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STABILITY OF THREE-WHEELED AND TWO WHEEL BICYCLES

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2012

THE STABILITY
OF THREE-
WHEELED
VEHICLES & TWO
WHEEL BICYCLES

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Abstract

The goal of this project is the investigation of the stability of several non-trivial transportation devices, such as a motorcycle with a sidecar and a two-wheel bicycle, under various dynamic conditions.

The first approach is a theoretical study of the stability in the motion of a motorcycle with a sidecar. Its specific feature is its lack of symmetry. In the project, conditions were obtained that guarantee directional, lateral, and, most importantly, rollover stability. Under certain dynamic conditions, tipping may be possible about a line through the ground contacts of the side wheel and the front or rear wheels.

In this problem, stabilization is permanently fixed through design choices and manufacturing expertise. The project also studies the dynamic control maintained over a bicycle through the course of its motion.

The first dynamic topic concerns the self-stability of a bicycle where the gyroscopic effects have been removed. The problem is more than a century old, but only recently (2011) has it been realized that an *uncontrolled* bicycle without any gyro effects can demonstrate self-stability due to an appropriate feedback applied through the steering column alone. In this project, the values of some constructive parameters were determined that maximize the forward velocity range for uncontrolled stability.

The second dynamic topic of the project concerns the *controlled* stability of a moving bicycle. Its vertical position is then maintained via special action taken by a rider who properly operates the handlebar. A bicycle is similar to an inverted pendulum acted upon by an additional force created through rotation of the handlebar. The main idea is to operate the handlebar so as to make this force identical to the force that appears when fast vertical oscillations of a pivot stabilize an inverted pendulum. As a consequence, the bicycle can be stabilized for forward velocities above a certain limit.

1. Motorcycle with a Sidecar

There is some literature [1,2] on the theoretical analysis of stability for symmetric tricycles, such as the delta and tadpole motorcycles (Fig.1).

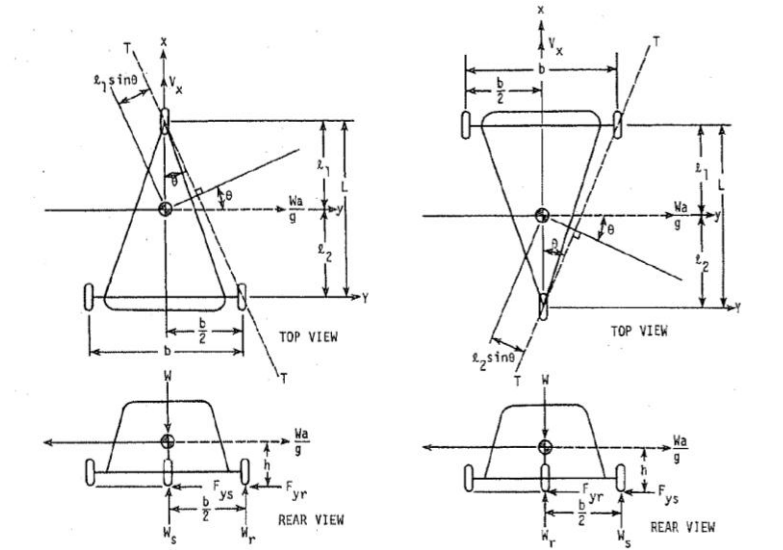


Figure 1. Top and rear view of delta (left) and tadpole (right) tricycle models

In this project, we consider, as an alternative, a sidecar motorcycle outfit. Unlike delta and tadpole, this outfit is lacking symmetry, which affects its directional and lateral stability. To the best of our knowledge, the stability analysis has not been conducted for a non-symmetric case. In the following section we are providing an analysis of the directional stability of a sidecar outfit negotiating a turn.

1.1 Directional Stability

A scheme for the sidecar vehicle is given in Fig. 2.

