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Higher-Order Compact Finite Element Method

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Major Qualifying Project
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Abstract

The Finite Element Method (FEM) is a scheme that can approximate the solution of boundary value problems. We study the fundamentals of FEM and construct a MATLAB code to approximate the error of the solution for each refinement and compute the rate of convergence of this discretization. Then, we study the article of Bramble and Schatz [2] to construct MATLAB code to approximate better solution by averaging the FEM's solution.

Acknowledgement

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Chapter 1

Introduction

The Finite Element Method is a scheme that can approximate the solution of boundary value problem described by a partial differential equation (PDE). In this paper, we focus on polygonal $\mathbb{R}^2$ domain. To compute the solution, we divide our domain into many subdomains called finite element. The simplest elements to compute are triangle and square, and they are the ones we consider here. Then we construct stiffness matrices associated to the Poisson problem.

1.1 The Model Problem

In our model problem, we start with definition of our domain and boundaries. Then, we introduce basis function to solve the Poisson problem.

1.1.1 Domain and Boundary

Denote the polygon domain in $\mathbb{R}^2$ by Ω. Two boundary conditions, Dirichlet Γ_D and Neumann Γ_N.

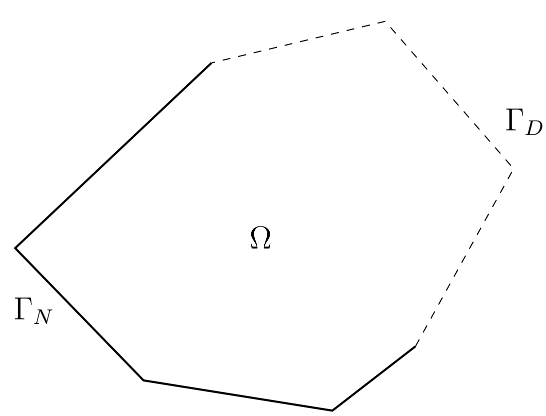


Figure 1.1: Domain Ω

- Γ_D is the Dirichlet boundary where the solution is given.

- Γ_N is the Neumann boundary where the normal derivative is given

1.1.2 Boundary Value Problem

Consider the Laplace operator, or Laplacian

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

The boundary value problem is

$$-\Delta u + cu = f \quad in\,\Omega$$

$$u = g_0 \quad on\,\Gamma_D$$

$$\partial_n u = g_1 \quad on\,\Gamma_N$$

- u is an unknown function defined on the domain Ω
- c is a non-negative constant value. In this paper, we set $c = 0$
- f is a given function on Ω
- g_0 , g_1 are given on two different parts of boundary

1.1.3 Weak form of FEM

We now use the weak form of FEM. To get start, we introduce the Green's Theorem. The theorem states that

$$\int_\Omega (\Delta u)v + \int_\Omega \nabla u \cdot \nabla v = \int_\Gamma (\partial_n u)v$$

Our boundary contains with Dirichlet and Neumann

$$\int_\Omega (\Delta u)v + \int_\Omega \nabla u \cdot \nabla v = \int_{\Gamma_N} (\partial_n u)v + \int_{\Gamma_D} (\partial_n u)v$$

Since $c = 0$ so that $-\Delta u = f$

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv + \int_{\Gamma_N} g_1 v + \int_{\Gamma_D} (\partial_n u)v$$

Since the value on Γ_D is given, we set $v = 0$ on Γ_D
Therefore, we have a new formula of our problem

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv + \int_{\Gamma_N} g_1 v$$

In this project we assume that $\Gamma_N = \emptyset$, that is, the boundary of Ω is all Dirichlet boundary.

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv$$

1.2 Triangulation

As we mentioned in domain and boundary, we have to divide our domain into subdomains. We focus on linear and quadratic function of two variables P_1 and P_2.

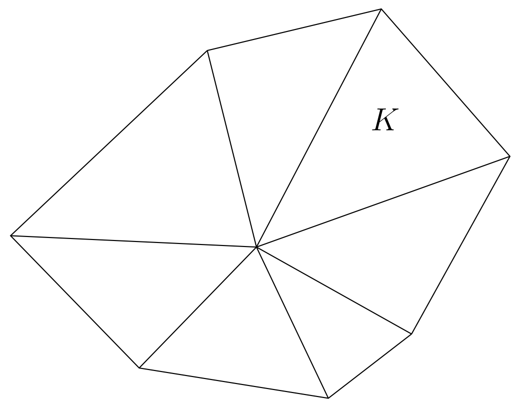


Figure 1.2: Subdomains of Ω

1.2.1 Linear and Quadratic triangular

Define P_1 : Linear Triangular and P_2 : Quadratic Triangular
A linear function of two variables is

$$p(x,y) = a_i + b_i x + c_i y$$

$$p \in P_1 = \{a_i + b_i x + c_i y \,|\, a_i, b_i, c_i \in \mathbb{R}\}$$

A quadratic function of two variables is

$$p(x,y) = a_i + b_i x + c_i y + d_i xy + e_i x^2 + f_i y^2$$

$$p \in P_2 = \{a_i + b_i x + c_i y + d_i xy + e_i x^2 + f_i y^2 \,|\, a_i, b_i, c_i, d_i, e_i, f_i \in \mathbb{R}\}$$

1.2.2 The Reference Triangle

Given a subdomain K in Ω, define

- F_k is a function mapping $\hat{K} \to K$
- K is a triangle on Ω
- $\hat{K}$ is a reference triangle with $\hat{v}_1$=(0,0), $\hat{v}_2$=(1,0), and $\hat{v}_3$=(0,1)

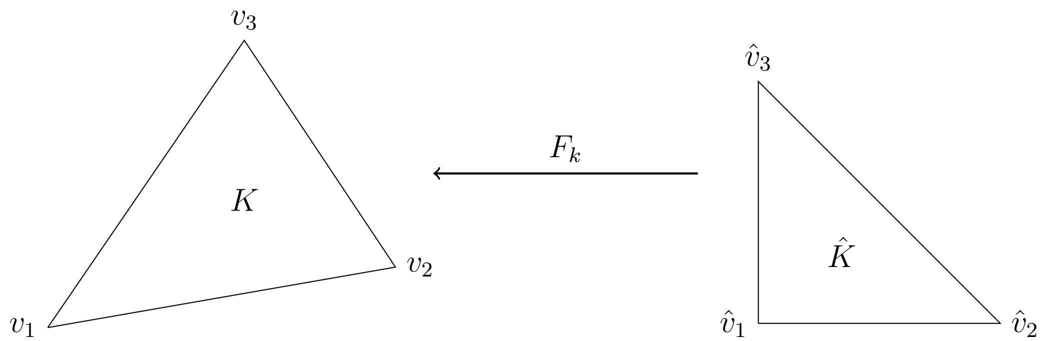


Figure 1.3: The function map to reference triangle

For any point (x,y) in K, $(x,y) = F_k(\hat{x},\hat{y})$

$$\begin{pmatrix} x \\ y \end{pmatrix} = F_k \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

Define $B_k = \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} (1 - \hat{x} - \hat{y}) + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \hat{x} + \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \hat{y}$$

The area of the triangle $K = |detB_k|/2$

1.2.3 Basis function

The polynomials are defined on each triangle in the mesh, which form a set of basis functions for the triangular element. Each basis function has a value of 1 at its node and a value of 0 at the other nodes giving it a pyramid shape over the triangle. Define v_h be a piecewise linear and continuous function on Ω so we can also define a basis function

$$\varphi_i(x_i) = \delta_{ij} = \begin{cases} 0 & \text{i} \neq j \\ 1 & \text{i} = j \end{cases}$$

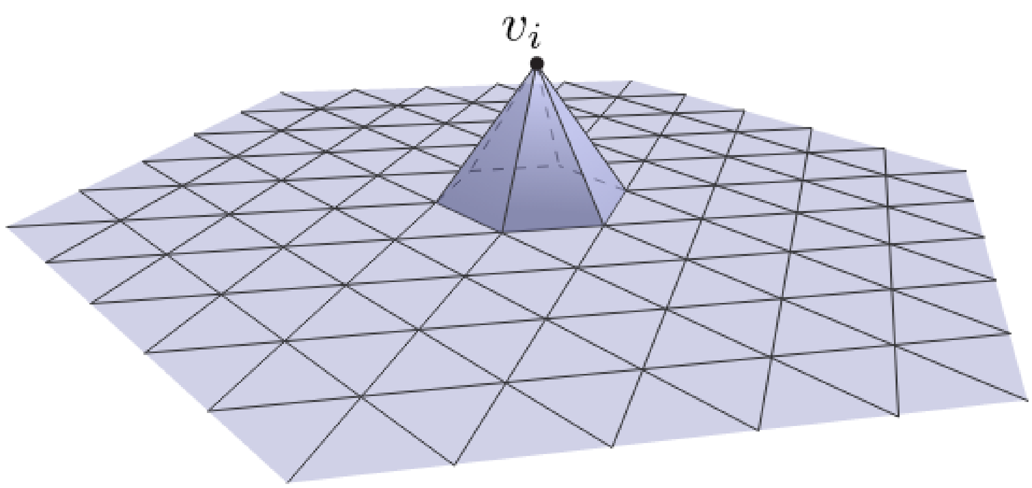


Figure 1.4: The hat function
(image credit: http://brickisland.net/cs177fa12/?p=302)

It is easy to see that

$$w_h(x) = \sum_{i \in \mathcal{N}} w_h(v_i)\varphi_i(x)$$

where $\mathcal{N}$ is the index set of interior nodes, since we consider $w_h(x) = 0$ for $x \in \partial\Omega$.

1.3 Mesh Grid

Our domain is a square 1-by-1 to be easier to compute. We refine our domain into mesh grid that we choose L = 1,2,3,... Here is the mesh grid for $L = 1$

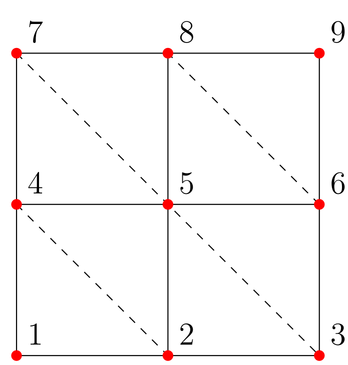


Figure 1.5: The global vertices for P_1 for $L = 1$

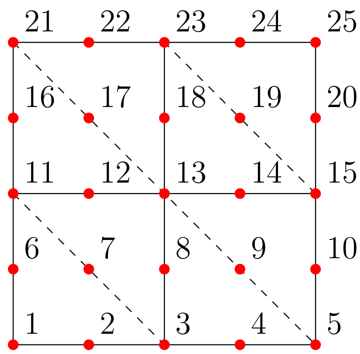


Figure 1.6: The global vertices for P_2 for $L = 1$

1.3.1 Local vertices

To order the number of vertices, we have to choose the direction clockwise or counter-clockwise. In this report, we use counter-clockwise for P_1. In P_2, three vertices in the middle of each edges are ordered in opposite of vertices.

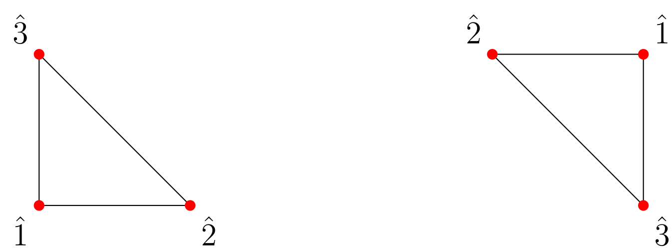


Figure 1.7: The ordering of vertices for P_1

Figure 1.8: The ordering of vertices for P_2

For example, we map vertices of triangle in global to local by

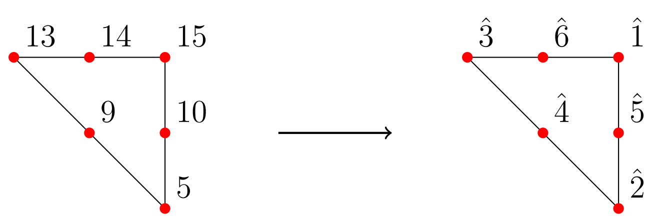


Figure 1.9: Example of mapping vertices

1.3.2 Elements

To compute each element, we order our element at the figure below.

