_{1,1} & t_{1,2} & t_{1,3} \\
    t_{2,1} & t_{2,2} & t_{2,3} \\
    t_{3,1} & t_{3,2} & t_{3,3}
\end{bmatrix}$$

The largest value from this set

$$t_{\text{max}}$$

Calculation of optimal delay matrix

$$t_{i,j}^{Opt} = t_{\text{max}} - t_{i,j}$$

For e.g., $t_{1,1}^{Opt} = t_{\text{max}} - t_{1,1}$

Illustrative representation of optimal delay matrix $T^{Opt}$

$$\begin{bmatrix}
    t_{1,1}^{Opt} & t_{1,2}^{Opt} & t_{1,3}^{Opt} \\
    t_{2,1}^{Opt} & t_{2,2}^{Opt} & t_{2,3}^{Opt} \\
    t_{3,1}^{Opt} & t_{3,2}^{Opt} & t_{3,3}^{Opt}
\end{bmatrix}$$
We evaluated the performance of our algorithm by using the delay values we obtained using the method above, in one of our pulse-echo system simulations.

### 6.4 Pulse-echo system simulation

The Waveform Alignment Algorithm described above was tested for a specific ultrasound pulse-echo system simulation.

For our simulation, we considered a 3-ring annular array transducer, with rings of outer radii of 5.1mm, 7.3mm, 9mm. These radii were selected so that the areas of the 3 rings of the annular array are approximately equal. The transducer response $w(t)$, was modeled in the form of a bandpass filter with 2.5MHz center frequency and 2.5MHz bandwidth at the -3dB level. This is the filter which is used to obtain the received signals from the echo signals.

Three different types of reflector surfaces were considered, placed 50 mm away from the annular array transducer:

- **Reflector A**: A 15mm x 15mm flat planar reflector tilted at 6 degrees
- **Reflector B**: A cylindrical reflector surface with radius of curvature 10mm
- **Reflector C**: A sinusoidal reflector surface

(To produce reflector B and C, the 15mm x 15mm flat reflector surface shaped into a cylindrical and sinusoidal surface.)
As per the explanation in the previous sections, we derived the optimal delay matrix using the algorithms described above, and then used equations (1) and (8) to obtain the following Energy Table:

**Table 6.1:** Energy Table obtained using the Waveform Alignment Algorithm

<table>
<thead>
<tr>
<th></th>
<th>Reflector A</th>
<th>Reflector B</th>
<th>Reflector C</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{A,\text{Opt}}$</td>
<td>$2.1014\text{e}11$</td>
<td>$2.1327\text{e}10$</td>
<td>$5.5067\text{e}10$</td>
</tr>
<tr>
<td>$T_{B,\text{Opt}}$</td>
<td>$1.255\text{e}11$</td>
<td>$2.9971\text{e}10$</td>
<td>$4.0225\text{e}10$</td>
</tr>
<tr>
<td>$T_{C,\text{Opt}}$</td>
<td>$1.3974\text{e}11$</td>
<td>$4.5363\text{e}10$</td>
<td>$4.2361\text{e}10$</td>
</tr>
</tbody>
</table>

Comparing the values we obtained in this energy table with the format of a standard energy table as discussed in Chapter 5 section 5.3, we can infer that the optimal delay matrices using this algorithm have given erroneous results for Reflector B and Reflector C. Instead of using $T_{B,\text{Opt}}$ to obtain the maximum energy from the received signal for Reflector B, we can see that $T_{C,\text{Opt}}$, which is in fact the optimal delay matrix obtained specific to Reflector C, needs to be used. Similarly in the case of Reflector C, $T_{A,\text{Opt}}$ is used instead of $T_{C,\text{Opt}}$ to obtain the total received signal from reflector C with the maximum energy.

Also, in the case of Reflector A the relative energy difference between the maximum energy value and the values obtained using the other optimal delay matrices is fairly low. If we do not see the actual numerical energy value for reflector A, the differentiation between reflectors A, B and C is not very reliable. Thus, we felt the need to find a more reliable and effective algorithm. In order to do so, we will closely look at the underlying objective that we need to achieve in order to obtain the total received signal. We need to
align the received signal waveforms obtained using different rings of the annular array transducer so that their summation produces the maximum energy as per (6.25).

### 6.5 Waveform Correlation Method

While investigating the reason for the poor performance of the algorithm proposed above, we formulated the approach that given two waveforms of received signals, we can consider one waveform as a reference and apply a time shift to the other waveform (non-reference waveform). We can then plot an energy curve, \( E(t_{\text{shift}}) \), for the energy of the combined reference waveform and non-reference waveform by using the following formula:

\[
E(t_{\text{shift}}) = \int_{-\infty}^{\infty} \left[ u_{\text{ref}}(t) + u_{\text{non-ref}}(t - t_{\text{shift}}) \right]^2 dt
\]  

(6.26)

The time shift \( t_{\text{shift}} \) for which the peak of the energy curve that is obtained by aligning the two waveforms is reached, is the optimal delay value. It is also important to note that the time shifts need to be applied only in the range where the waveforms overlap.

On expanding (6.26), we get

\[
E(t_{\text{shift}}) = \int_{-\infty}^{\infty} \left[ u_{\text{ref}}^2(t) + u_{\text{non-ref}}^2(t) + 2u_{\text{ref}}(t)u_{\text{non-ref}}(t - t_{\text{shift}}) \right] dt
\]

\[
= \int_{-\infty}^{\infty} u_{\text{ref}}^2(t) dt + \int_{-\infty}^{\infty} u_{\text{non-ref}}^2(t) dt + 2u_{\text{ref}}(t) \otimes u_{\text{non-ref}}(-t)
\]

\[= \text{Energy}_{\text{ref}} + \text{Energy}_{\text{non-ref}} + 2E_{\text{corr}}(t_{\text{shift}})\]
which implies that the energy curve is a function of \( t_{\text{shift}} \) and mainly depends on the correlation between the reference and the shifted non-reference waveforms i.e. \( u_{\text{ref}}(t) \otimes u_{\text{non-ref}}(-t) \), since \( \text{Energy}_{\text{reference}} + \text{Energy}_{\text{non-reference}} \) are constant positive numbers. Hence, the energy curve is a function of the cross-correlation term \( E_{\text{corr}}(t_{\text{shift}}) \) and this method is termed as the \textit{Waveform Correlation method}.

Looking at the received signal matrix \( U^{\text{REFL}}(t) \), one of the received signals in \( U^{\text{REFL}}(t) \) is to be considered as a reference signal. In the previous algorithm, the maximum delay value among the elements in \( T^{\text{REFL,shift}} \) was selected as the reference and the additional delay values \( t_{i,j}^{\text{REFL,Opt}} \) necessary for the waveforms to be aligned were calculated in (6.23) above.

We consider the received signal obtained using ring 1 as transmitter and ring 1 as receiver, \( u_{1,1}(t) \), as reference \( u_{\text{ref}}(t) \), and apply time shifts \( t_{\text{shift}} \) within a defined range of \( t_{\text{shift}} \in [-1 \mu s, 1 \mu s] \) to the remaining elements \( u_{i,j}(t) \) of the received signal matrix \( U^{\text{REFL}}(t) \).

The energy curve \( E(t_{\text{shift}}) \), for the different elements of the received signal elements is obtained using (6.26). The time shift \( t_{\text{shift}} \) producing the highest peak in the energy curve is considered as the optimal delay \( t_{\text{opt}} \) for that element. Hence an optimal delay matrix is obtained with the \( t_{\text{opt}} \) values corresponding to each element of the received signal matrix.
These values are further used in (6.19) and integrated as in (6.20) to obtain the corresponding set of energy values for different types of reflectors.

Just as stated above, let us consider this algorithm given a reflector, with a specific geometry, location and orientation and a 3 elements annular array transducer. The steps in the algorithm are illustrated on the next page for clarity.
For different values of $t_{shift}$ (in a given range, e.g. $-5ms < t_{shift} < 5ms$), we can obtain energy curves using the following formula, and obtain a matrix of energy curves $E_{i,j}(t)$ as illustrated below:

$$
\text{Energy curve, } E_{i,j}(t_{shift}) = \int_{-\infty}^{\infty} \left[ u_{ref}(t) + u_{i,j}(t-t_{shift}) \right]^2 dt
$$

Hence, we can obtain an optimal delay matrix $T^{opt}$. 

Time at which peak (point of highest magnitude) of signal $E_{i,j}(t)$ in $E(t)$ is reached = $t^{opt}_{i,j}$
We also evaluated this algorithm by using the optimal delay values we obtained, in the simulation discussed above. Given on the next page are the set of results obtained for the algorithm discussed in Section 6.3, which will further clearly illustrate the steps described above.
(A) For reflector A, $V^{REFL}(t) = V^{A}(t)$

Above is an illustration of the echo signal matrix for a tilted flat reflector surface. The X-axis for all signals is time [in $\mu$sec] and the Y-axis for all signals is amplitude [in $10^7 m^4/s^3$].
Above is an illustration of the echo signal matrix for a cylindrical reflector surface. The X-axis for all signals is time [in $\mu$sec] and the Y-axis for all signals is amplitude [in $10^8 m^4 / s^3$].
(C) For reflector $C$, $V_{REFL}^t(t) = V^C(t)$

Above is an illustration of the echo signal matrix for a sinusoidal reflector surface. The X-axis for all signals is time [in $\mu$sec] and the Y-axis for all signals is amplitude [in $10^8 m^4 / s^3$].
Obtaining the **received signal matrix** $U^{REFL}(t)$ which is also a $3 \times 3$ matrix

(A) For reflector A, $U^{REFL}(t) = U^A(t) = T^{B,Opt} \otimes V^A(t) \otimes w(t)$

Above is an illustration of the received signal matrix for a tilted flat reflector surface. The X-axis for all signals is time [in $\mu$sec] and the Y-axis for all signals is amplitude [in $10^7 m^4 / s^3$].
(B) For reflector B, \( U^{REFL} (t) = U^B (t) = V^B (t) \otimes w(t) \)

Above is an illustration of the received signal matrix for a cylindrical reflector surface. The X-axis for all signals is time [in \( \mu \text{sec} \)] and the Y-axis for all signals is amplitude [in \( 10^8 m^4/s^3 \)].
(C) For reflector C,  \( U^{REFL}(t) = U^C(t) = V^C(t) \otimes w(t) \)

Above is an illustration of the received signal matrix for a sinusoidal reflector surface. The X-axis for all signals is time \([\text{in } \mu\text{sec}]\) and the Y-axis for all signals is amplitude \([\text{in } 10^7 \text{m}^4/\text{s}^3]\).
Correlation Curves

(A) For Reflector A, \(2xcorr^4 \{u_{i,1}(t), u_{i,j}(t - t_{\text{shift}})\}\)

Above is an illustration of the cross-correlation signal matrix for a tilted flat reflector surface.

The X-axis for all signals is time [in \(\mu\text{sec}\)] and the Y-axis for all signals is amplitude [in \(10^7 m^4/s^3\)].
Above is an illustration of the cross-correlation signal matrix for a cylindrical reflector surface. The X-axis for all signals is time [in \( \mu \text{sec} \)] and the Y-axis for all signals is amplitude [in \( 10^9 m^4 / s^3 \)].
(C) For reflector C, \( 2 \text{corr}^C \{ u_{i,1}, u_{i,j} (t - t_{\text{shift}}) \} \)

Above is an illustration of the cross-correlation signal matrix for a sinusoidal reflector surface.

The X-axis for all signals is time [in \( \mu \text{sec} \)] and the Y-axis for all signals is amplitude

[\text{in} 10^9 m^4 / s^3].
The optimal delay values are obtained for each type of the reflector with different transmitting and receiving transducers. Hence we obtain a $3 \times 3$ matrix in these cases too.

