

January 2012

# Q-Polynomial Association schemes with Irrational Eigenvalues

George Matthew Shakan  
*Worcester Polytechnic Institute*

Yizhou Xia  
*Worcester Polytechnic Institute*

Follow this and additional works at: <https://digitalcommons.wpi.edu/mqp-all>

---

## Repository Citation

Shakan, G. M., & Xia, Y. (2012). *Q-Polynomial Association schemes with Irrational Eigenvalues*. Retrieved from <https://digitalcommons.wpi.edu/mqp-all/4105>

This Unrestricted is brought to you for free and open access by the Major Qualifying Projects at Digital WPI. It has been accepted for inclusion in Major Qualifying Projects (All Years) by an authorized administrator of Digital WPI. For more information, please contact [digitalwpi@wpi.edu](mailto:digitalwpi@wpi.edu).

Project Code: WJM2000



# Q-Polynomial Association Schemes with Irrational Eigenvalues

A Major Qualifying Project Report  
Submitted to The Faculty  
of the

**Worcester Polytechnic Institute**

In partial fulfillment of the requirements for the

**Degree of Bachelor of Science in  
Mathematical Science**

by

---

George Shakan

---

Yizhou Xia

Approved:

---

Dr. William J. Martin

## **Abstract**

We work towards classifying the feasible parameter sets of irrational  $Q$ -polynomial association schemes with three classes. We aim to provide a synopsis of the subject as well as provide examples, theorems and conjectures to understand these combinatorial objects.

## Acknowledgements

We would like to thank our project advisor, Professor Martin, who provided our team with countless hours of his time, providing helpful and patient advice.

We would also like to thank Alyssa Gottshall who provided our team with useful advice in the writing of our report.

Finally, we thank all of the mathematics professors at Worcester Polytechnic Institute, who properly prepared us for research.

# Contents

|  |           |
|--|-----------|
| <b>Abstract</b>  | <b>1</b>  |
| <b>Acknowledgements</b>                                  | <b>2</b>  |
| <b>1 Context</b>   | <b>4</b>  |
| <b>2 Math Background</b>                                 | <b>7</b>  |
| 2.1 The Krein Parameters . . . . .                       | 16        |
| 2.2 The Galois Automorphism . . . . .                    | 18        |
| 2.3 The Eigenvalues of Strongly Regular Graphs . . . . . | 20        |
| 2.4 The Bannai-Ito Conjectures . . . . .                 | 23        |
| <b>3 Examples</b>  | <b>24</b> |
| 3.1 The Polygons . . . . .                               | 25        |
| 3.2 The Conference Graphs . . . . .                      | 28        |
| 3.3 Symmetric Designs and Linked Systems . . . . .       | 30        |
| 3.4 The Q-Bipartite Case . . . . .                       | 36        |
| 3.4.1 Taylor Graphs . . . . .                            | 37        |
| 3.4.2 Hadamard Graphs . . . . .                          | 41        |
| <b>4 Our Results</b>                                     | <b>42</b> |
| 4.1 The First Case . . . . .                             | 42        |
| 4.2 The Second Case . . . . .                            | 46        |

# 1 Context

How is linear algebra related to graph theory? This question inspired us to explore the realm of algebraic graph theory. As we can expect, graphs can sometimes be very complicated so it can be useful to encode the information contained in a graph into matrix form. Once we input the description of the graph into matrix form, we can then apply all the tools of linear algebra.

Given a graph  $G$  with  $n$  vertices  $v_1, \dots, v_n$ , we define the adjacency matrix of  $G$  as the  $n \times n$  matrix  $A$  where

$$A_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

For an undirected graph, the adjacency matrix is always symmetric, that is  $A_{ij} = A_{ji}$ , since  $v_i$  is adjacent to  $v_j$  if and only if  $v_j$  is adjacent to  $v_i$ . It is natural to interpret the eigenvalues and eigenvectors of the adjacency matrices to gather information about  $G$ . Consider the icosahedron as our first example, depicted in Figure 1. Here we present an adjacency matrix of this graph:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

In our research, the icosahedron is perhaps the simplest nontrivial example of a much bigger class of objects that have beautiful symmetries and combinatorial properties called association schemes.

After encoding the graph as a matrix, we can apply the tools from linear algebra. We consider the eigenvalues of  $A$  and present them in the following table:

|                       |   |            |             |    |
|-----------------------|---|------------|-------------|----|
| $\theta$              | 5 | $\sqrt{5}$ | $-\sqrt{5}$ | -1 |
| $\text{mult}(\theta)$ | 1 | 3          | 3           | 5  |

Observe that the icosahedron has irrational eigenvalues,  $\sqrt{5}$  and  $-\sqrt{5}$ . For any simple graph, the adjacency matrix  $A$  is an integer matrix so its characteristic polynomial has integer coefficients; and thus the minimal polynomial also has integer coefficients. Therefore, if the

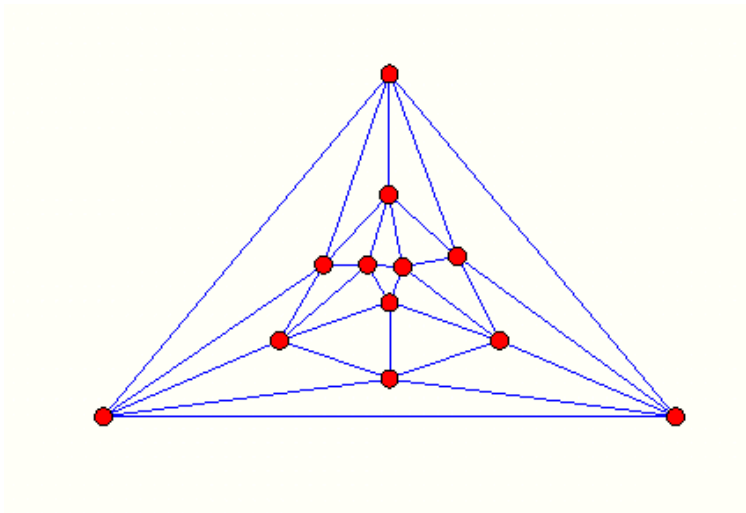


Figure 1: The graph of the icosahedron.

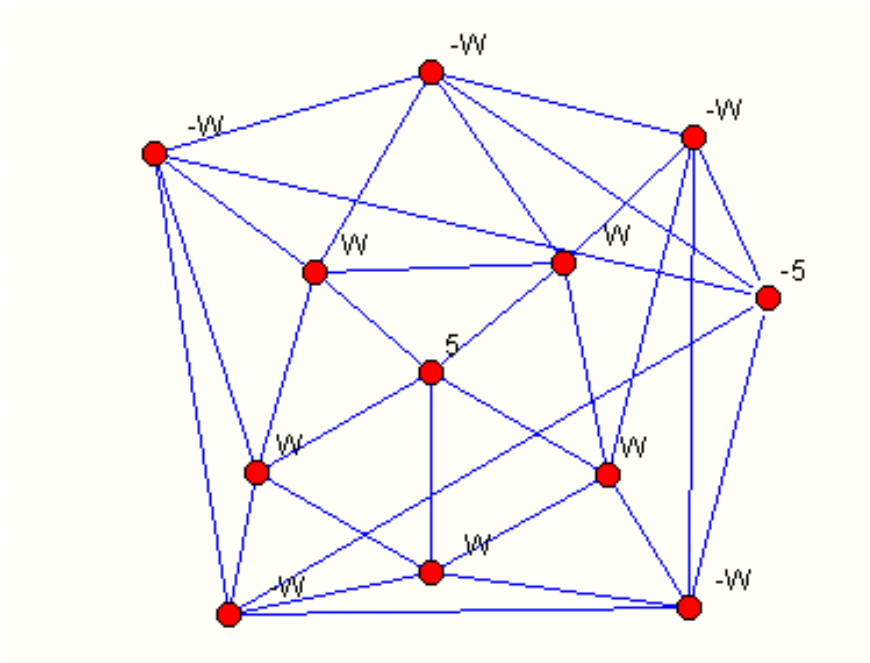


Figure 2: The graph of the icosahedron with the eigenvalues where  $W = \sqrt{5}$  and  $\lambda = \sqrt{5}$ .

graph spectrum contains an irrational eigenvalue then it must contain all conjugates of that eigenvalue with the same multiplicity.

We will give a brief explanation of what eigenvalues mean in a graph by using the icosahedron as an example. Consider the definition of eigenvalues of a matrix:  $Au = \lambda u$  for some vector  $u$ . For its  $i^{\text{th}}$  row, let  $u_i$  be the  $i^{\text{th}}$  component of  $u$ . Also for two vertices,  $v_i$  and  $v_j$ , we write  $v_i \sim v_j$  if  $v_i$  is adjacent to  $v_j$ . Then we have,

$$\sum_{v_j \sim v_i} u_j = \lambda u_i$$

We will derive a non-trivial eigenvector for  $\lambda = \sqrt{5}$  for the icosahedron as shown in Figure 2. We interpret the components of the eigenvector as vertex-weights. We can pick any vertex and find that the weight of the vertex times its eigenvalue equals the summation of eigenvector entries over all adjacent vertices. From Figure 2, we can see that

$$[5, \sqrt{5}, \sqrt{5}, \sqrt{5}, \sqrt{5}, \sqrt{5}, -\sqrt{5}, -\sqrt{5}, -\sqrt{5}, -\sqrt{5}, -\sqrt{5}, -5]^T$$

is an eigenvector of the adjacency matrix of the icosahedron corresponding to  $\lambda = \sqrt{5}$ .

The adjacency matrix of the icosahedron only records the information of vertices who are distance 1 related. Can we encode the information about the vertices that are distance 2 or 3 from each other? This idea corresponds to the notion of an association scheme [2].

Here we want to introduce the basis relations of an association scheme which, in the case of the icosahedron, are naturally defined by distance. For our research,  $R_1$  is only one relation which captures all pairs of vertices that are distance 1 from each other, in a collection of relations naturally defined on its vertex set. A pair of vertices is in the relation  $R_0$ ,  $R_2$  or  $R_3$  if they are distance 0, 2 or 3 from each other, respectively. We consider  $R_0, R_1, R_2$  and  $R_3$  and their corresponding adjacency matrices  $A_0=I$ ,  $A_1=A$ ,  $A_2$ , and  $A_3$ . Two vertices are in  $R_2$  if they are at distance 2 in the icosahedron, depicted in Figure 3. We form  $R_3$  in a similar manner which when represented as a graph, just 6 copies of  $K_2$ . Since there are 3 non-identity relations,  $R_1$ ,  $R_2$ , and  $R_3$ , the icosahedron is said to be a 3-class association scheme. In our project we consider 3-class schemes in which  $A_1$  has irrational eigenvalues.

The icosahedron has the intriguing property that the adjacency matrices of  $A_0$ ,  $A_1$ ,  $A_2$ , and  $A_3$ , are each expressible as a polynomial in  $A_1$  hence they can be diagonalized simultaneously. Let  $V_0$ ,  $V_1$ ,  $V_2$ , and  $V_3$  denote the eigenspaces for  $\theta = 5, \sqrt{5}, -\sqrt{5}, -1$ , respectively. From here, form the matrices  $U_0$ ,  $U_1$ ,  $U_2$ , and  $U_3$  where the columns of  $U_i$  form an orthonormal basis for  $V_i$ , for  $0 \leq i \leq 3$ . Amazingly, the columns of the four  $U_i$  correspond to a basis to each of the four eigenvalues of  $A_2$ ! Furthermore, the set of vectors from the columns of  $U_0$  and  $U_3$  form a basis for the eigenspace induced by the eigenvalue  $\lambda = 1$  of  $A_3$ , while those of  $U_1$  and  $U_2$  do the same for  $\lambda = -1$ .

