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Higher-Order Compact Finite Element Method

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Higher-Order Compact Finite Element Method

Major Qualifying Project
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Abstract

The Finite Element Method (FEM) is a scheme that can approximate the solution of boundary value problems. We study the fundamentals of FEM and construct a MATLAB code to approximate the error of the solution for each refinement and compute the rate of convergence of this discretization. Then, we study the article of Bramble and Schatz [2] to construct MATLAB code to approximate better solution by averaging the FEM’s solution.
Acknowledgement

I want to thank Professor Marcus Sarkis for his guidance, his continual support and his teaching. I would like to thank my friend, Thanacha Pi Choopojcharoen, for his guidance in MATLAB. And I want to thank my mother for her support all four terms.
3.3  Averaging $Q_3$  

4  Numerical Experiments
  4.1  Numerical experiments of $P_1$  
  4.2  Numerical experiments of $P_2$  
  4.3  Numerical experiments of $Q_3$

5  Conclusion
Chapter 1

Introduction

The Finite Element Method is a scheme that can approximate the solution of boundary value problem described by a partial differential equation (PDE). In this paper, we focus on polygonal $\mathbb{R}^2$ domain. To compute the solution, we divide our domain into many subdomains called finite element. The simplest elements to compute are triangle and square, and they are the ones we consider here. Then we construct stiffness matrices associated to the Poisson problem.

1.1 The Model Problem

In our model problem, we start with definition of our domain and boundaries. Then, we introduce basis function to solve the Poisson problem.

1.1.1 Domain and Boundary

Denote the polygon domain in $\mathbb{R}^2$ by $\Omega$. Two boundary conditions, Dirichlet $\Gamma_D$ and Neumann $\Gamma_N$.

![Figure 1.1: Domain $\Omega$](image)

- $\Gamma_D$ is the Dirichlet boundary where the solution is given.
• $\Gamma_N$ is the Neumann boundary where the normal derivative is given

### 1.1.2 Boundary Value Problem

Consider the Laplace operator, or Laplacian

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

The boundary value problem is

$$-\Delta u + cu = f \quad \text{in} \, \Omega$$

$$u = g_0 \quad \text{on} \, \Gamma_D$$

$$\partial_n u = g_1 \quad \text{on} \, \Gamma_N$$

- $u$ is an unknown function defined on the domain $\Omega$
- $c$ is a non-negative constant value. In this paper, we set $c = 0$
- $f$ is a given function on $\Omega$
- $g_0 \, , \, g_1$ are given on two different parts of boundary

### 1.1.3 Weak form of FEM

We now use the weak form of FEM. To get start, we introduce the Green’s Theorem. The theorem states that

$$\int_{\Omega} (\Delta u)v + \int_{\Omega} \nabla u \cdot \nabla v = \int_{\Gamma} (\partial_n u)v$$

Our boundary contains with Dirichlet and Neumann

$$\int_{\Omega} (\Delta u)v + \int_{\Omega} \nabla u \cdot \nabla v = \int_{\Gamma_N} (\partial_n u)v + \int_{\Gamma_D} (\partial_n u)v$$

Since $c = 0$ so that $-\Delta u = f$

$$\int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} f v + \int_{\Gamma_N} g_1 v + \int_{\Gamma_D} (\partial_n u)v$$

Since the value on $\Gamma_D$ is given, we set $v = 0$ on $\Gamma_D$

Therefore, we have a new formula of our problem

$$\int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} f v + \int_{\Gamma_N} g_1 v$$

In this project we assume that $\Gamma_N = \emptyset$, that is, the boundary of $\Omega$ is all Dirichlet boundary.

$$\int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} f v$$
1.2 Triangulation

As we mentioned in domain and boundary, we have to divide our domain into subdomains. We focus on linear and quadratic function of two variables $P_1$ and $P_2$.

![Subdomains of Ω](image)

Figure 1.2: Subdomains of Ω

1.2.1 Linear and Quadratic triangular

Define $P_1$: Linear Triangular and $P_2$: Quadratic Triangular

A linear function of two variables is

$$p(x, y) = a_i + b_i x + c_i y$$

$$p \in P_1 = \{a_i + b_i x + c_i y \mid a_i, b_i, c_i \in \mathbb{R}\}$$

A quadratic function of two variables is

$$p(x, y) = a_i + b_i x + c_i y + d_i xy + e_i x^2 + f_i y^2$$

$$p \in P_2 = \{a_i + b_i x + c_i y + d_i xy + e_i x^2 + f_i y^2 \mid a_i, b_i, c_i, d_i, e_i, f_i \in \mathbb{R}\}$$

1.2.2 The Reference Triangle

Given a subdomain $K$ in $\Omega$, define

- $F_k$ is a function mapping $\hat{K} \rightarrow K$
- $K$ is a triangle on $\Omega$
- $\hat{K}$ is a reference triangle with $\hat{v}_1=(0,0)$, $\hat{v}_2=(1,0)$, and $\hat{v}_3=(0,1)$
For any point \((x,y)\) in \(K\), \((x,y) = F_k(\hat{x}, \hat{y})\)

\[
\begin{pmatrix}
  x \\
y
\end{pmatrix} = F_k \begin{pmatrix}
  \hat{x} \\
\hat{y}
\end{pmatrix}
\]

\[
\begin{pmatrix}
  x \\
y
\end{pmatrix} = \begin{pmatrix}
  x_2 - x_1 & x_3 - x_1 \\
y_2 - y_1 & y_3 - y_1
\end{pmatrix} \begin{pmatrix}
  \hat{x} \\
\hat{y}
\end{pmatrix} + \begin{pmatrix}
  x_1 \\
y_1
\end{pmatrix}
\]

Define \(B_k = \begin{pmatrix}
  x_2 - x_1 & x_3 - x_1 \\
y_2 - y_1 & y_3 - y_1
\end{pmatrix}\)

\[
\begin{pmatrix}
  x \\
y
\end{pmatrix} = \begin{pmatrix}
  x_1 \\
y_1
\end{pmatrix} (1 - \hat{x} - \hat{y}) + \begin{pmatrix}
  x_2 \\
y_2
\end{pmatrix} \hat{x} + \begin{pmatrix}
  x_3 \\
y_3
\end{pmatrix} \hat{y}
\]

The area of the triangle \(K = |\text{det}B_k|/2\)

### 1.2.3 Basis function

The polynomials are defined on each triangle in the mesh, which form a set of basis functions for the triangular element. Each basis function has a value of 1 at its node and a value of 0 at the other nodes giving it a pyramid shape over the triangle. Define \(v_h\) be a piecewise linear and continuous function on \(\Omega\) so we can also define a basis function

\[
\varphi_i(x_i) = \delta_{ij} = \begin{cases}
  0 & i \neq j \\
  1 & i = j
\end{cases}
\]
It is easy to see that

\[ w_h(x) = \sum_{i \in \mathcal{N}} w_h(v_i)\varphi_i(x) \]

where \( \mathcal{N} \) is the index set of interior nodes, since we consider \( w_h(x) = 0 \) for \( x \in \partial \Omega \).