(A) For reflector A

$$T^{A,\text{Opt}} = \begin{bmatrix} 0 & 2.5253e-7 & 4.4444e-7 \\ 2.5253e-7 & 4.3434e-7 & 5.7576e-7 \\ 4.4444e-7 & 5.7576e-7 & 6.5657e-7 \end{bmatrix}$$

(B) For reflector B

$$T^{B,\text{Opt}} = \begin{bmatrix} 0 & 1.0101e-8 & 3.0303e-8 \\ 1.0101e-8 & -1.0101e-7 & 0 \\ 3.0303e-8 & 0 & 6.1616e-7 \end{bmatrix}$$

(C) For reflector C

$$T^{C,\text{Opt}} = \begin{bmatrix} 0 & 5.0505e-8 & 1.0101e-7 \\ 5.0505e-8 & 5.0505e-8 & 8.0808e-8 \\ 1.0101e-7 & 8.0808e-8 & 7.0707e-8 \end{bmatrix}$$

**Table 6.2:** The energy table calculated using the Waveform Correlation Method

<table>
<thead>
<tr>
<th></th>
<th>Reflector A</th>
<th>Reflector B</th>
<th>Reflector C</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T^{A,\text{Opt}}$</td>
<td>$8.2057e8$</td>
<td>3.0959e10</td>
<td>3.0554e10</td>
</tr>
<tr>
<td>$T^{B,\text{Opt}}$</td>
<td>2.2616e8</td>
<td>$9.0262e10$</td>
<td>2.8611e10</td>
</tr>
<tr>
<td>$T^{C,\text{Opt}}$</td>
<td>3.5167e8</td>
<td>2.0010e10</td>
<td>$1.2892e11$</td>
</tr>
</tbody>
</table>

By examining the energy values in Table 6.2, we observe that on obtaining the optimal delay matrices for the different reflectors using the Waveform Correlation algorithm and on obtaining the corresponding energies using each of these optimal delay matrices for
different reflector surfaces, the maximum energy value corresponds to the reflector type whose optimal delay matrix we are using for the energy calculation. This complies with the basic energy optimization concept discussed in Chapter 5.

We have investigated a potential further improvement to the performance of this algorithm by using an adaptive technique. By improving the performance, we will achieve a better differentiation in the energy values, i.e. between the maximum energy value for a given reflector and the other energy values for that reflector. This method will be presented in Section 6.6.

6.6 Adaptive Waveform Correlation Method

In the adaptive technique, it is not a priori assumed that \( u_{1,1}(t) \) is the most appropriate signal to represent the reference signal \( u_{ref}(t) \), but where instead the reference signal is stepwise updated as the correlation steps are carried out. To describe the adaptive technique, consider an \( N \times N \) received signal matrix for a reflector \( \text{REFL} \), as shown in (6.27):

\[
U^{\text{REFL}}(t) = \begin{bmatrix}
  u_{i,j}(t) & \ldots & u_{i,j}(t) & \ldots & u_{i,N}(t) \\
  \ldots & \ddots & \ldots & \ddots & \ldots \\
  u_{N,j}(t) & \ldots & u_{N,j}(t) & \ldots & u_{N,N}(t)
\end{bmatrix}
\]  (6.27)
Given that $U_{REFL}(t)$ is symmetric, there are $N(N+1)/2$ unique signals in $U_{REFL}(t)$. The first step is to calculate the energy of these $N(N+1)/2$ signals, and then order the signals in terms of descending energy.

As an example let us consider as illustration the case for $N = 4$, giving 10 unique signals. Let the signals listed in terms of descending energy be as follows:

$$u_{2,2}^{REFL}(t), u_{1,3}^{REFL}(t), u_{1,2}^{REFL}(t), u_{4,4}^{REFL}(t), \ldots, u_{4,3}^{REFL}(t)$$

The signal with the highest energy, $u_{2,2}^{REFL}(t)$, is chosen to represent the initial reference signal, $u_{ref,1}^{REFL}(t)$ where the subscript ‘$\text{ref,1}$’ indicates the first reference signal. The first step is then to cross-correlate $u_{ref,1}^{REFL}(t) = u_{2,2}^{REFL}(t)$ with the signal with the next highest energy, in this case $u_{1,3}^{REFL}(t)$, producing the correlation signal $r_1^{REFL}(t)$:

$$r_1^{REFL}(t) = u_{ref,1}^{REFL}(t) \otimes u_{1,3}^{REFL}(-t)$$

The time shift, $t_{1,3}^{REFL,shift}$, is the time occurrence of the positive peak amplitude in $r_1^{REFL}(t)$, allowing $u_{1,3}^{REFL}(t)$ to be approximated as

$$u_{1,3}^{REFL}(t) \simeq A_{1,3} u_{ref,1}^{REFL}(t - t_{1,3}^{REFL,shift}) \equiv A_{1,3} u_{2,2}^{REFL}(t - t_{1,3}^{REFL,shift})$$

The time shift $t_{1,3}^{REFL,shift} = t_{3,1}^{REFL,shift}$ will thus be the first time shift value determined for the time shift matrix for reflector $REFL$. This time shift can be used in two ways:

$t_{1,3}^{REFL,shift}$ is one of the time shift values in the time shift matrix for the Adaptive Waveform
Correlation Method as shown in (6.22), where \( t_{2,2}^{\text{REFL,shift}} = 0 \), and this time shift will also be used to generate a modified reference signal, as is described below. The scale factor \( A_{1,3} \) is not going to be considered in this work. The next step is to add \( u_{2,2}^{\text{REFL}}(t) \) and the shifted \( u_{1,3}^{\text{REFL}}(t) \) to form the new reference signal, \( u_{2,2}^{\text{REFL}}(t) \); once again, ‘ref.2’ indicates the second reference signal:

\[
u_{\text{ref.2}}^{\text{REFL}}(t) = u_{\text{ref.1}}^{\text{REFL}}(t) + u_{1,3}^{\text{REFL}}(t + t_{1,3}^{\text{REFL,shift}}) = u_{2,2}^{\text{REFL}}(t) + u_{1,3}^{\text{REFL}}(t + t_{1,3}^{\text{REFL,shift}})
\]

The next step is to correlate \( u_{\text{ref.2}}^{\text{REFL}}(t) \) with the signal with the third highest energy, \( u_{1,1}^{\text{REFL}}(t) \), producing the correlation signal \( r_{2}^{\text{REFL}}(t) \):

\[
r_{2}^{\text{REFL}}(t) = u_{\text{ref.2}}^{\text{REFL}}(t) \otimes u_{1,1}^{\text{REFL}}(-t)
\]

The time shift, \( t_{1,1}^{\text{REFL,shift}} \), is defined as the time occurrence of the positive peak amplitude in \( r_{2}^{\text{REFL}}(t) \). The shift \( t_{1,1}^{\text{REFL,shift}} \) is thus the next time shift value determined for the time shift matrix for reflector \( \text{REFL} \). The reference signal, \( u_{\text{ref.2}}^{\text{REFL}}(t) \), is then added to the shifted \( u_{1,1}^{\text{REFL}}(t) \) to produce the next following reference signal, \( u_{\text{ref.3}}^{\text{REFL}}(t) \):

\[
u_{\text{ref.3}}^{\text{REFL}}(t) = u_{\text{ref.2}}^{\text{REFL}}(t) + u_{1,1}^{\text{REFL}}(t + t_{1,1}^{\text{REFL,shift}})
\]

This process of correlating, determining time shift and producing an updated reference signal is continued until the time shift associated with the last signal (with the lowest energy) has been determined. The signal processing steps of the Waveform Correlation Method described so far, based on \( \text{Reflector A} \), are described in the following.
From the time location of the peak in each of the correlation signals, $r_i^{REFL}(t), i \in 0, \frac{N(N+1)}{2} - 1$, a symmetric time shift matrix for the reflector $REFL$, $T_{REFL,shift}^{REFL}$, can be created as shown in (6.28).

$$T_{REFL,shift}^{REFL} = \begin{bmatrix}
  t_{1,1}^{REFL,shift} & \cdots & t_{1,j}^{REFL,shift} & \cdots & t_{1,N}^{REFL,shift} \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  t_{i,1}^{REFL,shift} & \cdots & t_{i,j}^{REFL,shift} & \cdots & t_{i,N}^{REFL,shift} \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  t_{N,1}^{REFL,shift} & \cdots & t_{N,j}^{REFL,shift} & \cdots & t_{N,N}^{REFL,shift}
\end{bmatrix}$$

(6.28)

To maximize the energy of the received signal from the array transducer due to the reflector, the individual received signals in $U^{REFL}(t)$ need to be aligned as well as possible. As the received signals in $U^{REFL}(t)$ are similar, but not identical, the alignment can never be perfect. For a practical implementation, we have to see to it that the alignment is accomplished by applying positive time shifts or positive delays. Considering these two factors of aligning the received signals and applying positive time shifts, we defined the maximum delay value among the elements in $T_{REFL,shift}^{REFL}$ as $t_{max}^A$.

Similar to the Waveform Correlation Method described previously, we then added a positive time shift, $t_{i,j}^{REFL,Opt}$ so that $(t_{i,j}^{REFL,Opt} + t_{i,j}^{REFL,shift})$ is the same for all cells in the matrix in (6.28). Hence, the additional delay values $t_{i,j}^{REFL,Opt}$, necessary for aligning the waveforms in $U^{REFL}(t)$ are found as

$$t_{i,j}^{REFL,Opt} = t_{max}^A - t_{i,j}^{REFL,shift}$$

(6.29)
As an example, if $t_{i,j}^{REFL,shift}$ is the highest time shift value among all the elements in (6.28), the corresponding positive time shift calculated for the signal received with ring 1 as transmitter and receiver would be zero.

With $t_{i,j}^{REFL,Opt}$ determined for $i, j \in [1, N]$, an optimal delay matrix, $T^{REFL,Opt}$, can be formulated as shown in (6.30)

$$
T^{REFL,Opt} = 
\begin{bmatrix}
    t_{i,1}^{REFL,Opt} & \cdots & t_{i,j}^{REFL,Opt} & \cdots & t_{i,N}^{REFL,Opt} \\
    \vdots & \ddots & \vdots & \ddots & \vdots \\
    \vdots & \vdots & \ddots & \ddots & \vdots \\
    t_{N,1}^{REFL,Opt} & \cdots & t_{N,j}^{REFL,Opt} & \cdots & t_{N,N}^{REFL,Opt}
\end{bmatrix}
$$

When the positive delay values thus calculated are used in (6.19), we obtain the total received signal with the maximum energy, which is our objective. Hence, this received signal which maximizes the energy from Reflectors $REFL$ is denoted as $u_{max}^{REFL}(t)$, and is obtained as follows:

$$
u_{max}^{REFL}(t) = \sum_{i=1}^{N} \sum_{j=1}^{N} v_{i,j}^{REFL}(t) \otimes \delta(t - t_{i,j}^{REFL,Opt}) \otimes w(t)
$$

(6.31)

The illustration of this algorithm is given below.
Next, we calculate the energies of each received signal above and arrange them in order of their energies. As an example, let us consider:

\[
\text{Energy}(u_{2,3}(t)) = \text{Energy}(u_{3,2}(t)) > \text{Energy}(u_{3,3}(t)) > \text{Energy}(u_{1,3}(t)) = \text{Energy}(u_{3,1}(t)) > \text{Energy}(u_{2,2}(t)) > \ldots \]

Now, \( u_{2,3}(t) = u_{ref,j}(t) \)

As per the *Adaptive Waveform Correlation Method*, we then calculate energy curves \( E_{i,j}(t) \) as follows, and the time at which peak (point of highest magnitude) of signal \( E_{i,j}(t) \) is reached = \( t_{i,j}^{opt} \)
\[ E_{2,3}(t) = \int_{-\infty}^{\infty} \left[ u_{ref,0}(t) + u_{2,3}(t - t_{shift}) \right]^2 \]

It can easily be interpreted that for \( E_{2,3}\left(t_{shift}\right) = E_{2,3}^{max}\left(t_{shift}\right), t_{shift} = 0\), thus \( t_{Opt}^{2,3} = t_{Opt}^{2,3} = 0 \)

Now, since the signal with the next highest energy is \( u_{3,3}(t) \),

\[ E_{3,3}\left(t_{shift}\right) = \int_{-\infty}^{\infty} \left[ u_{ref,0}(t) + u_{3,3}(t - t_{shift}) \right]^2 \]

For \( E_{3,3}\left(t_{shift}\right) = E_{3,3}^{max}\left(t_{shift}\right), t_{shift} = t_{Opt}^{3,3} \)

Now, \( u_{ref,0}(t) + u_{3,3}(t - t_{Opt}^{3,3}) = u_{ref,3}(t) \)

Since the signal with the next highest energy is \( u_{1,3}(t) \),

\[ E_{1,3}\left(t_{shift}\right) = \int_{-\infty}^{\infty} \left[ u_{ref,3}(t) + u_{1,3}(t - t_{shift}) \right]^2 \]

For \( E_{1,3}\left(t_{shift}\right) = E_{1,3}^{max}\left(t_{shift}\right), t_{shift} = t_{Opt}^{1,3} \)

Now, \( u_{ref,3}(t) + u_{1,3}(t - t_{Opt}^{1,3}) = u_{ref,1}(t) \)

Since the signal with the next highest energy is \( u_{2,2}(t) \),

\[ E_{2,2}\left(t_{shift}\right) = \int_{-\infty}^{\infty} \left[ u_{ref,2}(t) + u_{2,2}(t - t_{shift}) \right]^2 \]

For \( E_{2,2}\left(t_{shift}\right) = E_{2,2}^{max}\left(t_{shift}\right), t_{shift} = t_{Opt}^{2,2} \)

Now, \( u_{ref,2}(t) + u_{2,2}(t - t_{Opt}^{2,2}) = u_{ref,3}(t) \)

By moving ahead in this manner, we get an energy curve matrix \( E(t_{shift}) \), comprising of individual energy curves \( E_{i,j}\left(t_{shift}\right) \), calculated as explained above.