We now observe an intriguing connection between graph theory and geometry. Consider  $U_1$  and plot the rows as points in 3-dimensional euclidean space. Taking the convex hull of these 12 points, we obtain the classical drawing of the icosahedron. That is, we started with the icosahedron as a combinatorial object and gathered information that led to a precise geometric drawing of the icosahedron! It is quite amazing that we could study a polyhedron



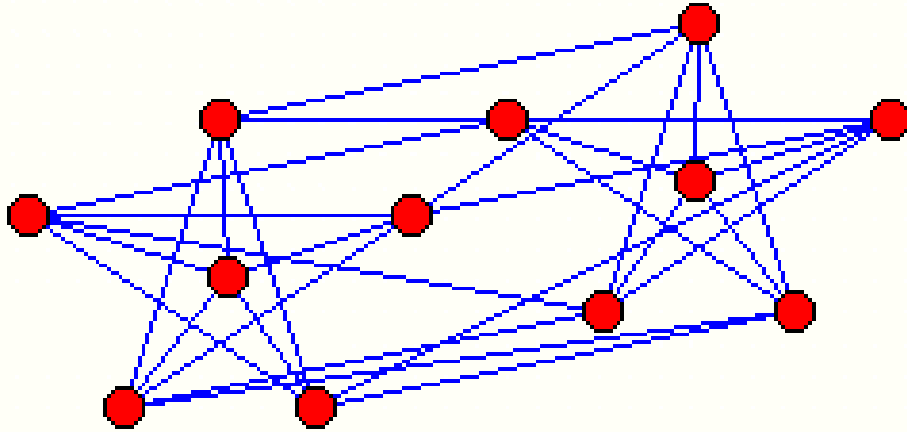


Figure 3: Graph of the distance 2 relation of the icosahedron

via its association scheme and rebuild the polyhedron from the information we collected from the association scheme.

Now, if we take the orthonormal basis for the eigenspaces and consider  $E_j = U_j U_j^\top$ , we obtain four  $12 \times 12$  matrices. It is known that  $\{E_j\}_{j=0}^3$  form a second vector space basis for the algebra spanned by the  $\{A_i\}_{i=0}^3$ . We are interested in Q-polynomial schemes. An association scheme is Q-polynomial if there exists an ordering of the eigenspaces and polynomials  $v_j$  such that  $\deg(v_j) \leq j$  and  $v_j(E_1) = E_j$  where multiplication is entrywise.

It has been conjectured that for  $d$  large enough, every association scheme without any disconnected relations is Q-polynomial if and only if  $R_1$  is a distance-regular graph for some ordering of the relations. This is still an open problem, and its proof would have implications that would reach to group theory, coding theory, and statistics. In this paper, we work towards classifying all association schemes having  $d = 3$  that are Q-polynomial with irrational eigenvalues. Such a classification may lend insight into the aforementioned conjecture for irrational schemes.

## 2 Math Background

Here, we aim to formally introduce the notion of an association scheme, which is fundamental to our project.

**Definition 2.1.** A symmetric  $d$ -class association scheme is a pair  $(X, \{R_i\}_{i=0}^d)$ ,  $X$  finite, satisfying the following:

(i)  $R_0 = \{(x, x) \mid x \in X\}$

- (ii)  $X \times X = R_0 \cup R_1 \cup R_2 \cup \dots \cup R_d$  and  $R_i \cap R_j = \emptyset$ , when  $i \neq j$
- (iii)  $(x, y) \in R_i$  if and only if  $(y, x) \in R_i, 0 \leq i \leq d$
- (iv) There exist  $p_{ij}^k \in \mathbb{Z}^+ \cup \{0\}$ , such that for all  $x, y \in X$  with  $(x, y) \in R_k$ ,  
 $p_{ij}^k = |\{z \in X : (x, z) \in R_i, (z, y) \in R_j\}|$ .

Let  $|X| = v$  and let  $k_i$  denote the valency of  $(X, R_i)$ . Thus  $v = \sum_{i=0}^d k_i$  since the relations  $(X, R_i)$  partition  $X \times X$ .

The relations  $(X, R_i)$  of a symmetric association scheme can be expressed by their adjacency matrices defined by:

$$[A_i]_{jk} = \begin{cases} 1 & \text{if } v_j \text{ is } i\text{-related to } v_k, \\ 0 & \text{otherwise.} \end{cases}$$

The following are known results for these matrices, where  $J$  is the all ones matrix [8, pg 44]:

- (i)  $\sum_{i=0}^d A_i = J$ ,
- (ii)  $A_0 = I$ ,
- (iii)  $A_i = A_i^\top$ , and
- (iv)  $A_i A_j = \sum_{k=0}^d p_{ij}^k A_k$ .

As a result of these properties,  $\{A_0, A_1, A_2, \dots, A_d\}$  generate a  $(d+1)$ -dimensional commutative algebra of symmetric matrices, called the Bose-Mesner algebra [17]. Since the  $A_i$ 's commute, they can be simultaneously diagonalized [8]. That is, there exists an orthogonal decomposition  $\mathbb{R}^n = V_0 \perp V_1 \perp \dots \perp V_d$  such that  $V_j$  is an eigenspace for each of the  $A_i$ . Let  $U_j$  be a matrix whose columns form a basis for  $V_j$ , that is  $\text{col}(U_j) = V_j$ . That is for all  $0 \leq i, j \leq d$ , there exists some eigenvalue  $\lambda_{ij}$  such that  $A_i U_j = \lambda_{ij} U_j$  for some  $\lambda_{ij}$ , an eigenvalue of  $A_i$  [2].

Clearly, the matrices  $A_i$  contain the same information as the relations  $R_i$ , therefore we sometimes write  $(X, \{A_i\}_{i=0}^d)$  in place of  $(X, \{R_i\}_{i=0}^d)$

At this point, it is instructive to introduce another example. We will consider the association scheme induced by the points and blocks of the Fano plane. The Fano plane, depicted in Figure 4, is a geometry in which every point is incident to 3 lines and every line is incident to 3 points. Furthermore, any two points are joined by a unique line and any two lines meet in a unique point. The point-line incidence graph of the Fano plane is called the Heawood

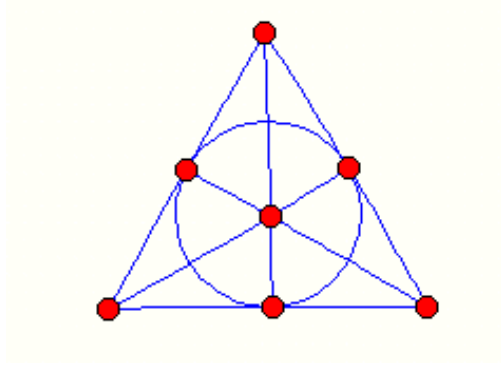


Figure 4: The Fano plane, Desarguesian projective plane  $PG(2,2)$ .

graph and is depicted in Figure 5. Relations 2 and 3 are depicted in Figure 6 and Figure 7, respectively.

In our example of the Heawood graph, we present the  $A_i$  and the  $U_i$ .  $A_0$  is the  $14 \times 14$  identity matrix, which we omit. We have:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

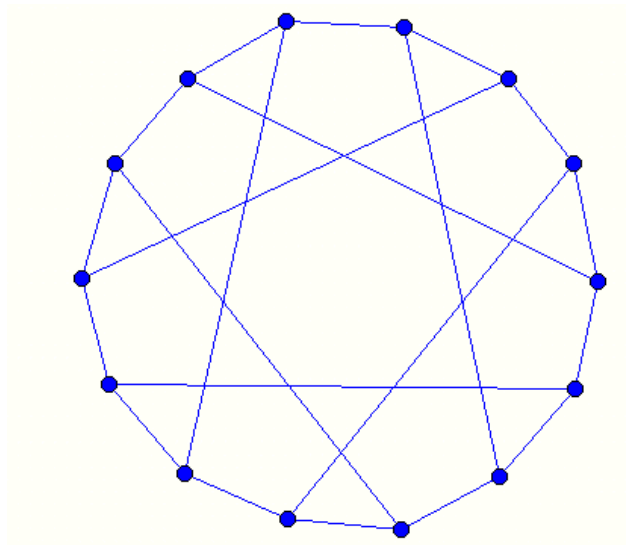


Figure 5: The Heawood graph.

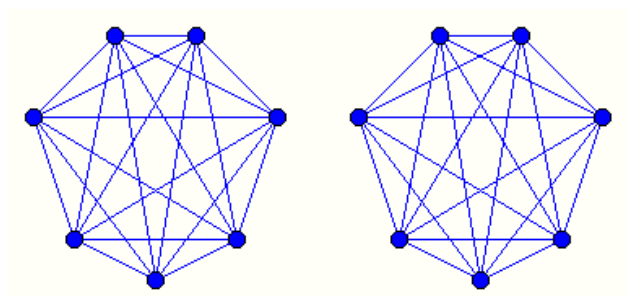


Figure 6: Relation 2 of the Heawood graph,  $2K_7$ .

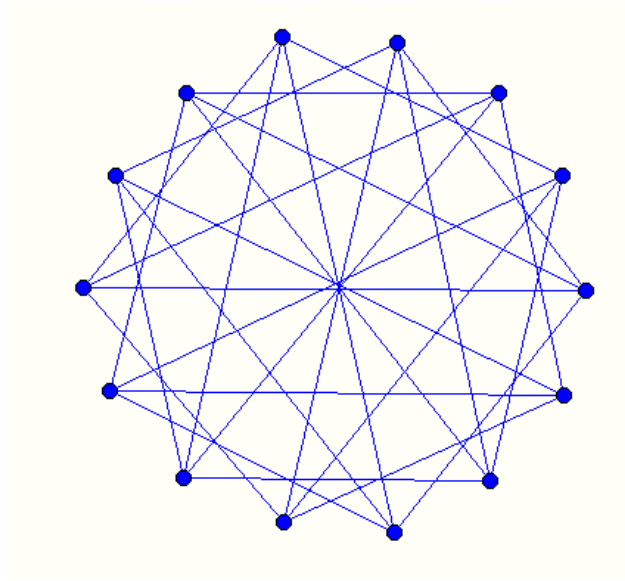


Figure 7: Relation 3 of the Heawood graph.

$$A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix},$$



$$U_2 = \begin{bmatrix} 0 & 0 & 0 & -1 & -\sqrt{2} & -1 \\ -1 & 0 & 0 & \sqrt{2} & 1 & \sqrt{2} \\ 0 & -1 & -\sqrt{2} & -1 & 0 & 0 \\ 1 & \sqrt{2} & 1 & 0 & -1 & -\sqrt{2} \\ -\sqrt{2} & -1 & 0 & 1 & \sqrt{2} & 1 \\ 0 & 0 & -1 & -\sqrt{2} & -1 & 0 \\ \sqrt{2} & 1 & \sqrt{2} & 0 & 0 & -1 \\ -1 & -\sqrt{2} & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$U_3 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}.$$

We denote  $m_j = \dim(V_j) = \dim(\text{col}(U_j))$ . Now  $(X, R_i)$  has valency  $k_i$ , so each row of  $A_i$  has exactly  $k_i$  entries of 1. Thus the all ones vector is an eigenvector for all  $A_i$  corresponding to the eigenvalue  $k_i$ . We know that one of the eigenspaces has dimension 1, since  $J$  is in the algebra of the  $\{A_i\}_{i=0}^d$  and  $J$  has  $v$  as an eigenvalue of the all ones eigenvector with multiplicity 1. We follow the convention that  $m_0 = 1$  and  $U_0$  is always a matrix with one column, a multiple of the all ones vector. Now after we make each  $U_j$  into an orthonormal basis for its respective eigenspace, we can consider  $E_j = U_j U_j^\top$  for  $j = 0, 1, \dots, d$ . The  $\{E_j\}_{j=0}^d$  form a vector space basis for the same Bose-Mesner algebra that the  $A_i$ 's do! Furthermore, the  $E_j$  are orthogonal projectors onto the eigenspaces  $V_j$ !