### 1.3 Mesh Grid

Our domain is a square 1-by-1 to be easier to compute. We refine our domain into mesh grid that we choose \( L = 1, 2, 3, \ldots \). Here is the mesh grid for \( L = 1 \).
1.3.1 Local vertices

To order the number of vertices, we have to choose the direction clockwise or counter-clockwise. In this report, we use counter-clockwise for $P_1$. In $P_2$, three vertices in the middle of each edges are ordered in opposite of vertices.

For example, we map vertices of triangle in global to local by

Figure 1.6: The global vertices for $P_2$ for $L = 1$

Figure 1.7: The ordering of vertices for $P_1$

Figure 1.8: The ordering of vertices for $P_2$

Figure 1.9: Example of mapping vertices
1.3.2 Elements

To compute each element, we order our element at the figure below.

\begin{center}
\begin{tabular}{|c|c|}
\hline
5 & 6 \\
\hline
2 & 7 \\
\hline
1 & 8 \\
\hline
\end{tabular}
\end{center}

Figure 1.10: The ordering of element for $L = 1$

By using the ordering of vertices above, we get the ordering of vertices in each element.

\begin{center}
\begin{tabular}{|c|c|c|c|}
\hline
Element & $v_1$ & $v_2$ & $v_3$ \\
\hline
1 & 1 & 2 & 4 \\
2 & 5 & 4 & 2 \\
3 & 2 & 3 & 5 \\
4 & 6 & 5 & 3 \\
5 & 7 & 5 & 6 \\
6 & 8 & 7 & 5 \\
7 & 8 & 6 & 6 \\
\hline
\end{tabular}
\end{center}

Table 1.1: The order of vertices for $P_1$

\begin{center}
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
Element & $v_1$ & $v_2$ & $v_3$ & $v_4$ & $v_5$ & $v_6$ \\
\hline
1 & 1 & 3 & 11 & 7 & 6 & 2 \\
2 & 13 & 11 & 3 & 7 & 8 & 12 \\
3 & 3 & 5 & 13 & 9 & 8 & 4 \\
4 & 15 & 13 & 5 & 9 & 10 & 14 \\
5 & 11 & 13 & 21 & 17 & 16 & 12 \\
6 & 23 & 21 & 13 & 17 & 18 & 22 \\
7 & 13 & 15 & 23 & 19 & 18 & 14 \\
8 & 25 & 23 & 15 & 19 & 20 & 24 \\
\hline
\end{tabular}
\end{center}

Table 1.2: The order of vertices for $P_2$
1.3.3 Dirichlet boundary

As we mentioned above, we are focusing only zero Dirichlet boundary condition, so all the vertices are zero at the boundary. However, we can adapt to have Neumann boundary if we want to. Here is the ordering of Dirichlet global boundary nodes and the Tables 1.3 and 1.4 of the ordering of vertices

![Figure 1.11: The ordering of Dirichlet boundaries for $L = 1$](image)

<table>
<thead>
<tr>
<th>Dirichlet</th>
<th>$v_1$</th>
<th>$v_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
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<td>3</td>
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<td>7</td>
<td>4</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1.3: The ordering of vertices for $P_1$

<table>
<thead>
<tr>
<th>Dirichlet</th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
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<td>4</td>
<td>5</td>
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<tr>
<td>8</td>
<td>11</td>
<td>6</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1.4: The ordering of vertices for $P_2$
1.4 $Q_3$

We introduce new element which is easy to compute for square domain. It is a square element. As we define our mesh grid, coordinate, element, and Dirichlet boundary above. We do the same thing in our $Q_3$. Here is the global ordering of vertices for $L = 1$

![Figure 1.12: The ordering of vertices for $Q_3$](image)

1.4.1 Local vertices

We also define the order of each element in the same way. Here is the example of first element for $L = 1$

![Figure 1.13: The ordering of vertices for $Q_3$](image)

1.4.2 Element

Now our element is a square with four points on boundary of each element.
1.4.3 Dirichlet boundary

For the Dirichlet boundary, we also order in counter-clockwise.

Table 1.5: The ordering of vertices for $Q_3$

<table>
<thead>
<tr>
<th>Element</th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
<th>$v_4$</th>
<th>$v_5$</th>
<th>$v_6$</th>
<th>$v_7$</th>
<th>$v_8$</th>
<th>$v_9$</th>
<th>$v_{10}$</th>
<th>$v_{11}$</th>
<th>$v_{12}$</th>
<th>$v_{13}$</th>
<th>$v_{14}$</th>
<th>$v_{15}$</th>
<th>$v_{16}$</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
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<td>47</td>
<td>48</td>
<td>49</td>
</tr>
</tbody>
</table>

Figure 1.15: The ordering of Dirichlet boundaries for $L = 1$ for $Q_3$
Table 1.6: The ordering of vertices for $Q_3$

<table>
<thead>
<tr>
<th>Element</th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
<th>$v_4$</th>
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Chapter 2

Finite Element Method

In this section, we decide to set all the boundary to be zero Dirichlet. Therefore, we get: Find $u \in V_h(\Omega)$ such that

$$\int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} f v \quad \forall v \in V_h(\Omega),$$

where $V_h(\Omega)$ is the space of piecewise linear and continuous functions which are vanish on $\partial \Omega$. We study this equation in two parts, left-hand side and right-hand side.

2.1 Left-hand side

We denote the $i, j$ position of the stiffness matrix by

$$\int_{\Omega} \nabla \varphi_i \cdot \nabla \varphi_j$$

Recall our solution $u$ is of the form

$$u = \sum_{j \in N} u_j \varphi_j$$

then substitute $\nabla u$

$$\int_{\Omega} \nabla u \cdot \nabla \varphi_i = \sum_{j \in N} u_j \int_{\Omega} \nabla \varphi_j \cdot \nabla \varphi_i$$

We compute element $K$ in $\Omega$

$$\int_K \nabla \varphi_i^K \cdot \nabla \varphi_j^K = |det B_K| \int_K (\nabla \varphi_i^K \circ F_K) \cdot (\nabla \varphi_j^K \circ F_K)$$