Hence, we get an energy curve matrix and then the optimal delay matrix.
Energy curves

\[ E(t) \]

Time at which peak (point of highest magnitude) of signal \( E_{i,j}(t) \) in \( E(t) \) is reached = \( t_{i,j}^{Opt} \). Hence, we can obtain an optimal delay matrix \( T^{Opt} \).

**Optimal delay matrix**

Illustrative representation of optimal delay matrix \( T^{Opt} \)

\[
\begin{bmatrix}
  t_{1,1}^{Opt} & t_{1,2}^{Opt} & t_{1,3}^{Opt} \\
  t_{2,1}^{Opt} & t_{2,2}^{Opt} & t_{2,3}^{Opt} \\
  t_{3,1}^{Opt} & t_{3,2}^{Opt} & t_{3,3}^{Opt}
\end{bmatrix}
\]

**Table 6.3**: The Energy table obtained using the Adaptive Waveform Correlation algorithm

<table>
<thead>
<tr>
<th></th>
<th>Reflector A</th>
<th>Reflector B</th>
<th>Reflector C</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_{A,Opt} )</td>
<td>1.301e11</td>
<td>2.6562e11</td>
<td>2.6448e11</td>
</tr>
<tr>
<td>( T_{B,Opt} )</td>
<td>7.9509e10</td>
<td>9.3497e11</td>
<td>4.8166e11</td>
</tr>
<tr>
<td>( T_{C,Opt} )</td>
<td>5.2329e10</td>
<td>8.2506e11</td>
<td>7.8844e11</td>
</tr>
</tbody>
</table>
6.7 Observations
It would be interesting to plot these energy values as energy bars and compare them to actually deduce that the Adaptive Waveform Correlation method can be used to provide accurate results, and the highest values of energy. The red energy bar is obtained using $T^{A,\text{Opt}}$ as the optimal delay matrix, while the blue and green are obtained using $T^{A,\text{Opt}}$ and $T^{A,\text{Opt}}$ respectively. Method 1 is the Waveform Alignment Method discussed in 6.5, while Method 2 is the Waveform Correlation Method discussed in 6.6 and Method 3 is the Adaptive Waveform Correlation Method discussed in 6.7.

6.8 Inference
As can be seen from the energy bars in Table 6.4 using the Adaptive Waveform Correlation Method, the optimal delay matrix calculated for a given reflector geometry delivers the highest energy for that reflector geometry. Also, there is a considerable difference between the energies obtained for a given reflector using the optimal delay matrix calculated for that reflector as opposed to the energy obtained for that reflector using the optimal delay matrix calculated for another reflector.
Table 6.4: Energy bar graphs to compare the energy values calculated using the non-implementable energy algorithms

<table>
<thead>
<tr>
<th>Method</th>
<th>Reflector A</th>
<th>Reflector B</th>
<th>Reflector C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Method 1</td>
<td><img src="image1" alt="Graph" /></td>
<td><img src="image2" alt="Graph" /></td>
<td><img src="image3" alt="Graph" /></td>
</tr>
<tr>
<td>Method 2</td>
<td><img src="image4" alt="Graph" /></td>
<td><img src="image5" alt="Graph" /></td>
<td><img src="image6" alt="Graph" /></td>
</tr>
<tr>
<td>Method 3</td>
<td><img src="image7" alt="Graph" /></td>
<td><img src="image8" alt="Graph" /></td>
<td><img src="image9" alt="Graph" /></td>
</tr>
</tbody>
</table>
Chapter 7

The implementable energy optimization algorithm

The non-implementable optimization, discussed in the previous chapter is of value, in order to determine the optimal performance when the pulse echo system is operating with maximum flexibility. This means that every transducer element can be fired one at a time independent of one another. However, most practical implementations require that all elements in the transmitting transducer are fired together although each with its own unique delay. In Chapter 6, we have seen how the implementation of a system based on the delay values in a non-implementable delay set requires that one element of an array transducer be fired at a time, rather than firing all array elements simultaneously, with appropriate time shifts applied, as is done conventionally. The condition of firing one element at a time makes the system slow, impractical to implement and subject to poor signal to noise ratio. Hence, the associated delay set was termed the non-implementable delay set, and we now need to come up with an implementable delay set for practical purposes. Just as in the previous chapter, this chapter evaluates several different ways to obtain these delay sets; however, based on specific advantages and drawbacks of each of these methods, we finalized on one set of methods to obtain the implementable delay values.

Before proceeding further, we must analyze the formulation of an implementable delay matrix just as we did in Chapter 6, Section 6.1 for the non-implementable delay matrix.
7.1. Formulation of delay matrices for the implementable optimization method

It is important to recall that in this formulation, we transmit with all the array elements together and receive with all the array elements at together. It is assumed that the excitation signal to each element can have its own unique delay at transmit, and that likewise, after the echo has been detected by each element, a unique delay can be assigned to each received signal before summation.

This form of operating the array transducer gives less flexibility relative to the operation for the non-implementable optimization. However, since this is the standard way that a pulse-echo systems operates, this form lends itself much better to implementation in practical pulse echo array systems.

Let us reconsider the four element array transducer described in Chapter 6, Section 6.1. We can illustrate the transmit delays and the receive delays graphically, as is done in Fig 7.1. Note that we are starting the analysis with the assumption that we will need independent sets of transmit delays and receive delays.
The transmit delays can be arranged in a column matrix (or array), $T^{\text{TRANS}}$, while the receive delays can be arranged in a row matrix (or array), $T^{\text{REC}}$, as given in (7.1).

$$T^{\text{TRANS}} = \begin{bmatrix} \tau_1 & \tau_2 & \tau_3 & \tau_4 \end{bmatrix}; \quad T^{\text{REC}} = \begin{bmatrix} \tau_1' \\ \tau_2' \\ \tau_3' \\ \tau_4' \end{bmatrix}$$ (7.1)

By repeating the rows in $T^{\text{TRANS}}$ and the columns in $T^{\text{REC}}$, a delay matrix for the implementable optimization, $T^{\text{IMP}}$, can be created, as shown in (7.2).

$$T^{\text{IMP}} = \begin{bmatrix} \tau_1 & \tau_2 & \tau_3 & \tau_4 \\ \tau_1 & \tau_2 & \tau_3 & \tau_4 \\ \tau_1 & \tau_2 & \tau_3 & \tau_4 \\ \tau_1 & \tau_2 & \tau_3 & \tau_4 \end{bmatrix} + \begin{bmatrix} \tau_1' & \tau_1' & \tau_1' & \tau_1' \\ \tau_2' & \tau_2' & \tau_2' & \tau_2' \\ \tau_3' & \tau_3' & \tau_3' & \tau_3' \\ \tau_4' & \tau_4' & \tau_4' & \tau_4' \end{bmatrix}$$ (7.2)
In (7.3), $T^{\text{IMP}}$ is converted into a single matrix. Clearly, (7.3) is not a symmetrical matrix, as we had stipulated in Chapter 6, Section 6.1. This issue will be addressed later.

\[
T^{\text{IMP}} = \begin{bmatrix}
\tau_1 + \tau_1' & \tau_2 + \tau_1' & \tau_3 + \tau_1' & \tau_4 + \tau_1' \\
\tau_1 + \tau_2' & \tau_2 + \tau_2' & \tau_3 + \tau_2' & \tau_4 + \tau_2' \\
\tau_1 + \tau_3' & \tau_2 + \tau_3' & \tau_3 + \tau_3' & \tau_4 + \tau_3' \\
\tau_1 + \tau_4' & \tau_2 + \tau_4' & \tau_3 + \tau_4' & \tau_4 + \tau_4'
\end{bmatrix}
\]  

Now we will again make use of the fact that the beamforming is determined by the relative delays among the transmit delays and among the receive delays, and we can therefore set one of the transmit delay values and one of the receive delay values equal to zero. Without any loss in generality, we will choose to set $\tau_1$ and $\tau_1'$ equal to zero. (To be technically correct, $\tau_2$ should be replaced with $\tau_2 - \tau_1$, $\tau_3$ should be replaced with $\tau_3 - \tau_1$ etc., and the same for the receive delays). This results in the following delay matrix for implementable optimization.

\[
T^{\text{IMP}} = \begin{bmatrix}
0 & \tau_2 & \tau_3 & \tau_4 \\
\tau_2 & \tau_2 + \tau_2' & \tau_3 + \tau_2' & \tau_4 + \tau_2' \\
\tau_3 & \tau_2 + \tau_3' & \tau_3 + \tau_3' & \tau_4 + \tau_3' \\
\tau_4 & \tau_2 + \tau_4' & \tau_3 + \tau_4' & \tau_4 + \tau_4'
\end{bmatrix}
\]  

We observe that $t_{i+1,j} = t_{i,j} + t_{i+1,j+1} - t_{i,j+1}$ is now fulfilled. However, we also note that (7.4) is not a symmetrical matrix. To make (7.4) symmetrical, we must require that $\tau_2 = \tau_2'$; $\tau_3 = \tau_3'$; and $\tau_4 = \tau_4'$. This states that the transmit delays and the receive delays
are not independent, but must in fact all be the same. For simplicity, we will express (7.4) in terms of the transmit delays, as given in (7.5). For formulating the received beamformed signal, (7.5) also contains $T^{\text{IMP}}$ in the form of the $t_{i,j}$ elements.

$$
T^{\text{IMP}} = \begin{bmatrix}
0 & t_{1,2} & t_{1,3} & t_{1,4} \\
 t_{1,2} & t_{2,2} & t_{2,3} & t_{2,4} \\
 t_{1,3} & t_{2,3} & t_{3,3} & t_{3,4} \\
 t_{1,4} & t_{2,4} & t_{3,4} & t_{4,4}
\end{bmatrix} = \begin{bmatrix}
\tau_2 & \tau_3 & \tau_4 \\
2\tau_2 & \tau_3 + \tau_2 & \tau_4 + \tau_2 \\
\tau_3 & \tau_2 + \tau_3 & 2\tau_3 & \tau_4 + \tau_3 \\
\tau_4 & \tau_2 + \tau_4 & \tau_3 + \tau_4 & 2\tau_4
\end{bmatrix}
$$

Now the matrix in (7.5) is clearly symmetrical, and what we can further observe is that the formulation $t_{i+1,j} = t_{i,j} + t_{i+1,j+1} - t_{i,j+1}$ is fulfilled. This formulation will be used in one of the methods that we will discuss in the next section to obtain the implementable delay matrix. There are now only $(N-1)$ unique delay values needed. This is illustrated in (7.6), which describes the transmit delay matrix and the receive delay matrix.

$$
T^{\text{TRANS}} = \begin{bmatrix}
0 & \tau_2 & \tau_3 & \tau_4
\end{bmatrix}; \quad T^{\text{REC}} = \begin{bmatrix}
0 \\
\tau_2 \\
\tau_3 \\
\tau_4
\end{bmatrix}
$$

The received beamformed signal from the reflector $REFL$ is obtained as given earlier in Chapter 6, eq.(6.2), which is repeated here as (7.7) for a four element transducer.

$$
u^{\text{SUM}}(t) = \sum_{i=1}^{4} \sum_{j=1}^{4} u_{i,j}(t - t_{i,j}).$$
7.2. Implementable methods

The first two implementable methods, (the methods which produce delay matrices that can be used in practice in an actual pulse-echo system) that are discussed here, are the practical adaptations of the waveform correlation concept.

To involve use of fewer variables, the algorithms below will be denoted based on a 6 ring annular array transducer i.e. $N = 6$. Therefore we have a $6 \times 6$ received signal matrix.