Here are the  $E_i$  from the Heawood graph, where  $\delta = 2\sqrt{2}$ :

$$E_0 = \frac{1}{14}J,$$





Here are some properties of the  $E_j$  [8, pg 45]:

- (i)  $m_j = \text{rank}(E_j)$ ,
- (ii)  $\sum_{i=0}^d E_j = I$ ,
- (iii)  $E_0 = \frac{1}{n}J$ , and
- (iv)  $E_i E_j = \delta_{ij} E_i$ .

Furthermore, we introduce the change-of-basis matrices P and Q defined by the following:

$$A_j = \sum_{i=0}^d P_{ij} E_i \quad (2.1)$$

and

$$E_j = \frac{1}{n} \sum_{i=0}^d Q_{ij} A_i. \quad (2.2)$$

Now, it turns out that  $P_{ij}$  is the eigenvalue of  $A_j$  for the eigenspace  $V_i$ , since  $A_j E_i = P_{ij} E_i$ . Now, since the  $A_i$  have integer entries, their minimal polynomials have integer coefficients with leading coefficient 1. Thus the entries of  $P_{ij} \in \mathbb{B}$  where  $\mathbb{B}$  is the set of algebraic integers [27, pg 14].

From, 2.1 and 2.2 we can derive the orthogonality relations of P and Q, that is  $PQ = nI$  and  $\Delta_k Q = P^\top \Delta_m^{-1}$  where  $\Delta_m$  and  $\Delta_k$  are diagonal matrices with  $[\Delta_m]_{ii} = m_i$  and  $[\Delta_k]_{ii} = k_i$ .

The following are the  $P$  and  $Q$  eigenmatrices for the Heawood graph. These are called, respectively, the first and second eigenmatrix of the association scheme. They are:

$$P = \begin{bmatrix} 1 & 3 & 6 & 4 \\ 1 & \sqrt{2} & -1 & -\sqrt{2} \\ 1 & -\sqrt{2} & -1 & \sqrt{2} \\ 1 & -3 & 6 & -4 \end{bmatrix}, Q = \begin{bmatrix} 1 & 6 & 6 & 1 \\ 1 & 2\sqrt{2} & -2\sqrt{2} & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -\frac{3}{2}\sqrt{2} & \frac{3}{2}\sqrt{2} & -1 \end{bmatrix}.$$

---

<sup>1</sup>For any two matrices,  $M$  and  $N$ , of the same dimensions,  $\text{trace}(MN^\top) = \text{SUM}(M \cdot N)$  where  $\text{SUM}(\cdot)$  is the sum of all entries in a matrix. Now observe  $A_i E_j^\top = A_i E_j = P_{ji} E_j$  While  $A_i \circ E_j = \frac{1}{n} Q_{ij} A_i$ .

## 2.1 The Krein Parameters

The  $p_{ij}^k$  values, or intersection numbers, that come from the adjacency matrices are non-negative integers that contain combinatorial information. We define similar parameters,  $q_{ij}^k$ , or dual intersection numbers, for the primitive idempotents, and investigate what information they can provide. The dual intersection numbers are not always integers or even rational, but they are always nonnegative and they contain information about the association scheme.

The Bose-Mesner algebra is closed under entrywise multiplication, since the  $A_i$  form a vector space basis and  $A_i \circ A_j = \delta_{ij}A_i$ . Thus the entrywise product of two idempotents must be expressible as a linear combination of idempotents. In this way, we define the Krein parameters to be

$$E_i \circ E_j = \frac{1}{n} \sum_{k=0}^d q_{ij}^k E_k.$$

So,  $\frac{q_{ij}^k}{n}$  is an eigenvalue of  $E_i \circ E_j$ .

**Theorem 2.2.** [8, pg 49] *The Krein parameters satisfy the following properties:*

- (i)  $q_{0j}^k = \delta_{jk}$
- (ii)  $q_{ij}^0 = \delta_{ij}m_j$
- (iii)  $q_{ij}^k = q_{ji}^k$
- (iv)  $q_{ij}^k m_k = q_{ik}^j m_j$
- (v)  $Q_{ij}Q_{ik} = \sum_l q_{jk}^l Q_{il}$

We arrange the  $q_{ij}^k$  values in  $d + 1$  matrices  $L_0^*, L_1^*, \dots, L_d^*$  defined by  $[L_i^*]_{kj} = q_{ij}^k$ , which we will call the dual intersection matrices.

For the Heawood graph, we have the following dual intersection matrices:  $L_0^* = I_4$ ,

$$L_1^* = \begin{bmatrix} 0 & 6 & 0 & 0 \\ 1 & \frac{10+\sqrt{2}}{4} & \frac{10-\sqrt{2}}{4} & 0 \\ 0 & \frac{10-\sqrt{2}}{4} & \frac{10+\sqrt{2}}{4} & 1 \\ 0 & 0 & 6 & 0 \end{bmatrix}, L_2^* = \begin{bmatrix} 0 & 0 & 6 & 0 \\ 1 & \frac{10-\sqrt{2}}{4} & \frac{10+\sqrt{2}}{4} & 1 \\ 1 & \frac{10+\sqrt{2}}{4} & \frac{10-\sqrt{2}}{4} & 0 \\ 0 & 6 & 0 & 0 \end{bmatrix}, L_3^* = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Also, the following are the dual intersection matrices for the icosahedron:

$$L_1^* = \begin{bmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & \frac{6}{5} & 0 & \frac{9}{5} \\ 0 & 0 & 3 & 0 \end{bmatrix}, L_2^* = \begin{bmatrix} 0 & 0 & 5 & 0 \\ 0 & 2 & 0 & 3 \\ 1 & 0 & 4 & 0 \\ 0 & 3 & 0 & 2 \end{bmatrix}, L_3^* = \begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 \\ 0 & \frac{9}{5} & 0 & \frac{6}{5} \\ 1 & 0 & 2 & 0 \end{bmatrix}.$$

The following theorem demonstrates the important property that the Krein parameters are nonnegative; these are called the Krein conditions.

**Theorem 2.3.** [8]: For any association scheme with second eigenmatrix  $Q$ ,

$$vm_k q_{ij}^k = \sum_{l=0}^d k_l Q_{li} Q_{lj} Q_{lk} \geq 0$$

It is common to study special types of association schemes. Two important classes are P-polynomial and Q-polynomial association schemes. Here is the definition of metric association scheme, which we will see later is equivalent to P-polynomial:

An association scheme is metric provided for some ordering of its relations, the following holds:

$$p_{ij}^k \neq 0 \text{ implies } |i - j| \leq k \leq i + j \text{ and for all } i, j \text{ such that } i + j \leq d, p_{ij}^{i+j} \neq 0 .$$

It is not hard to see that a scheme is metric with respect to some fixed ordering of its relations precisely when  $(X, R_1)$  is a distance-regular graph. In this case,  $R_i$  is exactly the distance  $i$  relation of this graph.

The following was proven by Delsarte.

**Theorem 2.4.** [8] Let  $(X, \{R_i\}_{i=0}^d)$  be an association scheme with some ordering of its  $d + 1$  relations. The following are equivalent:

- (i)  $(X, \{R_i\}_{i=0}^d)$  is metric;
- (ii)  $p_{1i}^{i+1} \neq 0, p_{1i}^k = 0$  when  $k > i + 1$  ( $i = 0, \dots, d - 1$ );
- (iii) there are polynomials  $p_i$  of degree  $i$  such that  $A_i = p_i(A_1)$ ;
- (iv)  $(X, \{R_i\}_{i=0}^d)$  is P-polynomial i.e. there exist  $z_0, \dots, z_d \in \mathbb{R}$  and polynomials  $p_i$  of degree  $i$  such that  $P_{ji} = p_i(z_j)$  for all  $0 \leq i, j \leq d$ .

It is interesting to note that the previous theorem allows us to view the P-polynomial property in terms of distance-regular graphs. Partly because of this connection, much is known about P-polynomial schemes. On the other hand, there is no similar combinatorial or geometric interpretation of the cometric property; much less is known about these objects. An association scheme is cometric provided the following holds for some ordering of its primitive idempotents:

$$q_{ij}^k \neq 0 \text{ implies } |i - j| \leq k \leq i + j \text{ and for all } i, j \text{ such that } i + j \leq d, q_{ij}^{i+j} \neq 0 .$$

The following, also due to Delsarte, essentially gives four equivalent definitions of Q-polynomial association schemes.

**Theorem 2.5.** [8] The following are equivalent, where  $(X, \{R_i\}_{i=0}^d)$  is an association scheme with some ordering of its  $d + 1$  primitive idempotents:

- (i)  $(X, \{R_i\}_{i=0}^d)$  is cometric;
- (ii)  $q_{1i}^{i+1} \neq 0, q_{1i}^k = 0$  when  $k > i + 1$  ( $i = 0, \dots, d - 1$ );
- (iii) there are polynomials  $q_i$  of degree  $i$  such that  $E_i = q_i(E_1)$ , where the polynomial in this case is applied entrywise;
- (iv)  $(X, \{R_i\}_{i=0}^d)$  is Q-polynomial i.e. there exist  $z_0, \dots, z_d \in \mathbb{R}$  and polynomials  $q_i$  of degree  $i$  such that  $Q_{ji} = q_i(z_j)$  for all  $0 \leq i, j \leq d$ .

**Remark 2.6.** For the rest of this section, when  $E_j$  is an idempotent and  $f(t)$  is a polynomial, the expression  $f(E_j)$  shall be interpreted as entrywise application of the polynomial.

Recall that  $[L_i^*]_{kj} = q_{ij}^k$ . From (ii) we see that in a Q-polynomial association scheme,  $L_1^*$  is a tridiagonal matrix with nonzero entries directly above and below the diagonal. We denote the diagonal entries of  $L_1^*$  by  $a_i^* = q_{1i}^i$ , the entries just above the diagonal are denoted  $b_i^* = q_{1,i+1}^i$  and those just below the diagonal are denoted  $c_i^* = q_{1,i-1}^i$ .

We remind the reader that, just as with the  $L_i^*$ , the the rows and the columns of the matrix  $Q$  are indexed from 0 up to  $d$ . From the previous theorem we know that, when the scheme is Q-polynomial, there exist polynomials  $q_i$  of degree  $i$  that map column 1 of  $Q$ , entrywise, to column  $i$  of  $Q$ . In the Heawood graph example, these polynomials are:

$$q_0(t) = 1, \quad q_1(t) = t, \quad q_2(t) = \frac{t^2 - (\frac{5}{2} + \frac{1}{4}\sqrt{2})t - 6}{\frac{5}{2} - \frac{1}{4}\sqrt{2}} = \frac{t^2 - a_1^*t - m}{c_2^*}, \text{ and}$$

$$\begin{aligned} q_3(t) &= \frac{1}{6} \frac{t^3 - (5 + \frac{1}{2}\sqrt{2})t^2 + \left(\left(\frac{5}{2} + \frac{1}{4}\sqrt{2}\right)^2 - 6\right)t + 15 + \frac{3}{2}\sqrt{2}}{\frac{5}{2} - \frac{1}{4}\sqrt{2}} - \frac{1}{6} \left(\frac{5}{2} - \frac{1}{4}\sqrt{2}\right)t \\ &= \frac{t^3 - (a_1^* + a_2^*)t^2 + a_1^*a_2^* - m}{c_3^*c_2^*}t + \frac{b_1^*}{c_3^*}. \end{aligned}$$

The  $d + 1$  polynomials  $q_j(t)$  are found in general by the following recursive formula:

$$q_0(t) = 1, \quad q_1(t) = t, \text{ and} \quad q_{j+1}(t) = \frac{1}{c_{j+1}^*}[(t - a_j^*)q_j(t) - b_{j-1}^*q_{j-1}(t)].$$

## 2.2 The Galois Automorphism

We will now consider the splitting field of a symmetric association scheme. The splitting field  $K$  of a commutative association scheme, introduced by Munemasa [32], is defined to be the extension of the rationals by the adjunction of all the entries of the P matrix (or the Q matrix since  $\Delta_k Q = P^\top \Delta_m$ ). Let  $L$  be the field obtained by adjoining all Krein parameters to the field of rational numbers. From Theorem 2.3,  $L$  is a subfield of  $K$ . Munemasa proved the following.