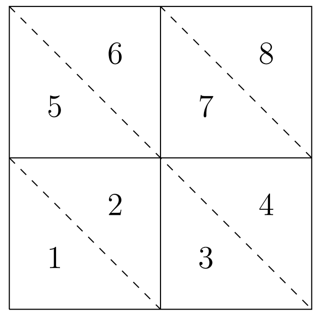


Figure 1.10: The ordering of element for $L = 1$

By using the ordering of vertices above, we get the ordering of vertices in each element.

Element	v_1	v_2	v_3
1	1	2	4
2	5	4	2
3	2	3	5
4	6	5	3
5	4	5	7
6	8	7	5
7	5	6	8
8	9	8	6

Table 1.1: The order of vertices for P_1

Element	v_1	v_2	v_3	v_4	v_5	v_6
1	1	3	11	7	6	2
2	13	11	3	7	8	12
3	3	5	13	9	8	4
4	15	13	5	9	10	14
5	11	13	21	17	16	12
6	23	21	13	17	18	22
7	13	15	23	19	18	14
8	25	23	15	19	20	24

Table 1.2: The order of vertices for P_2

1.3.3 Dirichlet boundary

As we mentioned above, we are focusing only zero Dirichlet boundary condition, so all the vertices are zero at the boundary. However, we can adapt to have Neumann boundary if we want to. Here is the ordering of Dirichlet global boundary nodes and the Tables 1.3 and 1.4 of the ordering of vertices

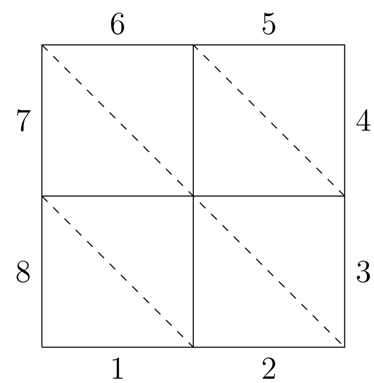


Figure 1.11: The ordering of Dirichlet boundaries for $L = 1$

Dirichlet	v_1	v_2
1	1	2
2	2	3
3	3	6
4	6	9
5	9	8
6	8	7
7	7	4
8	4	1

Table 1.3: The ordering of vertices for P_1

Dirichlet	v_1	v_2	v_3
1	1	2	3
2	3	4	5
3	5	10	15
4	15	20	25
5	25	24	23
6	23	22	21
7	21	16	11
8	11	6	1

Table 1.4: The ordering of vertices for P_2

1.4 Q_3

We introduce new element which is easy to compute for square domain. It is a square element. As we define our mesh grid, coordinate, element, and Dirichlet boundary above. We do the same thing in our Q_3. Here is the global ordering of vertices for $L = 1$

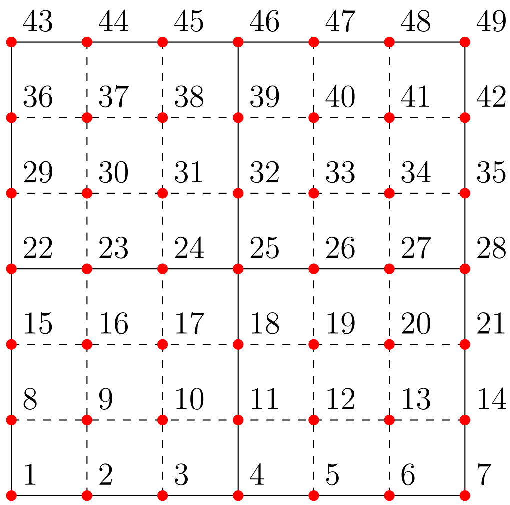


Figure 1.12: The ordering of vertices for Q_3

1.4.1 Loval vertices

We also define the order of each element in the same way. Here is the example of first element for $L = 1$

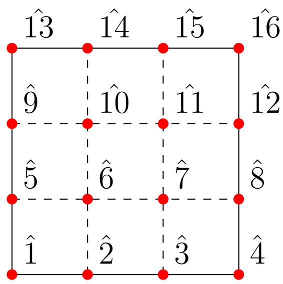


Figure 1.13: The ordering of vertices for Q_3

1.4.2 Element

Now our element is a square with four points on boundary of each element.

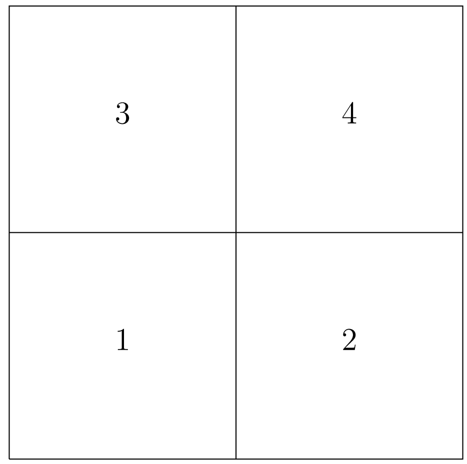


Figure 1.14: The ordering of element for $L = 1$

Element	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}
1	1	2	3	4	8	9	10	11	15	16	17	18	22	23	24	25
2	4	5	6	7	11	12	13	14	18	19	20	21	25	26	27	28
3	22	23	24	25	29	30	31	32	36	37	38	39	43	44	45	46
4	25	26	27	28	32	33	34	35	39	40	41	42	46	47	48	49

Table 1.5: The ordering of vertices for Q_3

1.4.3 Dirichlet boundary

For the Dirichlet boundary, we also order in counter-clockwise.

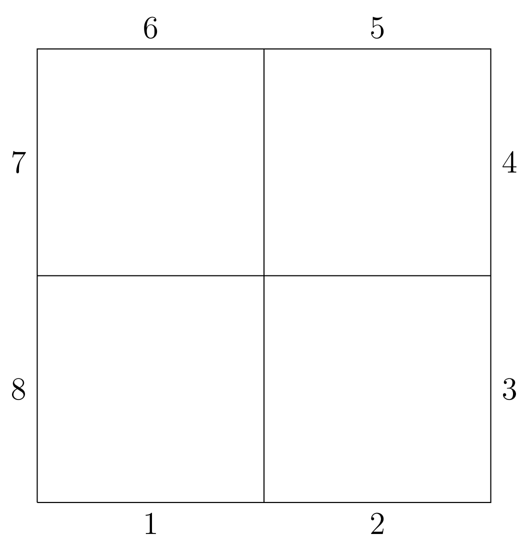


Figure 1.15: The ordering of Dirichlet boundaries for $L = 1$ for Q_3

Element	v_1	v_2	v_3	v_4
1	1	2	3	4
2	4	5	6	7
3	7	14	21	28
4	28	35	42	49
5	49	48	47	46
6	46	45	44	43
7	43	36	29	22
8	22	15	8	1

Table 1.6: The ordering of vertices for Q_3

Chapter 2

Finite Element Method

In this section, we decide to set all the boundary to be zero Dirichlet. Therefore, we get: Find $u \in V_h(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv \qquad \forall v \in V_h(\Omega),$$

where $V_h(\Omega)$ is the space of piecewise linear and continuous functions which are vanish on $\partial\Omega$. We study this equation in two parts, left-hand side and right-hand side.

2.1 Left-hand side

We denote the i, j position of the stiffness matrix by

$$\int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j$$

Recall our solution u is of the form

$$u = \sum_{j\in\mathcal{N}} u_j\varphi_j$$

then substitute ∇u

$$\int_\Omega \nabla u \cdot \nabla\varphi_i = \sum_{j\in\mathcal{N}} u_j \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i$$

We compute element K in Ω

$$\int_K \nabla\varphi_i^K \cdot \nabla\varphi_j^K = |detB_K| \int_{\hat{K}} (\nabla\varphi_i^K \circ F_K) \cdot (\nabla\varphi_j^K \circ F_K)$$

Then, we use a change of variable to obtain

$$\int_K \nabla\varphi_i^K \cdot \nabla\varphi_j^K = |detB_K| \int_{\hat{K}} (B_K^{-T}\hat{\nabla}\hat{\varphi}_{\hat{i}}) \cdot (B_K^{-T}\hat{\nabla}\hat{\varphi}_{\hat{j}})$$

Therefore, the entries of the stiffness matrix is given by

$$\int_K \nabla\varphi_i^K \cdot \nabla\varphi_j^K = |detB_K| \int_{\hat{K}} C_K \hat{\nabla}\hat{\varphi}_{\hat{i}} \cdot \hat{\nabla}\hat{\varphi}_{\hat{j}}$$

where

$$C_K = B_K^{-1} B_K^{-T} = \begin{pmatrix} c_{11}^K & c_{12}^K \\ c_{21}^K & c_{22}^K \end{pmatrix}$$

2.1.1 Hat functions of P_1

The basis function on $\hat{K}$ is of the form:

$$\hat{\varphi}_{\hat{i}}(\hat{x}, \hat{y}) = a_1 + b_1\hat{x} + c_1\hat{y}$$

Each $\hat{\varphi}_{\hat{i}}$ has three unknown coefficients a_i, b_i, c_i. For instance, for $\hat{\varphi}_{\hat{1}}$ we want $\hat{\varphi}_{\hat{1}}(0,0) = 1$, $\hat{\varphi}_{\hat{1}}(1,0) = 0$, and $\hat{\varphi}_{\hat{1}}(0,1) = 0$. So, this implies that

$$a_1 + b_1\hat{x}_{\hat{1}} + c_1\hat{y}_{\hat{1}} = 1$$
$$a_1 + b_1\hat{x}_{\hat{2}} + c_1\hat{y}_{\hat{2}} = 0$$
$$a_1 + b_1\hat{x}_{\hat{3}} + c_1\hat{y}_{\hat{3}} = 0$$

so we can write it in matrix form as

$$\begin{pmatrix} 1 & \hat{x}_{\hat{1}} & \hat{y}_{\hat{1}} \\ 1 & \hat{x}_{\hat{2}} & \hat{y}_{\hat{2}} \\ 1 & \hat{x}_{\hat{3}} & \hat{y}_{\hat{3}} \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

For all the three basis function in triangle, we get the equation of the form

$$\begin{pmatrix} 1 & \hat{x}_{\hat{1}} & \hat{y}_{\hat{1}} \\ 1 & \hat{x}_{\hat{2}} & \hat{y}_{\hat{2}} \\ 1 & \hat{x}_{\hat{3}} & \hat{y}_{\hat{3}} \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Solve for coefficient matrix, we get each $\hat{\varphi}_i$ in the form

$$\hat{\varphi}_{\hat{1}} = 1 - \hat{x} - \hat{y}$$
$$\hat{\varphi}_{\hat{2}} = \hat{x}$$
$$\hat{\varphi}_{\hat{3}} = \hat{y}$$

2.1.2 Hat functions of P_2

The basis function on $\hat{K}$ is of the form:

$$\hat{\varphi}_{\hat{i}}(\hat{x},\hat{y}) = a_1 + b_1\hat{x} + c_1\hat{y} + d_1\hat{x}\hat{y} + e_1\hat{x}^2 + f_1\hat{y}^2$$