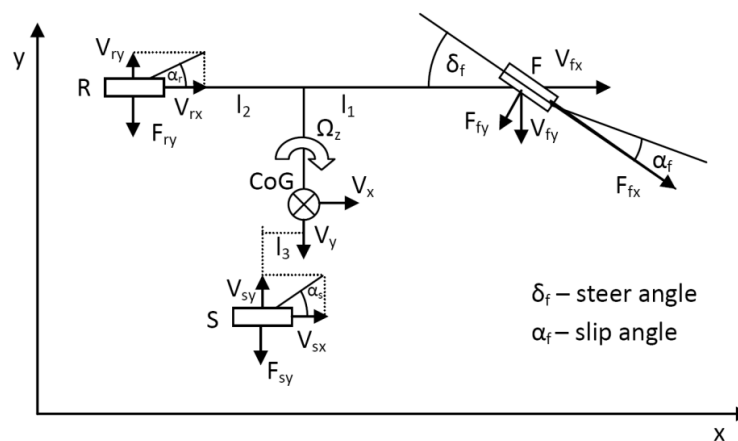


Figure 2. Schematic model of the sidecar outfit

Weights applied to wheels are designated as W_1, W_2, W_3 (index 1 for front wheel F, index 2 for rear wheel R, and index 3 for side wheel S); $W = W_1 + W_2 + W_3$.

Usually, the coordinates of wheels are known, whereas the coordinates of the CoG depend on the distribution of weight among the wheels, that is, upon two independent parameters. The lateral forces F_{fy} , F_{ry} , F_{sy} are linked with the slip angles (Fig. 2) by the formulae:

$$F_{fy} = C_f \alpha_f, \quad F_{ry} = C_r \alpha_r, \quad F_{sy} = C_s \alpha_s,$$

with stiffnesses C_f, C_r, C_s . Here, we assume that the two planes through the rear wheels and side wheels remain parallel to each other. Also, V_x is assumed to be constant and significantly larger than the contribution due to rotational velocity Ω_z . Differential equations of motion take the form [1,2]:

$$\frac{W}{g} \dot{V}_y + a_1 V_y + a_2 \Omega_z = C_f \delta_f, \quad I_z \dot{\Omega}_z + a_3 V_y + a_4 \Omega_z = l_1 C_f \delta_f,$$

where I_z is the moment of inertia about CoG, and the symbols a_1, \dots, a_4 are combinations of lateral stiffnesses. The geometric parameters were introduced in Fig. 2., including weight W , and forward velocity V_x :

$$\begin{aligned} a_1 &= \frac{C_f + C_r + C_s}{V_x}, \\ a_2 &= \frac{l_1 C_f - l_2 C_r - l_3 C_s}{V_x} + \frac{W}{g} V_x, \\ a_3 &= \frac{l_1 C_f - l_2 C_r - l_3 C_s}{V_x}, \\ a_4 &= \frac{l_1^2 C_f - l_2^2 C_r - l_3^2 C_s}{V_x}. \end{aligned}$$

From geometry, and by small-angle assumption ($\delta_f \ll 1$, $\alpha_f \ll 1$, $\alpha_r \ll 1$, $\alpha_s \ll 1$)

$$\alpha_f = \delta_f - \tan^{-1} \frac{V_{fy}}{V_{fx}} = \delta_f - \frac{V_{fy}}{V_{fx}},$$

$$\alpha_r = \frac{V_{ry}}{V_{rx}}, \quad \alpha_s = \frac{V_{sy}}{V_{sx}},$$

$$V_{fy} = l_1 \Omega_z + V_y, \quad V_{fx} = V_x,$$

$$V_{ry} = l_2 \Omega_z - V_y, \quad V_{rx} = V_x,$$

$$V_{sy} = l_3 \Omega_z - V_y, \quad V_{sx} = V_x.$$

A sidecar outfit negotiating a turn is *directionally* stable, i.e., it isn't brought to uncontrolled rotation as long as $a_1 a_4 - a_2 a_3 > 0$, or, after transformations,

$$-V_x^2 \frac{W}{g} (l_1 C_f - l_2 C_r - l_3 C_s) + C_f C_r L_{12}^2 + C_f C_s L_{13}^2 + C_r C_s L_{23}^2 \geq 0, \quad (1)$$

where $L_{12} = l_1 + l_2$, $L_{13} = l_1 + l_3$, $L_{23} = l_2 - l_3$.

To study how the weight applied to every wheel affects directional stability, we show in Fig. 3 how the CoG (point O) splits the triangle 123 into three triangles 203 , 103 and 102 . Each is proportional, respectively, to the weights W_1 , W_2 , W_3 .