**Theorem 2.7.** [32] *Let  $K$  be the splitting field of a symmetric association scheme. If  $L$  is any subfield of  $K$  containing all the Krein parameters, then  $\text{Gal}(K/L)$  is contained in the center of  $\text{Gal}(K/\mathbb{Q})$ .*

Recall that a number field is a subfield of  $\mathbb{C}$ . Observe that both the splitting field  $L$  and the field  $K$ , which we shall call the Krein field, are both number fields.

**Definition 2.8.** *A cyclotomic field is a number field obtained by adjoining a complex primitive root of unity to  $\mathbb{Q}$ .*

**Corollary 2.9.** *If the Krein parameters are all rational, then the splitting field  $K$  is contained in a cyclotomic number field.*

Another tool that will be of use to us is Galois theory [25, Chap. 32] which establishes a correspondence between subfields of  $K$  and subgroups of a group called the Galois group. Consider the Galois group,  $\text{Gal}(L/\mathbb{Q})$  of the field extension  $K/L$ ; this is the group of automorphisms of  $K$  which fix every rational number. Note that  $\text{Gal}(L/K)$  is a subgroup of  $\text{Gal}(L/\mathbb{Q})$ .

Let  $\sigma \in \text{Gal}(L/\mathbb{Q})$  and  $M \in \text{span}(\{A_i\}_{i=0}^d)$  the algebra generated by the adjacency matrices. Define  $\bar{\sigma}(M) = M^\sigma$  to be matrix obtained by applying  $\sigma$  to each entry of  $M$ .

**Theorem 2.10.** [32] *Let  $(X, \{A_i\}_{i=0}^d)$  be an association scheme with splitting field  $L$  and Krein field  $K$ . Let  $\sigma \in \text{Gal}(L/\mathbb{Q})$ . Then  $\bar{\sigma}(M)$  is an algebra automorphism if and only if  $\sigma$  fixes each element of  $K$ .*

From Suzuki [33], we have the following results, very important to our research:

**Theorem 2.11.** [33] *Let  $(X, \{R_i\}_{i=0}^d)$  with  $k_1^* > 2$  be a  $\mathbb{Q}$ -polynomial association scheme with respect to the ordering  $E_0, E_1, \dots, E_d$  of the primitive idempotents.*

(1) *Suppose the association scheme is  $\mathbb{Q}$ -polynomial with respect to another ordering. Then the new ordering is one of the following:*

- (I)  $E_0, E_2, E_4, E_6, \dots, E_5, E_3, E_1,$
- (II)  $E_0, E_d, E_1, E_{d-1}, E_2, E_{d-2}, E_3, E_{d-3}, \dots,$
- (III)  $E_0, E_d, E_2, E_{d-2}, E_4, E_{d-4}, \dots, E_{d-5}, E_5, E_{d-3}, E_3, E_{d-1}, E_1,$
- (IV)  $E_0, E_{d-1}, E_2, E_{d-3}, E_4, E_{d-5}, \dots, E_5, E_{d-4}, E_3, E_{d-2}, E_1, E_d,$  or
- (V)  $d = 5$  and  $E_0, E_5, E_3, E_2, E_4, E_1.$

(2) *Let  $q_{i,j}^h$  be the Krein parameters with respect to the original ordering. Suppose  $d \geq 3$ . Then :*

- (I) *holds if and only if  $q_{1,1}^1 = \dots = q_{1,d-1}^{d-1} = 0 \neq q_{1,d}^d.$*

(II) holds if and only if  $q_{1,d}^d \neq 0 = q_{2,d}^d = \dots = q_{d,d}^d$ .

(III) holds if and only if one of the following holds:

(i)  $d = 3$ , and  $q_{1,3}^3 = 0 \neq q_{2,3}^3$ .

(ii)  $d = 4$ ,  $q_{1,4}^4 = q_{3,4}^4 = 0$ , and  $q_{2,4}^4 \neq 0 \neq q_{2,3}^4$ , or

(iii)  $d \geq 5$ ,  $q_{2,d}^d \neq 0 = q_{1,d}^d = q_{3,d}^d = \dots = q_{d,d}^d$ . Moreover if  $d = 2e - 1$ , then  $q_{i,j}^j \neq 0$  implies  $j = e$  and if  $d = 2e$ , then  $q_{i,j}^j \neq 0$  if and only if  $j = e$  or  $j = e + 1$ .

(IV) holds if and only if one of the following holds:

(i)  $d = 3$ ,  $q_{1,2}^2 \neq 0 = q_{3,2}^2$ , or

(ii)  $d \geq 4$ ,  $q_{2,d}^{d-1} = \dots = q_{d,d}^{d-1} = 0$ . Moreover, if  $d = 2e$ , then  $q_{i,j}^j \neq 0$  implies  $j = e$  and if  $d = 2e + 1$ , then  $q_{i,j}^j \neq 0$  if and only if  $j = e$  or  $j = e + 1$ .

(V) holds if and only if  $q_{1,5}^5 = q_{2,5}^5 = q_{4,5}^5 = q_{5,5}^5 = 0 \neq q_{3,5}^5$  and  $q_{3,4}^5 = 0$ .

(3) The association scheme has at most two  $Q$ -polynomial structures.

Note that any Galois automorphism of any  $Q$ -polynomial association scheme has group size at most two immediately follows from the fact that the scheme has at most two  $Q$ -polynomial structures.

## 2.3 The Eigenvalues of Strongly Regular Graphs

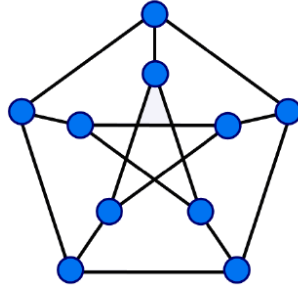
In Section 2.1, we introduced distance-regular graphs. Here we consider only distance-regular graphs of diameter two and derive their eigenvalues. These graphs are called strongly regular graphs. A graph  $\Gamma$  with vertex set  $X$  and adjacency relation  $R$  is strongly regular with parameters  $v, k, a$ , and  $c$  if:

- $\Gamma$  has  $v$  vertices;
- $\Gamma$  is regular with valency  $k$ ;
- any two adjacent vertices have exactly  $a$  common neighbors;
- any two distinct non-adjacent vertices have exactly  $c$  common neighbors.

In this case, we say  $\Gamma$  has parameters  $(v, k, a, c)$ . In fact, strongly regular graphs are exactly distance-regular graphs with diameter two.

Here are some of the small strongly regular graphs and their parameters:

- five cycle  $(5, 2, 0, 1)$



Petersen Graph: parameters  $(10, 3, 0, 1)$

Figure 8: Petersen graph.

- graph for  $GF(9)$   $(9,4,1,2)$
- Petersen graph (Figure 8)  $(10,3,0,1)$
- Shrikhande graph (Figure 9)  $(16,6,2,2)$
- Cartesian product  $K_4 \times K_4$   $(16,6,2,2)$
- complete bipartite graph  $K_{m,m}$   $(2m,m,0,m)$   $(m \geq 2)$

If  $A$  is the adjacency matrix of  $\Gamma$ , the above defining conditions are equivalent to the two matrix equations

$$AJ = kJ, \tag{2.3}$$

$$A^2 = kI + aA + c(J - I - A). \tag{2.4}$$

Here, we derive the eigenvalues for strongly regular graphs from the above equations. From (2.3) we have,

$$(A - kI)J = 0$$

and from (2.4) we also have,

$$A^2 + (c - a)A + (c - k)I = cJ.$$

Then multiplying both sides on the left by  $A - kI$ , we obtain

$$(A - kI)(A^2 + (c - a)A + (c - k)I) = c(A - kI)J = 0.$$

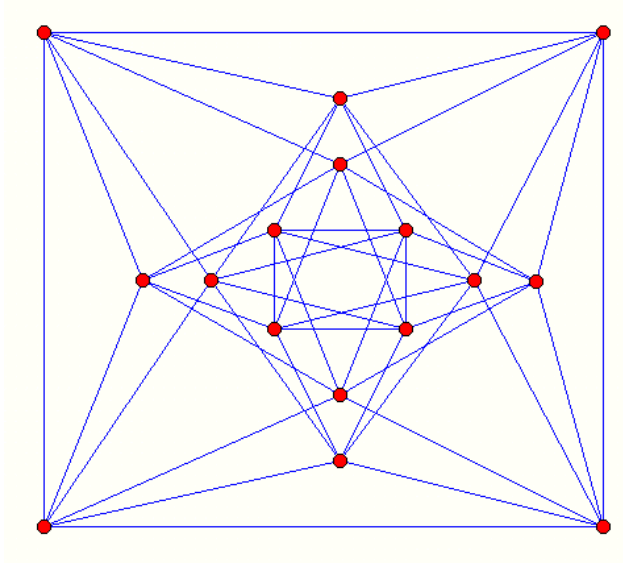


Figure 9: Shrikande graph.

This gives us the minimal polynomial for  $\Gamma$ :  $m_A(t) = (t - kI)(t^2 + (c - a)t + (c - k)I)$ . Thus the valency  $k$  is an eigenvalue of  $\Gamma$ . If  $\theta$  and  $\tau$  denote the other two eigenvalues for  $\Gamma$ , we find

$$\theta, \tau = \frac{a - c \pm \sqrt{(a - c)^2 + 4(k - c)}}{2}.$$

It is straightforward to use the above matrix equations to show that if  $A$  is the adjacency matrix of a strongly regular graph, then  $I, A, J - I - A$  are the adjacency matrices of a two-class association scheme on vertex set  $X$ . Conversely, any association scheme with two classes arises from a strongly regular graph, typically in two ways. Suppose we have a two-class association scheme with minimal idempotents  $E_0, E_1$  and  $E_2$ . Then

$$A_0 = E_0 + E_1 + E_2 \quad \text{and} \quad A_1 = kE_0 + \theta E_1 + \tau E_2.$$

These equations determine two columns of the eigenmatrix  $P$ . Since  $A_2 = J - I - A_1$ , we also have  $A_2 = (v - 1 - k)E_0 - (\theta + 1)E_1 - (\tau + 1)E_2$ . Therefore

$$P = \begin{bmatrix} 1 & k & v - 1 - k \\ 1 & \theta & -\theta - 1 \\ 1 & \tau & -\tau - 1 \end{bmatrix}$$

from which we compute that

$$Q = \frac{1}{\theta - \tau} \begin{bmatrix} \theta - \tau & k + (v - 1)\tau & k + (v - 1)\theta \\ \theta - \tau & v - k + \tau & k - v - \theta \\ \theta - \tau & \tau - k & k - \theta \end{bmatrix}.$$



Now we observe that a strongly regular graph always generates a Q-polynomial association scheme. Since,  $d = 2$  we only have to prove that there exists a quadratic polynomial that maps the second column of  $Q$  to the third column. If the entries of the second column are distinct, we observe that there must exist a parabola that goes through all three points. Since the eigenvalues are distinct for all connected strongly regular graphs, every primitive 2-class scheme is Q-polynomial.

## 2.4 The Bannai-Ito Conjectures

We just saw that strongly regular graphs provide association schemes that are both P-polynomial and Q-polynomial. We say that the association scheme has (P and Q)-polynomial structure. However, many association schemes are known that have only a P-polynomial structure, and likewise for Q-polynomial structures.