7.2.1. Method 1: Simple Waveform Correlation Method

In an attempt to obtain the individual optimal delay matrices for the transmitter and receiver rings, it was decided to calculate the delay value or shift that needs to be applied to each annular array ring while transmitting and receiving so that they can all be fired at a time. A method to find out this specific delay value would be to obtain a characteristic received signal for each transmitter or receiver ring and then find the time shift which when applied to this signal, maximizes the energy of the total received signal from the reflector. Hence, we formulated the idea of using a column matrix as a representation of the $6 \times 6$ received signal matrix, where each element is the sum of the received signals transmitted by the same transducer ring but received by each of the six rings, as we have already seen in Section 7.1.

Going by this concept, we first obtain a column matrix ($6 \times 1$ matrix in our case) using the elements from the $6 \times 6$ received signal matrix for a reflector $REFL$ as described in Chapter 6, Section 6.1, equation (6.1), and re-stated for the convenience of the reader in
(7.9) below. Each element of the column matrix is calculated as the sum of the 6 elements in the corresponding row of the received signal matrix. Hence, the \( m^{th} \) row of the column matrix will contain the element \( u_m(t) = \sum_{i=m}^{6} u_{i,j}(t) \), where \( u_{i,j}(t) \) is the received signal obtained using the \( i^{th} \) ring and the \( j^{th} \) ring of the annular array transducer as transmitter and receiver, respectively.

The top-most element of the column matrix: \( u_1(t) = \sum_{j=1}^{6} u_{1,j}(t) \), which is the sum of the individual received signals obtained on transmitting with the innermost ring and receiving with every ring of the annular array transducer respectively is selected as a reference signal. A cross correlation between the reference signal and other elements of the column matrix gives rise to what we term as a set of energy curves as discussed in Chapter 6, Section 6.4, and formulated in (6.26). Since there are 6 elements in the column matrix, \( u_m(t), m \in [1,6] \), we obtain 6 energy curves considering the reference element is cross correlated with each of the other 5 elements and with itself. The time shift at which the peak of the energy curve \( E_m(t) = u_{\text{ref}}(t) \ast u_m(t), m \in [1,6] \) is reached, is noted as the implementable delay value \( \tau_m \). Also, it is important to note that the ‘\( \ast \)’ denotes cross-correlation. It is obvious that on cross-correlating a signal with itself, the peak of the energy curve is reached when there is no time shift and the time signals coincide; thus the implementable delay value becomes \( \tau_1 = 0 \), for \( u_1(t) \). The column matrix consisting of the six delay values is termed as the implementable delay matrix \( \Gamma^{\text{REFL.Opt}} \). This column
matrix of delay values can be used on the received signal column matrix to obtain the total received signal as:

$$u_{\text{sum}}^{\text{REFL}}(t) = \sum_{m=1}^{6} u_m(t) \otimes \delta(t - \tau_m)$$  \hspace{1cm} (7.8)

After this general introduction, we will now give a more formal mathematical presentation. We started with a 6 x 6 received signal matrix, as described in (5.1).

$$U^{\text{REFL}}(t) =
\begin{bmatrix}
u_{1,1}(t) & u_{1,2}(t) & u_{1,3}(t) & u_{1,4}(t) & u_{1,5}(t) & u_{1,6}(t) \\
u_{2,1}(t) & u_{2,2}(t) & u_{2,3}(t) & u_{2,4}(t) & u_{2,5}(t) & u_{2,6}(t) \\
u_{3,1}(t) & u_{3,2}(t) & u_{3,3}(t) & u_{3,4}(t) & u_{3,5}(t) & u_{3,6}(t) \\
u_{4,1}(t) & u_{4,2}(t) & u_{4,3}(t) & u_{4,4}(t) & u_{4,5}(t) & u_{4,6}(t) \\
u_{5,1}(t) & u_{5,2}(t) & u_{5,3}(t) & u_{5,4}(t) & u_{5,5}(t) & u_{5,6}(t) \\
u_{6,1}(t) & u_{6,2}(t) & u_{6,3}(t) & u_{6,4}(t) & u_{6,5}(t) & u_{6,6}(t)
\end{bmatrix}$$  \hspace{1cm} (7.9)

and obtained a column matrix from it for that reflector

$$U_{\text{col}}^{\text{REFL}}(t) =
\begin{bmatrix}
u_1(t) \\
u_2(t) \\
u_3(t) \\
u_4(t) \\
u_5(t) \\
u_6(t)
\end{bmatrix}$$

where $$u_m(t) = \sum_{i=m}^{j=10b} u_{i,j}(t)$$, $$u_{i,j}(t) \in U^{\text{REFL}}(t)$$ \hspace{1cm} (7.10)

$$u_1(t)$$ is the reference signal; we cross-correlate it with the other signals and obtain the time at which peak of the curves is reached denoted and term this as the implementable delay matrix $$\Gamma^{\text{REFL,Opt}}$$. 

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When different kinds of reflectors are used, we obtain different combinations of received signal column matrices \(( \mathbf{U}_{col}^{REFL} (t) )\) as calculated in (7.10) and delay matrices \(( \mathbf{\Gamma}^{REFL,Opt} )\) as calculated in (7.11). The elements in these column matrices are substituted in (7.8) to obtain \(u_{sum}^{REFL} (t)\), which is further used to calculate the energies for these reflectors as follows:

\[
E^{REFL} (t) = \int (u_{sum}^{REFL} (t))^2 dt
\]  

(7.12)

The set of delays obtained in (7.11) is used to obtain the Energy Table (defined in Chapter 5, Table 1) using (7.8) and (7.12). The pulse-echo system and reflectors are same as described in section 6.3 of chapter 6.

**Table 7.1:** Energy table using the Simple Waveform Correlation Method

<table>
<thead>
<tr>
<th></th>
<th>Reflector A</th>
<th>Reflector B</th>
<th>Reflector C</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mathbf{T}^{A,Opt})</td>
<td>\textbf{4.7939e9}</td>
<td>4.2359e10</td>
<td>4.1148e10</td>
</tr>
<tr>
<td>(\mathbf{T}^{B,Opt})</td>
<td>2.1834e9</td>
<td>\textbf{9.43371e10}</td>
<td>3.9631e10</td>
</tr>
<tr>
<td>(\mathbf{T}^{C,Opt})</td>
<td>2.6451e9</td>
<td>3.5976e10</td>
<td>\textbf{4.7759e10}</td>
</tr>
</tbody>
</table>
Advantages:

We did meet our objective to come up with an implementable delay set, which validates the energy table.

Disadvantages:

As we will see, the energy values obtained are low compared to those obtained by other methods, which we investigated in order to obtain higher energy values.

7.2.2. Method 2: Adaptive waveform correlation method

In the method discussed above, we decided to have the topmost element in the received signal column matrix, i.e. the signal received on transmitting with all the elements and receiving with the innermost ring of the transducer, as the reference signal. Just as we did in the non-implementable adaptive waveform correlation method, described in Chapter 6, Section 6.5, it would be interesting to see the energy table values obtained by considering the signal with the highest energy as reference and aligning the signal with the next highest energy to calculate the corresponding time shift or optimal delay. Since we primarily give importance to the energy of the signals, calculate the time shifts that align these signals in the order of their energy, and further maximize the energy of the combined signals (reference signal and the aligned signal) at each step as explained in Chapter 6, Section 6.4, we are bound to get higher energy numbers in the energy table.

This method makes use of the same received signal row matrix that has been discussed in the previous method. However, instead of considering the first element of this column
matrix as the reference signal, we calculate the energies of all the signals in the column
matrix and the signal with the highest energy is considered as the reference signal.

It is easier to comprehend this method with the help of an example:

Let us consider a received signal column matrix, $U_{col}^{REFL}(t)$, as given in (7.10).

\[
U_{col}^{REFL}(t) = \begin{bmatrix}
    u_1(t) \\
    u_2(t) \\
    u_4(t) \\
    u_5(t) \\
    u_6(t)
\end{bmatrix}, \text{ where } u_m(t) = \sum_{i=m}^{j=1} u_{i,j}(t), u_{i,j}(t) \in U_{REFL}(t)
\]

Also, let us assume that $u_3(t)$ is the signal with the highest energy and the remaining
signals are arranged in the following manner in order of their energies:

\[
\text{Energy}(u_2(t)) > \text{Energy}(u_4(t)) > \text{Energy}(u_5(t)) > \text{Energy}(u_6(t))
\]

Hence, the reference signal $u_{ref1}(t) = u_3(t)$, is first cross-correlated with itself, and the
time at which the peak of the energy curve obtained is reached on the cross-correlation, is
noted as the optimal delay value $\tau_3$. It is obvious that in this case, the peak of the
correlation will reach when the signals are aligned i.e. $\tau_3 = 0$. This value of $\tau_3$ will take
the position of the signal with the highest energy, which is the third row in the column
matrix. The signal with the next highest energy value, $u_2(t)$, is then cross correlated
with the reference signal to obtain the energy curve. The time at which the peak of this
curve is reached, is noted as the optimal delay value, $\tau_2$, which replaces this signal with
the second highest energy. This signal is then shifted by the delay value obtained, added
to the reference signal and considered as the new reference signal, i.e. \( u_{ref_2}(t) = u_2(t - \tau_2) + u_3(t) \). Now the signal with the next highest energy, \( u_1(t) \), is cross-correlated with \( u_{ref_2}(t) \). The time shift thus obtained, \( \tau_1 \), is then used in obtaining the next reference signal \( u_{ref_3}(t) = u_{ref_2}(t - \tau_2) + u_1(t) \).

This “shifting and adding” sequence stated above is continued for the other signals in the descending order of their energy values and each of these values is replaced by the corresponding optimal delay value. In this manner, we will end up with a column matrix of delay values, which in this case will be our implementable delay matrix

\[
\Gamma^{REFL,Opt} = \begin{bmatrix}
\tau_1 \\
\tau_2 \\
0 \\
\tau_4 \\
\tau_5 \\
\tau_6
\end{bmatrix}
\]  

(7.13)

The values in this delay matrix are plugged in (7.8) to obtain the total received signal \( u^{REFL}_{num}(t) \), and the corresponding energy value \( E^{REFL}(t) \) is obtained using (7.12).

The Energy Table obtained using this method, is as follows:

**Table 7.2:** Energy table obtained using the Adaptive Waveform Correlation Method

<table>
<thead>
<tr>
<th></th>
<th>Reflector A</th>
<th>Reflector B</th>
<th>Reflector C</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T^{A,Opt} )</td>
<td>6.9581e11</td>
<td>1.0963e12</td>
<td>8.8963e11</td>
</tr>
<tr>
<td>( T^{B,Opt} )</td>
<td>3.8714e11</td>
<td>1.7989e12</td>
<td>1.8229e12</td>
</tr>
<tr>
<td>( T^{C,Opt} )</td>
<td>2.5953e11</td>
<td>1.6979e12</td>
<td>1.9015e12</td>
</tr>
</tbody>
</table>
Advantages:

We did manage to improve the energy values; in fact, this is probably the best method in terms of getting the highest energy values for the reflectors we used.

This will also be apparent in the “Energy bar graphs” portion of section 7.3, where we can view an illustration of these energy tables.

Disadvantages:

None in particular. However, one factor that did raise concern was the small difference between the energy values obtained on using the different optimal delay matrices for the same reflector. We would like to see a distinct difference as far as identification of a reflector surface goes, although this method serves well to enhance the reflector surface.

In spite of all the advantages and drawbacks, the Simple Waveform Correlation Method and the Adaptive Waveform Correlation Method made us wonder whether it was justifiable to use each element of the received signal column matrix to represent the characteristics of the corresponding row of received signals i.e. to use the sum of all the energy magnitudes of signals transmitted by a specific ring and received by different receivers to represent the characteristics of that particular transmitter ring. Hence we decided to come up with a practical or rather implementable method wherein we can obtain a set of delay values which helps to build a complete 6x6 matrix of delays.
7.2.3. Method 3: Inverse Fourier Transform Method

On implementing the methods discussed in 7.2.1 and 7.2.2, we felt that we might be able to come up with a more appropriate representation of the received signal column matrix. We decided to formulate a received signal column matrix, where each received signal element \( u_i(t), \ i \in [1,6], \) is specifically dependent on the corresponding \( i^{th} \) transducer ring when it is used as the transmitter and the receiver. Thus, we decided to focus mainly on the received signals obtained by transmitting and receiving using the same transducer ring. We decided to stick to Method 2 to obtain the optimal delay matrix for the received signal column matrix thus obtained, as a latter part of this method.