Then, we use a change of variable to obtain

$$\int_K \nabla \varphi_i^K \cdot \nabla \varphi_j^K = |det B_K| \int_K (B_K^{-T} \hat{\nabla} \hat{\varphi}_i) \cdot (B_K^{-T} \hat{\nabla} \hat{\varphi}_j)$$
Therefore, the entries of the stiffness matrix is given by

\[
\int_{K} \nabla \varphi_i^K \cdot \nabla \varphi_j^K = |\text{det} B_K| \int_{K} C_K \nabla \hat{\varphi}_i \cdot \nabla \hat{\varphi}_j
\]

where

\[
C_K = B_K^{-1} B_K^{-T} = \begin{pmatrix} c_{11}^K & c_{12}^K \\ c_{21}^K & c_{22}^K \end{pmatrix}
\]

### 2.1.1 Hat functions of \( P_1 \)

The basis function on \( \hat{K} \) is of the form:

\[
\hat{\varphi}_1(x, y) = a_1 + b_1 x + c_1 y
\]

Each \( \hat{\varphi}_i \) has three unknown coefficients \( a_i, b_i, c_i \). For instance, for \( \hat{\varphi}_1 \) we want \( \hat{\varphi}_1(0, 0) = 1 \), \( \hat{\varphi}_1(1, 0) = 0 \), and \( \hat{\varphi}_1(0, 1) = 0 \). So, this implies that

\[
\begin{align*}
1 + b_1 \hat{x}_1 + c_1 \hat{y}_1 &= 1 \\
1 + b_1 \hat{x}_2 + c_1 \hat{y}_2 &= 0 \\
1 + b_1 \hat{x}_3 + c_1 \hat{y}_3 &= 0
\end{align*}
\]

so we can write it in matrix form as

\[
\begin{pmatrix}
1 & \hat{x}_1 & \hat{y}_1 \\
1 & \hat{x}_2 & \hat{y}_2 \\
1 & \hat{x}_3 & \hat{y}_3
\end{pmatrix}
\begin{pmatrix}
a_1 \\
b_1 \\
c_1
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
\]

For all the three basis function in triangle, we get the equation of the form

\[
\begin{pmatrix}
1 & \hat{x}_1 & \hat{y}_1 \\
1 & \hat{x}_2 & \hat{y}_2 \\
1 & \hat{x}_3 & \hat{y}_3
\end{pmatrix}
\begin{pmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

Solve for coefficient matrix, we get each \( \hat{\varphi}_i \) in the form

\[
\begin{align*}
\hat{\varphi}_1 &= 1 - \hat{x} - \hat{y} \\
\hat{\varphi}_2 &= \hat{x} \\
\hat{\varphi}_3 &= \hat{y}
\end{align*}
\]
2.1.2 Hat functions of $P_2$

The basis function on $\hat{K}$ is of the form:

$$\hat{\phi}_i(\hat{x}, \hat{y}) = a_1 + b_1\hat{x} + c_1\hat{y} + d_1\hat{x}\hat{y} + e_1\hat{x}^2 + f_1\hat{y}^2$$

Each $\hat{\phi}_i$ has three unknown coefficients $a_i, b_i, c_i$. For instance, for $\hat{\phi}_1$ we want $\hat{\phi}_1(0, 0) = 1$, $\hat{\phi}_1(1, 0) = 0$, $\hat{\phi}_1(0, 1) = 0$, $\hat{\phi}_1(1/2, 1/2) = 0$, $\hat{\phi}_1(0, 1/2) = 0$, and $\hat{\phi}_1(1/2, 0) = 0$. So, this implies that

$$a_1 + b_1\hat{x}_1 + c_1\hat{y}_1 + d_1\hat{x}_1\hat{y}_1 + e_1\hat{x}_1^2 + f_1\hat{y}_1^2 = 1$$
$$a_1 + b_1\hat{x}_2 + c_1\hat{y}_2 + d_1\hat{x}_2\hat{y}_2 + e_1\hat{x}_2^2 + f_1\hat{y}_2^2 = 0$$
$$a_1 + b_1\hat{x}_3 + c_1\hat{y}_3 + d_1\hat{x}_3\hat{y}_3 + e_1\hat{x}_3^2 + f_1\hat{y}_3^2 = 0$$
$$a_1 + b_1\hat{x}_4 + c_1\hat{y}_4 + d_1\hat{x}_4\hat{y}_4 + e_1\hat{x}_4^2 + f_1\hat{y}_4^2 = 0$$
$$a_1 + b_1\hat{x}_5 + c_1\hat{y}_5 + d_1\hat{x}_5\hat{y}_5 + e_1\hat{x}_5^2 + f_1\hat{y}_5^2 = 0$$
$$a_1 + b_1\hat{x}_6 + c_1\hat{y}_6 + d_1\hat{x}_6\hat{y}_6 + e_1\hat{x}_6^2 + f_1\hat{y}_6^2 = 0$$

so we can write it in matrix form as

$$\begin{pmatrix}
1 & \hat{x}_1 & \hat{y}_1 & \hat{x}_1\hat{y}_1 & \hat{x}_1^2 & \hat{y}_1^2 \\
1 & \hat{x}_2 & \hat{y}_2 & \hat{x}_2\hat{y}_2 & \hat{x}_2^2 & \hat{y}_2^2 \\
1 & \hat{x}_3 & \hat{y}_3 & \hat{x}_3\hat{y}_3 & \hat{x}_3^2 & \hat{y}_3^2 \\
1 & \hat{x}_4 & \hat{y}_4 & \hat{x}_4\hat{y}_4 & \hat{x}_4^2 & \hat{y}_4^2 \\
1 & \hat{x}_5 & \hat{y}_5 & \hat{x}_5\hat{y}_5 & \hat{x}_5^2 & \hat{y}_5^2 \\
1 & \hat{x}_6 & \hat{y}_6 & \hat{x}_6\hat{y}_6 & \hat{x}_6^2 & \hat{y}_6^2 \\
\end{pmatrix}
\begin{pmatrix}
a_1 \\
b_1 \\
c_1 \\
d_1 \\
e_1 \\
f_1 \\
\end{pmatrix}
= \begin{pmatrix}1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