In 1984, Bannai and Ito conjectured the following [2, p312]:

- (1) If  $d$  is sufficiently large, a primitive association scheme is P-polynomial if and only if it is Q-polynomial.
- (2) If an association scheme  $(X, \{R_i\}_{i=0}^d)$  has a (P and Q)-polynomial structure with sufficiently large diameter, then  $(X, \{R_i\}_{i=0}^d)$  is known or, in some sense, can be attained from a known example.
- (3) For fixed valency  $k > 2$ , there are finitely many distance-regular graphs of valency  $k$ .

It is this last conjecture that is widely referred to as *the* Bannai-Ito Conjecture. We will discuss these three conjectures in turn.

The first conjecture is the most relevant to our investigation. We know of no Q-polynomial scheme with more than 5 classes which is primitive and yet not P-polynomial. In view of the following theorem, it seems as though a association scheme having the P-polynomial and Q-polynomial property with sufficiently large class number will have rational eigenvalues.

**Theorem 2.12.** [2, Theorem III.7.11] *All (P and Q)-polynomial association schemes of diameter  $d \geq 34$  are rational.*

We learned from Tatsuro Ito [pers. comm.] that this theorem has been improved to  $d \geq 6$  by Garth Dickie in his PhD thesis. If the conjecture is true, then there should be no primitive Q-polynomial irrational schemes with  $d \geq 6$ . This motivates us to study the irrational schemes with  $d \leq 5$  and find the structures of the examples that arise.

Significant progress toward the second conjecture appears in the celebrated theorem of Leonard [8, Sec. 8.1] and this conjecture is the focus “Terwilliger Program.” We do not discuss this further in our report.

Now, let us discuss the third conjecture. Let’s first give the definition for distance-transitive graphs.

**Definition 2.13.** A distance-transitive graph is a connected graph  $\Gamma$  such that for every four (not necessarily distinct) vertices  $x, y, u, v$  in  $V(\Gamma)$  with  $d(x, y) = d(u, v)$ , there exists an automorphism  $\tau$  of  $\Gamma$  such that  $\tau(x) = u$  and  $\tau(y) = v$  both hold.

It is straight-forward to see that distance-transitive graphs are distance-regular graphs. Cameron, Praeger, Saxl and Seitz [11] proved that there are only finitely many finite distance-transitive graphs of fixed valency greater than two. The first class of distance-regular graphs for which the Bannai-Ito conjecture was shown is the class of regular generalized  $n$ -gons. Feit and Higman [19] showed that a regular generalized  $n$ -gon has either valency 2 or  $n \in \{3, 4, 6, 8, 12\}$ . In addition, Damerell [16], and Bannai and Ito [3] have independently shown that there are only finitely many Moore graphs with valency at least three. Bannai and Ito [4–7] showed that their conjecture holds for valencies  $k = 3, 4$ , as well as for the special class of bipartite distance-regular graphs. Koolen and Moulton [23] also showed that the conjecture holds for distance-regular graphs of fixed valency  $k = 5, 6$  or  $7$  and for triangle-free distance-regular graphs of fixed valency  $k = 8, 9$  or  $10$  [24]. After 25 years of progress, a complete proof was given by Bang, Dubickas, Koolen, and Moulton [1].

In addition to the Bannai-Ito Conjecture, we also have a dual conjecture that for any fixed  $m_1 > 2$ , there are only finitely many cometric association schemes  $(X, \{R_i\}_{i=0}^d)$  with the property that the first idempotent in a  $\mathbb{Q}$ -polynomial ordering has rank  $m_1$ . This conjecture was proved by Martin and Williford in 2007. A key step in the proof is to control the irrational eigenvalues of the scheme. The following is their main theorem.

**Theorem 2.14.** [28] *Let  $m \geq 2$  and  $n \geq 1$ . Then there are, up to isomorphism, only finitely many symmetric association schemes  $(X, \{R_i\}_{i=0}^d)$  with the following properties*

- (i) *for some  $1 \leq j \leq d$ ,  $Q_{0j} = m > Q_{ij}$  for all  $i = 1, \dots, d$ , and*
- (ii) *each eigenvalue  $P_j i$  of each adjacency matrix  $A_i$  has minimal polynomial (over the rationals) of degree at most  $n$ .*

Even though the proof is finished, cometric association schemes have not been widely studied. We know only a few examples of  $\mathbb{Q}$ -polynomial association schemes that are not  $\mathbb{P}$ -polynomial.

### 3 Examples

We present the known examples of  $\mathbb{Q}$ -polynomial association schemes with irrational eigenvalues in the following table. The column on the left is the number  $d$  of classes of the association scheme while the row on the top gives the multiplicity of the first eigenvalue, denoted  $m_1$ .

### Examples of schemes with irrational eigenvalues

| $d$   | $m_1 = 2$       | $m_1 > 2$                              |
|-------|-----------------|--|
| 2     | 5-gon           | conference graphs                      |
| 3     | 7-gon           | Taylor graphs and symmetric designs    |
| 4     | 8-gon and 9-gon | Hadamard graphs                        |
| 5     | 10 and 11-gon   | infinite family of open parameter sets |
| $> 6$ | polygons        | no known examples                      |

At the end of this report, we delve much deeper into the  $d = 3$  case, but in this section, we will discuss polygons, conference graphs, symmetric designs, Taylor graphs, and Hadamard graphs.

We say that an association scheme  $(X, \{R_i\}_{i=0}^d)$  is imprimitive if some graph of the scheme  $(X, R_i)$  ( $i \neq 0$ ) is disconnected. All of the examples in the above table, except for the conference graphs and the polygons of prime order, are imprimitive. In fact, the polygons of prime order (with  $m_1 = 2$ ) and the conference graphs (with  $d = 2$ ) give us two infinite families of primitive irrational Q-polynomial schemes. We conjecture that any irrational Q-polynomial association scheme with  $d, m_1 > 2$  is imprimitive.

We now discuss two particular kinds of imprimitive association schemes. They are called and Q-bipartite. Let  $(X, \{R_i\}_{i=0}^d)$  be a Q-polynomial association scheme, with the natural ordering of its eigenspaces and second eigenmatrix  $Q$ . Since  $Q$  is invertible and each column of  $Q$  is a polynomial the first column of  $Q$ , we have that the entries of the first column of  $Q$  are distinct. Order the relations so that  $Q_{01} > Q_{11} > \dots > Q_{d1}$ . We say  $(X, \{R_i\}_{i=0}^d)$  is if  $R_i$  is disconnected for all even  $i$  and Q-bipartite if  $R_d = wK_2$  for some  $w \in \mathbb{Z}^+$  where  $K_2$  denotes the complete graph on two vertices. As we will explain shortly, symmetric designs give us a infinite family of Q-antipodal schemes and Taylor graphs give us an infinite family of Q-bipartite schemes.

Recall the diagonal entries of  $L_1^*$  are denoted by  $a_i^* = q_{1i}^i$ , the entries just above the diagonal by  $b_i^* = q_{1,i+1}^i$  and those just below the diagonal by  $c_i^* = q_{1,i-1}^i$ .

**Theorem 3.1.** ([33],[15],[36]) *Let  $(X, \{R_i\}_{i=0}^d)$  be an imprimitive Q-polynomial association scheme with respect to the ordering  $E_0, E_1, \dots, E_d$  of its primitive idempotents. If  $m_1 > 2$ , then at least one of the following holds:*

- (i)  $(X, \{R_i\}_{i=0}^d)$  is Q-bipartite: i.e.,  $a_i^* = 0$  for all  $i = 0, 1, \dots, d$ .
- (ii)  $(X, \{R_i\}_{i=0}^d)$  is Q-antipodal: i.e.,  $b_i^* = c_{d-i}^*$  for all  $i = 0, 1, \dots, d$  except possibly for  $i = \lfloor \frac{d}{2} \rfloor$ .

In the following sections, we consider each family of examples appearing in the table above. In each case, we describe how Theorem 3.1 applies to these examples.

## 3.1 The Polygons

The polygons induce association schemes which frequently have irrational eigenvalues. Let  $C_n$  be the cycle of length  $n$ . The  $i^{\text{th}}$  relation of the association scheme induced by the

polygon of order  $n$  is the  $i^{\text{th}}$  distance relation of  $C_n$ . For  $n = 3, 4, 6$  the polygons have rational eigenvalues, but in all other cases some irrational eigenvalues arise. On the other hand, a polygon of length  $n$  will induce a primitive association scheme if and only if  $n$  is prime. In this subsection, let us take on the task of calculating all of the eigenvalues of the polygons.

We can think of  $C_n$ , with adjacency matrix  $A$ , as the union of two directed cycles, one oriented clockwise and one oriented counterclockwise,  $C_{cw}$  and  $C_{ccw}$  respectively <sup>2</sup>. Let  $A_{cw}$  be the adjacency matrix for  $C_{cw}$  and let  $A_{ccw}$  be the adjacency matrix for  $C_{ccw}$ . Note that  $A = A_{cw} + A_{ccw}$ . Also note that  $A_{cw}$  and  $A_{ccw}$  have the same eigenspaces since they are adjacency matrices of isomorphic directed graphs. It is easy to see that the characteristic equation for  $A_{cw}$  is  $t^n = 1$  (likewise for  $A_{ccw}$ ). Let  $\lambda$  be a primitive  $n^{\text{th}}$  root of unity in  $\mathbb{C}$ , so that  $\lambda^i$  satisfies the characteristic equation for  $0 \leq i \leq n - 1$ . Notice that  $A_{ccw} = A_{cw}^{n-1}$  and so it is easy to check that every eigenvector of  $A_{cw}$  is also an eigenvector of  $A$ ; indeed for an eigenvector  $x \in \mathbb{R}^n$  of  $A_{cw}$ ,  $Ax = (A_{cw} + A_{ccw})x = (A_{cw} + A_{cw}^{n-1})x = (\lambda^i + \lambda^{n-i})x$  for some  $i = 0, 1, \dots, n - 1$ . Thus  $x$  is an eigenvector for  $A$  with eigenvalue  $\lambda^i + \lambda^{n-i}$ !

So all of the eigenvalues of  $A_1$ , including multiplicities, are  $\lambda^i + \lambda^{n-i}$  for  $i = 0, 1, \dots, n - 1$ . For example, the 6-cycle has eigenvalues  $2, 1, -1, -2, -1, 1$ . Following Theorem 2.4(iv) and dual to the recurrence relation at the end of Section 2.1, we have the polynomials:  $p_0(t) = 1, p_1(t) = t$ , and  $p_{j+1}(t) = \frac{1}{c_{j+1}}[(t - a_j)p_j(t) - b_{j-1}p_{j-1}(t)]$  giving  $A_i = p_i(A_1)$ . For the polygon, this simplifies to  $p_{j+1}(t) = tp_j(t) - p_{j-1}(t)$  for  $j = 2, 3, 4, \dots, d - 1$  since for these values of  $j$ ,  $a_j = 0, b_j = 1 = c_j$ . We can use this recurrence to calculate the eigenvalues for any  $A_i$ , since we know that the polygons are distance-regular graphs. In particular, if  $A_1$  has rational eigenvalues, so do  $A_2, \dots, A_d$ . We are therefore content to consider only the spectrum of  $A_1$ .