Just like the previous methods, here, we have a 6 x 6 received signal matrix, \( U_{REFL}^t \) as given in (7.9). We then find the Fourier transform of each of the elements of this matrix separately. This matrix would appear as follows:

\[
\begin{bmatrix}
    u_{11}(\omega) & u_{12}(\omega) & u_{13}(\omega) & u_{14}(\omega) & u_{15}(\omega) & u_{16}(\omega) \\
    u_{21}(\omega) & u_{22}(\omega) & u_{23}(\omega) & u_{24}(\omega) & u_{25}(\omega) & u_{26}(\omega) \\
    u_{31}(\omega) & u_{32}(\omega) & u_{33}(\omega) & u_{34}(\omega) & u_{35}(\omega) & u_{36}(\omega) \\
    u_{41}(\omega) & u_{42}(\omega) & u_{43}(\omega) & u_{44}(\omega) & u_{45}(\omega) & u_{46}(\omega) \\
    u_{51}(\omega) & u_{52}(\omega) & u_{53}(\omega) & u_{54}(\omega) & u_{55}(\omega) & u_{56}(\omega) \\
    u_{61}(\omega) & u_{62}(\omega) & u_{63}(\omega) & u_{64}(\omega) & u_{65}(\omega) & u_{66}(\omega)
\end{bmatrix} = \]

\[
\begin{bmatrix}
    u_1^2(\omega) & u_1(\omega)u_2(\omega) & \cdots & \cdots & \cdots & \cdots \\
    u_2(\omega)u_1(\omega) & u_2^2(\omega) & \cdots & \cdots & \cdots & \cdots \\
    \cdots & \cdots & u_3^2(\omega) & \cdots & \cdots & \cdots \\
    \cdots & \cdots & \cdots & u_4^2(\omega) & \cdots & \cdots \\
    \cdots & \cdots & \cdots & \cdots & u_5^2(\omega) & \cdots \\
    \cdots & \cdots & \cdots & \cdots & \cdots & u_6^2(\omega)
\end{bmatrix} \quad (7.14)
\]
Each component of the first matrix in (7.14) can be represented as a correlation of two individual signal components i.e. \( u_{mn} (\omega) = u_m (\omega) u_n (\omega) \), as has been shown in the second matrix in (7.14) above.

Now, we can take the diagonal elements of the second matrix in (7.14) and form a column matrix \( \mathbf{U}^D (\omega) \) as shown below. These are the signals that are obtained using the same transducer ring as transmitter and receiver.

\[
\mathbf{U}^D (\omega) = \begin{bmatrix} u_1^2 (\omega) & u_1 (\omega) \\ u_2^2 (\omega) & u_2 (\omega) \\ u_3^2 (\omega) & u_3 (\omega) \\ u_4^2 (\omega) & u_4 (\omega) \\ u_5^2 (\omega) & u_5 (\omega) \\ u_6^2 (\omega) & u_6 (\omega) \end{bmatrix} = \begin{bmatrix} |u_1 (\omega)|^2 \angle 2\phi_1 (\omega) \\ |u_2 (\omega)|^2 \angle 2\phi_2 (\omega) \\ |u_3 (\omega)|^2 \angle 2\phi_3 (\omega) \\ |u_4 (\omega)|^2 \angle 2\phi_4 (\omega) \\ |u_5 (\omega)|^2 \angle 2\phi_5 (\omega) \\ |u_6 (\omega)|^2 \angle 2\phi_6 (\omega) \end{bmatrix} \quad (7.15)
\]

The second matrix in (7.14) is the first matrix, written in terms of its amplitude and phase. Let \( \mathbf{U}^F (\omega) \) be a column matrix where each term is the square root of the terms in \( \mathbf{U}^D (\omega) \).

\[
\mathbf{U}^F (\omega) = \begin{bmatrix} \sqrt{|u_1 (\omega)|^2} \angle \phi_1 (\omega) \\ \sqrt{|u_2 (\omega)|^2} \angle \phi_2 (\omega) \\ \sqrt{|u_3 (\omega)|^2} \angle \phi_3 (\omega) \\ \sqrt{|u_4 (\omega)|^2} \angle \phi_4 (\omega) \\ \sqrt{|u_5 (\omega)|^2} \angle \phi_5 (\omega) \\ \sqrt{|u_6 (\omega)|^2} \angle \phi_6 (\omega) \end{bmatrix} \quad (7.16)
\]
Finally we take the inverse Fourier transform IFT of each term in $U^F(\omega)$, to get $U^F(t)$ where

$$U^F(t) = \begin{bmatrix}
u_1(t) \\ 
u_2(t) \\ 
u_3(t) \\ 
u_4(t) \\ 
u_5(t) \\ 
u_6(t) \end{bmatrix}$$

(7.17)

We then treat the signal with the maximum energy from this column matrix $U^F(t)$ as the reference signal and shift and add the other signals as per the description in Method 2 to obtain a corresponding column matrix of delays.

We expand this column matrix of delays into a 6 x 6 matrix of delays by reproducing the column six times and denote this as the transmit delay matrix. The transpose of this matrix would be denoted as the receive delay matrix. The individual delay values in the resultant matrix, obtained by summing the transmit and receive delay matrices are applied to the corresponding individual signals in the 6 x 6 received signal matrix, as given in (7.9) and the energy values are calculated using (7.12).

The Energy Table obtained using the delay values derived using this method is as follows.
Table 7.3: Energy table obtained using the Inverse Fourier Transform Method

<table>
<thead>
<tr>
<th></th>
<th>Reflector A</th>
<th>Reflector B</th>
<th>Reflector C</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T^A_{Opt}$</td>
<td>7.0088e10</td>
<td>2.5684e11</td>
<td>1.8986e10</td>
</tr>
<tr>
<td>$T^B_{Opt}$</td>
<td>5.0303e10</td>
<td>1.9214e11</td>
<td>1.074e10</td>
</tr>
<tr>
<td>$T^C_{Opt}$</td>
<td>4.8108e10</td>
<td>2.0148e11</td>
<td>1.0977e10</td>
</tr>
</tbody>
</table>

The evaluation of this technique has revealed that this method in fact does not work.

Disadvantage:

As we see, this produces an erroneous energy table. The errors observed were produced because of signals transmitted by a certain annular ring but received by a different annular ring of the same annular array transducer since these aren’t accounted for by the algorithm.

7.2.4. Method 4: Top-row-left-column method

With this technique, we wish to investigate whether the column delay matrix obtained using the best method we have discussed so far: Method 2: “Adaptive waveform correlation” can be expanded to a 6x6 delay matrix, and further what the energy table calculated using this delay matrix on the received signal matrix would appear like. It seemed logical to build up the 6 x 6 optimal delay matrix, if we have the delay values in the leftmost column and the topmost row of the matrix. In order to find the leftmost column of delay values, we can treat the leftmost column of a 6x6 received signal matrix
as our received signal column matrix and apply Method 2 to find the corresponding delay values. The topmost row can be obtained by taking a transpose of the leftmost column.

To be a little more descriptive, in this method, only the top most row or left most column of the signals in the received signal matrix $U^{REFL}(t)$ is considered. The signal with the highest energy in this matrix is considered as the reference signal. The signal with the next highest energy value is then cross correlated with the reference signal to obtain the energy curve and the corresponding delay value. This signal is then shifted by the delay value obtained, added to the reference signal and considered as the new reference signal. The shifting and adding sequence stated in the two steps above is continued for the other signals: in order of their descending energy values.

Once we have the top-most row and hence left-most column of delay values in the delay matrix, we can obtain the other elements of the delay matrix, $\Gamma^{REFL,Opt}$, by using the following formula which we came across in Section 7.1:

$$t_{i,j} = t_{i+1,j+1} - t_{i+1,j} - t_{i,j+1}$$

(7.18)

The energy values are calculated using the formulation given by (7.12), and the Energy Table obtained using this method is as follows:
Table 7.4: Energy table obtained using the Top-row-left-column method

<table>
<thead>
<tr>
<th></th>
<th>Reflector A</th>
<th>Reflector B</th>
<th>Reflector C</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T^{A,Opt}$</td>
<td>1.1199e11</td>
<td>4.1597e11</td>
<td>2.4490e10</td>
</tr>
<tr>
<td>$T^{B,Opt}$</td>
<td>8.8281e10</td>
<td>1.0350e12</td>
<td>2.7704e10</td>
</tr>
<tr>
<td>$T^{C,Opt}$</td>
<td>1.06e11</td>
<td>6.2127e11</td>
<td>2.9500e10</td>
</tr>
</tbody>
</table>

**Advantage:**

This method gave the expected results unlike the previous method. Basically, it satisfied what the Energy table is expected to imply.

**Disadvantage:**

Compared to Method 2, there is not much variation in the optimal delay matrices for the different reflectors. Hence there is also little differentiation in the energy values obtained for a particular type of reflector using these delay matrices. This can be observed from the energy table above.

### 7.2.5. Method 5: CPLEX Method

We tested all the above methods in an effort to come up with an implementable and efficient algorithm to obtain the optimal delay matrix for different types of reflectors. We eventually realized that our objective to obtain a certain set of delays subject to some fixed constraints can be formulated as a linear programming problem. This linear programming (LP) problem can then be fed into some LP-optimizer software tool which can be used to output an optimum set of delays that meets all our requirements.
To be more specific, this implementable method makes use of the “Adaptive Waveform correlation method” discussed in Chapter 6, Section 6.6 to obtain the (non-implementable) optimal delay matrix set. The delay values in this matrix are then fed into an LP optimizer software (CPLEX, an ILOG Inc. product) with the constraints, to obtain the (implementable) set of delays.

After executing the “New Adaptive Waveform correlation method”, a LP problem is formulated to optimize or maximize \( z \): the set of delay values scaled with the corresponding energy values as shown below:

\[
\text{LP Problem: Maximize } z = \sum_{i,j} E[i,j] \cdot |t_{i,j} - d[i,j]|
\]

where \( E[i,j] \) is the energy matrix, \( d[i,j] \) is the delay matrix and '.' denotes multiplication, subject to:

\[
t_{i,j} = t_{i+1,j+1} - t_{i+1,j} - t_{i,j+1}.
\]

This problem is fed into CPLEX which is an LP-optimizer software and it produces the implementable delay matrix as the output.

Appendix (Appendix B) at the end of this thesis has been dedicated to the CPLEX tool. This appendix basically covers the guidelines, codes and scripts to run CPLEX for our application.
7.3. Results

Since this is the method we have selected to execute the energy optimization method, it is pertinent to see a comparison between the optimal delay matrices, obtained using the non-implementable “New Adaptive Waveform Correlation method”, and the delay matrix sets obtained using the implementable “CPLEX method”. We will now see all the intermediate results from obtaining the echo signal matrix to calculating the energy curves, using this selected method and the simulation set-up discussed earlier.

The optimal delay matrices have been represented graphically below with the horizontal axes denoting the transmitter ring# and receiver ring# while the vertical axis represents the optimal delay value.
From these plots, we can infer that the implementable optimal delay matrix generally follows a pattern, in which the optimal delay values steadily increase from the transmitter ring #1, receiver ring #1 combination to the transmitter ring #6, receiver ring #6 combination. In the case of the
non-implementable delay matrices this pattern is followed by Reflector A and Reflector B, however, Reflector C follows a different pattern. The LP-optimizer software tool changes this pattern, as we can see. The energy table obtained using this method is as follows:

**Table 7.5:** Energy table obtained using the CPLEX method

<table>
<thead>
<tr>
<th></th>
<th>Reflector A</th>
<th>Reflector B</th>
<th>Reflector C</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_{A,\text{Opt}} )</td>
<td>6.841e11</td>
<td>9.933e11</td>
<td>4.812e11</td>
</tr>
<tr>
<td>( T_{B,\text{Opt}} )</td>
<td>6.053e11</td>
<td>1.612e12</td>
<td>4.270e11</td>
</tr>
<tr>
<td>( T_{C,\text{Opt}} )</td>
<td>8.490e10</td>
<td>8.205e11</td>
<td>7.024e11</td>
</tr>
</tbody>
</table>

As we can see, this method does produce accurate results and there is a considerable variation between the energy numbers obtained for the different reflectors using the optimal time delay matrices calculated using the CPLEX method for the respective reflectors.