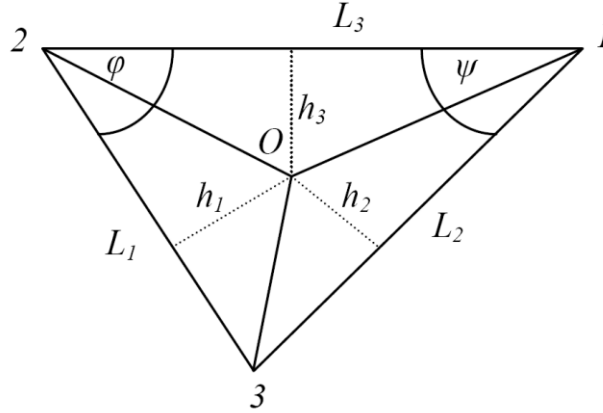


Figure 3. Location of the Center of Gravity

Directional stability will take place regardless of V_x if the factor of V_x^2 in (1) is negative, that is,

$$(W_1 L_3 + W_3 L_1 \cos \phi)(C_f + C_r + C_s) \geq W(C_f L_3 + C_s L_1 \cos \phi). \quad (2)$$

This means that a sufficient load on the front and side wheels guarantees directional stability for any V_x . If (2) is violated then this stability will be preserved if V_x^2 remains below the critical value

$$V_x^2 \leq g \frac{C_f C_r L_{12}^2 + C_f C_s L_{13}^2 + C_r C_s L_{23}^2}{W(C_f L_3 + C_s L_1 \cos \varphi) - (W_1 L_3 + W_3 L_1 \cos \varphi)(C_f + C_r + C_s)}. \quad (3)$$

1.2 Rollover Instability

The sidecar outfit may also demonstrate *rollover instability* as a result of lateral acceleration caused for any reason (e.g., a wind gust), but also while it is accelerating or breaking in a turn. An accident or rollover may occur either about the axis 3-1 or about the axis 2-3 in Fig. 3. For example, when lateral acceleration a applies perpendicular to the line 2-1 then no rollover happens if

$$a \leq \frac{g}{H} \min\left(\frac{h_1}{\cos \varphi}, \frac{h_2}{\cos \psi}\right).$$

Here, H is the height of the CoG over the ground. If a rollover occurs, then it will require less acceleration to happen around the 3-1 axis if

$$m_2 L_1 \cos \varphi \leq m_1 L_2 \cos \psi.$$

The smaller is the load m_2 on the rear wheel, and the closer are the axes 2 and 3 to coincidence, the easier will be the chance of a turnover about the line 3-1.

2. Self-Stability of a Bicycle

2.1 Statement of the problem

While a bicycle is a complicated mechanical assembly, its structure does allow for satisfactory modeling. For schematic versions, see Figs. 4 and 5 [3]. The tilt angle (positive or negative) for the steering axis is given by λ_s . In this model, the wheels are replaced by skates (“two-mass-skate” bicycle - TMS), which is reasonable if the gyro effect is removed thanks to a compensating wheel mounted on the frame. This removal was purposefully introduced in [3] to eliminate the stabilizing action of gyro effect and to determine how the steering wheel alone would work as a stabilizing factor. The differential equation

$$M\ddot{q} + vC\dot{q} + [gK_0 + v^2K_2]q = 0$$

governs the vector $\mathbf{q} = (\varphi, \delta)$ of the lean angle φ between the frame and the vertical plane, and the steer angle δ of the front wheel. \mathbf{M} is a positive-definite matrix of masses, and $v\mathbf{C}$ characterizes gyroscopic torques due to the rates of φ, δ . The symmetric matrix $g\mathbf{K}_0$ reflects the acceleration of gravity, while $v^2\mathbf{K}_2$ is due to gyroscopic and centrifugal effects. The elements of all matrices are lengthy expressions [3] involving constructive parameters (see Figs. 4,5), among them the coordinates (x_H, z_H) and (x_B, z_B) of the front and rear frame masses, m_H and m_B . These masses are introduced into the model as replacements for the frame and steering column masses, respectively. It is known that for a range of v the system remains stable on its own (self-stability). The goal of this project is to find position (x_H, z_H) of the front mass that makes the lower boundary of the velocity range of self-stability as small as possible.