Let us begin with the example of the 12-cycle,  $C_{12}$ . Choosing  $\lambda = e^{\sqrt{-1}\pi/6}$  we have eigenvalues

|                              |   |            |   |   |    |             |    |
|------------------------------|---|------------|---|---|----|-------------|----|
| $i$                          | 0 | 1          | 2 | 3 | 4  | 5           | 6  |
| $\lambda^i + \lambda^{12-i}$ | 2 | $\sqrt{3}$ | 1 | 0 | -1 | $-\sqrt{3}$ | -2 |

and we have

$$L_1 = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{bmatrix},$$

---

<sup>2</sup>In the association scheme of  $C_n$ , the relation  $R_i$  is the adjacency relation of a copy of  $C_{\frac{n}{a}}$  where  $a = \gcd(n, i)$ . This is easily seen by choosing  $Z_n$  as vertex set for  $C_n$ .

$$P = \begin{bmatrix} 1 & 2 & 2 & 2 & 2 & 2 & 1 \\ 1 & \sqrt{3} & 1 & 0 & -1 & -\sqrt{3} & -1 \\ 1 & 1 & -1 & -2 & -1 & 1 & 1 \\ 1 & 0 & -2 & 0 & 2 & 0 & -1 \\ 1 & -1 & -1 & 2 & -1 & -1 & 1 \\ 1 & -\sqrt{3} & 1 & 0 & -1 & \sqrt{3} & -1 \\ 1 & -2 & 2 & -2 & 2 & -2 & 1 \end{bmatrix}.$$

Here we present the the minimal polynomials of the first few polygons  $C_n$ ,  $4 \leq n \leq 30$ , where  $\phi_n(t)$  denotes the minimal polynomial of  $C_n$ .

| order( $n$ ) | minimal polynomial $\phi_n(t)$   |
|--------------|--|
| 4            | $t(t-2)(t+2)$  |
| 5            | $(t-2)(t^2+t-1)$   |
| 6            | $(t-1)(t-2)(t+2)(t+1)$   |
| 7            | $(t-2)(t^3+t^2-2t-1)$  |
| 8            | $\phi_4(t)(t^2-2)$   |
| 9            | $(t-2)(t+1)(t^3-3t+1)$   |
| 10           | $\phi_5(t)(t+2)(t^2-t-1)$  |
| 11           | $(t-2)(t^5+t^4-4t^3-3t^2+3t+1)$  |
| 12           | $\phi_6(t)t(-3+t^2)$   |
| 13           | $(t-2)(t^6+t^5-5t^4-4t^3+6t^2+3t-1)$                                       |
| 14           | $\phi_7(t)(t+2)(t^3-t^2-2t+1)$   |
| 15           | $\phi_5(t)(t+1)(t^4-t^3-4t^2+4t+1)$  |
| 16           | $\phi_8(t)(t^4-4t^2+2)$  |
| 17           | $(t-2)(t^8+t^7-7t^6-6t^5+15t^4+10t^3-10t^2-4t+1)$                          |
| 18           | $\phi_9(t)(t-1)(t+2)(t^3-3t-1)$  |
| 19           | $(t-2)(t^9+t^8-8t^7-7t^6+21t^5+15t^4-20t^3-10t^2+5t+1)$                    |
| 20           | $\phi_t(t)t(t^4-5t^2+5)$   |
| 21           | $\phi_7(t)(t+1)(t^6-t^5-6t^4+6t^3+8t^2-8t+1)$                              |
| 22           | $\phi_{11}(t)(t+2)(t^5-t^4-4t^3+3t^2+3t-1)$                                |
| 23           | $(t-2)(t^{11}+t^{10}-10t^9-9t^8+36t^7+28t^6-56t^5-35t^4+35t^3+15t^2-6t-1)$ |
| 24           | $\phi_{12}(t)(t^2-2)(t^4-4t^2+1)$  |
| 25           | $\phi_5(t)(t^{10}-10t^8+35t^6+t^5-50t^4-5t^3+25t^2+5t-1)$                  |
| 26           | $\phi_{13}(t)(t+2)(t^6-t^5-5t^4+4t^3+6t^2-3t-1)$                           |
| 27           | $\phi_9(t)(t^9-9t^7+27t^5-30t^3+9t+1)$                                     |
| 28           | $\phi_{14}(t)t(t^6-7t^4+14t^2-7)$  |
| 29           | $(t-2)(t^{14}+t^{13}-13t^{12}-120t^6+126t^{11}+, \dots, +t^2+7t-1)$        |
| 30           | $\phi_{15}(t)(t-1)(t+2)(t^2-t-1)(t^4+t^3-4t^2-4t+1)$                       |

Notice, since the valency of each polygon is 2, we have that  $t-2$  is a factor in every minimal polynomial. Also, a polygon is bipartite if and only if it has even order. So  $-2$  is an eigenvalue of a polygon when it has even order. That is,  $t+2$  is a factor if and only if

the polygons has even order. The characteristic polynomial of the polygon can be obtained by squaring every irreducible factor of the minimal polynomial, except the terms  $(t \pm 2)$ .

As observed in the table, if  $n|m$ , then the minimal polynomial of  $C_n$  divides the minimal polynomial of  $C_m$ . Also, if  $n$  is prime, then the minimal polynomial of  $C_n$  has an irreducible factor of degree  $\frac{n-1}{2}$ . More generally, for any  $C_n$  the sum of the degrees of the irreducible polynomials is  $\lceil \frac{n+1}{2} \rceil$ .

### 3.2 The Conference Graphs

A conference graph is a strongly regular graph with parameters  $v$ ,  $k = \frac{(v-1)}{2}$ ,  $a = \frac{(v-5)}{4}$ , and  $c = \frac{(v-1)}{4}$ . It is known in this case that  $v$  must be congruent to 1 (mod 4) and a sum of two squares. Conference graphs are known to exist for all small values of  $v$  satisfying these restrictions:  $v = 5, 9, 13, 17, 25, 29, 37, 41$ . And as we will discuss later in this section, there is a Paley graph of order  $v$  for every prime power,  $v$ , congruent to 1 (mod 4); these are all conference graphs. However, for many values of  $v$  the existence of a conference graphs is unknown.

Recall that in Section 2.3, we found the eigenvalues of strongly regular graphs. All conference graphs are strongly regular graphs, so we can easily find the eigenvalues for the conference graphs. The eigenvalues are  $k$  with multiplicity 1 and

$$r, s = \frac{-1 \pm \sqrt{v}}{2},$$

each with multiplicity  $(v-1)/2$ .

It is worth noting that the eigenvalues of a conference graph need not be integers and are in fact irrational whenever  $v$  is not a perfect square.

Haemers [21], for example, gives the intersection matrices of the association schemes induced by conference graphs, as well as the first and second eigenmatrix. Recall that  $[L_i]_{kj} = p_{ij}^k$ .

$$L_0 = I_3$$

$$L_1 = \begin{bmatrix} 0 & \frac{(v-1)}{2} & 0 \\ 1 & \frac{(v-5)}{4} & \frac{(v-1)}{4} \\ 0 & \frac{(v-1)}{4} & \frac{(v-1)}{4} \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 0 & 0 & \frac{(v-1)}{2} \\ 0 & \frac{(v-1)}{4} & \frac{(v-1)}{4} \\ 1 & \frac{(v-1)}{4} & \frac{(v-5)}{4} \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & \frac{(v-1)}{2} & \frac{(v-1)}{2} \\ 1 & r & -r - 1 \\ 1 & s & -s - 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & f & g \\ 1 & \frac{2fr}{v-1} & \frac{2gs}{v-1} \\ 1 & -\frac{f(r+1)}{v-k-1} & -\frac{g(s+1)}{v-k-1} \end{bmatrix}$$

The Paley graphs are an infinite family of conference graphs. Let  $q$  be a prime power. Let  $\Gamma$  be a graph with  $q$  vertices and label the vertices each with a distinct element of the finite field of order  $q$ . Join two vertices if the difference of their labels is a square in  $\text{GF}(q)$ . We say that  $\Gamma$  is a Paley graph of order  $q$ , which we will denote  $\text{Paley}(q)$ .  $\text{Paley}(q)$  is undirected when  $q \equiv 1 \pmod{4}$ , and in this case is a conference graph, i.e. it is strongly regular with parameters  $(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4})$ .

Here, we construct  $\text{Paley}(9)$ . We will first build the multiplication table of  $\text{GF}(9)$ . Since  $q = 9$ , we may take  $\text{GF}(9) = \{0, 1, 2, \alpha, \alpha + 1, \alpha + 2, 2\alpha, 2\alpha + 1, 2\alpha + 2\}$  with  $\alpha^2 = 2$ .

**The multiplication table of the finite field of order nine.**

|             |   |             |             |             |             |             |             |               |              |
|-------------|---|-------------|-------------|-------------|-------------|-------------|-------------|---------------|--------------|
| *           | 0 | 1           | 2           | $\alpha$    | $\alpha+1$  | $\alpha+2$  | $2\alpha$   | $2\alpha + 1$ | $2\alpha+2$  |
| 0           | 0 | 0           | 0           | 0           | 0           | 0           | 0           | 0             | 0            |
| 1           | 0 | 1           | 2           | $\alpha$    | $\alpha+1$  | $\alpha+2$  | $2\alpha$   | $2\alpha + 1$ | $2\alpha+2$  |
| 2           | 0 | 2           | 1           | $2\alpha$   | $2\alpha+2$ | $2\alpha+1$ | $\alpha$    | $\alpha+2$    | $\alpha + 1$ |
| $\alpha$    | 0 | $\alpha$    | $2\alpha$   | 2           | $\alpha+2$  | $2\alpha+2$ | 1           | $\alpha + 1$  | $2\alpha+1$  |
| $\alpha+1$  | 0 | $\alpha+1$  | $2\alpha+2$ | $\alpha+2$  | $2\alpha$   | 1           | $2\alpha+1$ | 2             | $\alpha$     |
| $\alpha+2$  | 0 | $\alpha+2$  | $2\alpha+1$ | $2\alpha+2$ | 1           | $\alpha$    | $\alpha+1$  | $2\alpha$     | 2            |
| $2\alpha$   | 0 | $2\alpha$   | $\alpha$    | 1           | $2\alpha+1$ | $\alpha+1$  | 2           | $2\alpha + 2$ | $\alpha+2$   |
| $2\alpha+1$ | 0 | $2\alpha+1$ | $\alpha+2$  | $\alpha+1$  | 2           | $2\alpha$   | $2\alpha+2$ | $\alpha$      | 1            |
| $2\alpha+2$ | 0 | $2\alpha+2$ | $\alpha+1$  | $2\alpha+1$ | $\alpha$    | 2           | $\alpha+2$  | 1             | $2\alpha$    |

Notice that the diagonal of this table gives us the quadratic residues:  $\{1, 2, \alpha, 2\alpha\}$ . Thus we can construct the  $\text{Paley}(9)$  with the rule that we join two vertices when the difference between their labels belongs to this set (Figure 10).

*9-Paley graph  
generalized quadrangle  
(2,1)*



Figure 10:  $\text{Paley}(9)$  [9].

Just to illustrate what happens when  $q \not\equiv 1 \pmod{4}$ , we consider Paley(7). The multiplication table of  $\text{GF}(7)$  is

|   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|
| * | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 0 | 2 | 4 | 6 | 1 | 3 | 5 |
| 3 | 0 | 3 | 6 | 2 | 5 | 1 | 4 |
| 4 | 0 | 4 | 1 | 5 | 2 | 6 | 3 |
| 5 | 0 | 5 | 3 | 1 | 6 | 4 | 2 |
| 6 | 0 | 6 | 5 | 4 | 3 | 2 | 1 |

Again we extract the quadratic residues from the diagonal:  $\{1, 2, 4\}$ . Thus we can construct the directed Paley graph for  $\text{GF}(7)$ . We will use this example to illustrate that Paley graphs of order  $q \equiv 3 \pmod{4}$  are directed. Notice, in  $\text{GF}(7)$ ,  $5 - 3 = 2 \in \{1, 2, 4\}$  while  $3 - 5 = 5 \notin \{1, 2, 4\}$ . Thus there will be a directed edge from vertex 3 to vertex 5. This construction produces an infinite family of doubly-regular tournaments, which are beyond the scope of our paper.

The conference graphs include an infinite family of primitive Q-polynomial irrational association schemes with  $d = 2$ . We now move on to  $d > 2$  and  $m_1 > 2$  and study more combinatorial objects that induce irrational Q-polynomial schemes.