Each $\hat{\varphi}_{\hat{i}}$ has three unknown coefficients a_i, b_i, c_i. For instance, for $\hat{\varphi}_{\hat{1}}$ we want $\hat{\varphi}_{\hat{1}}(0,0) = 1$, $\hat{\varphi}_{\hat{1}}(1,0) = 0$, $\hat{\varphi}_{\hat{1}}(0,1) = 0$, $\hat{\varphi}_{\hat{1}}(1/2,1/2) = 0$, $\hat{\varphi}_{\hat{1}}(0,1/2) = 0$, and $\hat{\varphi}_{\hat{1}}(1/2,0) = 0$. So, this implies that

$$\begin{aligned}
a_1 + b_1\hat{x}_{\hat{1}} + c_1\hat{y}_{\hat{1}} + d_1\hat{x}_{\hat{1}}\hat{y}_{\hat{1}} + e_1\hat{x}_{\hat{1}}^2 + f_1\hat{y}_{\hat{1}}^2 &= 1\\
a_1 + b_1\hat{x}_{\hat{2}} + c_1\hat{y}_{\hat{2}} + d_1\hat{x}_{\hat{2}}\hat{y}_{\hat{2}} + e_1\hat{x}_{\hat{2}}^2 + f_1\hat{y}_{\hat{2}}^2 &= 0\\
a_1 + b_1\hat{x}_{\hat{3}} + c_1\hat{y}_{\hat{3}} + d_1\hat{x}_{\hat{3}}\hat{y}_{\hat{3}} + e_1\hat{x}_{\hat{3}}^2 + f_1\hat{y}_{\hat{3}}^2 &= 0\\
a_1 + b_1\hat{x}_{\hat{4}} + c_1\hat{y}_{\hat{4}} + d_1\hat{x}_{\hat{4}}\hat{y}_{\hat{4}} + e_1\hat{x}_{\hat{4}}^2 + f_1\hat{y}_{\hat{4}}^2 &= 0\\
a_1 + b_1\hat{x}_{\hat{5}} + c_1\hat{y}_{\hat{5}} + d_1\hat{x}_{\hat{5}}\hat{y}_{\hat{5}} + e_1\hat{x}_{\hat{5}}^2 + f_1\hat{y}_{\hat{5}}^2 &= 0\\
a_1 + b_1\hat{x}_{\hat{6}} + c_1\hat{y}_{\hat{6}} + d_1\hat{x}_{\hat{6}}\hat{y}_{\hat{6}} + e_1\hat{x}_{\hat{6}}^2 + f_1\hat{y}_{\hat{6}}^2 &= 0
\end{aligned}$$

so we can write it in matrix form as

$$\begin{pmatrix}
1 & \hat{x}_{\hat{1}} & \hat{y}_{\hat{1}} & \hat{x}_{\hat{1}}\hat{y}_{\hat{1}} & \hat{x}_{\hat{1}}^2 & \hat{y}_{\hat{1}}^2\\
1 & \hat{x}_2 & \hat{y}_2 & \hat{x}_2\hat{y}_2 & \hat{x}_{\hat{2}}^2 & \hat{y}_{\hat{2}}^2\\
1 & \hat{x}_3 & \hat{y}_3 & \hat{x}_3\hat{y}_3 & \hat{x}_{\hat{3}}^2 & \hat{y}_{\hat{3}}^2\\
1 & \hat{x}_4 & \hat{y}_4 & \hat{x}_4\hat{y}_4 & \hat{x}_{\hat{4}}^2 & \hat{y}_{\hat{4}}^2\\
1 & \hat{x}_5 & \hat{y}_5 & \hat{x}_5\hat{y}_5 & \hat{x}_{\hat{5}}^2 & \hat{y}_{\hat{5}}^2\\
1 & \hat{x}_6 & \hat{y}_6 & \hat{x}_6\hat{y}_6 & \hat{x}_{\hat{6}}^2 & \hat{y}_{\hat{6}}^2
\end{pmatrix}
\begin{pmatrix} a_1\\ b_1\\ c_1\\ d_1\\ e_1\\ f_1 \end{pmatrix}
=
\begin{pmatrix} 1\\ 0\\ 0\\ 0\\ 0\\ 0 \end{pmatrix}$$

For all the six basis function in triangle, we get the equation of the form

$$\begin{pmatrix}
1 & \hat{x}_{\hat{1}} & \hat{y}_{\hat{1}} & \hat{x}_{\hat{1}}\hat{y}_{\hat{1}} & \hat{x}_{\hat{1}}^2 & \hat{y}_{\hat{1}}^2\\
1 & \hat{x}_2 & \hat{y}_2 & \hat{x}_2\hat{y}_2 & \hat{x}_{\hat{2}}^2 & \hat{y}_{\hat{2}}^2\\
1 & \hat{x}_3 & \hat{y}_3 & \hat{x}_3\hat{y}_3 & \hat{x}_{\hat{3}}^2 & \hat{y}_{\hat{3}}^2\\
1 & \hat{x}_4 & \hat{y}_4 & \hat{x}_4\hat{y}_4 & \hat{x}_{\hat{4}}^2 & \hat{y}_{\hat{4}}^2\\
1 & \hat{x}_5 & \hat{y}_5 & \hat{x}_5\hat{y}_5 & \hat{x}_{\hat{5}}^2 & \hat{y}_{\hat{5}}^2\\
1 & \hat{x}_6 & \hat{y}_6 & \hat{x}_6\hat{y}_6 & \hat{x}_{\hat{6}}^2 & \hat{y}_{\hat{6}}^2
\end{pmatrix}
\begin{pmatrix}
a_1 & a_2 & a_3 & a_4 & a_5 & a_6\\
b_1 & b_2 & b_3 & b_4 & b_5 & b_6\\
c_1 & c_2 & c_3 & c_4 & c_5 & c_6\\
d_1 & d_2 & d_3 & d_4 & d_5 & d_6\\
e_1 & e_2 & e_3 & e_4 & e_5 & e_6\\
f_1 & f_2 & f_3 & f_4 & f_5 & f_6
\end{pmatrix}
=
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

Solve for coefficient matrix, we get each $\hat{\varphi}_i$ in the form

$$\begin{aligned}
\hat{\varphi}_{\hat{1}} &= (1-\hat{x}-\hat{y})(1-2\hat{x}-2\hat{y})\\
\hat{\varphi}_{\hat{2}} &= \hat{x}(2\hat{x}-1)\\
\hat{\varphi}_{\hat{3}} &= \hat{y}(2\hat{y}-1)\\
\hat{\varphi}_{\hat{4}} &= 4\hat{x}\hat{y}\\
\hat{\varphi}_{\hat{5}} &= 4\hat{x}(1-\hat{x}-\hat{y})\\
\hat{\varphi}_{\hat{6}} &= 4\hat{y}(1-\hat{x}-\hat{y})
\end{aligned}$$

2.1.3 Hat function of Q_3

For Q_3, it is different from P_1 and P_2. We now have an element that is a square which in each direction x and y is a polynomial degree 3 in x and y respectively. We first focus in 1D.

$$\hat{\varphi}_{\hat{i}}(\hat{x}) = a_1 + b_1\hat{x} + c_1\hat{x}^2 + d_1\hat{x}^3$$

Each $\hat{\varphi}_{\hat{i}}$ has four unknown coefficients a_i, b_i, c_i, d_i For instance, for $\hat{\varphi}_{\hat{i}}$ we want $\hat{\varphi}_{\hat{1}}(0) = 1$, $\hat{\varphi}_{\hat{1}}(1/3) = 0$, $\hat{\varphi}_{\hat{1}}(2/3) = 0$, and $\hat{\varphi}_{\hat{1}}(1) = 0$. So, this implies that

$$\begin{aligned}
a_1 + b_1\hat{x}_{\hat{1}} + c_1\hat{x}_{\hat{1}}^2 + d_1\hat{x}_{\hat{1}}^3 &= 1 \\
a_1 + b_1\hat{x}_{\hat{2}} + c_1\hat{x}_{\hat{2}}^2 + d_1\hat{x}_{\hat{2}}^3 &= 0 \\
a_1 + b_1\hat{x}_{\hat{3}} + c_1\hat{x}_{\hat{3}}^2 + d_1\hat{x}_{\hat{3}}^3 &= 0 \\
a_1 + b_1\hat{x}_{\hat{4}} + c_1\hat{x}_{\hat{4}}^2 + d_1\hat{x}_{\hat{4}}^3 &= 0
\end{aligned}$$

so we can write it in matrix form as

$$\begin{pmatrix} 1 & \hat{x}_{\hat{1}} & \hat{x}_{\hat{1}}^2 & \hat{x}_{\hat{1}}^3 \\ 1 & \hat{x}_{\hat{2}} & \hat{x}_{\hat{2}}^2 & \hat{x}_{\hat{2}}^3 \\ 1 & \hat{x}_{\hat{3}} & \hat{x}_{\hat{3}}^2 & \hat{x}_{\hat{3}}^3 \\ 1 & \hat{x}_{\hat{4}} & \hat{x}_{\hat{4}}^2 & \hat{x}_{\hat{4}}^3 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

For all the four basis function in x-axis, we get the equation of the form

$$\begin{pmatrix} 1 & \hat{x}_{\hat{1}} & \hat{x}_{\hat{1}}^2 & \hat{x}_{\hat{1}}^3 \\ 1 & \hat{x}_{\hat{2}} & \hat{x}_{\hat{2}}^2 & \hat{x}_{\hat{2}}^3 \\ 1 & \hat{x}_{\hat{3}} & \hat{x}_{\hat{3}}^2 & \hat{x}_{\hat{3}}^3 \\ 1 & \hat{x}_{\hat{4}} & \hat{x}_{\hat{4}}^2 & \hat{x}_{\hat{4}}^3 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solve for coefficient matrix, we get each $\hat{\varphi}_i$ in the form

$$\begin{aligned}
\hat{\varphi}_{\hat{1}} &= 9/2(1-\hat{x})(1/3-\hat{x})(2/3-\hat{x}) \\
\hat{\varphi}_{\hat{2}} &= 27/2(\hat{x})(1-\hat{x})(2/3-\hat{x}) \\
\hat{\varphi}_{\hat{3}} &= 27/2(\hat{x})(1-\hat{x})(\hat{x}-1/3) \\
\hat{\varphi}_{\hat{4}} &= 9/2(\hat{x})(\hat{x}-1/3)(\hat{x}-2/3)
\end{aligned}$$

Also for y-axis,

$$\begin{aligned}
\hat{\varphi}_{\hat{1}} &= 9/2(1-\hat{y})(1/3-\hat{y})(2/3-\hat{y}) \\
\hat{\varphi}_{\hat{2}} &= 27/2(\hat{y})(1-\hat{y})(2/3-\hat{y}) \\
\hat{\varphi}_{\hat{3}} &= 27/2(\hat{y})(1-\hat{y})(\hat{y}-1/3) \\
\hat{\varphi}_{\hat{4}} &= 9/2(\hat{y})(\hat{y}-1/3)(\hat{y}-2/3)
\end{aligned}$$

For 2D problem, there are 16 nodes in each elements which are $\hat{\varphi}_{\hat{i}\hat{j}}(\hat{x},\hat{y}) = \hat{\varphi}_{\hat{i}}(\hat{x})\hat{\varphi}_{\hat{j}}(\hat{y})$ for $\hat{i},\hat{j} = \hat{1},\hat{2},\hat{3},\hat{4}$

2.2 Right-hand side

Our computation of the right-hand side is

$$\int_\Omega fv$$

We approximate this integration by

$$\int_\Omega fv \sim \sum_K c_i \int_K \varphi_i \cdot \varphi_j$$

where

$$c_i = f(x_i)$$

Then,

$$\int_K \varphi_i^K \cdot \varphi_j^K = |detB_K| \int_{\hat{K}} (\varphi_i^K \circ F_K) \cdot (\varphi_j^K \circ F_K)$$

Therefore, our Finite Element is

$$\sum_j u_j \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i = \sum_K c_i \int_K \varphi_i \cdot \varphi_j$$

or

$$Au = b$$

where

$$A = \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i \quad and \quad b = \sum_K c_i \int_K \varphi_i \cdot \varphi_j$$

2.3 FEM of P_1

Recall our stiffness matrix

$$\int_K \nabla\varphi_i^K \cdot \nabla\varphi_j^K = |detB_K| \int_{\hat{K}} C_K \hat{\nabla}\hat{\varphi}_i \cdot \hat{\nabla}\hat{\varphi}_j$$

where

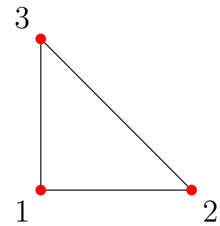
$$C_K = B_K^{-1} B_K^{-T} = \begin{pmatrix} c_{11}^K & c_{12}^K \\ c_{21}^K & c_{22}^K \end{pmatrix}$$

$$\hat{K}_{xx} = \int_{\hat{K}} \partial_x\hat{\varphi}_i \partial_x\hat{\varphi}_j \;\; , \;\; \hat{K}_{yy} = \int_{\hat{K}} \partial_y\hat{\varphi}_i \partial_y\hat{\varphi}_j \;\; , \;\; \hat{K}_{xy} = \int_{\hat{K}} \partial_x\hat{\varphi}_i \partial_y\hat{\varphi}_j \;\; , \;\; \hat{K}_{yx} = \int_{\hat{K}} \partial_y\hat{\varphi}_i \partial_x\hat{\varphi}_j$$