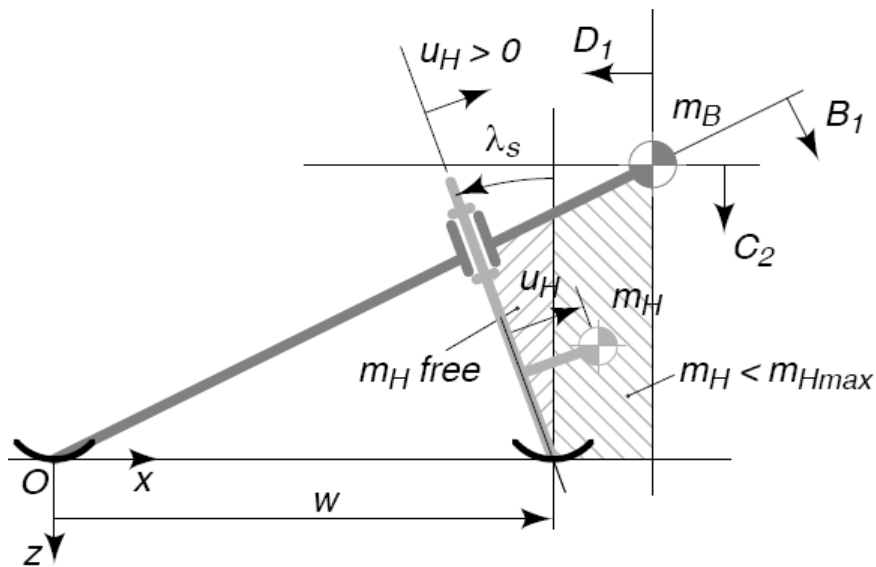


Figure 4. Theoretical scheme of a TMS bicycle, $\lambda_s > 0$.

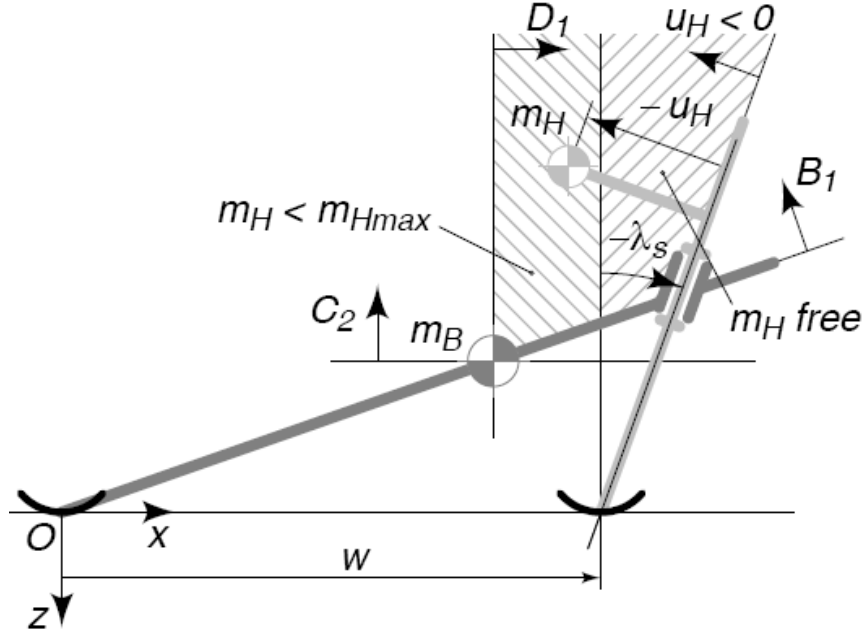


Figure 5. Theoretical scheme of a TMS bicycle, $\lambda_s < 0$.