For all the six basis function in triangle, we get the equation of the form

$$\begin{pmatrix}
1 & \hat{x}_1 & \hat{y}_1 & \hat{x}_1\hat{y}_1 & \hat{x}_1^2 & \hat{y}_1^2 \\
1 & \hat{x}_2 & \hat{y}_2 & \hat{x}_2\hat{y}_2 & \hat{x}_2^2 & \hat{y}_2^2 \\
1 & \hat{x}_3 & \hat{y}_3 & \hat{x}_3\hat{y}_3 & \hat{x}_3^2 & \hat{y}_3^2 \\
1 & \hat{x}_4 & \hat{y}_4 & \hat{x}_4\hat{y}_4 & \hat{x}_4^2 & \hat{y}_4^2 \\
1 & \hat{x}_5 & \hat{y}_5 & \hat{x}_5\hat{y}_5 & \hat{x}_5^2 & \hat{y}_5^2 \\
1 & \hat{x}_6 & \hat{y}_6 & \hat{x}_6\hat{y}_6 & \hat{x}_6^2 & \hat{y}_6^2 \\
\end{pmatrix}
\begin{pmatrix}
a_1 \\
b_1 \\
c_1 \\
d_1 \\
e_1 \\
f_1 \\
\end{pmatrix}
= \begin{pmatrix}0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solve for coefficient matrix, we get each $\hat{\phi}_i$ in the form

$$\hat{\phi}_1 = (1 - \hat{x} - \hat{y})(1 - 2\hat{x} - 2\hat{y})$$
$$\hat{\phi}_2 = \hat{x}(2\hat{x} - 1)$$
$$\hat{\phi}_3 = \hat{y}(2\hat{y} - 1)$$
$$\hat{\phi}_4 = 4\hat{x}\hat{y}$$
$$\hat{\phi}_5 = 4\hat{x}(1 - \hat{x} - \hat{y})$$
$$\hat{\phi}_6 = 4\hat{y}(1 - \hat{x} - \hat{y})$$
2.1.3 Hat function of $Q_3$

For $Q_3$, it is different from $P_1$ and $P_2$. We now have an element that is a square which in each direction $x$ and $y$ is a polynomial degree 3 in $x$ and $y$ respectively. We first focus in 1D.

Each $\hat{\varphi}_i$ has four unknown coefficients $a_i, b_i, c_i, d_i$. For instance, for $\hat{\varphi}_i$, we want $\hat{\varphi}_i(0) = 1$, $\hat{\varphi}_i(1/3) = 0$, $\hat{\varphi}_i(2/3) = 0$, and $\hat{\varphi}_i(1) = 0$. So, this implies that

$$\begin{align*}
a_1 + b_1 \hat{x}_1 + c_1 \hat{x}_1^2 + d_1 \hat{x}_1^3 &= 1 \\
a_1 + b_1 \hat{x}_2 + c_1 \hat{x}_2^2 + d_1 \hat{x}_2^3 &= 0 \\
a_1 + b_1 \hat{x}_3 + c_1 \hat{x}_3^2 + d_1 \hat{x}_3^3 &= 0 \\
a_1 + b_1 \hat{x}_4 + c_1 \hat{x}_4^2 + d_1 \hat{x}_4^3 &= 0
\end{align*}$$

so we can write it in matrix form as

$$\begin{pmatrix}
1 & \hat{x}_1 & \hat{x}_1^2 & \hat{x}_1^3 \\
1 & \hat{x}_2 & \hat{x}_2^2 & \hat{x}_2^3 \\
1 & \hat{x}_3 & \hat{x}_3^2 & \hat{x}_3^3 \\
1 & \hat{x}_4 & \hat{x}_4^2 & \hat{x}_4^3 \\
\end{pmatrix} \begin{pmatrix}
a_1 \\
b_1 \\
c_1 \\
d_1 \\
\end{pmatrix} = \begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
\end{pmatrix}$$

For all the four basis function in x-axis, we get the equation of the form

$$\begin{pmatrix}
1 & \hat{x}_1 & \hat{x}_1^2 & \hat{x}_1^3 \\
1 & \hat{x}_2 & \hat{x}_2^2 & \hat{x}_2^3 \\
1 & \hat{x}_3 & \hat{x}_3^2 & \hat{x}_3^3 \\
1 & \hat{x}_4 & \hat{x}_4^2 & \hat{x}_4^3 \\
\end{pmatrix} \begin{pmatrix}
a_1 & a_2 & a_3 & a_4 \\
b_1 & b_2 & b_3 & b_4 \\
c_1 & c_2 & c_3 & c_4 \\
d_1 & d_2 & d_3 & d_4 \\
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}$$

Solve for coefficient matrix, we get each $\hat{\varphi}_i$ in the form

$$\begin{align*}
\hat{\varphi}_1 &= 9/2(1 - \hat{x})(1/3 - \hat{x})(2/3 - \hat{x}) \\
\hat{\varphi}_2 &= 27/2(\hat{x})(1 - \hat{x})(2/3 - \hat{x}) \\
\hat{\varphi}_3 &= 27/2(\hat{x})(1 - \hat{x})(\hat{x} - 1/3) \\
\hat{\varphi}_4 &= 9/2(\hat{x})(\hat{x} - 1/3)(\hat{x} - 2/3)
\end{align*}$$

Also for y-axis,

$$\begin{align*}
\hat{\varphi}_1 &= 9/2(1 - \hat{y})(1/3 - \hat{y})(2/3 - \hat{y}) \\
\hat{\varphi}_2 &= 27/2(\hat{y})(1 - \hat{y})(2/3 - \hat{y}) \\
\hat{\varphi}_3 &= 27/2(\hat{y})(1 - \hat{y})(\hat{y} - 1/3) \\
\hat{\varphi}_4 &= 9/2(\hat{y})(\hat{y} - 1/3)(\hat{y} - 2/3)
\end{align*}$$

For 2D problem, there are 16 nodes in each elements which are $\hat{\varphi}_{ij}(\hat{x}, \hat{y}) = \hat{\varphi}_i(\hat{x})\hat{\varphi}_j(\hat{y})$ for $i, j = 1, 2, 3, 4$
2.2 Right-hand side

Our computation of the right-hand side is

$$\int_{\Omega} f v$$

We approximate this integration by

$$\int_{\Omega} f v \sim \sum_{K} c_i \int_{K} \varphi_i \cdot \varphi_j$$

where

$$c_i = f(x_i)$$

Then,

$$\int_{K} \varphi^K_i \cdot \varphi^K_j = |detB_K| \int_{K} (\varphi^K_i \circ F_K) \cdot (\varphi^K_j \circ F_K)$$

Therefore, our Finite Element is

$$\sum_{j} u_j \int_{\Omega} \nabla \varphi_j \cdot \nabla \varphi_i = \sum_{K} c_i \int_{K} \varphi_i \cdot \varphi_j$$

or

$$Au = b$$

where

$$A = \int_{\Omega} \nabla \varphi_j \cdot \nabla \varphi_i \quad \text{and} \quad b = \sum_{K} c_i \int_{K} \varphi_i \cdot \varphi_j$$