### 3.3 Symmetric Designs and Linked Systems

In this section we will discuss an infinite family of association schemes that arise from  $(v, k, \lambda)$  symmetric designs. Symmetric designs give rise to the only known examples of 3-class Q-polynomial irrational association schemes that satisfy the conditions of the first case in Chapter 4.

**Definition 3.2.** *Given a set of points,  $\mathcal{P}$ , and a set of blocks,  $\mathcal{B}$ , the ordered pair  $(\mathcal{P}, \mathcal{B})$  is a  $(v, k, \lambda)$  symmetric design if the following hold:*

- (i)  $\mathcal{B}$  is a subset of the power set of  $\mathcal{P}$
- (ii)  $|\mathcal{P}| = |\mathcal{B}| = v$
- (iii) For all  $b \in \mathcal{B}$ ,  $|b| = k$
- (iv) For all  $p \in \mathcal{P}$ ,  $|\{b \in \mathcal{B} : p \in b\}| = k$ .
- (v) For all distinct  $p, q \in \mathcal{P}$   $|\{b \in \mathcal{B} : p, q \in b\}| = \lambda$
- (vi) For all distinct  $b, c \in \mathcal{B}$   $|\{p \in \mathcal{P} : p \in b, p \in c\}| = \lambda$



For each  $(v, k, \lambda)$  symmetric design, there are  $v$  points and  $v$  blocks, every point is contained in  $k$  blocks, every block contains  $k$  points, every two distinct points are contained in  $\lambda$  common blocks and every two distinct blocks contain  $\lambda$  common points. For instance, the Fano plane, discussed earlier, is a  $(7, 3, 1)$  symmetric design. Every line of the Fano plane corresponds to a block of the symmetric design; that is, there are 7 points and 7 lines, every point lies in 3 lines, every line contains 3 points, and every two distinct points lie in exactly 1 common line while every two distinct lines contain exactly 1 common point.

Here we demonstrate how to construct an association scheme given a  $(v, k, \lambda)$  symmetric design. Recall that a symmetric design is an ordered pair of sets,  $(\mathcal{P}, \mathcal{B})$ , satisfying the axioms of Definition 3.2; where  $\mathcal{P}$  and  $\mathcal{B}$  are disjoint. Let  $X = \mathcal{P} \cup \mathcal{B}$ . Define the following relations on  $X \times X$ :

- (i)  $R_0 = \{(x, x) : x \in X\}$
- (ii)  $R_1 = \{(x, y) \in \mathcal{P} \times \mathcal{B} : x \in y\} \cup \{(y, x) \in \mathcal{B} \times \mathcal{P} : x \in y\}$
- (iii)  $R_2 = \{(x, y) \in \mathcal{P} \times \mathcal{P} : x \neq y\} \cup \{(x, y) \in \mathcal{B} \times \mathcal{B} : x \neq y\}$
- (iv)  $R_3 = \{(x, y) \in \mathcal{P} \times \mathcal{B} : x \notin y\} \cup \{(y, x) \in \mathcal{B} \times \mathcal{P} : x \notin y\}$ .

Using the definition of a symmetric design one can verify the parameters in the matrices below to show that  $(X, \{R_i\}_{i=0}^d)$  is indeed a 3-class association scheme. We can see that this association scheme is imprimitive since the relation  $R_2$  is disconnected. For a given  $(v, k, \lambda)$  symmetric design, we called the graph  $IG(v, k, \lambda)$  generated by  $R_1$  defined above by simply treating both points and blocks to be vertices and connect two vertices if they are in relation  $R_1$ . Note that this graph is not necessary well-defined. It refers to the incidence graph of a specific design with these parameters. But in most cases when we use it in this paper, any such symmetric design will do.

Here we provide the intersection matrices, dual intersection matrices, and eigenmatrices for the association scheme, as defined above, arising from a given  $(v, k, \lambda)$  symmetric design:

$$L_0 = I_4,$$

$$L_1 = \begin{bmatrix} 0 & k & 0 & 0 \\ 1 & 0 & k-1 & 0 \\ 0 & \lambda & 0 & k-\lambda \\ 0 & 0 & k & 0 \end{bmatrix}, L_2 = \begin{bmatrix} 0 & 0 & v-1 & 0 \\ 0 & k-1 & 0 & v-k \\ 1 & 0 & v-2 & 0 \\ 0 & k & 0 & v-k-1 \end{bmatrix},$$

$$L_3 = \begin{bmatrix} 0 & 0 & 0 & v-k \\ 0 & 0 & v-k & 0 \\ 0 & k-\lambda & 0 & v-2k+\lambda \\ 1 & 0 & v-k-1 & 0 \end{bmatrix},$$

$$P = \begin{bmatrix} 1 & k & v-1 & v-k \\ 1 & \sqrt{k-\lambda} & -1 & -\sqrt{k-\lambda} \\ 1 & -\sqrt{k-\lambda} & -1 & \sqrt{k-\lambda} \\ 1 & -k & v-1 & -(v-k) \end{bmatrix},$$

$$Q = \begin{bmatrix} 1 & v-1 & v-1 & 1 \\ 1 & \frac{k}{\sqrt{k-\lambda}} & -\frac{k}{\sqrt{k-\lambda}} & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -\frac{k-v}{\sqrt{k-\lambda}} & \frac{k-v}{\sqrt{k-\lambda}} & -1 \end{bmatrix},$$

$$L_0^* = I_4,$$

$$L_1^* = \begin{bmatrix} 0 & v-1 & 0 & 0 \\ 1 & \bar{b}_1^* & b_1^* & 0 \\ 0 & b_1^* & \bar{b}_1^* & 1 \\ 0 & 0 & v-1 & 0 \end{bmatrix}, L_2^* = \begin{bmatrix} 0 & 0 & v-1 & 0 \\ 0 & b_1^* & \bar{b}_1^* & 1 \\ 1 & \bar{b}_1^* & b_1^* & 0 \\ 0 & v-1 & 0 & 0 \end{bmatrix}, L_3^* = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Note that  $b_1^* = \frac{v-2}{2} + \frac{-k^2+(v-1)(k-\lambda)}{2k\sqrt{k-\lambda}}$  and  $\bar{b}_1^* = v-2-b_1^*$ .

In this paper, we are only interested in the symmetric designs when they give rise to association schemes with irrational eigenvalues. From the  $P$  matrix, we see that the association scheme of a symmetric design has irrational eigenvalues if and only if  $k-\lambda$  is not a perfect square.

Next, we will begin discussing the properties of symmetric designs. Finding sufficient conditions for the existence of a  $(v, k, \lambda)$  symmetric design is still an unsolved problem, although there are some partial results, as we will see later when we discuss projective planes. In the following section, we state and prove some necessary conditions. Let  $(v, k, \lambda)$  be a symmetric design.

**Theorem 3.3.**  $k(k-1) = \lambda(v-1)$

(Pf) Let  $p \in \mathcal{P}$ . Our goal is to double count the cardinality of  $A = \{(q, c) \in \mathcal{P} \times \mathcal{B} : q \neq p \text{ and } \exists c \in \mathcal{B} \text{ such that } q, p \in c\}$ . First, there are  $k$  blocks in  $\mathcal{B}$  that contain  $p$ . Each one of these blocks contain  $k-1$  points that are not  $p$ , so  $|A| = k(k-1)$ . On the other hand, there are  $v-1$  points in  $\mathcal{P}$  different from  $p$ . Now for each point  $q \in \mathcal{P}$  satisfying  $q \neq p$ , there are exactly  $\lambda$  blocks that contain both  $p$  and  $q$ . Thus  $|A| = \lambda(v-1)$ . We have shown that  $k(k-1) = \lambda(v-1)$ .

**Theorem 3.4.** (Bruck-Ryser-Chowla [30], [35])

(i) If  $v$  is even, then  $k-\lambda$  is a square.

(ii) If  $v$  is odd, then the Diophantine equation  $z^2 = (k - \lambda)x^2 + (-1)^{\frac{v-1}{2}} \lambda y^2$  has a nontrivial solution.

The previous theorem is of particular interest to us because it tells us that no irrational association schemes arise from a symmetric design where  $v$  is even.

Now we will discuss a particular class of symmetric designs, called projective planes.

**Definition 3.5.** A  $(v, k, \lambda)$  symmetric design  $(\mathcal{P}, \mathcal{B})$  is a projective plane if  $\lambda = 1$ .

In every projective plane, we see that any two distinct points are contained in a unique common block and any two distinct blocks contain a unique common point. Let  $n = k - 1$  be the order of a projective plane. From Theorem 3.3 we see that  $v = n^2 + n + 1$ . Therefore, a projective plane is a  $(n^2 + n + 1, n + 1, 1)$  symmetric design. For every prime power  $n$ , there exists a projective plane of order  $n$ . So sufficient conditions are known for the existence of a projective plane of order  $n$ , but necessary conditions are not. A long-standing conjecture in finite geometry is that a projective plane of order  $n$  exists if and only if  $n$  is a prime power. One example of a projective plane is depicted in Figure 11. It is also interesting to note that it has been conjectured that, for each  $\lambda \geq 2$ , there are only finitely many  $(v, k, \lambda)$  symmetric designs.

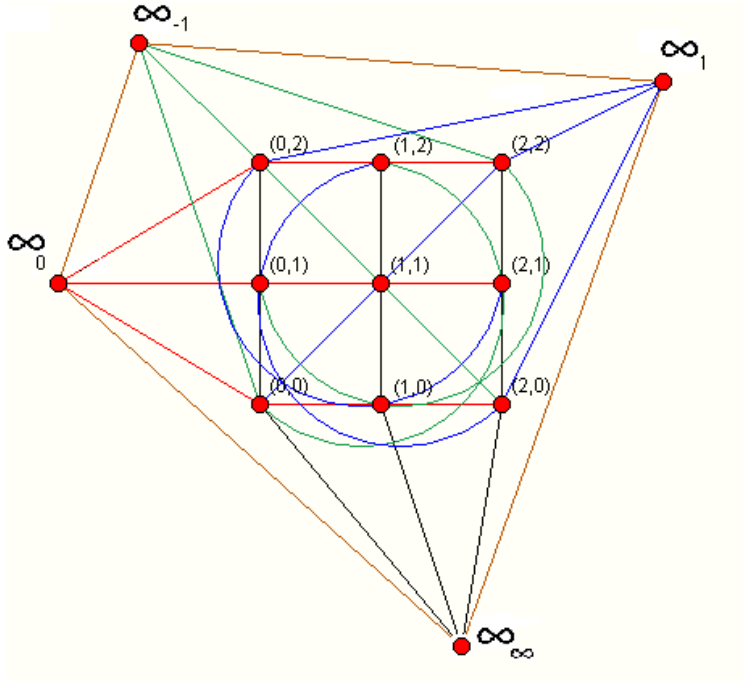


Figure 11: The projective plane of order 2

Consider the set  $\text{GF}(3) \times \text{GF}(3)$ . For each slope  $s \in \{0, 1, -1, \infty\}$ , we define an equivalence relation  $\sim_s$  on  $\text{GF}(3) \times \text{GF}(3)$ . For the slope  $\infty$ , we say  $(x, y) \sim_\infty (u, v)$  if  $u - x = 0$ .

Given a slope,  $s \neq \infty$  we say  $(x, y) \sim_s (u, v)$  if  $\frac{(v-y)}{(u-x)} = s$ . Now, we label 13 points in Figure 11 by the elements of  $\text{GF}(3) \times \text{GF}(3)$  and  $\infty_s$  where  $s \in \{0, 1, -1, \infty\}$ . We form a blue line to connect all points in a maximal subset of  $\text{GF}(3) \times \text{GF}(3)$  satisfying every two points in this subset are in an equivalence relation of slope 1. We use red, green and black lines in a same manner respect to the equivalence relations of slope 0,  $-1, \infty$ . For each slope, we have three lines in the respect color in Figure 11. Then we connected point  $\infty_1, \infty_0, \infty_{-1}, \infty_\infty$  with all blue, red, green and black lines respectively. Also, the four points labeled  $\infty_s$  form a line, which is represented by the brown line in Figure 11.