2.2 Method

We apply Routh-Hurwitz (RH) criterion that guarantees the negativeness of the real parts of all roots of the characteristic equation

$$\det (\mathbf{M}\lambda^2 + v\mathbf{C}\lambda + g\mathbf{K}_0 + v^2\mathbf{K}_2) = 0.$$

An equivalent form for it

$$A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0 \quad (4)$$

introduces the symbols A, \dots, E defined as

$$A = A_0, \quad B = B_1v, \quad C = C_0 + C_2v^2, \quad D = D_1v + D_3v^3, \quad E = E_0 + E_2v^2. \quad (5)$$

The coefficients A_0, \dots, E_2 in these expressions are functions of material properties and constructive parameters [3], among them (x_H, z_H) . The RH criterion requires that all coefficients A, B, C, D, E , and the polynomial $X = BCD - ADD - EBB = X_2v^2 + X_4v^4 + X_6v^6$ should be negative. These conditions are tested below for both schemes represented in Figs. 4 and 5, and optimal locations (x_H, z_H) and the smallest critical velocities found for these schemes. The following formulae for A_0, \dots, X_6 are given in [3]:

$$A_0 = m_B m_H u_H^2 z_B^2,$$

$$B_1 = -m_B m_H u_H z_B (x_B z_H - x_H z_B) / \bar{w},$$

$$\begin{aligned}
C_0 &= -gm_H u_H (m_B \sin \lambda_s z_B^2 - m_B u_H z_B + m_H \sin \lambda_s z_H^2 + m_H u_H z_H), \\
C_2 &= m_B m_H u_H z_B (z_B - z_H) / \bar{w}, \\
D_1 &= -gm_B m_H u_H z_B (x_B - x_H) / \bar{w}, \\
D_3 &= 0, \\
E_0 &= -g^2 m_H u_H (m_H (x_H - w) \cos \lambda_s + m_B z_B \sin \lambda_s), \\
E_2 &= 0, \\
X_2 &= -g^2 (m_B^2 m_H^3 u_H^3 z_B^2 (z_B - z_H) (m_B \sin \lambda_s x_B^2 z_B z_H + m_B u_H x_B^2 z_B - m_B \sin \lambda_s x_B x_H z_B^2 \\
&\quad - m_B u_H x_B x_H z_B + m_H \sin \lambda_s x_B x_H z_H^2 + m_H u_H x_B x_H z_H - m_H \sin \lambda_s x_H^2 z_B z_H \\
&\quad - m_H u_H x_H^2 z_B)) / \bar{w}^2, \\
X_4 &= gm_B^3 m_H^3 u_H^3 z_B^3 (x_B z_H - x_H z_B) (x_B - x_H) (z_B - z_H) / \bar{w}^3, \\
X_6 &= 0.
\end{aligned} \tag{6}$$

Here and below, the symbols u_H and \bar{w} have the following meaning:

$$u_H = (x_H - w) \cos \lambda_s - z_H \sin \lambda_s, \quad \bar{w} = w / \cos \lambda_s. \tag{7}$$

The RH inequalities now become (notice that both z_H, z_B are negative):

- (i) $A_0 > 0$;
- (ii) $B_1 > 0$ for $v > 0$, that is,
either $u_H > 0$, and $x_B z_H - x_H z_B > 0$,
or $u_H < 0$, and $x_B z_H - x_H z_B < 0$;
- (iii) $C_2 > 0$, that is,
either $u_H > 0$, and $z_B < z_H$,
or $u_H < 0$, and $z_B > z_H$;
- (iv) $D_1 > 0$, that is,
either $u_H > 0$, and $x_B > x_H$,
or $u_H < 0$, and $x_B < x_H$;
- (v) $E_0 > 0$, that is,
either $u_H > 0$, and $\tan \lambda_s > m_H (w - x_H) / m_B z_B$,
or $u_H < 0$, and $\tan \lambda_s < m_H (w - x_H) / m_B z_B$.

We see that inequalities $B_1 > 0$, $C_2 > 0$, $D_1 > 0$, $X_4 > 0$ take place simultaneously for both positive and negative values of u_H .