Then, our stiffness matrix with respect to the reference triangle is

$$\int_K \nabla\varphi_i^K \cdot \nabla\varphi_j^K = |detB_K|(c_{11}^K \hat{K}_{xx} + c_{22}^K \hat{K}_{yy} + c_{12}^K \hat{K}_{xy} + c_{21}^K \hat{K}_{yx})$$

Recall our P_1 basis functions has the following ordering



$$\hat{\varphi}_{\hat{1}} = 1 - \hat{x} - \hat{y}, \quad \hat{\varphi}_{\hat{2}} = \hat{x}, \quad \hat{\varphi}_{\hat{3}} = \hat{y}$$

We start with construct Stiffness matrix.

$$\hat{\nabla}\hat{\varphi}_{\hat{1}} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \qquad \hat{\nabla}\hat{\varphi}_{\hat{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \hat{\nabla}\hat{\varphi}_{\hat{3}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\hat{K}_{xx} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \hat{K}_{yy} = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\hat{K}_{xy} = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \hat{K}_{yx} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 0 \end{pmatrix}$$

The part $|detB_K|$ and c_{ij}^K depend on element K For the right-hand side of FEM, we can compute the Mass matrix

$$Mass^{\hat{K}} = \int_{\hat{K}} \hat{\varphi}_i \cdot \hat{\varphi}_j$$

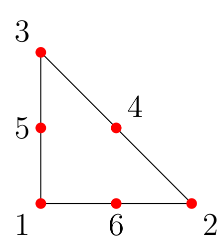
Then, for P_1, it is

$$Mass^{\hat{K}} = \frac{1}{24}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

And we compute for each element and sum in our matrix b

2.4 FEM of P_2

Recall our basis functions of P_2



$$\hat{\varphi}_{\hat{1}} = (1 - \hat{x} - \hat{y})(1 - 2\hat{x} - 2\hat{y}), \quad \hat{\varphi}_{\hat{2}} = \hat{x}(2\hat{x} - 1), \quad \hat{\varphi}_{\hat{3}} = \hat{y}(2\hat{y} - 1)$$

$$\hat{\varphi}_{\hat{4}} = 4\hat{x}\hat{y}, \quad \hat{\varphi}_{\hat{5}} = 4\hat{x}(1-\hat{x}-\hat{y}), \quad \hat{\varphi}_{\hat{6}} = 4\hat{y}(1-\hat{x}-\hat{y})$$

We take the gradient of our basis functions and construct stiffness matrix as P_1

$$\hat{\nabla}\hat{\varphi}_{\hat{1}} = \begin{pmatrix} 4\hat{x}+4\hat{y}-3 \\ 4\hat{x}+4\hat{y}-3 \end{pmatrix} \qquad \hat{\nabla}\hat{\varphi}_{\hat{2}} = \begin{pmatrix} 4\hat{x}-1 \\ 0 \end{pmatrix} \qquad \hat{\nabla}\hat{\varphi}_{\hat{3}} = \begin{pmatrix} 0 \\ 4\hat{y}-1 \end{pmatrix}$$

$$\hat{\nabla}\hat{\varphi}_{\hat{4}} = \begin{pmatrix} 4\hat{y} \\ 4\hat{x} \end{pmatrix} \qquad \hat{\nabla}\hat{\varphi}_{\hat{5}} = \begin{pmatrix} -4\hat{y} \\ 4-4\hat{x}-8\hat{y} \end{pmatrix} \qquad \hat{\nabla}\hat{\varphi}_{\hat{6}} = \begin{pmatrix} 4-8\hat{x}-4\hat{y} \\ -4\hat{x} \end{pmatrix}$$

$$\hat{K}_{xx} = \frac{1}{6}\begin{pmatrix} 3 & 1 & 0 & 0 & 0 & -4 \\ 1 & 3 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & -8 & 0 \\ 0 & 0 & 0 & -8 & 8 & 0 \\ -4 & -4 & 0 & 0 & 0 & 8 \end{pmatrix} \qquad \hat{K}_{yy} = \frac{1}{6}\begin{pmatrix} 3 & 0 & 1 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & -4 & 0 \\ 0 & 0 & 0 & 8 & 0 & -8 \\ -4 & 0 & -4 & 0 & 8 & 0 \\ 0 & 0 & 0 & -8 & 0 & 8 \end{pmatrix}$$

$$\hat{K}_{xy} = \frac{1}{6}\begin{pmatrix} 3 & 0 & 1 & 0 & -4 & 0 \\ 0 & 0 & -1 & 4 & 0 & -4 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 & -4 & -4 \\ -4 & 0 & 0 & -4 & 4 & 4 \\ 0 & -4 & 0 & -4 & 4 & 4 \end{pmatrix} \qquad \hat{K}_{yx} = \frac{1}{6}\begin{pmatrix} 3 & 1 & 0 & 0 & 0 & -4 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & -4 & 0 \\ 0 & 0 & 4 & 4 & -4 & -4 \\ 0 & 0 & -4 & -4 & 4 & 4 \\ -4 & 0 & 0 & -4 & 4 & 4 \end{pmatrix}$$

For the Right-hand side of FEM, we can compute by integrate Mass matrix

$$Mass^{\hat{K}} = \int_{\hat{K}} \hat{\varphi}_i \cdot \hat{\varphi}_j$$

Then, for P_2, it is

$$Mass^{\hat{K}} = \frac{1}{360}\begin{pmatrix} 6 & -1 & -1 & -4 & 0 & 0 \\ -1 & 6 & -1 & 0 & -4 & 0 \\ -1 & -1 & 6 & 0 & 0 & -4 \\ -4 & 0 & 0 & 32 & 16 & 16 \\ 0 & -4 & 0 & 16 & 32 & 16 \\ 0 & 0 & -4 & 16 & 16 & 32 \end{pmatrix}$$

And we compute for each element and sum in our matrix b

2.5 FEM of Q_3

Recall our basis functions of Q_3

$$\hat{\varphi}_{\hat{i}\hat{j}}(\hat{x},\hat{y}) = \hat{\varphi}_{\hat{i}}(\hat{x})\hat{\varphi}_{\hat{j}}(\hat{y})$$

where

$$\hat{\varphi}_{\hat{i}}(\hat{x}) = \begin{cases} 9/2(1-\hat{x})(1/3-\hat{x})(2/3-\hat{x}) & \hat{i}\text{=}1 \\ 27/2(\hat{x})(1-\hat{x})(2/3-\hat{x}) & \hat{i}\text{=}2 \\ 27/2(\hat{x})(1-\hat{x})(\hat{x}-1/3) & \hat{i}\text{=}3 \\ 9/2(\hat{x})(\hat{x}-1/3)(\hat{x}-2/3) & \hat{i}\text{=}4 \end{cases}$$

$$\hat{\varphi}_{\hat{j}}(\hat{y}) = \begin{cases} 9/2(1-\hat{y})(1/3-\hat{y})(2/3-\hat{y}) & \hat{j}\text{=}1 \\ 27/2(\hat{y})(1-\hat{y})(2/3-\hat{y}) & \hat{j}\text{=}2 \\ 27/2(\hat{y})(1-\hat{y})(\hat{y}-1/3) & \hat{j}\text{=}3 \\ 9/2(\hat{y})(\hat{y}-1/3)(\hat{y}-2/3) & \hat{j}\text{=}4 \end{cases}$$

The gradient of Q_3 is

$$\hat{\nabla}\hat{\varphi}_{\hat{i}\hat{j}}(\hat{x},\hat{y}) = \begin{pmatrix} \partial_{\hat{x}}\hat{\varphi}_{\hat{i}}(\hat{x})\hat{\varphi}_{\hat{j}}(\hat{y}) \\ \hat{\varphi}_{\hat{i}}(\hat{x})\partial_{\hat{y}}\hat{\varphi}_{\hat{j}}(\hat{y}) \end{pmatrix} \quad \hat{i},\hat{j} \text{ =1,2,3,4}$$

To compute stiffness matrix

$$\int \hat{\nabla}\hat{\varphi}_{\hat{i}\hat{j}}(\hat{x},\hat{y}) \cdot \hat{\nabla}\hat{\varphi}_{\hat{m}\hat{n}}(\hat{x},\hat{y}) = \int \begin{pmatrix} \partial_{\hat{x}}\hat{\varphi}_{\hat{i}}(\hat{x})\hat{\varphi}_{\hat{j}}(\hat{y}) \\ \hat{\varphi}_{\hat{i}}(\hat{x})\partial_{\hat{y}}\hat{\varphi}_{\hat{j}}(\hat{y}) \end{pmatrix} \cdot \begin{pmatrix} \partial_{\hat{x}}\hat{\varphi}_{\hat{m}}(\hat{x})\hat{\varphi}_{\hat{n}}(\hat{y}) \\ \hat{\varphi}_{\hat{m}}(\hat{x})\partial_{\hat{y}}\hat{\varphi}_{\hat{n}}(\hat{y}) \end{pmatrix}$$

$$= \int (\partial_{\hat{x}}\hat{\varphi}_{\hat{i}}(\hat{x})\hat{\varphi}_{\hat{j}}(\hat{y}))(\partial_{\hat{x}}\hat{\varphi}_{\hat{m}}(\hat{x})\hat{\varphi}_{\hat{n}}(\hat{y})) + (\hat{\varphi}_{\hat{i}}(\hat{x})\partial_{\hat{y}}\hat{\varphi}_{\hat{j}}(y))(\hat{\varphi}_{\hat{m}}(\hat{x})\partial_{\hat{y}}\hat{\varphi}_{\hat{n}}(\hat{y}))$$

$$= \int \partial_{\hat{x}}\hat{\varphi}_{\hat{i}}(\hat{x})\partial_{\hat{x}}\hat{\varphi}_{\hat{m}}(\hat{x}) + \int \hat{\varphi}_{\hat{j}}(\hat{y})\hat{\varphi}_{\hat{n}}(\hat{y}) + \int \hat{\varphi}_{\hat{i}}(\hat{x})\hat{\varphi}_{\hat{m}}(\hat{x}) + \int \partial_{\hat{y}}\hat{\varphi}_{\hat{j}}(\hat{y})\partial_{\hat{y}}\hat{\varphi}_{\hat{n}}(\hat{y})$$

Therefore our stiffness matrix of Q_3 is 16-by16.
For the right-hand side, we can directly integrate our equation with the basis functions

$$\int_{\Omega} fv \sim \sum_{K} c_{ij} \int_{K} f\hat{\varphi}_{ij}$$

where

$$c_{ij} = f(x_i, y_j)$$

There will be 16-by-16 as well as our stiffness matrix for Q_3.

Chapter 3

More accuracy by averaging

By the article of J.H. Bramble and A.H. Schatz [2], Higher Order Local Accuracy by Averaging in the Finite Element Method, we can define a better approximation of any given point x in Ω by averaging $u_h(x)$ of the neighborhood of that point. We denote this approximation by $\widetilde{u}_h$.