Below are the energy bar graphs, which are an illustration of the energy tables we saw above. It is easier to realize the drawbacks and disadvantages of the methods discussed above by looking at the bar graphs.
**Energy bar graphs**

- The energy obtained using optimal delay matrix $\Gamma^{A,\text{Opt}}$ calculated using the respective method.
- The energy obtained using optimal delay matrix $\Gamma^{B,\text{Opt}}$ calculated using the respective method.
- The energy obtained using optimal delay matrix $T^{C,\text{Opt}}$ calculated using the respective method.

**Method 1: Simple waveform correlation method**

![Energy bar graphs for three reflectors]
Method 2: Adaptive waveform correlation method

Method 3: Inverse fourier transform method
Method 4: Top-row-left-column method

Method 5: CPLEX method
The graphs obtained using the CPLEX method are distinctly accurate with the red bar considerably longer for Reflector A than the bars obtained using the implementable optimal delay matrices calculated for the other, blue bar longest for Reflector B and the green bar longest for Reflector C. The steps executed in the CPLEX method are explained in Appendix B.
Chapter 8

Robustness of the Energy Optimization Algorithm

In Chapter 7, we have investigated implementable energy optimization algorithms which can be used to obtain a set of optimal delay values for specific reflector geometries. These optimal delay values are further used to optimize the energy of the received signal from that respective reflector geometry. Next, it will be interesting to investigate to which extent the same optimal delay set, specific to a given reflector geometry, can be used to optimize the received signal energy of that geometry with certain modifications in its dimensions or physical positioning. Basically, we want to test how robust our optimization algorithm was, and whether the delay matrix obtained using this algorithm for a specific reflector geometry can be applied to obtain the received signal with maximum energy from the same reflector with some differences in its physical attributes or its lateral or angular position with respect to the transducer. This concept of testing the robustness of our algorithm to optimize the received signal from a given reflector will be more clearer as we go through this chapter.

8.1 Robustness test scenarios

So far, in Chapters 6 and 7, we have seen how the selected non-implementable method, the Adaptive Waveform Correlation Method, and the implementable method, that is the CPLEX Method is used for the following reflector geometry specifications:

1. Reflector A: A 25mm x 25mm flat reflector tilted at 6 degrees to the plane of the

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transducer surface and located at 50mm from the transducer, with center on transducer axis.

2. **Reflector B:** The above 25mm x 25mm flat reflector surface (located in the same position as Reflector A) curved in a manner that resembles a section of a cylinder. The radius of curvature of this curved surface was selected to be 86mm from the center of the transducer.

3. **Reflector C:** A 25mm x 25mm flat reflector surface (located in the same position as Reflector A), but with a sinusoidal shape. On one side of the transducer axis, one half of the reflector is bent into a concave surface with respect to the transducer plane while the other half is bent to a convex plane. Both the curves have a 10mm radius of curvature from a point located 10 mm from a point on the transducer axis, which again is 50 mm away from the center of the transducer and located on the normal parallel to the transducer axis as illustrated in the Fig 8.2.(c).

Figure 8.2 illustrates the geometries described above, as seen from the top. As far as the actual positioning goes, Fig 8.1 depicts how each one of these transducer-reflector geometry pairs would appear, and applies to all geometries.

![Figure 8.1: A transducer-reflector system](image)

Figure 8.1: A transducer-reflector system
Figure 8.2. (a) Top View of the transducer-reflector system (shown in Fig 8.1) with a 6 degrees tilted flat reflector. (b) Top View of the transducer-reflector system (shown in Fig 8.1) with curved reflector surface with radius of curvature = 86 mm. (c) Top View of the above transducer-reflector system (shown in Fig 8.1) with a sinusoidal reflector surface with its center on transducer axis.
As stated in the introduction to this chapter, we wish to investigate the effect of modifying the existing physical location and attributes of these reflector geometries. In the case of the flat reflector surface, we will change its tilt angle by decreasing it and increased it in steps of one degree from the normal tilt angle of 6 degrees. Likewise, we will modify (increasing and decreasing) the radius of curvature of the second curved surface in steps of one millimeter from the nominal radius of 86 mm. In the case of the sinusoidal reflector surface, we will change the positioning of the reflector surface. Specifically, we will shift the center of the reflector surface by one millimeter in either direction from the nominal position on the transducer axis in the horizontal plane.

Below is a table which lists the modifications in the geometry and positioning if any, for the reflectors. The original specifications are denoted in bold.

### Table 8.1: Modifications in the geometry and physical positioning of the reflectors

<table>
<thead>
<tr>
<th>Tilt in degrees for Reflector A</th>
<th>Radius of curvature in mm for the Reflector B</th>
<th>x-coordinate of the center of the Reflector C in mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>82</td>
<td>-4</td>
</tr>
<tr>
<td>4.5</td>
<td>83</td>
<td>-3</td>
</tr>
<tr>
<td>5</td>
<td>84</td>
<td>-2</td>
</tr>
<tr>
<td>5.5</td>
<td>85</td>
<td>-1</td>
</tr>
<tr>
<td><strong>6</strong></td>
<td><strong>86</strong></td>
<td><strong>0</strong></td>
</tr>
<tr>
<td>6.5</td>
<td>87</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>88</td>
<td>2</td>
</tr>
<tr>
<td>7.5</td>
<td>89</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>90</td>
<td>4</td>
</tr>
</tbody>
</table>
8.2 Results

Given below are the results of our simulations. Figure 3 illustrates the first section of the results which shows the implementable and non-implementable optimal set of delays that we obtained for each of the reflector surfaces, in order to test the robustness of the selected non-implementable (Adaptive Waveform Correlation Method) and the selected implementable (CPLEX Method) algorithm, in Chapter 6 and 7 respectively. Each of these three sets of delays, specific to the flat, the cylindrical and the sinusoidal reflector surfaces, respectively, is then used to calculate the energies of the received signals obtained from a flat reflector surface tilted at varying angles, from the cylindrical reflector surface with varying radii of curvature and from the sinusoidal reflector surface with its center shifted at varying distances from its original position as shown in the second section of results below. The third section shows a plot of the energy values tabulated in tables 8.2, 8.3, 8.4, 8.5, 8.6 and 8.7 in section two of the results, each for the non-implementable and implementable delay sets. The observations made and the inference drawn from these set of results are discussed in section 8.3.
Section I: Optimal delay matrices for different types of reflector geometries

(a) Optimal delay matrices for Reflector A obtained using

Non-implementable method

Implementable Method

(b) Optimal delay matrices for Reflector B obtained using

Non-implementable method

Implementable Method
Section II: Energy values obtained using the specific reflector geometry, implementable and non-implementable optimal delay matrices, for modified reflector geometries

Table 8.2: Energy values obtained using the non-implementable delay matrices obtained for Reflector A, B and C, on varying the tilt angle from the standard 6 degrees for Reflector A

<table>
<thead>
<tr>
<th>Tilt in degrees</th>
<th>$T^{A,\text{Opt}}$</th>
<th>$T^{B,\text{Opt}}$</th>
<th>$T^{C,\text{Opt}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>9.06e11</td>
<td>4.47e11</td>
<td>2.55e11</td>
</tr>
<tr>
<td>4.5</td>
<td>9.07e11</td>
<td>3.94e11</td>
<td>1.88e11</td>
</tr>
<tr>
<td>5</td>
<td>9.08e11</td>
<td>3.43e11</td>
<td>1.26e11</td>
</tr>
<tr>
<td>5.5</td>
<td>8.67e11</td>
<td>3.05e11</td>
<td>9.42e10</td>
</tr>
<tr>
<td>6</td>
<td>7.63e11</td>
<td>2.65e11</td>
<td>7.36e10</td>
</tr>
<tr>
<td>6.5</td>
<td>5.99e11</td>
<td>2.22e11</td>
<td>6.15e10</td>
</tr>
<tr>
<td>7</td>
<td>4.68e11</td>
<td>1.95e11</td>
<td>4.93e10</td>
</tr>
<tr>
<td>7.5</td>
<td>3.41e11</td>
<td>1.62e11</td>
<td>4.07e10</td>
</tr>
<tr>
<td>8</td>
<td>2.40e11</td>
<td>1.32e11</td>
<td>3.35e10</td>
</tr>
</tbody>
</table>
Table 8.3: Energy values obtained using the non-implementable delay matrices obtained for Reflector A, B and C, on varying the radius of curvature from the standard 86mm for Reflector B.

<table>
<thead>
<tr>
<th>Radius of Curvature (mm)</th>
<th>$T_{A,Opt}$</th>
<th>$T_{B,Opt}$</th>
<th>$T_{C,Opt}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>82</td>
<td>9.02e11</td>
<td>2.87e12</td>
<td>1.00e12</td>
</tr>
<tr>
<td>83</td>
<td>8.89e11</td>
<td>2.87e12</td>
<td>9.99e11</td>
</tr>
<tr>
<td>84</td>
<td>8.86e11</td>
<td>2.87e12</td>
<td>9.75e11</td>
</tr>
<tr>
<td>85</td>
<td>8.91e11</td>
<td>2.89e12</td>
<td>9.86e11</td>
</tr>
<tr>
<td>86</td>
<td>8.97e11</td>
<td>2.91e12</td>
<td>9.92e11</td>
</tr>
<tr>
<td>87</td>
<td>9.00e11</td>
<td>2.92e12</td>
<td>9.95e11</td>
</tr>
<tr>
<td>88</td>
<td>9.02e11</td>
<td>2.94e12</td>
<td>1.00e12</td>
</tr>
<tr>
<td>89</td>
<td>9.06e11</td>
<td>2.95e12</td>
<td>1.01e12</td>
</tr>
<tr>
<td>90</td>
<td>9.08e11</td>
<td>2.97e12</td>
<td>1.02e12</td>
</tr>
</tbody>
</table>

Table 8.4: Energy values obtained using the non-implementable delay matrices obtained for Reflector A, B and C, on shifting the centre point of Reflector C, which lies on the transducer axis, at specific distances in the horizontal plane (x-coordinate).

<table>
<thead>
<tr>
<th>x-coordinate of center of reflector (mm)</th>
<th>$T_{A,Opt}$</th>
<th>$T_{B,Opt}$</th>
<th>$T_{C,Opt}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>4.60e11</td>
<td>4.20e11</td>
<td>1.46e12</td>
</tr>
<tr>
<td>-3</td>
<td>4.45e11</td>
<td>4.37e11</td>
<td>1.54e12</td>
</tr>
<tr>
<td>-2</td>
<td>4.30e11</td>
<td>4.09e11</td>
<td>1.49e12</td>
</tr>
<tr>
<td>-1</td>
<td>4.23e11</td>
<td>4.79e11</td>
<td>1.49e12</td>
</tr>
<tr>
<td>0</td>
<td>4.13e11</td>
<td>4.27e11</td>
<td>1.72e12</td>
</tr>
<tr>
<td>1</td>
<td>4.10e11</td>
<td>4.12e11</td>
<td>1.48e12</td>
</tr>
<tr>
<td>2</td>
<td>4.02e11</td>
<td>3.62e11</td>
<td>1.55e12</td>
</tr>
<tr>
<td>3</td>
<td>3.91e11</td>
<td>3.79e11</td>
<td>1.53e12</td>
</tr>
<tr>
<td>4</td>
<td>3.63e11</td>
<td>3.83e11</td>
<td>1.51e12</td>
</tr>
</tbody>
</table>
Table 8.5: Energy values obtained using the *implementable* delay matrices obtained for *Reflector A, B* and *C*, on varying the tilt angle from the standard 6 degrees for *Reflector A*.

<table>
<thead>
<tr>
<th>Tilt in degrees</th>
<th>$\Gamma_{A,\text{Opt}}$</th>
<th>$\Gamma_{B,\text{Opt}}$</th>
<th>$\Gamma_{C,\text{Opt}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1.03e12</td>
<td>9.06e11</td>
<td>2.55e11</td>
</tr>
<tr>
<td>4.5</td>
<td>9.69e11</td>
<td>8.43e11</td>
<td>1.88e11</td>
</tr>
<tr>
<td>5</td>
<td>9.11e11</td>
<td>7.91e11</td>
<td>1.26e11</td>
</tr>
<tr>
<td>5.5</td>
<td>8.11e11</td>
<td>7.13e11</td>
<td>9.42e10</td>
</tr>
<tr>
<td>6</td>
<td>6.84e11</td>
<td>6.05e11</td>
<td>7.36e10</td>
</tr>
<tr>
<td>6.5</td>
<td>5.11e11</td>
<td>4.13e11</td>
<td>6.15e10</td>
</tr>
<tr>
<td>7</td>
<td>4.35e11</td>
<td>3.92e11</td>
<td>4.93e10</td>
</tr>
<tr>
<td>7.5</td>
<td>3.25e11</td>
<td>3.02e11</td>
<td>4.07e10</td>
</tr>
<tr>
<td>8</td>
<td>2.36e11</td>
<td>2.31e11</td>
<td>3.35e10</td>
</tr>
</tbody>
</table>

Table 8.6: Energy values obtained using the *implementable* delay matrices obtained for *Reflector A, B* and *C*, on varying the radius of curvature from the standard 86mm for *Reflector B*.