2.3 FEM of $P_1$

Recall our stiffness matrix

$$\int_{K} \nabla \varphi^K_i \cdot \nabla \varphi^K_j = |detB_K| \int_{K} C_K \hat{\nabla} \hat{\varphi}_i \cdot \hat{\nabla} \hat{\varphi}_j$$

where

$$C_K = B^{-1}_K B^{-T}_K = \begin{pmatrix} c_{11}^K & c_{12}^K \\ c_{21}^K & c_{22}^K \end{pmatrix}$$

$$\hat{K}_{xx} = \int_{K} \partial_x \hat{\varphi}_i \partial_x \hat{\varphi}_j , \quad \hat{K}_{yy} = \int_{K} \partial_y \hat{\varphi}_i \partial_y \hat{\varphi}_j , \quad \hat{K}_{xy} = \int_{K} \partial_x \hat{\varphi}_i \partial_y \hat{\varphi}_j , \quad \hat{K}_{yx} = \int_{K} \partial_y \hat{\varphi}_i \partial_x \hat{\varphi}_j$$

Then, our stiffness matrix with respect to the reference triangle is

$$\int_{K} \nabla \varphi^K_i \cdot \nabla \varphi^K_j = |detB_K| (c_{11}^K \hat{K}_{xx} + c_{12}^K \hat{K}_{xy} + c_{12}^K \hat{K}_{yx} + c_{21}^K \hat{K}_{yy})$$
Recall our $P_1$ basis functions has the following ordering

\[
\phi_1 = 1 - \hat{x} - \hat{y}, \quad \phi_2 = \hat{x}, \quad \phi_3 = \hat{y}
\]

We start with construct Stiffness matrix.

\[
\hat{\nabla} \phi_1 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad \hat{\nabla} \phi_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \hat{\nabla} \phi_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

\[
\hat{K}_{xx} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{K}_{yy} = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad \hat{K}_{xy} = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{K}_{yx} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 0 \end{pmatrix}
\]

The part $|\text{det}B_K|$ and $c_{ij}^K$ depend on element $K$ For the right-hand side of FEM, we can compute the Mass matrix

\[
\text{Mass}^K = \int_K \phi_i \cdot \phi_j
\]

Then, for $P_1$, it is

\[
\text{Mass}^K = \frac{1}{24} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}
\]

And we compute for each element and sum in our matrix $b$

### 2.4 FEM of $P_2$

Recall our basis functions of $P_2$

\[
\phi_1 = (1 - \hat{x} - \hat{y})(1 - 2\hat{x} - 2\hat{y}), \quad \phi_2 = \hat{x}(2\hat{x} - 1), \quad \phi_3 = \hat{y}(2\hat{y} - 1)
\]
φ_4 = 4\tilde{x}\tilde{y}, \quad \phi_5 = 4\tilde{x}(1 - \tilde{x} - \tilde{y}), \quad \phi_6 = 4\tilde{y}(1 - \tilde{x} - \tilde{y})

We take the gradient of our basis functions and construct stiffness matrix as \( P_1 \)

\[
\hat{\nabla}\phi_1 = \begin{pmatrix} 4\hat{x} + 4\tilde{y} - 3 \\ 4\hat{x} + 4\tilde{y} - 3 \end{pmatrix}, \quad \hat{\nabla}\phi_2 = \begin{pmatrix} 4\hat{x} - 1 \\ 0 \end{pmatrix}, \quad \hat{\nabla}\phi_3 = \begin{pmatrix} 0 \\ 4\tilde{y} - 1 \end{pmatrix}
\]

\[
\hat{\nabla}\phi_4 = \begin{pmatrix} 4\tilde{y} \\ 4\hat{x} \end{pmatrix}, \quad \hat{\nabla}\phi_5 = \begin{pmatrix} -4\tilde{y} \\ 4 - 4\hat{x} - 8\tilde{y} \end{pmatrix}, \quad \hat{\nabla}\phi_6 = \begin{pmatrix} 4 - 8\hat{x} - 4\tilde{y} \\ -4\hat{x} \end{pmatrix}
\]

\[
\hat{K}_{xx} = \frac{1}{6} \begin{pmatrix} 3 & 1 & 0 & 0 & 0 & -4 \\
1 & 3 & 0 & 0 & 0 & -4 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -8 & 8 & 0 \\
-4 & -4 & 0 & 0 & 0 & 8 \\
\end{pmatrix}, \quad \hat{K}_{yy} = \frac{1}{6} \begin{pmatrix} 3 & 0 & 1 & 0 & -4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 3 & 0 & -4 & 0 \\
0 & 0 & 0 & 8 & 0 & -8 \\
-4 & 0 & -4 & 0 & 8 & 0 \\
0 & 0 & 0 & -8 & 0 & 8 \\
\end{pmatrix}
\]

\[
\hat{K}_{xy} = \frac{1}{6} \begin{pmatrix} 3 & 0 & 1 & 0 & -4 & 0 \\
0 & 0 & -1 & 4 & 0 & -4 \\
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 4 & -4 & -4 \\
-4 & 0 & 0 & -4 & 4 & 4 \\
0 & -4 & 0 & -4 & 4 & 4 \\
\end{pmatrix}, \quad \hat{K}_{yx} = \frac{1}{6} \begin{pmatrix} 3 & 1 & 0 & 0 & 0 & -4 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -4 & 0 & -4 \\
0 & 0 & 0 & 4 & 4 & -4 \\
-4 & 0 & -4 & 4 & 4 & 4 \\
-4 & 0 & -4 & -4 & 4 & 4 \\
\end{pmatrix}
\]

For the Right-hand side of FEM, we can compute by integrate Mass matrix

\[
Mass\hat{K} = \int_{K} \hat{\phi}_i \cdot \hat{\phi}_j
\]

Then, for \( P_2 \), it is

\[
Mass\hat{K} = \frac{1}{360} \begin{pmatrix} 6 & -1 & -1 & -4 & 0 & 0 \\
-1 & 6 & -1 & 0 & -4 & 0 \\
-1 & -1 & 6 & 0 & 0 & -4 \\
-4 & 0 & 0 & 32 & 16 & 16 \\
0 & -4 & 0 & 16 & 32 & 16 \\
0 & 0 & -4 & 16 & 16 & 32 \\
\end{pmatrix}
\]

And we compute for each element and sum in our matrix \( b \)