3.1 Bramble and Schatz

In the article of Bramble and Schatz [2], they give an example of subspaces generated by the B-splines of Schoenberg.
For t is real number, define

$$\chi = \begin{cases} 1 & |t| \leq 1/2, \\ 0 & |t| > 1/2 \end{cases}$$

and for l an integer, set convolution $l-1$ times. Example, for $l=2$ see Figure 3.1. For $l=3$ see Figure 3.2.

$$\psi_1^{(l)}(t) = \chi * \chi * \cdots * \chi.$$

Let $\psi_1^{(l)}$ be the one-dimensional smooth spline of order l defined by above equation. For $l = r-2$, $r \geq 2$ given, find k_0,k_1, $\ldots$, k_{r-2} by

$$\sum_{j=0}^{r-2} k_j \int_{R_1} \psi_1^{(r-2)}(y)(y+j)^{2m} dy = \begin{cases} 1 & \text{if} \quad m = 0, \\ 0 & \text{if} \quad m = 1, \ldots, r-2. \end{cases}$$

The constants k'_j are defined as

- $k'_{-j} = k'_j$, $j = 0, \ldots, r-2$.
- $k'_0 = k_0$ and $k'_j = k_{j/2}$, $j = 0, \ldots, r-2$,

where the k_j , $j = 0, \ldots, r-2$.
We can compute new solution by averaging as

$$(K_h * u_h)(h\gamma) = \sum_{\gamma,\delta \in Z^N} a^j_{\gamma-\delta} d^j_\delta$$

where

$$d^j_\delta = \sum_{\beta \in Z} k'_\beta \int_{\mathbb{R}} \psi^{r-2}(\delta - \beta - \eta)\varphi_j(\eta)d\eta$$

In the article of Bramble and Schatz [2], they already gave us the Table of k'_j and l

$j \backslash r$	3	4	5
0	13/12	37/30	346517/241920
1	-1/24	-23/180	-81329/322560
2		1/90	6337/161280
3			-3229/967680

Table 3.1: k'_j, $l = r - 2$, $t = r - 1$

3.2 Averaging P_2 in 2D

By the Table k'_j of Bramble and Schatz, we can construct a coefficient area $k'_{i,j}$ where $k'_{i,j} = k'_i k'_j$. For P_2, we set $r = 3$, then our $k'_0 = 13/12$ and $k'_1 = k'_{-1} = -1/24$ for defining $\psi_1^{(1)}$.

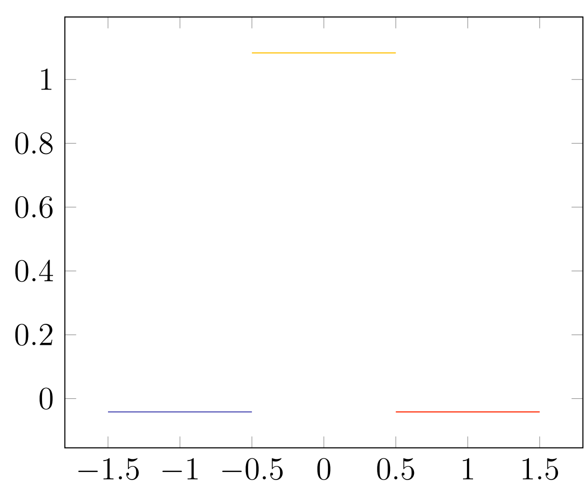


Figure 3.1: The graph of coefficients in $\sum_{j=-1}^{1} k'_j \psi_1^{(1)}(x - j)$

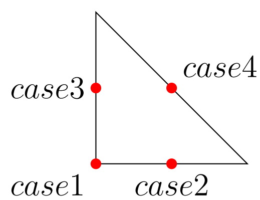
In 2D, we define the area of coefficients $k'_{i,j}$ as

$$\sum_{i=-1}^{1} \sum_{j=-1}^{1} k'_{i,j} \psi_1^{(1)}(x - i)\psi_1^{(1)}(x - j)$$

where $k'_{i,j} = k'_i k'_j$ and $\psi_1^{(1)}$ is 1 for $-\frac{1}{2} \leq x \leq \frac{1}{2}$ and zero otherwise.

$k'_{-1,1}$	$k'_{0,1}$	$k'_{1,1}$
$k'_{-1,0}$	$k'_{0,0}$	$k'_{1,0}$
$k'_{-1,-1}$	$k'_{0,-1}$	$k'_{1,-1}$

To compute the weights for averaging the solution, we have consider 4 cases,



3.2.1 Case 1

As we can see in Figure 3.3, we show that to average the point $(0, 0)$, we must find all the weight $d(m, n)$.

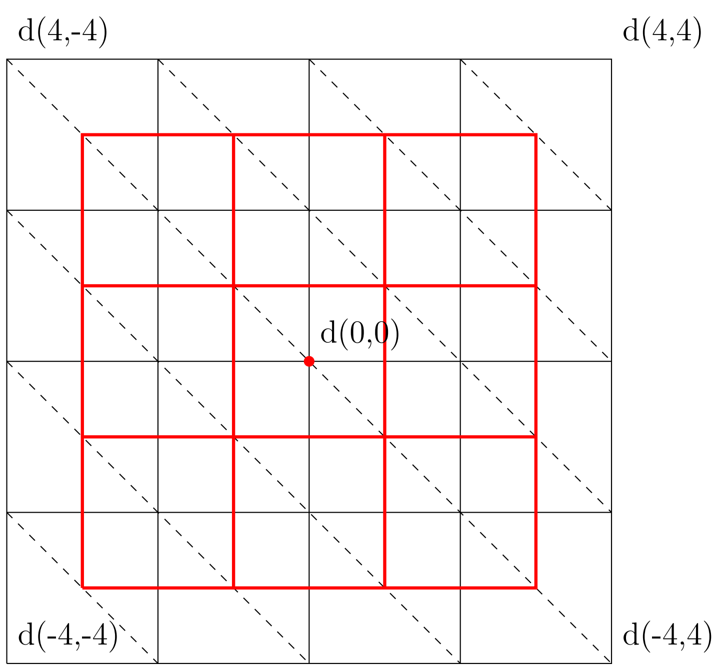


Figure 3.3: The first case of the weights $d(m,n)$

We define $d(m,n)$ as

$$d(m,n) = \int_{\infty}^{-\infty}\int_{\infty}^{-\infty} k_i' k_j' \psi(\hat{x}-i)\psi(\hat{y}-j)\hat{\varphi}_{m,n}(\hat{x},\hat{y})d\hat{x}d\hat{y},$$

where $\hat{\varphi}_{m,n}$ is a basis function of $\hat{P}_2$.
Example: to find $d(0,0)$, we use triangles that are connected to the point $d(0,0)$

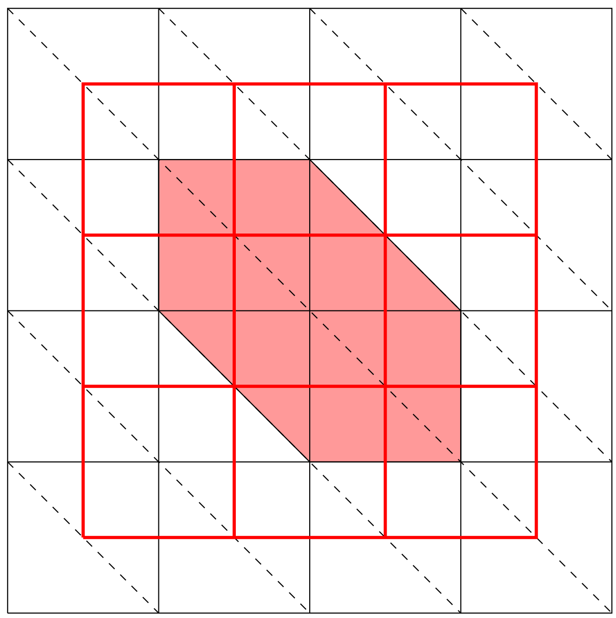


Figure 3.4: The area to find $d(0,0)$

Each part of triangle can be computed by mapping $\hat{\varphi}_{m,n}$ to basis functions $\hat{\varphi}_i$.

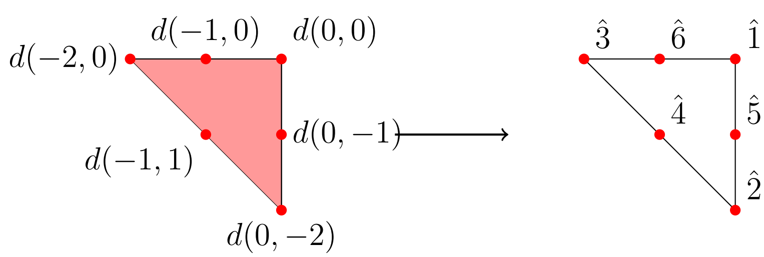


Figure 3.5: The mapping from $\hat{\varphi}_{m,n}$ to basis functions $\hat{\varphi}_i$

By Figure 3.4, we can see the support of $\hat{\varphi}_{0,0}$. We note that the sum of $d(m,n) = 1$.

3.2.2 Case 2

As we can see in Figure 3.6, we can compute $d(m,n)$ in the same method as case 1.

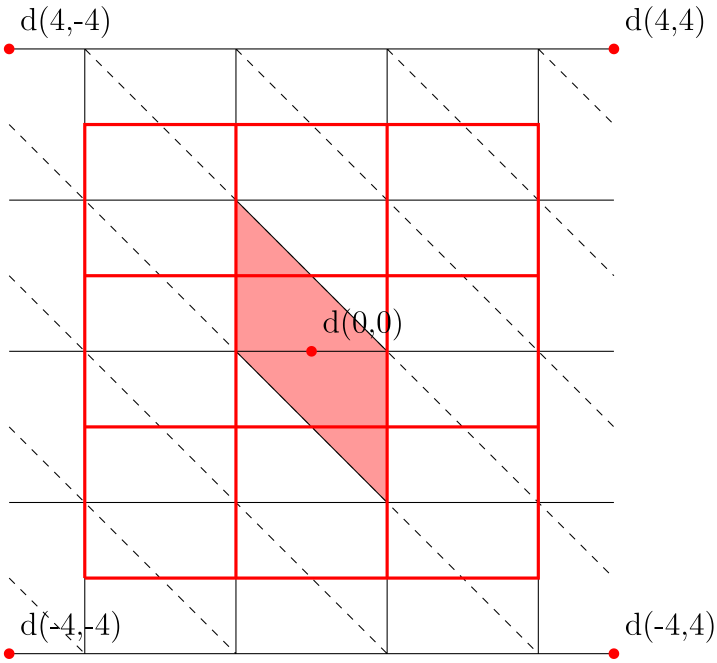


Figure 3.6: The second case of the weights $d(m,n)$ and show the support of $\hat{\varphi}_{0,0}$

3.2.3 Case 3

As we can see in Figure 3.7, we can compute $d(m,n)$ in the same method as case 1.

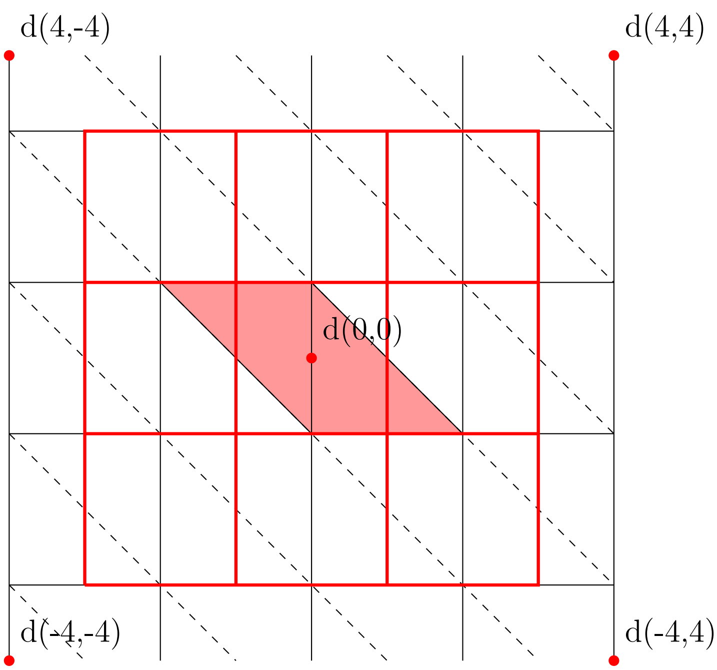


Figure 3.7: The third case of the weights $d(m,n)$ and show the support of $\hat{\varphi}_{0,0}$

3.2.4 Case 4

As we can see in Figure 3.8, we can compute $d(m, n)$ in the same method as case 1.