<table>
<thead>
<tr>
<th>Radius of Curvature</th>
<th>$\Gamma_{A,\text{Opt}}$</th>
<th>$\Gamma_{B,\text{Opt}}$</th>
<th>$\Gamma_{C,\text{Opt}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>82</td>
<td>9.89e11</td>
<td>1.60e12</td>
<td>7.92e11</td>
</tr>
<tr>
<td>83</td>
<td>9.84e11</td>
<td>1.59e12</td>
<td>7.92e11</td>
</tr>
<tr>
<td>84</td>
<td>9.82e11</td>
<td>1.59e12</td>
<td>7.91e11</td>
</tr>
<tr>
<td>85</td>
<td>9.88e11</td>
<td>1.60e12</td>
<td>8.08e11</td>
</tr>
<tr>
<td>86</td>
<td>9.93e11</td>
<td>1.61e12</td>
<td>8.20e11</td>
</tr>
<tr>
<td>87</td>
<td>9.95e11</td>
<td>1.62e12</td>
<td>8.29e11</td>
</tr>
<tr>
<td>88</td>
<td>1.00e12</td>
<td>1.63e12</td>
<td>8.44e11</td>
</tr>
<tr>
<td>89</td>
<td>1.00e12</td>
<td>1.64e12</td>
<td>8.55e11</td>
</tr>
<tr>
<td>90</td>
<td>1.01e12</td>
<td>1.65e12</td>
<td>8.70e11</td>
</tr>
</tbody>
</table>
Table 8.7: Energy values obtained using the \textit{implementable} delay matrices obtained for \textit{Reflector A, B and C}, on shifting the centre point of \textit{Reflector C}, which lies on the transducer axis, at specific distances in the horizontal plane (x-coordinate).

<table>
<thead>
<tr>
<th>x-coordinate of center of reflector (mm)</th>
<th>$\Gamma_{A,\text{Opt}}$</th>
<th>$\Gamma_{B,\text{Opt}}$</th>
<th>$\Gamma_{C,\text{Opt}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>4.62e11</td>
<td>4.19e11</td>
<td>7.80e11</td>
</tr>
<tr>
<td>-3</td>
<td>4.61e11</td>
<td>4.37e11</td>
<td>7.34e11</td>
</tr>
<tr>
<td>-2</td>
<td>4.33e11</td>
<td>4.08e11</td>
<td>7.15e11</td>
</tr>
<tr>
<td>-1</td>
<td>4.62e11</td>
<td>4.29e11</td>
<td>6.98e11</td>
</tr>
<tr>
<td>0</td>
<td>4.81e11</td>
<td>4.27e11</td>
<td>7.02e11</td>
</tr>
<tr>
<td>1</td>
<td>4.37e11</td>
<td>4.16e11</td>
<td>6.98e11</td>
</tr>
<tr>
<td>2</td>
<td>3.96e11</td>
<td>3.81e11</td>
<td>7.12e11</td>
</tr>
<tr>
<td>3</td>
<td>4.32e11</td>
<td>4.11e11</td>
<td>7.10e11</td>
</tr>
<tr>
<td>4</td>
<td>4.78e11</td>
<td>4.51e11</td>
<td>7.03e11</td>
</tr>
</tbody>
</table>

Section III: Energy value plots corresponding to the energy tables given above

For all the plots below Series 1 specifies the energy numbers obtained using $T_{A,\text{Opt}}$, Series 2 specifies the energy numbers obtained using $T_{B,\text{Opt}}$, and Series 3 specifies the energy numbers obtained using $T_{C,\text{Opt}}$.

![Robustness results for flat reflector surface](image)

**Fig 8.4 (a):** Energy values obtained using the \textit{non-implementable} delay matrices obtained for \textit{Reflector A, B and C}, on varying the tilt angle from the nominal 6 degrees for \textit{Reflector A}
Figure 8.4(b): Energy values obtained using the *non-implementable* delay matrices obtained for Reflector A, B and C, on varying the radius of curvature from the nominal 86mm for Reflector B.

Figure 8.4(c): Energy values obtained using the *non-implementable* delay matrices obtained for Reflector A, B and C, on shifting the centre point of Reflector C, which lies on the transducer axis, at specific distances in the horizontal plane (x-coordinate).

**Figure 8.4:** Energy plots obtained using the non-implementable delay matrices. For all the plots below Series 1 specifies the energy numbers obtained using $\Gamma^{A,\text{Opt}}$, Series 2 specifies the energy numbers obtained using $\Gamma^{B,\text{Opt}}$, and Series 3 specifies the energy numbers obtained using $\Gamma^{C,\text{Opt}}$. 

```
Figure 8.5(a): Energy values obtained using the implementable delay matrices obtained for Reflector A, B and C, on varying the tilt angle from the nominal 6 degrees for Reflector A.

Figure 8.5(b): Energy values obtained using the implementable delay matrices obtained for Reflector A, B and C, on varying the radius of curvature from the nominal 86mm for Reflector B.
Figure 8.5(c): Energy values obtained using the implementable delay matrices obtained for Reflectors A, B, and C, on shifting the centre point of Reflector C, which lies on the transducer axis, at specific distances in the horizontal plane (x-coordinate).

Figure 8.5: Energy plots obtained using the implementable delay matrices

8.3 Observations and Inference from the Results in Section 8.2

The optimal energy values obtained from the flat reflector with varying tilt angles was obtained by using the delay set designed for the flat reflector tilted at 6 degrees. Similarly, the optimum values of energy obtained from the cylindrical reflector with varying radii of curvature, and from the sinusoidal reflector with modifications in its positioning were obtained by using the delay set designed for the cylindrical reflector with a 86mm radius of curvature and a sinusoidal reflector surface with center on the transducer axis at 50 mm from the center of the transducer, respectively.

However, one would assume that the energy values obtained with the use of the non-implementable delay matrix which is actually calculated using the given transducer-
reflector geometry, would be the largest or optimal as compared to the values obtained where modifying the geometries. This is however, not always the case. For example, in the case of Reflector A, the optimal non-implementable delay matrix was obtained for a flat reflector with a tilt of 6 degrees, hence, the energy value obtained for this setting would be assumed to be more than the energy value obtained for a tilt of 4 degrees using this delay matrix. However, the results in Fig 8.5(a) show the energy at 4 degrees to be about 50% larger than the energy at 6 degrees.

On carefully studying the trend of the energy values with the change in geometry, what one can observe that the optimal delay matrix obtained is moderately robust as far as the geometrical and positioning variations go for a specific reflector type, and then, the variation of energy numbers can be justified logically as depending on the geometry and positioning variations of the reflector surface. These concepts are easier to explain based on an example. Hence, in the case of the flat reflector surface, for the given non-implementable delay matrix $T_{A,Opt}^A$, the energy values obtained keeps reducing as the tilt of the reflector surface increases from 4 degrees to 8 degrees which can be logically justified considering the received signal energy would decrease with increase in the tilt angle. For the cylindrical surface, with increase in the radius of curvature, the received signal energy will increase and this is demonstrated in the energy values obtained using the non-implementable delay matrix $T_{B,Opt}^B$. Similarly, for the sinusoidal reflector surface the energy is maximum when the reflector is positioned with its center on the transducer axis. Although we obtained the non-implementable optimal delay matrix for a specific geometry and positioning, in all cases the variations in the energy values were
more pertinent to the variations in the geometry and not on the delay matrix values, thus making this matrix more generic in its use, for a given type of reflector. It is difficult to make such predictions or assumptions on processing the non-implementable delay matrix to obtain the implementable delay matrix and using this matrix to obtain the energy numbers for the different types of reflectors.
Chapter 9

Conclusion

This thesis describes a technique that can be used to optimally design an ultrasound pulse-echo system with annular array transducers, for tasks such as identifying objects of specified shapes, determining surface topology or alignment of surfaces. The thesis can be divided into three main sections to achieve this objective.

- The first part of the thesis discusses a fast numerical modeling method, the *Diffraction Response from Extended Area Method* (DREAM), for calculation of the received signal using a planar piston transducer in a pulse-echo system.

- In the second part, this modeling concept has been extended to calculate the received signal from individual elements in an annular array transducer. Basically, this received signal is expressed as a superposition of the received signals from planar piston transducers. An annular array was used for this research since it provides the flexibility to customize and thus study different acoustic fields.

- In the last part, we considered three types of specific reflector geometries and investigated various methods to create customized transducer and receiver acoustic fields that can maximize the energy of the total received signal from the respective reflector geometry. We assumed a specific set of dimensions for the annular array transducers that were used to create these customized fields.
Let us now discuss in brief what each of the above three steps actually comprised of in terms of the options we had on hand, the assumptions we made, and the algorithm or the methods that we finally appointed to achieve the objective.

The DREAM method, initially developed by Prof. Pedersen and Li Wan was evaluated using rectangular and triangular tiles. This tessellation method worked appropriately as per the design: it was faster and quite accurate when compared to the reference method based on Huygens principle where the reflector is divided into microtiles one-tenth the size of the tiles used for DREAM. In either case the total received signal from the reflector surface is calculated as the sum of the received signals from each individual tile. However, the right angled isosceles triangular tile or the rectangular tiles can be used effectively to tessellate a flat reflector surface with straight edges, but cannot be used for reflector surfaces with arbitrary geometries. Hence, we felt the need to evaluate a robust tessellation technique which can be used for different kinds of reflector geometries. On studying some of the universally used tessellation techniques, we decided to appoint the Delaunay tessellation method. We selected Delaunay equilateral triangles with specific dimensions to tessellate the reflector geometries that we used in our simulations, so as to keep the results as accurate as the reference Huygens Method.

Just a quick recapitulation of the DREAM method: The DREAM method tessellates the surface into triangular tiles of moderate dimensions. The diffraction response is then evaluated at each corner of the tiles and the center of each tile. For points situated not too far away from each other, the diffraction responses are quite similar. They have a similar
shape or waveform and similar amplitude, but they differ in terms of their delays. Therefore, the diffraction response of the center of the triangular tile, without considering the propagation delay, can be used to approximate the diffraction responses within the small triangular tile area. The spatial integration of the diffraction response over surface of a given tile is replaced by an equivalent low pass filtering operation on the diffraction response at the center of the tile. The low pass filter is defined by the relative delays of the diffraction responses from the corners of the tile, and is referred to as the “delay filter”. The received signal from this tile is calculated as a function of this diffraction response. The total received signal from the reflector surface is then calculated as a sum of the received signals from the individual tiles.

Once again, the diffraction response or the received signal thus obtained is calculated for a planar piston transducer with a certain radius used as a transmitter and for a planar piston transducer with a similar or different radius used as a receiver. In an annular array transducer composed of rings with different diameters, the received signal from a reflector surface is calculated for a given ring as transmitter and a given ring as receiver. This annular array received signal can be expressed as a superposition of the respective received signals obtained using the planar piston transducers. The planar piston received signals that are used in this case are calculated using the outer and inner diameters of the transmitting and receiving planar pistons.
Using this concept for calculating the received signals from an annular array transducer, \(N \times N\) received signals are obtained for an \(N\)-ring annular array transducer. Thus, a 3-ring annular array transducer can be used to obtain \(3 \times 3 = 9\) received signals based on the different transmitter ring and receiver ring combinations. However, due to reciprocity, the received signal obtained using ring ‘\(i\)’ as transmitter and ring ‘\(j\)’ as receiver is the same as the received signal obtained using ring ‘\(j\)’ as transmitter and ring ‘\(i\)’ as the receiver. Thus, in actuality a 3-ring transducer can be used to obtain 6 distinct received signals; basically an N ring annular array transducer can be used to obtain \(N(N+1)/2\) distinct received signals. These received signals are then used to obtain a set of delay values which can be applied to the excitation and received signals obtained using the different transmitter and receiver rings so as to customize the acoustic field. We used these received signals to develop algorithms that can be used to obtain a set of delay values which can customize the acoustic field so as to obtain a received signal with maximum energy. These set of delay values are termed as the optimal delay values, and the methods used to maximize the energy of the combined received signal from a given reflector geometry are termed as the Energy Optimization Algorithms.

We came up with multiple analytical and numerical algorithms to obtain this set of optimal delays. As per the analytical methods, the delay values have to be applied to the annular array transducer rings one at a time which is not possible in reality. The numerical methods were thus designed (some of them based on the analytical methods) to obtain a set of delay values that can be applied at one time to all the annular array transducer rings as is done in a practical implementation.
Based on our evaluations and observations of the energy values of the combined received signals obtained from specific reflector geometries for given dimensions of annular array transducers, and the practical drawbacks of each algorithm, we selected one analytical method, and one numerical method (based on the selected analytical method) that can be used to maximize the energy of the combined received signal. The Adaptive Waveform Correlation Method discussed in Chapter 6, and the CPLEX method discussed in Chapter 7 are the selected algorithms to optimize the acoustic field from an annular array transducer.

9.1 Future Work

As a part of this thesis, we conceptualized ideas keeping the objective “to maximize the energy of the received signal from a given reflector geometry using annular array transducer based pulse-echo ultrasound system” in mind. At every step for ease of calculations, we made reasonable assumptions, defined a scope, and tried to consider all the factors that can affect real-time ultrasound pulse-echo systems while developing the simulation. Given that, the future work on this thesis mainly involves using these concepts in a real-time system. This would involve:

- **Conducting experiments on a real annular array based pulse-echo system and with the defined reflector geometries, so as to verify the simulation results.** More than verifying actual numbers it would be interesting to see the improvement in the energy and thus the intensity of the received signal by customizing the acoustic excitation field.
• Extending these concepts to cover different types of transducers such as linear array transducers and different reflector geometries. The Delaunay tessellation algorithm has also been tested in the industry on 3-Dimensional geometries. It would be interesting to see how 3-Dimensional reflector objects can be modeled and how received signals can be calculated for these objects.

• Including absorption media between the transducer and reflector. This can also be included in the simulation and accounted for. The absorption media would basically affect the phase and amplitude of the received signals we obtained in this thesis (in the absence of absorption media). A filter function can be designed to take this factor into account and can be applied to the individual received signals obtained by using the different annular array rings as transmitter and receiver.

Thus, the basic objective of the future work would be to make the optimization process developed in this thesis more robust and generally applicable to the real world.
References


[15] Li Wan, Peder C Pedersen and Soren K Jespersen, “Modeling of Received Signals from Annular Array Ultrasound Transducers due to Extended Reflectors”, Worcester Polytechnic Institute, 1999


Appendix A

Steps involved in the CPLEX method

Many researchers in computer science have stated that optimizing compilations can take great benefit from using a Linear Programming (LP) numerical tool. CPLEX from ILOG is an industrial and commercial tool that is well known to be the best (the most efficient) solver of linear programming problems.

The selected non-implementable energy optimization algorithm, the Adaptive Waveform Correlation Method, described in Section 6.6 was evaluated as the ideal method that meets the desired objectives and produces maximum energy values for the reflectors under consideration. However, this method cannot be implemented in a practical pulse-echo ultrasound system. Hence, we decided to design an LP problem with a constraint equation around the delay values obtained using the non-implementable method, which satisfies the theory around the formulation of implementable delay matrices discussed in Chapter 7, Section 7.1. The objective of this LP problem is to maximize the energy values of the received signals that will be obtained using this method. This has been formulated as discussed in Chapter 7, equations (7.19) and (7.20).

The details about the syntax and programming to devise this LP problem which can be fed into the CPLEX tool are beyond the scope of this thesis. I would like to acknowledge the work of Dr. William Martin, Associate Professor and Associate Dept. Head,
Mathematical Sciences Dept., WPI which helped me to implement this idea, and further evaluate and select the CPLEX method as the best implementable algorithm.

The building blocks towards obtaining the implementable delay values can be stated in the form of the following steps:

1. Formulating the general Linear Programming problem using MAPLE (a math  
software package) code.

2. Running the MAPLE code and feeding the following information to the code:
   (a) N x N delay values obtained using the Adaptive Waveform Correlation  
Algorithm for an annular array transducer with N rings.
   (b) N x N energy values, which are the energies of the N x N received signals  
calculated for an N-ring annular array transducer.

3. Saving the output of this MAPLE code as a *.lp file in the folder containing the  
CPLEX.exe file.

4. The CPLEX.exe file gets created on downloading ILOG’s CPLEX tool.

5. In order to run CPLEX, one needs to run the CPLEX.exe file.

6. Entering commands at the CPLEX command line to solve for the implementable  
delay variables.

Let us now consider an example for each of the steps above with simple numbers, that’ll help to understand the steps more easily. As per step 1, an LP problem is formulated using MAPLE code. Given below is the MAPLE code that was designed for the LP problem as defined in Chapter 7, (7.19) and (7.20).
## MAPLE code to set up integer linear program