### 2.5 FEM of \( Q_3 \)

Recall our basis functions of \( Q_3 \)

\[
\hat{\phi}_{ij}(\tilde{x}, \tilde{y}) = \hat{\phi}_i(\tilde{x})\hat{\phi}_j(\tilde{y})
\]
where

\[
\hat{\phi}_i(\hat{x}) = \begin{cases} 
   9/2(1 - \hat{x})(1/3 - \hat{x})(2/3 - \hat{x}) & \hat{i} = 1 \\
   27/2(\hat{x})(1 - \hat{x})(2/3 - \hat{x}) & \hat{i} = 2 \\
   27/2(\hat{x})(1 - \hat{x})(\hat{x} - 1/3) & \hat{i} = 3 \\
   9/2(\hat{x})(\hat{x} - 1/3)(\hat{x} - 2/3) & \hat{i} = 4 
\end{cases}
\]

\[
\hat{\phi}_j(\hat{y}) = \begin{cases} 
   9/2(1 - \hat{y})(1/3 - \hat{y})(2/3 - \hat{y}) & \hat{j} = 1 \\
   27/2(\hat{y})(1 - \hat{y})(2/3 - \hat{y}) & \hat{j} = 2 \\
   27/2(\hat{y})(1 - \hat{y})(\hat{y} - 1/3) & \hat{j} = 3 \\
   9/2(\hat{y})(\hat{y} - 1/3)(\hat{y} - 2/3) & \hat{j} = 4 
\end{cases}
\]

The gradient of \( Q_3 \) is

\[
\nabla \hat{\phi}_{ij}(\hat{x}, \hat{y}) = \begin{pmatrix} \partial_{\hat{x}} \hat{\phi}_i(\hat{x}) \partial_{\hat{y}} \hat{\phi}_j(\hat{y}) \\ \partial_{\hat{x}} \hat{\phi}_i(\hat{x}) \partial_{\hat{y}} \hat{\phi}_j(\hat{y}) \end{pmatrix} \quad \hat{i}, \hat{j} = 1,2,3,4
\]

To compute stiffness matrix

\[
\int \nabla \hat{\phi}_{ij}(\hat{x}, \hat{y}) \cdot \nabla \hat{\phi}_{\hat{m}\hat{n}}(\hat{x}, \hat{y}) = \int \begin{pmatrix} \partial_{\hat{x}} \hat{\phi}_i(\hat{x}) \partial_{\hat{y}} \hat{\phi}_j(\hat{y}) \\ \partial_{\hat{x}} \hat{\phi}_i(\hat{x}) \partial_{\hat{y}} \hat{\phi}_j(\hat{y}) \end{pmatrix} \cdot \begin{pmatrix} \partial_{\hat{x}} \hat{\phi}_{\hat{m}}(\hat{x}) \partial_{\hat{y}} \hat{\phi}_{\hat{n}}(\hat{y}) \\ \partial_{\hat{x}} \hat{\phi}_{\hat{m}}(\hat{x}) \partial_{\hat{y}} \hat{\phi}_{\hat{n}}(\hat{y}) \end{pmatrix}
\]

\[
= \int (\partial_{\hat{x}} \hat{\phi}_i(\hat{x}) \hat{\phi}_j(\hat{y})) (\partial_{\hat{x}} \hat{\phi}_{\hat{m}}(\hat{x}) \hat{\phi}_{\hat{n}}(\hat{y})) + (\hat{\phi}_i(\hat{x}) \partial_{\hat{y}} \hat{\phi}_j(\hat{y})) (\hat{\phi}_{\hat{m}}(\hat{x}) \partial_{\hat{y}} \hat{\phi}_{\hat{n}}(\hat{y}))
\]

\[
= \int \partial_{\hat{x}} \hat{\phi}_i(\hat{x}) \partial_{\hat{x}} \hat{\phi}_{\hat{m}}(\hat{x}) + \int \hat{\phi}_j(\hat{y}) \hat{\phi}_{\hat{n}}(\hat{y}) + \int \hat{\phi}_i(\hat{x}) \hat{\phi}_{\hat{m}}(\hat{x}) + \int \partial_{\hat{y}} \hat{\phi}_j(\hat{y}) \partial_{\hat{y}} \hat{\phi}_{\hat{n}}(\hat{y})
\]

Therefore our stiffness matrix of \( Q_3 \) is 16-by-16.

For the right-hand side, we can directly integrate our equation with the basis functions

\[
\int_{\Omega} fv \sim \sum_K c_{ij} \int_K f \hat{\phi}_{ij}
\]

where

\[
c_{ij} = f(x_i, y_j)
\]

There will be 16-by-16 as well as our stiffness matrix for \( Q_3 \).
Chapter 3

More accuracy by averaging

By the article of J.H. Bramble and A.H. Schatz [2], Higher Order Local Accuracy by Averag-ing in the Finite Element Method, we can define a better approximation of any given point $x$ in $\Omega$ by averaging $u_h(x)$ of the neighborhood of that point. We denote this approximation by $\tilde{u}_h$.

3.1 Bramble and Schatz

In the article of Bramble and Schatz [2], they give an example of subspaces generated by the $B$-splines of Schoenberg.

For $t$ is real number, define

$$
\chi = \begin{cases} 
1 & |t| \leq 1/2, \\
0 & |t| > 1/2 
\end{cases}
$$

and for $l$ an integer, set convolution $l-1$ times. Example, for $l = 2$ see Figure 3.1. For $l = 3$ see Figure 3.2.

$$
\psi_{1}^{(l)}(t) = \chi \ast \chi \ast \cdots \ast \chi.
$$

Let $\psi_{1}^{(l)}$ be the one-dimensional smooth spline of order $l$ defined by above equation. For $l = r - 2$, $r \geq 2$ given, find $k_0, k_1, \ldots, k_{r-2}$ by

$$
\sum_{j=0}^{r-2} k_j \int_{R_1} \psi_{1}^{(r-2)}(y)(y + j)^{2m} dy = \begin{cases} 
1 & \text{if } m = 0, \\
0 & \text{if } m = 1, \ldots, r - 2.
\end{cases}
$$

The constants $k'_j$ are defined as

- $k'_{r-j} = k'_j$, $j = 0, \ldots, r - 2$.
- $k'_0 = k_0$ and $k'_j = k_{r/2}$, $j = 0, \ldots, r - 2$.

where the $k_j$, $j = 0, \ldots, r - 2$.

We can compute new solution by averaging as

$$
(K_h * u_h)(h\gamma) = \sum_{\gamma, \delta \in \mathbb{Z}^N} a_{\gamma, \delta} d_{\delta}^{i},
$$
where
\[
d_{j}^{\delta} = \sum_{\beta \in \mathbb{Z}} k_{\beta}^{j} \int_{\mathbb{R}} \psi^{r-2}(\delta - \beta - \eta) \varphi_{j}(\eta) d\eta
\]

In the article of Bramble and Schatz [2], they already gave us the Table of \(k_{j}^{r}\) and \(l\).