By (7), the formula (6) for E_0 is rewritten as $E_0 = g^2 m_H u_H [-(m_B z_B + m_H z_H) \sin \lambda_s - m_H u_H]$. To guarantee $E_0 > 0$, it is necessary that $u_H \sin \lambda_s > 0$. For positive λ_s , this means $u_H > 0$. In other words, for a structure where the steering is in the “leaned back” orientation, the front mass should be in front of the steering axis. For positive values of u_H , we have, by (ii) – (iv), $x_B z_H - x_H z_B > 0$, and $z_B < z_H$, and $x_B > x_H$. The first of these inequalities means that the front mass m_H is below a line through O and the rear mass m_B (Fig. 4). Another two inequalities say that the shaded domain shown in Fig. 4 is consistent with (ii) – (iv). As to (v), this inequality is satisfied for all m_H if this mass is within the triangle in the portion of the shaded domain to the left of $x = w$. When we go to the rest of the shaded domain, then inequality (v) is satisfied only for

$$m_H < m_{Hmax} = \frac{-m_B z_B \tan \lambda_s}{x_h - w}.$$

The RH conditions are now reduced to

$$C_0 + C_2 v^2 > 0, \text{ and } X_2 + X_4 v^2 > 0. \quad (8)$$

These inequalities determine the critical velocity at which the bicycle loses stability. We give the numerical analysis in the following section. Specifically, we will be interested in the position (x_H, z_H) of the front mass that makes the critical velocity as low as possible.

2.3 Results

In Fig. 4, we assign the following parameters: $w = 1m$, $\lambda_s = 5^\circ$, $m_B = 10 \text{ kg}$, $(x_B, z_B) = (1.2, -0.4)m$, $m_H = 1 \text{ kg}$. The coefficients A, B, D, E are all positive for $v > 0$ if (x_H, z_H) falls into the shaded domain in Fig. 4. Adjusting the data for Fig. 4, we specify the parameters as: $w = 1m$, $\lambda_s = -5^\circ$, $m_B = 10 \text{ kg}$, $(x_B, z_B) = (0.85, -0.2)m$, $m_H = 1 \text{ kg}$. The coefficients A, B, D, E are all positive for $v > 0$, with (x_H, z_H) inside the shaded domain in Fig. 5. Because $X_6 = 0$ in both cases, the optimal critical value of v^2 is defined by (8) as

$$\min v_{cr}^2 = \min_{(x_H, z_H)} \left[\max \left(-\frac{C_0}{C_2}, -\frac{X_2}{X_4} \right) \right].$$

The ratios $-C_0/C_2$ and $-X_2/X_4$ depend on (x_H, z_H) taking values within the shaded regions. The graphs of the ratios (red for $-C_0/C_2$ and blue for $-X_2/X_4$) are reproduced in Fig. 6a for positive λ_s , and in Fig. 6b for negative λ_s . We see that in both cases, the blue surface is above the red one, so it is the ratio $-X_2/X_4$ that defines the critical velocity below which stability is lost.

When $\lambda_s > 0$ (see Fig. 4), the minimum critical velocity of 2.26 m/s is attained at $x_H = w = 1$ and $z_H = 0$. That is, when the load m_H is mounted on the front skate. Using $(x_H, z_H) = (1.02, -0.2)m$, the non-optimal critical velocity was found in [3] to be 2.8 m/s .

When $\lambda_s < 0$ (see Fig. 5), the minimum critical velocity of 2.16 m/s is attained at $(x_H, z_H) = (x_B, z_B)$. That is, when the front mass and the frame mass are located at the same point. Using $(x_H, z_H) = (1, -0.4)m$, the non-optimal critical velocity was found in [3] to be 2.6 m/s .

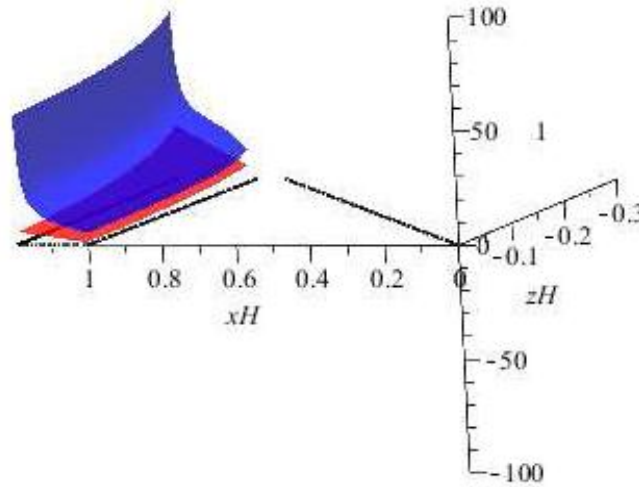


Figure 6a. Plot of $-C_0/C_2$ (red) and $-X_2/X_4$ (blue) as function of (x_H, z_H) for $\lambda_s > 0$.