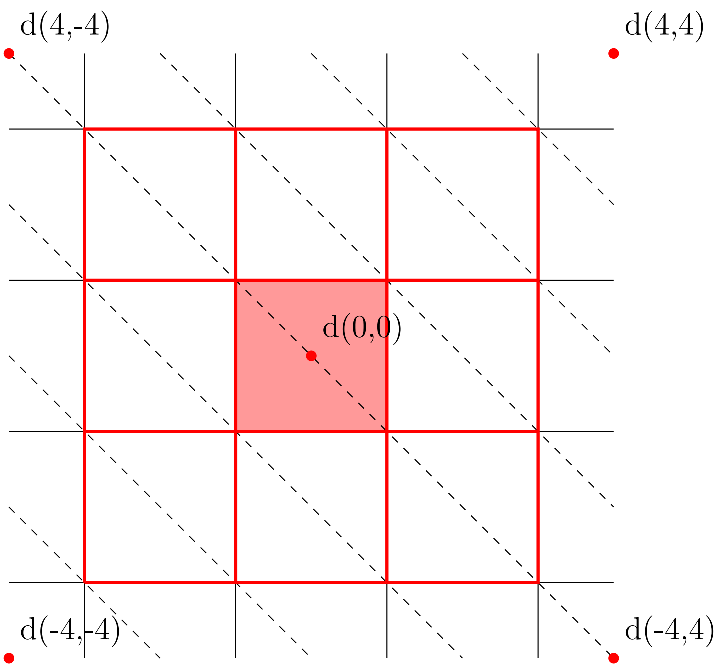


Figure 3.8: The fourth case of the weights $d(m, n)$ and show the support of $\hat{\varphi}_{0,0}$

3.3 Averaging Q_3

By the Table 3.1 k'_j of Bramble and Schatz [2], we can construct a coefficient graph of k'_j. In 2D, we have $r = 4$, our $k'_0 = 37/30$, $k'_1 = k'_{-1} = -23/180$ and $k'_2 = k'_{-2} = 1/90$ for defining $\psi_1^{(2)}$.

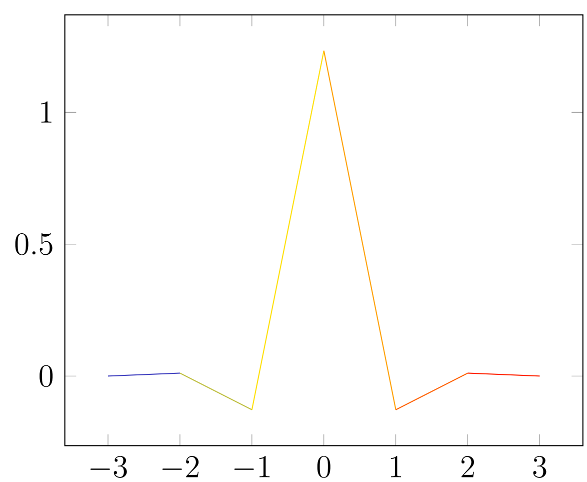


Figure 3.9: The graph of coefficients in $\sum_{j=-2}^{2} k'_j \psi_1^{(2)}(x - j)$

To compute the weight, we recall the basis function of Q_3.

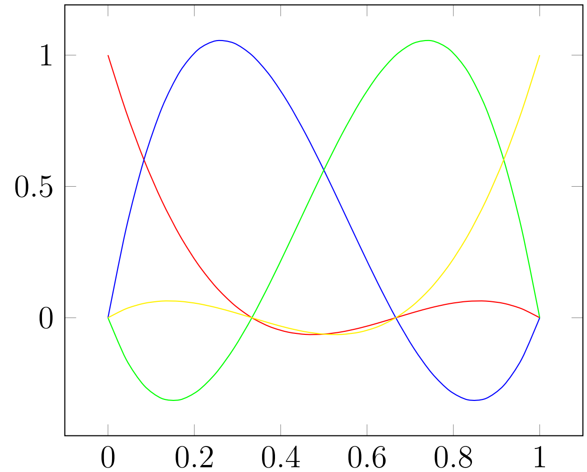


Figure 3.10: The basis function of Q_3

$$\hat{\varphi}_i(x) = \begin{cases} 9/2(1-x)(1/3-x)(2/3-x) & \text{i} = 1 \\ 27/2(x)(1-x)(2/3-x) & \text{i} = 2 \\ 27/2(x)(1-x)(x-1/3) & \text{i} = 3 \\ 9/2(x)(x-1/3)(x-2/3) & \text{i} = 4 \end{cases}$$

We consider the computation for 1D. Because of the square element of Q_3, we can find the weight in 2D by cross-product. We define the weight as

$$d_i = \int_{\mathbb{R}} k_i' \psi_1^{(2)}(\hat{x} - i) d\hat{x}$$

For example, in each element by the Figure 3.11, we show that $d(m, n)$ can be computed by cross-product. We map the node to the basis function of Q_3.

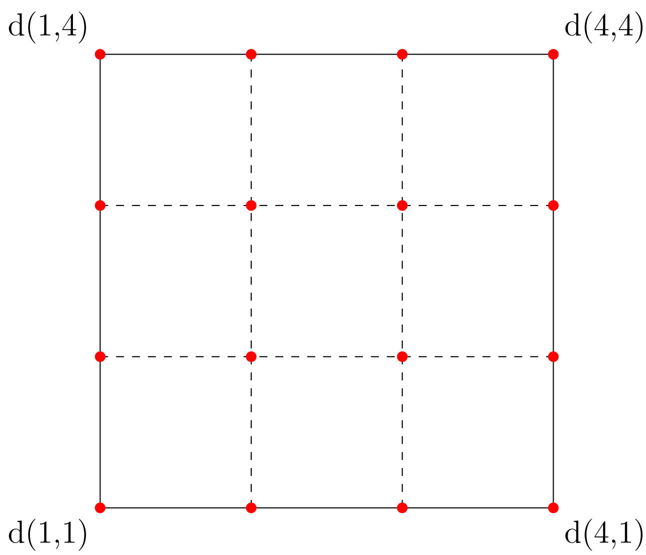


Figure 3.11: The weight of Q_3

Chapter 4

Numerical Experiments

In this section, we consider an example where the exact solution is given by

$$u = sin(\pi x)sin(\pi y)(x + \pi y)$$

We note that we can average the solution u_h to obtain $\widetilde{u}_h$ only at nodes which are not too close to the boundary.

4.1 Numerical experiments of P_1

The domain $\Omega = (0,1)^2$ and $h = \frac{1}{2^L}$. The Figures 4.1-4.5 are the finite element solutions with $L = 1, 2, \ldots, 5$ respectively and $f = -\Delta u$. By the Table 4.1, we denote N_1 to be a set of all nodes of P_1 and define the maximum error as

$$MaxErr = \max_{i \in N_1} |u_h(v_i) - u(v_i)|$$

where u_h is the FEM solution and u is the exact solution.

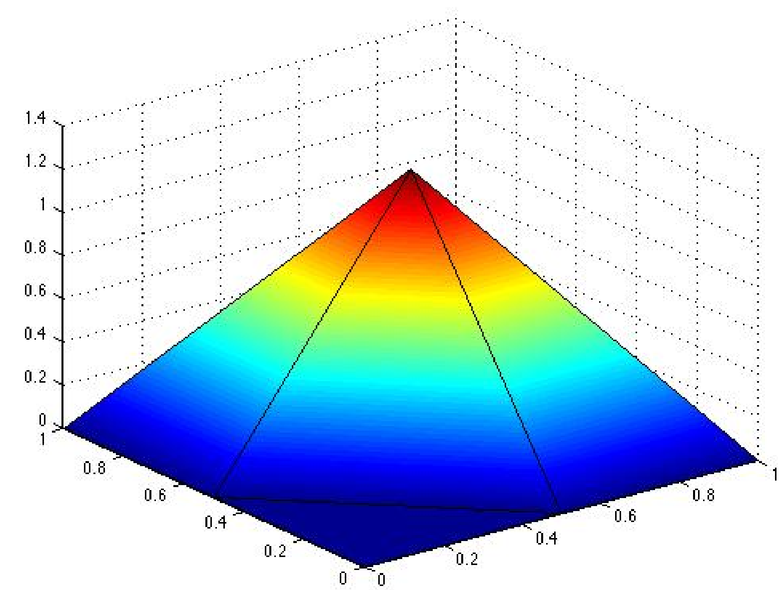


Figure 4.1: The solution of P_1 for $L = 1$. MaxErr = 0.7934.

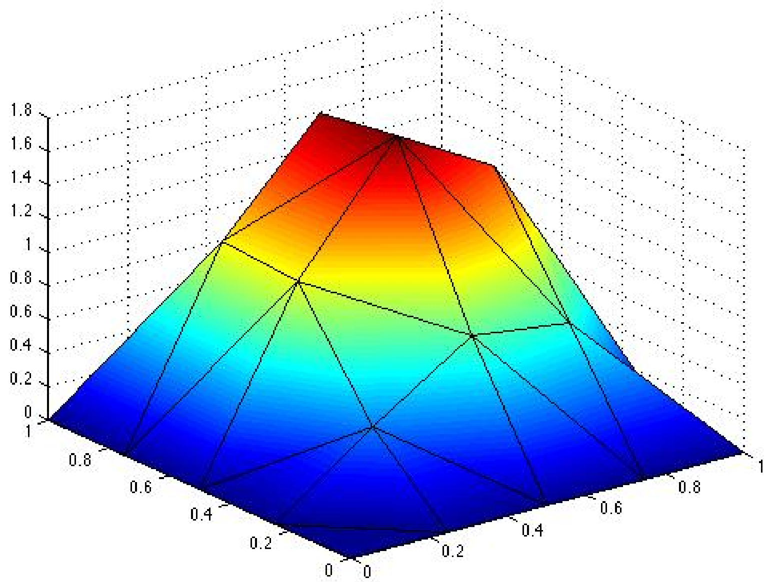


Figure 4.2: The solution of P_1 for $L = 2$. MaxErr = 0.3026.