<table>
<thead>
<tr>
<th>( j \backslash r )</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>13/12</td>
<td>37/30</td>
<td>346517/241920</td>
</tr>
<tr>
<td>1</td>
<td>-1/24</td>
<td>-23/180</td>
<td>-81329/322560</td>
</tr>
<tr>
<td>2</td>
<td>1/90</td>
<td>6337/161280</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>-3229/967680</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.1: \(k_{j}^{r}, l = r - 2, t = r - 1\)

### 3.2 Averaging \(P_{2}\) in 2D

By the Table \(k_{j}^{r}\) of Bramble and Schatz, we can construct a coefficient area \(k_{i,j}^{r}\) where \(k_{i,j}^{r} = k_{i}^{r}k_{j}^{r}\). For \(P_{2}\), we set \(r = 3\), then our \(k_{0}^{r} = 13/12\) and \(k_{1}^{r} = k_{-1}^{r} = -1/24\) for defining \(\psi_{1}^{(1)}\).

![Graph of coefficients in \(\sum_{j=-1}^{1} k_{j}^{r} \psi_{1}^{(1)}(x - j)\)](image)

Figure 3.1: The graph of coefficients in \(\sum_{j=-1}^{1} k_{j}^{r} \psi_{1}^{(1)}(x - j)\)

In 2D, we define the area of coefficients \(k_{i,j}^{r}\) as

\[
\sum_{i=-1}^{1} \sum_{j=-1}^{1} k_{i,j}^{r} \psi_{1}^{(i)}(x - i) \psi_{1}^{(1)}(x - j)
\]
where $k'_{i,j} = k'_i k'_j$ and $\psi_1^{(1)}$ is 1 for $-\frac{1}{2} \leq x \leq \frac{1}{2}$ and zero otherwise.

To compute the weights for averaging the solution, we have consider 4 cases,

3.2.1 Case 1

As we can see in Figure 3.3, we show that to average the point $(0, 0)$, we must find all the weight $d(m, n)$. 

3.2.1 Case 1

As we can see in Figure 3.3, we show that to average the point $(0, 0)$, we must find all the weight $d(m, n)$. 

3.2.1 Case 1

As we can see in Figure 3.3, we show that to average the point $(0, 0)$, we must find all the weight $d(m, n)$.
Figure 3.3: The first case of the weights $d(m, n)$

We define $d(m, n)$ as

$$d(m, n) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k'_i k'_j \psi(\hat{x} - i) \psi(\hat{y} - j) \hat{\varphi}_{m,n}(\hat{x}, \hat{y}) d\hat{x} d\hat{y},$$

where $\hat{\varphi}_{m,n}$ is a basis function of $\hat{P}_2$.

Example: to find $d(0, 0)$, we use triangles that are connected to the point $d(0, 0)$.

Figure 3.4: The area to find $d(0, 0)$

Each part of triangle can be computed by mapping $\hat{\varphi}_{m,n}$ to basis functions $\hat{\varphi}_i$.

![Diagram](image)

Figure 3.5: The mapping from $\hat{\varphi}_{m,n}$ to basis functions $\hat{\varphi}_i$

By Figure 3.4, we can see the support of $\hat{\varphi}_{0,0}$. We note that the sum of $d(m, n) = 1$. 

27
3.2.2 Case 2

As we can see in Figure 3.6, we can compute $d(m, n)$ in the same method as case 1.

![Figure 3.6: The second case of the weights $d(m, n)$ and show the support of $\hat{\varphi}_{0,0}$](image)

3.2.3 Case 3

As we can see in Figure 3.7, we can compute $d(m, n)$ in the same method as case 1.

![Figure 3.7: The third case of the weights $d(m, n)$ and show the support of $\hat{\varphi}_{0,0}$](image)
3.2.4 Case 4

As we can see in Figure 3.8, we can compute $d(m, n)$ in the same method as case 1.

![Figure 3.8: The fourth case of the weights $d(m, n)$ and show the support of $\varphi_{0,0}$](image)

3.3 Averaging $Q_3$

By the Table 3.1 $k_j'$ of Bramble and Schatz [2], we can construct a coefficient graph of $k_j'$. In 2D, we have $r = 4$, our $k_0' = 37/30$, $k_1' = k_{-1}' = -23/180$ and $k_2' = k_{-2}' = 1/90$ for defining $\psi_1^{(2)}$.

![Figure 3.9: The graph of coefficients in $\sum_{j=-2}^{2} k_j' \psi_1^{(2)}(x - j)$](image)
To compute the weight, we recall the basis function of $Q_3$.

\[
\hat{\phi}_i(x) = \begin{cases} 
9/2(1-x)(1/3-x)(2/3-x) & i = 1 \\
27/2(x)(1-x)(2/3-x) & i = 2 \\
27/2(x)(1-x)(x-1/3) & i = 3 \\
9/2(x)(x-1/3)(x-2/3) & i = 4 
\end{cases}
\]

We consider the computation for 1D. Because of the square element of $Q_3$, we can find the weight in 2D by cross-product. We define the weight as

\[
d_i = \int_{\mathbb{R}} k_i^j \psi_1^{(2)}(\hat{x} - i) d\hat{x}
\]

For example, in each element by the Figure 3.11, we show that $d(m, n)$ can be computed by cross-product. We map the node to the basis function of $Q_3$. 

![Figure 3.10: The basis function of $Q_3$](image)

![Figure 3.11: The weight of $Q_3$](image)
Chapter 4

Numerical Experiments

In this section, we consider an example where the exact solution is given by

\[ u = \sin(\pi x)\sin(\pi y)(x + \pi y) \]

We note that we can average the solution \( u_h \) to obtain \( \bar{u}_h \) only at nodes which are not too close to the boundary.

4.1 Numerical experiments of \( P_1 \)

The domain \( \Omega = (0,1)^2 \) and \( h = \frac{1}{2L} \). The Figures 4.1-4.5 are the finite element solutions with \( L = 1, 2, \ldots, 5 \) respectively and \( f = -\Delta u \). By the Table 4.1, we denote \( N_1 \) to be a set of all nodes of \( P_1 \) and define the maximum error as

\[ MaxErr = \max_{i \in N_1} |u_h(v_i) - u(v_i)| \]

where \( u_h \) is the FEM solution and \( u \) is the exact solution.
Figure 4.1: The solution of $P_1$ for $L = 1$. MaxErr = 0.7934.

Figure 4.2: The solution of $P_1$ for $L = 2$. MaxErr = 0.3026.