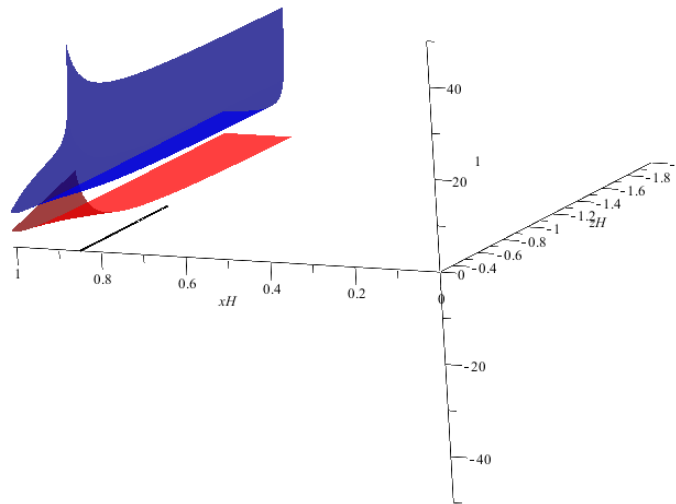


Figure 6b. Plot of $-C_0/C_2$ (red) and $-X_2/X_4$ (blue) as function of (x_H, z_H) for $\lambda_s < 0$.

3. Controlled Stability of a Bicycle

A simple model of a bicycle assumes that for a tilt angle $\lambda_s > 0$, the CoG of the front assembly will be located on the steering column. The front assembly is defined as the front wheel, the front forks and the handlebar. The rear assembly includes the frame, plus the rider, the rear forks and the rear wheel. For this typical arrangement, the lean and the steer angles φ, δ satisfy the equation [4]

$$a_0\ddot{\varphi} - a_1\varphi - a_2\ddot{\delta} - a_3V\dot{\delta} + (a_4 - a_5V^2)\delta = 0. \quad (9)$$

The coefficients a_0, \dots, a_5 then have the following meaning:

$$\begin{aligned} a_0 &= wJ_1, \\ a_1 &= wgh, \\ a_2 &= wJ_3 + w_1J_{12}, \\ a_3 &= (J_{12} + wC_2) \cos \lambda_s + w_1(h + C_1 + C_2), \\ a_4 &= wgm_2d, \\ a_5 &= (h + C_1 + C_2) \cos \lambda_s, \\ J_1 &= m_1h_1^2 + A_1 + m_2h_2^2 + A_2, \\ J_{12} &= m_1H_1l_1 + m_2h_2l_2 + D_1 - D_2, \\ J_3 &= m_2h_2d + A_2 \sin \lambda_s - D_2 \cos \lambda_s. \end{aligned}$$

The following symbols define the configuration of the bicycle. Through A_1, D_1 we denote, respectively, the central moment of inertia of the rear assembly relative to the horizontal axis, and the mixed central moment of inertia relative to this axis and the axis perpendicular to it in the plane of the rear wheel. The symbols A_2, D_2 denote the similar moments for the front assembly. $C_1, (C_2)$ are the moments of inertia of the rear (front) wheels about their axes of rotation; $h_1(h_2)$ are the distances from the CoG of the rear (front) wheels to the ground; $m_1(m_2)$ are the masses of the assemblies. The symbols $l_1, (l_2)$ denote the horizontal distances between the ground contact of the rear wheel and the horizontal projections of the CoG of the rear (front) assemblies. The symbol w is, as before, the base of the bicycle, and w_1 is equal to $\cos \lambda_s$ times the trailing¹ distance of the front wheel. Parameter d is defined as the shortest distance between CoG of the front assembly and the steering wheel.

¹ As taken from Wikipedia: "Trail, or caster, is the horizontal distance from where the steering axis intersects the ground to where the front wheel touches the ground."