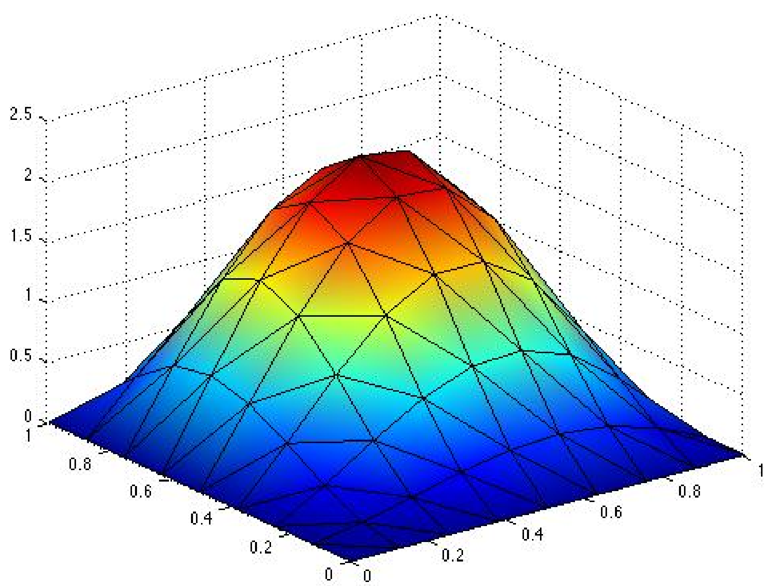


Figure 4.3: The solution of P_1 for $L = 3$. MaxErr = 0.09007

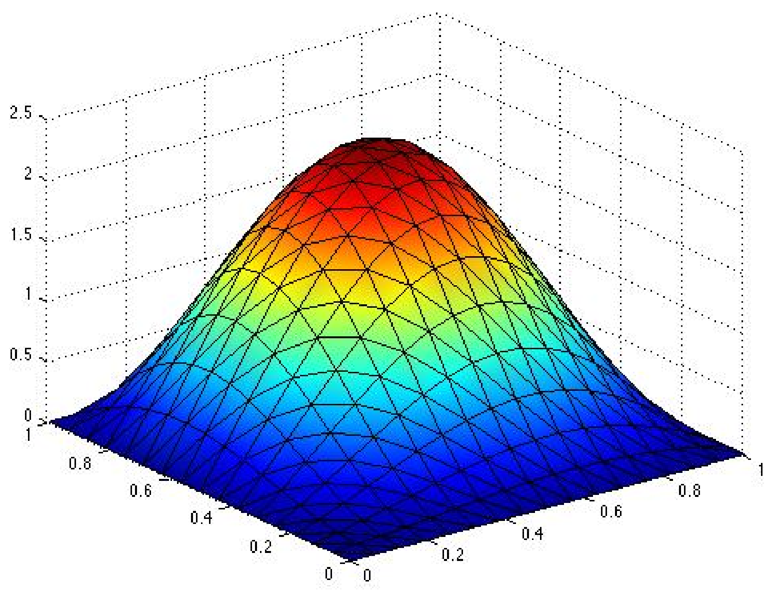


Figure 4.4: The solution of P_1 for $L = 4$. MaxErr $= 0.02304$

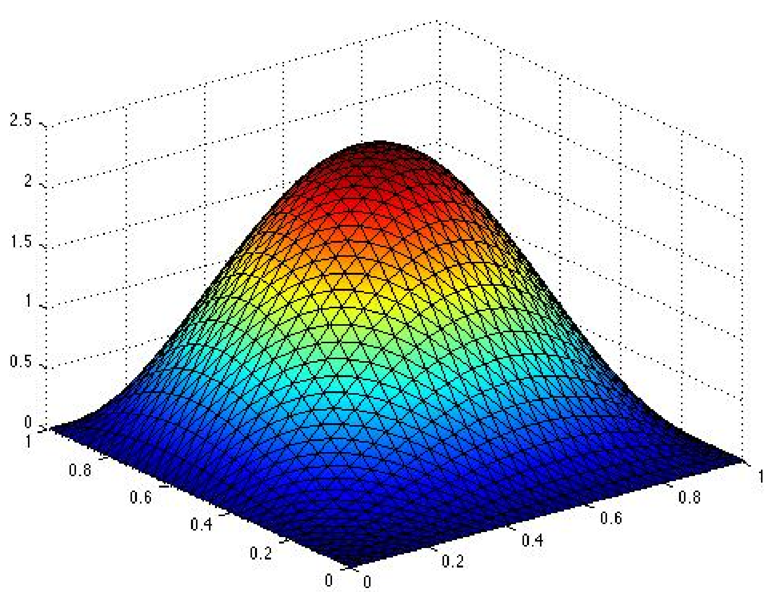


Figure 4.5: The solution of P_1 for $L = 5$. MaxErr $= 0.005834$

For the rate of convergence, we define that

$$Rate = \frac{MaxErr_{L-1}}{MaxErr_L}$$

$L \backslash Error$	$MaxErr$	$Rate$
1	0.7934	-
2	0.3026	2.622
3	0.09007	3.3596
4	0.02304	3.9093
5	0.005834	3.9493

Table 4.1: The error value of each L for P_1

4.2 Numerical experiments of P_2

For P_2, the domain $\Omega = (0,1)^2$ and $h = \frac{1}{2^L}$. The Figures 4.6-4.10 are the finite element solutions with $L = 1, 2, \ldots, 5$ respectively and $f = -\Delta u$. We denote N_2 to be a set of all nodes of P_2. The maximum error of P_2 is given by

$$MaxErr = \max_{i \in N_2} |u_h(v_i) - u(v_i)|$$

Since the averaging can be computed only the nodes that are not too close to the boundary, we denote $\widetilde{N}_2$ to be a set of the nodes of P_2 that can be averaged. By the Table 4.2, we define

$$MaxAvgErr = \max_{i \in \widetilde{N}_2} |\widetilde{u}_h(v_i) - u(v_i)|$$

$$MaxErrNode = \max_{i \in \widetilde{N}_2} |u_h(v_i) - u(v_i)|$$

where $\widetilde{u}_h$ is the solution by averaging

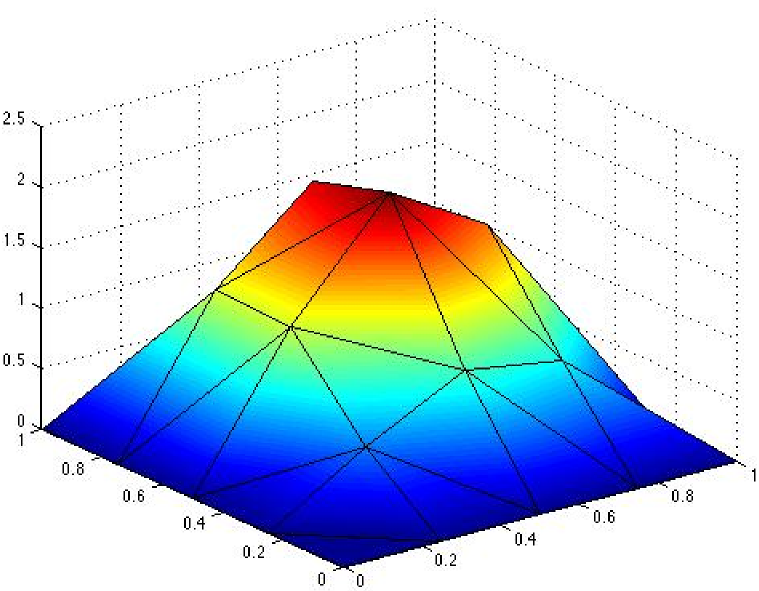


Figure 4.6: The solution of P_2 for $L = 1$. MaxErr = 0.1225.

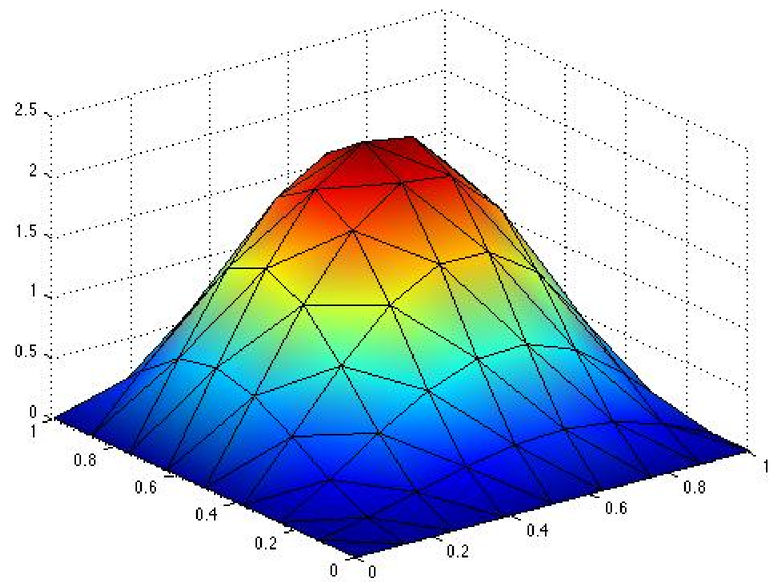


Figure 4.7: The solution of P_2 for $L = 2$. MaxErr = 0.001077.
MaxAvgErr = 0.02007. MaxErrNode = 0.002958.

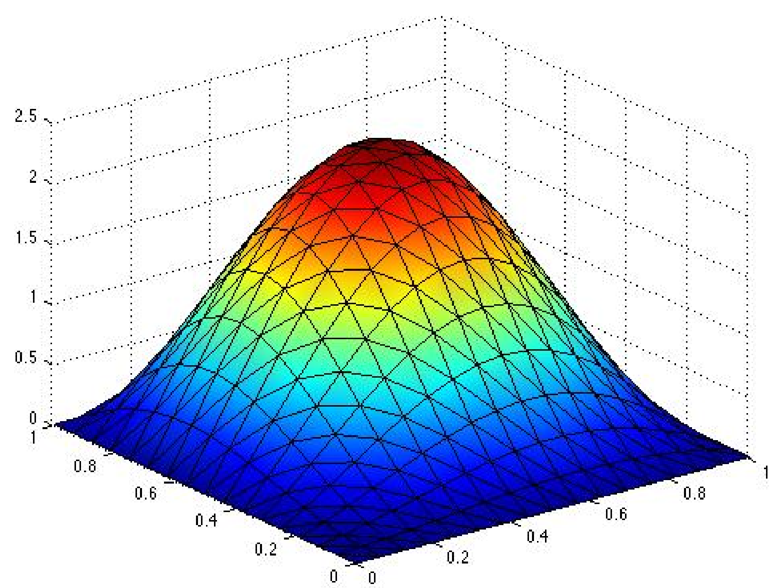


Figure 4.8: The solution of P_2 for $L = 3$. MaxErr = 8.1789e-4.
MaxAvgErr = 0.002054. MaxErrNode = 8.1789e-4.

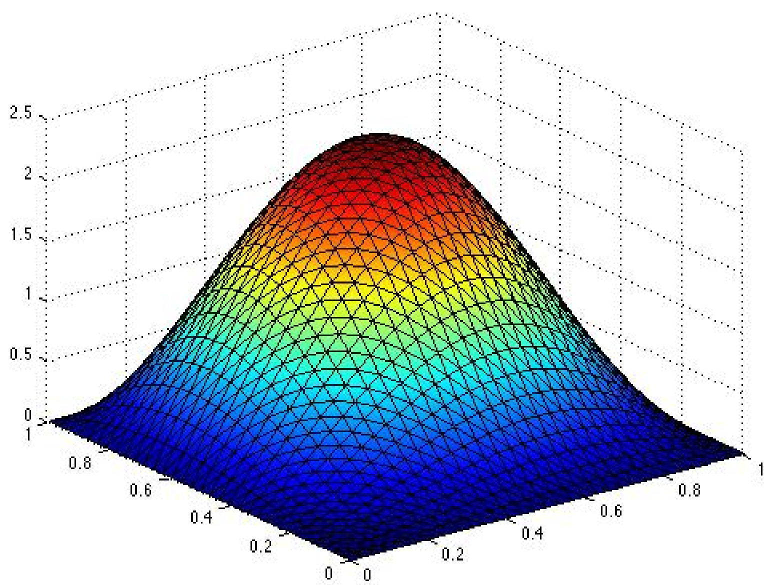


Figure 4.9: The solution of P_2 for $L = 4$. MaxErr = 5.2655e-5.
MaxAvgErr = 1.3146e-4. MaxErrNode = 5.2655e-5.

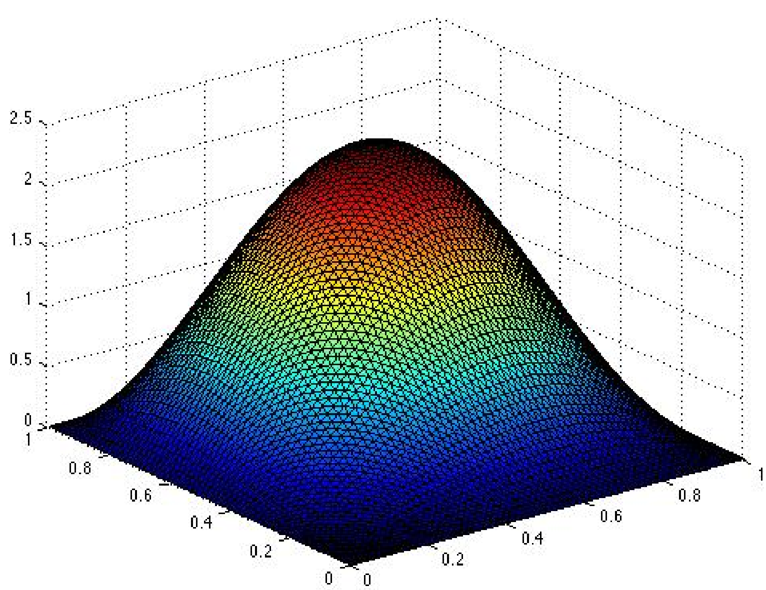


Figure 4.10: The solution of P_2 for $L = 5$. MaxErr = 3.3291e-6.
MaxAvgErr = 8.2886e-6. MaxErrNode = 3.3291e-6.