Figure 4.3: The solution of $P_1$ for $L = 3$. MaxErr = 0.09007
For the rate of convergence, we define that

\[ Rate = \frac{MaxErr_{L-1}}{MaxErr_L} \]
<table>
<thead>
<tr>
<th>$L\backslash Error$</th>
<th>$MaxErr$</th>
<th>$Rate$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.7934</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>0.3026</td>
<td>2.622</td>
</tr>
<tr>
<td>3</td>
<td>0.09007</td>
<td>3.3596</td>
</tr>
<tr>
<td>4</td>
<td>0.02304</td>
<td>3.9093</td>
</tr>
<tr>
<td>5</td>
<td>0.005834</td>
<td>3.9493</td>
</tr>
</tbody>
</table>

Table 4.1: The error value of each $L$ for $P_1$

### 4.2 Numerical experiments of $P_2$

For $P_2$, the domain $\Omega = (0,1)^2$ and $h = \frac{1}{2\pi}$. The Figures 4.6-4.10 are the finite element solutions with $L = 1, 2, \ldots, 5$ respectively and $f = -\Delta u$. We denote $N_2$ to be a set of all nodes of $P_2$. The maximum error of $P_2$ is given by

$$MaxErr = \max_{i \in N_2} |u_{h}(v_i) - u(v_i)|$$

Since the averaging can be computed only the nodes that are not too close to the boundary, we denote $\tilde{N}_2$ to be a set of the nodes of $P_2$ that can be averaged. By the Table 4.2, we define

$$MaxAvgErr = \max_{i \in \tilde{N}_2} |\tilde{u}_{h}(v_i) - u(v_i)|$$

$$MaxErrNode = \max_{i \in \tilde{N}_2} |u_{h}(v_i) - u(v_i)|$$

where $\tilde{u}_{h}$ is the solution by averaging
Figure 4.6: The solution of $P_2$ for $L = 1$. MaxErr = 0.1225.

Figure 4.7: The solution of $P_2$ for $L = 2$. MaxErr = 0.001077. MaxAvgErr = 0.02007. MaxErrNode = 0.002958.

Figure 4.8: The solution of $P_2$ for $L = 3$. MaxErr = 8.1789e-4. MaxAvgErr = 0.002054. MaxErrNode = 8.1789e-4.
For the rate of convergence, we define that

\[ RateAvg = \frac{MaxAvgErr_{L-1}}{MaxAvgErr_L} \]
\[ \text{RateNode} = \frac{\text{MaxErrNode}_{L-1}}{\text{MaxErrNode}_L} \]

<table>
<thead>
<tr>
<th>L \ Error</th>
<th>MaxErr</th>
<th>Rate</th>
<th>MaxAvgErr</th>
<th>RateAvg</th>
<th>MaxErrNode</th>
<th>RateNode</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1225</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>0.01077</td>
<td>11.374</td>
<td>0.02007</td>
<td>-</td>
<td>0.002958</td>
<td>-</td>
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<td>13.168</td>
<td>0.002054</td>
<td>9.771</td>
<td>8.1789e-4</td>
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<td>15.533</td>
<td>1.3146e-4</td>
<td>15.624</td>
<td>5.2655e-5</td>
<td>15.533</td>
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<tr>
<td>5</td>
<td>3.3291e-6</td>
<td>15.816</td>
<td>8.2886e-6</td>
<td>15.860</td>
<td>3.3291e-6</td>
<td>15.816</td>
</tr>
</tbody>
</table>

Table 4.2: The error value of each \( L \) for \( P_2 \)

### 4.3 Numerical experiments of \( Q_3 \)

For \( Q_3 \), the domain \( \Omega = (0,1)^2 \) and \( h = \frac{1}{2^L} \). The Figure 4.11-4.15 are the finite element solutions with \( L = 1, 2, \ldots, 5 \) respectively and \( f = -\Delta u \). We denote \( N_3 \) to be a set of all nodes of \( Q_3 \). The maximum error of \( Q_3 \) is given by

\[ \text{MaxErr} = \max_{i \in N_3} |u_h(v_i) - u(v_i)| \]

we denote \( \tilde{N}_3 \) to be a set of the nodes of \( Q_3 \) that can be averaged. By the Table 4.3, we define the maximum error of \( \tilde{N}_3 \) nodes

\[ \text{MaxAvgErr} = \max_{i \in \tilde{N}_3} |	ilde{u}_h(v_i) - u(v_i)| \]

\[ \text{MaxErrNode} = \max_{i \in \tilde{N}_3} |u_h(v_i) - u(v_i)| \]
Figure 4.11: The solution of $Q_3$ for $L = 1$. MaxErr = 0.007107.

Figure 4.12: The solution of $Q_3$ for $L = 2$. MaxErr = $5.00002522 \times 10^{-4}$.

Figure 4.13: The solution of $Q_3$ for $L = 3$. MaxErr = $3.3568 \times 10^{-5}$. MaxAvgErr = $4.5854 \times 10^{-5}$. MaxErrNode = $2.7828 \times 10^{-5}$.


The rate of convergence is given by

$$RateAvg = \frac{MaxAvgErr_{L-1}}{MaxAvgErr_L}$$

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\[ \text{RateNode} = \frac{\text{MaxErrNode}_{L-1}}{\text{MaxErrNode}_L} \]

<table>
<thead>
<tr>
<th>( L )</th>
<th>Error</th>
<th>MaxErr</th>
<th>Rate</th>
<th>MaxAvgErr</th>
<th>RateAvg</th>
<th>MaxErrNode</th>
<th>RateNode</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.007107</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
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<td>14.214</td>
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<td>4.5854e-5</td>
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<td>2.7828e-5</td>
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</tr>
<tr>
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<td>1.30589e-7</td>
<td>16.269</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.3: The error value of each \( L \) for \( Q_3 \)
Chapter 5

Conclusion

In this Chapter, we conclude our result of the errors in each refinement. In $P_1$, by the Table 4.1, we have the rate of convergence approximate to 4. Next, we compare the rate of convergence of $P_2$. The rate of convergence is 16 by the Table 4.2. That is, in $P_2$, the FEM generates a better solution at some nodes due to super-convergence for structured mesh. In the corollary of Bramble and Schatz states that let $\Omega_0 \subset \subset \Omega_1 \subset \subset \Omega$ and $N_0 = N/2 + 1$, at points $h\gamma \in \Omega_0$ and $\gamma \in Z^2$

$$\sup_{h\gamma \in \Omega_0, \gamma \in Z^2} |u(h\gamma) - \sum a_{\gamma-a}^j d_a^j| \leq C(h^{2r-2} \|u\|_{2r-2+N_0, \Omega_1})$$

In $P_2$, we have $r = 3$ so that the rate is $h^4$ which is the same result from our FEM. And it does not give us a better result. Therefore, we introduce the $Q_3$ refinement of FEM. By the Table 4.3, the rate of convergence is approximated to 16 and the rate of convergence by averaging is approximated to 64. As Bramble and Schatz’s theorem, we have $r = 4$ which give us $h^6$ as same as the rate of convergence that we have from the averaged $Q_3$ so that the averaging method shows that our test has more accuracy. Furthermore, we can expect that for $Q_4$ or $Q_5$ can give us a better result by using FEM solution.
Bibliography