Assume that $d = 0$ and define (l_0, h_0) as coordinates of the CoG of the bicycle; then $J_1 = mh_0^2$, $J_{12} = mh_0l_0$, $h = mh_0$ (here $m = m_1 + m_2$). Neglecting the moments of inertia C_1, C_2 compared to mh_0R , where R is the wheel's radius, we reduce Eq. (9) to

$$\ddot{\varphi} - \frac{g}{h_0}\varphi - \frac{k}{h_0w}\ddot{\delta} - \frac{bV}{h_0w}\dot{\delta} - \frac{V^2}{h_0w}\cos\lambda_s\delta = 0, \quad (10)$$

where

$$k = \frac{wJ_3}{mh_0} + w_1l_0, \quad b = l_0\cos\lambda_s + w_1,$$

and $V = \text{const}$ is the forward velocity. The steer angle δ serves here as a factor controlled by a rider to maintain stability. *The equation (10) is the same as the equation of an inverted pendulum acted upon by additional force represented by the last three terms on the left hand side.* We now introduce the feedback through defining these terms as

$$\frac{k}{h_0w}\ddot{\delta} + \frac{bV}{h_0w}\dot{\delta} + \frac{V^2}{h_0w}\cos\lambda_s\delta = \varphi\frac{A\omega^2}{h_0}\cos\omega t. \quad (11)$$

Eq. (10) then takes the form

$$\ddot{\varphi} - \left(\frac{g}{h_0} + \frac{A\omega^2}{h_0}\cos\omega t\right)\varphi = 0 \quad (12)$$

identical with the equation of the inverted pendulum with the pivot subjected to vertical oscillation $y = -A\cos\omega t$. Such oscillation is known to stabilize the pendulum if [5]

$$A\omega > \sqrt{2gh_0}.$$

Eq. (12) is the Mathieu equation, with its solution bounded once the last inequality holds. Now it is possible to find $\delta(t)$ by integrating (11), and this law should be enforced by the rider operating the handlebar. The function at the right hand side of (11) is bounded, and the relevant solution $\delta(t)$ has values that are also bounded. This bound is well-known [6]; it is defined by

$$|\delta(t)| \leq \delta_{st} \coth \frac{n\tau}{4}.$$

Here, δ_{st} is a static deviation of $\delta(t)$ from zero due to the action of the force equal to the maximum value of the rhs of (11):

$$\delta_{st} = \frac{A\omega^2w}{V^2} \max(\varphi \cos\omega t),$$

with parameters n and τ defined, respectively, as

$$n = \frac{bV}{2k}, \quad \tau = \frac{2\pi}{\sqrt{\frac{V^2}{k} \cos \lambda_s - n^2}}.$$

We assume here that parameter b in (11) is small enough to offer relatively low resistance.

Conclusions

1. The influence of weight distribution on directional stability of a sidecar motorcycle outfit is examined with the aid of the stability analysis applied to the differential equations governing the transverse and angular velocity of the vehicle negotiating the turn. The results specify the upper boundary of the forward velocity below which stability is lost. A similar analysis applies to the instability under the influence of external acceleration. It specifies conditions under which the turnover occurs either about the axis connecting the ground contacts of the side and front wheels, or about the axis between the ground contacts of the side and rear wheels.
2. The study of self-stability of the two-wheeled bicycle establishes conditions that minimize the critical velocity under which the self-stability is lost. We choose the position of the CoG of the front assembly as a control parameter, and find the optimal value of this parameter when the tilt angle of the front wheel is positive as well as negative.
3. Unlike the self-stability mode, the stability of the bicycle controlled by a rider is examined in the third section. Here, we assume that the rider acts to operate the handlebar such that a bicycle remains stable. To achieve this, we utilize an effect well-known to every bicycle rider: the stability is maintained when an oscillatory rotation is applied to the steering wheel. In our mathematical description, we argue mechanically by treating the bicycle as an inverted pendulum. An inverted pendulum can be stabilized if its pivot is subjected to high-frequency vertical oscillations. A similar mechanism works for the case of a bicycle where rotation of the handlebar produces vertical oscillations in the front forks. Using a feedback approach, we identify the bicycle equation with the equation of an inverted pendulum whose stability is maintained via an oscillating pivot. From this, we obtain the rule for operating the handlebar dictated by the lean angle at each instance of time, and the upper bound of the handlebar's oscillation is determined.

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