```maple
n := 3:
read <filename>:
printf(`enter <filename>
Minimize
obj: `);
for i to n
do
for j to n
do
    printf(` %5.4f p%1d%1d + %5.4f n%1d%1d `, E[i,j], i,j,E[i,j],i,j);
    if i<n or j<n then printf(`+`); fi;
    if n*(i-1)+j mod 3 = 0 then printf(`
`); fi
od;
od:
printf(`
Subject To
`);
for i to n-1
do
for j to n-1
do
    printf(` c%1d%1d:  t%1d%1d + t%1d%1d - t%1d%1d - t%1d%1d = 0
`, i,j,i,j,i+1,j+1,i,j+1,i+1,j);
    printf(` a%1d%1d:  p%1d%1d + t%1d%1d - n%1d%1d = %6.4f
`, i,j, i,j, i,j, d[i,j]);
    printf(` u%1d%1d:  p%1d%1d - 10 w%1d%1d <= 0 
`, i,j, i,j  ); # Assume no value > 10
    printf(` l%1d%1d:  n%1d%1d + 10 w%1d%1d <= 10 
`, i,j, i,j  ); # Assume no value < -10
    od;
od;
for i to n
do
for j to n
    printf(` t%1d%1d >= -inf
`, t[i,j]);
    printf(` p%1d%1d >= 0
`, p[i,j]);
    printf(` n%1d%1d >= 0
`, n[i,j]);
    od;
do;
```
```
```
```
As per step 2, the above code reads a file containing the delay matrix $d[i,j]$, where $i,j = 1$ to $N$, obtained using the Adaptive Waveform Correlation Method, and the energy matrix $E[i,j]$, where $i,j = 1$ to $N$, containing the energies of the $N \times N$ received signals obtained using an $N$-ring annular array transducer.

For simplicity let us consider the following examples for $d[i,j]$ and $E[i,j]$ for a 3-ring annular array transducer:

$$E[i,j] = \begin{bmatrix} 100 & 200 & 300 \\ 200 & 400 & 500 \\ 300 & 500 & 600 \end{bmatrix}$$

$$d[i,j] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$
Let us assume that these were the Energy and Delay values fed to the MAPLE code.

The output of the MAPLE code will appear as follows. The numbers in this output can be distinctly matched with the energy matrix and the delay matrix defined above:

Minimize

\[
\begin{align*}
\text{obj:} & \quad 100 \, p_{11} + 100 \, n_{11} + 200 \, p_{12} + 200 \, n_{12} + 300 \, p_{13} + 300 \, n_{13} + \\
& \quad 200 \, p_{21} + 200 \, n_{21} + 400 \, p_{22} + 400 \, n_{22} + 500 \, p_{23} + 500 \, n_{23} + \\
& \quad 300 \, p_{31} + 300 \, n_{31} + 500 \, p_{32} + 500 \, n_{32} + 600 \, p_{33} + 600 \, n_{33}
\end{align*}
\]

Subject To

\[
\begin{align*}
c_{11}: & \quad t_{11} + t_{22} - t_{12} - t_{21} = 0 \\
c_{12}: & \quad t_{12} + t_{23} - t_{13} - t_{22} = 0 \\
c_{21}: & \quad t_{21} + t_{32} - t_{22} - t_{31} = 0 \\
c_{22}: & \quad t_{22} + t_{33} - t_{23} - t_{32} = 0 \\
a_{11}: & \quad p_{11} + n_{11} = 1 \\
u_{11}: & \quad p_{11} - 10 \, w_{11} \leq 0 \\
l_{11}: & \quad n_{11} + 10 \, w_{11} \leq 10 \\
a_{12}: & \quad p_{12} + n_{12} = 2 \\
u_{12}: & \quad p_{12} - 10 \, w_{12} \leq 0 \\
l_{12}: & \quad n_{12} + 10 \, w_{12} \leq 10 \\
a_{13}: & \quad p_{13} + n_{13} = 3 \\
u_{13}: & \quad p_{13} - 10 \, w_{13} \leq 0 \\
l_{13}: & \quad n_{13} + 10 \, w_{13} \leq 10 \\
a_{21}: & \quad p_{21} + n_{21} = 2 \\
u_{21}: & \quad p_{21} - 10 \, w_{21} \leq 0 \\
l_{21}: & \quad n_{21} + 10 \, w_{21} \leq 10 \\
a_{22}: & \quad p_{22} + n_{22} = 4 \\
u_{22}: & \quad p_{22} - 10 \, w_{22} \leq 0 \\
l_{22}: & \quad n_{22} + 10 \, w_{22} \leq 10 \\
a_{23}: & \quad p_{23} + n_{23} = 5 \\
u_{23}: & \quad p_{23} - 10 \, w_{23} \leq 0 \\
l_{23}: & \quad n_{23} + 10 \, w_{23} \leq 10 \\
a_{31}: & \quad p_{31} + n_{31} = 3 \\
u_{31}: & \quad p_{31} - 10 \, w_{31} \leq 0 \\
l_{31}: & \quad n_{31} + 10 \, w_{31} \leq 10 \\
a_{32}: & \quad p_{32} + n_{32} = 5 \\
u_{32}: & \quad p_{32} - 10 \, w_{32} \leq 0 \\
l_{32}: & \quad n_{32} + 10 \, w_{32} \leq 10 \\
a_{33}: & \quad p_{33} + n_{33} = 6 \\
u_{33}: & \quad p_{33} - 10 \, w_{33} \leq 0 \\
l_{33}: & \quad n_{33} + 10 \, w_{33} \leq 10 \\
\text{Bounds} \\
t_{11} & \geq -\infty \\
p_{11} & \geq 0
\end{align*}
\]
As per step 3, this output can be saved as a *.lp file. Let us call it “output.lp”. This file is saved in the same folder as the CPLEX.exe, which was created on downloading the CPLEX tool from ILOG.

On running the CPLEX.exe file as per step 5, we get a window with the following command prompt:
At this command prompt, type the following commands:

CPLEX> read output.lp       // This command is executed to run the output.lp file

CPLEX>opt                         // This command is used to runs the optimal solution for the linear programming problem defined in the output.lp file

CPLEX>d sol var p11-p66 // This command displays the variable values between p11 and p66. The values corresponding to variables with a ‘p’ initial are considered to be positive, while the values corresponding to variables with an ‘n’ initial are considered to be negative.

Finally, these ‘p’ and ‘n’ numbers are substituted in the following equations defined in the constraints or “Subject To” section of ‘output.lp’ given above:

\[ p_{11} + t_{11} - n_{11} = 1 \]
\[ p_{12} + t_{12} - n_{12} = 2 \]
\[ p_{13} + t_{13} - n_{13} = 3 \]
\[ p_{21} + t_{21} - n_{21} = 2 \]
\[ p_{22} + t_{22} - n_{22} = 4 \]
\[ p_{23} + t_{23} - n_{23} = 5 \]
\[ p_{31} + t_{31} - n_{31} = 3 \]
\[ p_{32} + t_{32} - n_{32} = 5 \]
\[ p_{33} + t_{33} - n_{33} = 6 \]

Thus we can find the implementable delay values t11, t12, t13, t21, t22, t23, t31, t32, t33, and obtain an implementable delay matrix \([i,j]\) where \(i,j = 1\) to \(3\) using the CPLEX Method. This delay matrix can then be used to calculate the corresponding energy table.