For the rate of convergence, we define that

$$RateAvg = \frac{MaxAvgErr_{L-1}}{MaxAvgErr_L}$$

$$RateNode = \frac{MaxErrNode_{L-1}}{MaxErrNode_L}$$

$L\backslash Error$	$MaxErr$	$Rate$	$MaxAvgErr$	$RateAvg$	$MaxErrNode$	$RateNode$
1	0.1225	-	-	-	-	-
2	0.01077	11.374	0.02007	-	0.002958	-
3	8.1789e-4	13.168	0.002054	9.771	8.1789e-4	3.6166
4	5.2655e-5	15.533	1.3146e-4	15.624	5.2655e-5	15.533
5	3.3291e-6	15.816	8.2886e-6	15.860	3.3291e-6	15.816

Table 4.2: The error value of each L for P_2

4.3 Numerical experiments of Q_3

For Q_3, the domain $\Omega = (0,1)^2$ and $h = \frac{1}{2^L}$. The Figure 4.11-4.15 are the finite element solutions with $L = 1, 2, \ldots, 5$ respectively and $f = -\Delta u$. We denote N_3 to be a set of all nodes of Q_3. The maximum error of Q_3 is given by

$$MaxErr = \max_{i \in N_3} |u_h(v_i) - u(v_i)|$$

we denote $\widetilde{N}_3$ to be a set of the nodes of Q_3 that can be averaged. By the Table 4.3, we define the maximum error of $\widetilde{N}_3$ nodes

$$MaxAvgErr = \max_{i \in \widetilde{N}_3} |\widetilde{u}_h(v_i) - u(v_i)|$$

$$MaxErrNode = \max_{i \in \widetilde{N}_3} |u_h(v_i) - u(v_i)|$$

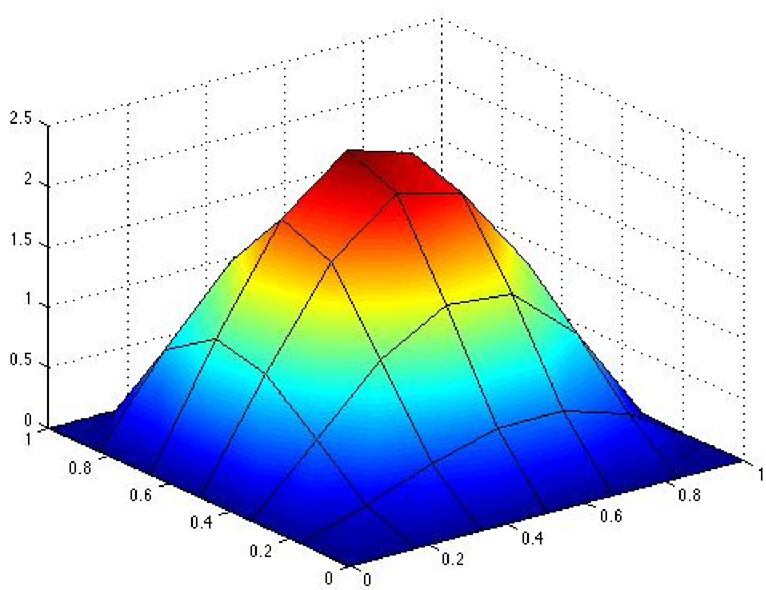


Figure 4.11: The solution of Q_3 for $L = 1$. MaxErr = 0.007107.

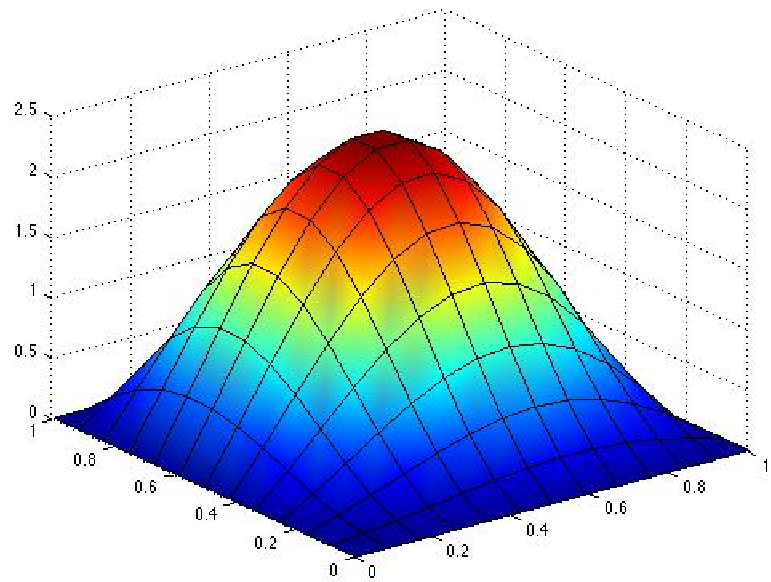


Figure 4.12: The solution of Q_3 for $L = 2$. MaxErr = 5.00002522e-4.

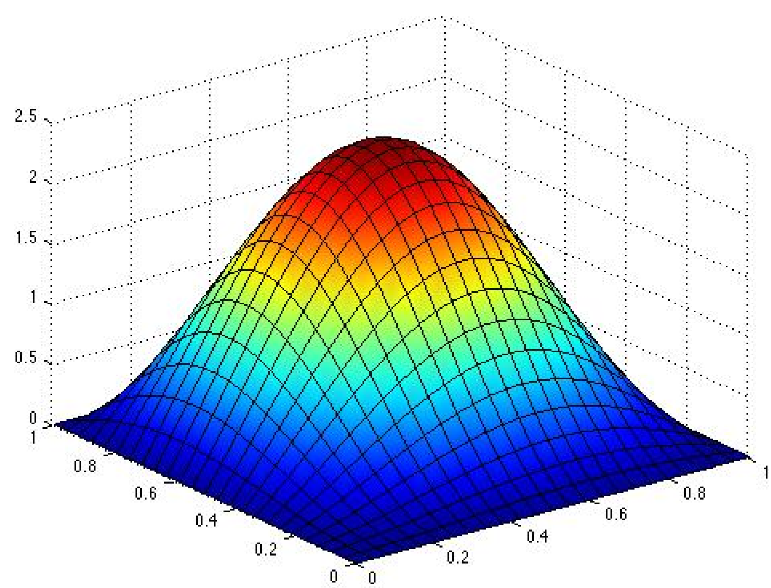


Figure 4.13: The solution of Q_3 for $L = 3$. MaxErr = 3.3568e-5.
MaxAvgErr = 4.5854e-5. MaxErrNode = 2.7828e-5.

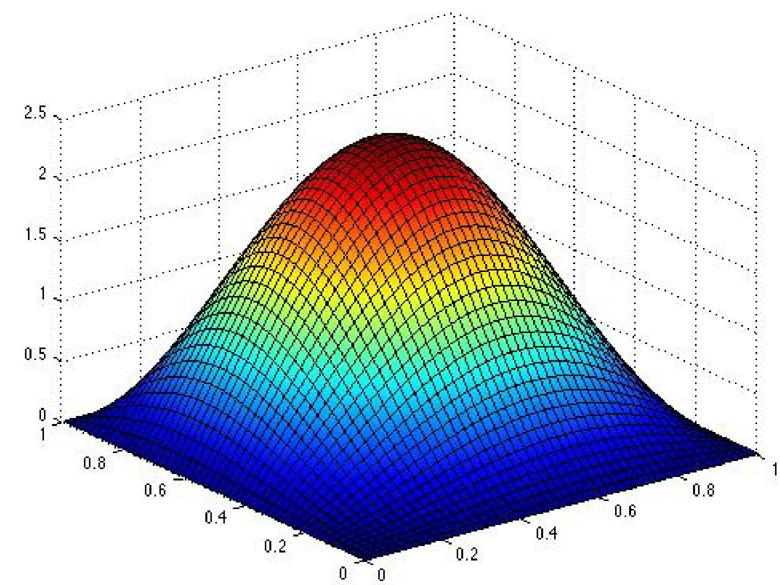


Figure 4.14: The solution of Q_3 for $L = 4$. MaxErr = 2.1258e-6. MaxAvgErr = 8.7115e-7. MaxErrNode = 2.1246e-6.

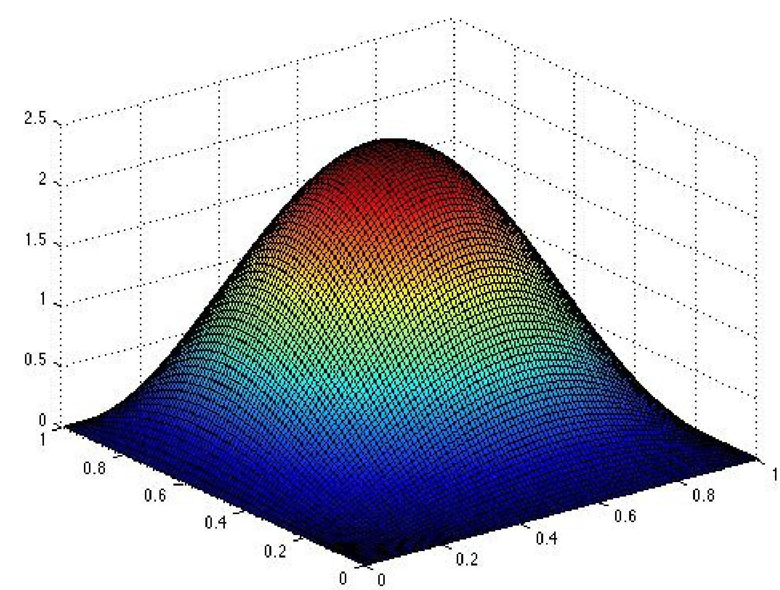


Figure 4.15: The solution of Q_3 for $L = 5$. MaxErr = 1.3059e-7. MaxAvgErr = 1.0924e-8. MaxErrNode = 1.30589e-7.

The rate of convergence is given by

$$RateAvg = \frac{MaxAvgErr_{L-1}}{MaxAvgErr_L}$$

$$RateNode = \frac{MaxErrNode_{L-1}}{MaxErrNode_L}$$

$L\backslash Error$	$MaxErr$	$Rate$	$MaxAvgErr$	$RateAvg$	$MaxErrNode$	$RateNode$
1	0.007107	-	-	-	-	-
2	5.000e-4	14.214	-	-	-	-
3	3.3568e-5	14.895	4.5854e-5	-	2.7828e-5	-
4	2.1258e-6	15.791	8.7115e-7	52.636	2.1246e-6	13.098
5	1.3059e-7	16.278	1.0924e-8	79.746	1.30589e-7	16.269

Table 4.3: The error value of each L for Q_3

Chapter 5

Conclusion

In this Chapter, we conclude our result of the errors in each refinement. In P_1, by the Table 4.1, we have the rate of convergence approximate to 4. Next, we compare the rate of convergence of P_2. The rate of convergence is 16 by the Table 4.2. That is, in P_2, the FEM generates a better solution at some nodes due to super-convergence for structured mesh. In the corollary of Bramble and Schatz states that let $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$ and $N_0 = N/2 + 1$, at points $h\gamma \in \Omega_0$ and $\gamma \in Z^2$

$$\sup_{h\gamma\in\Omega_0 \gamma\in Z^2} \left| u(h\gamma) - \sum_{\alpha} a^j_{\gamma-\alpha} d^j_{\alpha} \right| \leq C(h^{2r-2} \|u\|_{2r-2+N_0,\Omega_1})$$

In P_2, we have $r = 3$ so that the rate is h^4 which is the same result from our FEM. And it does not give us a better result. Therefore, we introduce the Q_3 refinement of FEM. By the Table 4.3, the rate of convergence is approximated to 16 and the rate of convergence by averaging is approximated to 64. As Bramble and Schatz's theorem, we have $r = 4$ which give us h^6 as same as the rate of convergence that we have from the averaged Q_3 so that the averaging method shows that our test has more accuracy. Furthermore, we can expect that for Q_4 or Q_5 can give us a better result by using FEM solution.

Bibliography

[1] Sayas Francisco-Javier. *A gentle introduction to the Finite Element Method*. 2008.

[2] J.H. Bramble, A.H. Schatz. *Higher order local accuracy by averaging in the finite element method*. Math. Comp., 31:94-111,1977.

[3] Braess, Dietrich. *Finite Elements Theory, Fast Solvers, and Applications in Elasticity Theory*. 3rd ed. Cambridge: Cambridge UP, 2007. Print