Each  $(v, k, \lambda)$  symmetric design satisfies the following inequality, where  $n = k - \lambda$ :

$$4n - 1 \leq v \leq n^2 + n + 1.$$

Any  $(v, k, \lambda)$  symmetric design such that  $v = n^2 + n + 1$  corresponds to a finite projective plane of order  $n$ , that is, a  $(n^2 + n + 1, n + 1, 1)$  symmetric design. On the other hand any  $(v, k, \lambda)$  symmetric design such that  $v = 4n - 1$  corresponds to a Hadamard Design of order  $n$ , that is, a  $(4n - 1, 2n - 1, n - 1)$ ; We will discuss these in the next section.

We refer to the ‘‘Handbook of Combinatorial Designs’’ [14] in our discussion of the known symmetric designs. In this paper, we only consider the symmetric designs that induce irrational association schemes. There are 21 known infinite families and 15 special constructions and we know which families and constructions contain at least one symmetric design that gives rise to an association scheme with irrational eigenvalues [14, pg 116-118]. The ones that do are Families 1, 2, 3, 4, 5, 8, 10, 11 and 20, as listed in pages 116 and 117 in [14], and the  $(79,13,2)$ ,  $(71,15,3)$ ,  $(41,16,6)$ ,  $(49,16,5)$  and  $(71,21,6)$  symmetric designs, as listed on page 118 [14].

Here we focus our discussion on difference sets in order to construct some of the families of symmetric designs on pages 116 and 117 in [14]. We start with Family 2 [14, pg 116].

**Definition 3.6.** Let  $G = \{0, 1, \dots, v - 1\}$  be an abelian group of order  $v$  with identity  $0 \in G$ . Let  $D = \{d_1, d_2, \dots, d_k\}$  be a  $k$ -element subset of  $G$ .  $D$  is called a  $(v, k, \lambda)$ -difference set if for every  $g \in G$  such that  $g \neq 0$  there are exactly  $\lambda$  pairs  $(d_i, d_j) \in D \times D$  such that  $g = d_i - d_j$ .

In particular, we are interested when  $G = \mathbb{Z}_v$ . To illustrate the previous definition, write  $\mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$  and let  $D = \{1, 2, 4\}$ .

|   |   |   |   |
|---|---|---|---|
| – | 1 | 2 | 4 |
| 1 | 0 | 6 | 4 |
| 2 | 1 | 0 | 5 |
| 4 | 3 | 2 | 0 |

The 0 element of  $\mathbb{Z}_7$  appears  $k = 3$  times and each nonzero element of  $\mathbb{Z}_7$  appears exactly  $\lambda = 1$  times. Therefore  $D$  is a  $(7, 3, 1)$ -difference set of  $\mathbb{Z}_7$ .

It is known that when  $v$  is a prime, the set of quadratic residues,  $\mathcal{Q}$ , is a difference set of size  $\frac{v-1}{2}$  for the ring  $\mathbb{Z}_v$  as a group under addition. It is also known that after constructing

this difference set, if we let  $\mathcal{P} = \mathbb{Z}_v$  and  $\mathcal{B} = \{g + \mathcal{Q} : g \in \mathbb{Z}_v\}$ , then  $(\mathcal{P}, \mathcal{B})$  forms a symmetric design, particularly one of the form in Family 2 [14, pg 116].

Let us take  $v = 23$ . Then  $\mathcal{P} = \mathbb{Z}_{23}$ ,  $\mathcal{Q} = \{1, 4, 9, 16, 2, 13, 3, 18, 12, 8, 6\}$  and  $\mathcal{B} = \{g + \mathcal{Q} : g \in \mathbb{Z}_{23}\}$ . We find that  $(\mathcal{P}, \mathcal{B})$  is a  $(23, 11, 5)$  symmetric design. Note that there are constructions from Family 2 that do not follow this method, namely when  $v$  is not a prime. When  $v$  is a prime power we can build a symmetric design in a similar manner, but we are required to work in  $\text{GF}(v)$  rather than  $\mathbb{Z}_v$ .

In Family 3, designs are of the form  $(v, k, \lambda)$  where  $v = 4t^2 + 1$ ,  $k = t^2$  and  $\lambda = \frac{3t^2 + 1}{4}$  where  $t$  is an odd integer such that  $4t^2 + 1$  is prime. These designs are constructed in a similar manner to Family 2, but instead we look at biquadratic residues, that is we let  $\mathcal{P} = \mathbb{Z}_v$ ,  $\mathcal{Q} = \{x^4 : x \in \mathbb{Z}_v\}$  and  $\mathcal{B} = \{g + \mathcal{Q} : g \in \mathbb{Z}_v\}$ . For example, the  $(37, 9, 2)$  symmetric design is in Family 3. In this example,  $\mathcal{P} = \mathbb{Z}_{37}$ ,  $\mathcal{Q} = \{1, 16, 7, 34, 33, 26, 12, 10, 9\}$  and  $\mathcal{B} = \{g + \mathcal{Q} : g \in \mathbb{Z}_{37}\}$ . Then  $(\mathcal{P}, \mathcal{B})$  is a  $(37, 9, 2)$  symmetric design.

Family 4b is similar to the construction above, but with octic residues, that is  $\mathcal{Q} = \{x^8 : x \in \mathbb{Z}_v\}$ . Also, Family 5 is another construction from difference sets, but there are only two known examples.

We end this section with a discussion of systems of linked symmetric designs. As with the symmetric designs, these systems give rise to association schemes, but our goal is to prove that these association schemes have rational eigenvalues.

**Definition 3.7.** *A collection of sets  $\Delta = \{\Omega_1, \dots, \Omega_f\}$ , ( $f \geq 3$ ) together with a collection of symmetric incidence relations on  $\Omega_i \times \Omega_j$ , for all  $i \neq j$ , is a system of linked symmetric designs provided that*

- (i) *Each pair  $\{\Omega_i, \Omega_j\}$  with its incidence relation is a symmetric design with parameters  $(v, k, \lambda)$*
- (ii) *Given any 3 sets  $\Omega_i, \Omega_j, \Omega_k$ , the number of elements  $\alpha \in \Omega_i$  that are incident with both  $\beta \in \Omega_j$  and  $\gamma \in \Omega_k$  only depends on whether or not  $\beta$  and  $\gamma$  are incident.*

Now notice that an incidence structure on a collection of sets,  $\Delta$ , is a system of linked symmetric designs only if every  $\Omega_i, \Omega_j, \Omega_k \in \Delta$  is a system of linked symmetric designs. Thus once we prove that the association scheme is rational for  $f = 3$ , we can easily extend the result to all  $f \geq 3$  with the use of induction.

Before we begin the proof for  $f = 3$ , it is useful to introduce some notation. For any symmetric design  $(\mathcal{P}, \mathcal{B})$ , define a matrix  $A$  where the rows are indexed by the points and the columns are indexed by blocks and for any  $p_i \in \mathcal{P}$  and  $b_j \in \mathcal{B}$ ,  $1 \leq i, j \leq v$ ,

$$A_{ij} = \begin{cases} 1 & \text{if } p_i \in b_j, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that  $AJ = A^T J = kJ$  since every point lies in  $k$  blocks and every block contains  $k$  points. Also,  $AA^T = A^T A = (k - \lambda)I + \lambda J$  follows from the fact that any two distinct points are contained in  $\lambda$  common blocks and any two distinct blocks contain  $\lambda$  common points.

**Theorem 3.8.** [12] Suppose that a system of linked symmetric designs with  $f = 3$  and each design has parameters  $(v, k, \lambda)$ . Then there exists an integer  $u$  such that  $k - \lambda = u^2$ .

(Pf) Let  $\{\Omega_1, \Omega_2, \Omega_3\}$  be any system of linked symmetric designs and let  $B_{23}, B_{31}, B_{12}$  be the incidence matrices for  $(\Omega_2, \Omega_3), (\Omega_3, \Omega_1), (\Omega_1, \Omega_2)$ , respectively. Notice that while  $B_{23}, B_{31}, B_{12}$  are incidence matrices for a  $(v, k, \lambda)$  symmetric design it does not follow that  $B_{ij} = B_{i'j'}$  for  $i \neq i'$  or  $j \neq j'$ .

Since we have a linked system of designs, we can define the integers  $x_1$  and  $u_1$  such that the number of elements in  $\Omega_1$  with  $\beta \in \Omega_2$  and  $\gamma \in \Omega_3$  is  $x_1$  or  $x_1 - u_1$  when  $\beta$  and  $\gamma$  are incident or not, respectively. Define  $x_2, u_2, x_3, u_3$  in a similar manner.

Consider  $B_{13}B_{12}$ . Now this matrix has rows indexed by  $\Omega_3$  and columns indexed by  $\Omega_2$ . Also,  $[B_{13}B_{12}]_{ij}$  only depends on whether or not the  $i^{\text{th}}$  element of  $\Omega_2$  is incident to the  $j^{\text{th}}$  element of  $\Omega_3$ . This is represented by the following equation:

$$B_{13}B_{12} = x_1J - u_1B_{23}^\top.$$

Now left multiply both sides by  $B_{13}^\top$  to obtain:

$$B_{13}^\top B_{13}B_{12} = x_1B_{13}^\top J - u_1B_{13}^\top B_{23}^\top$$

which simplifies to

$$((k - \lambda)I + \lambda J)B_{12} = x_1kJ - u_1(B_{23}B_{13})^\top$$

so that

$$(k - \lambda)B_{12} + \lambda kJ = x_1kJ - u_1(x_3J - u_3B_{12}^\top)^\top$$

to finally obtain

$$(k - \lambda)B_{12} + \lambda kJ = (x_1k - u_1x_3)J + u_1u_3B_{12}.$$

Now,  $B_{12}$  and  $J$  are linearly independent, since  $B_{12}$  has zero entries where  $J$  has nonzero entries. Thus we must have that the coefficients in front of  $B_{12}$  are equal, that is  $k - \lambda = u_1u_3$ . Now by using the same argument and permuting the indices of the incidence matrices, we can obtain  $k - \lambda = u_1u_3 = u_1u_2 = u_2u_3$ , that is  $u_1 = u_2 = u_3$ . Therefore  $k - \lambda = u_1^2$ , where  $u_1$  is an integer.

The following is the  $Q$  eigenmatrix for a system of linked symmetric designs:

$$Q = \begin{bmatrix} 1 & v - 1 & (f - 1)(v - 1) & f - 1 \\ 1 & \frac{k}{\sqrt{k - \lambda}} & -\frac{k}{\sqrt{k - \lambda}} & -1 \\ 1 & -1 & -(f - 1) & f - 1 \\ 1 & -\frac{k - v}{\sqrt{k - \lambda}} & \frac{k - v}{\sqrt{k - \lambda}} & -1 \end{bmatrix} [31].$$

Since  $k - \lambda$  is a square, the entries of the  $Q$  matrix are rational; and therefore, the association scheme that arises from a linked system of symmetric designs with  $f \geq 3$  is rational.

### 3.4 The Q-Bipartite Case

In this section, we will discuss the Taylor and Hadamard graphs, which give rise to infinite families of 3-class and 4-class Q-polynomial association schemes, respectively.

### 3.4.1 Taylor Graphs

The Taylor graphs give rise to the only known examples of 3-class Q-polynomial irrational association schemes that satisfy the conditions of the second case in Chapter 4. Before discussing the Taylor graphs, we first introduce the intersection array of a distance-regular graph. Let  $G$  be a connected, undirected graph with vertex set  $V(G)$ . For any two vertices  $x, y \in V(G)$ , let  $d(x, y)$  denote the distance between  $x$  and  $y$ . We say  $G$  is a distance-regular graph if for  $i = 0, \dots, d$  there exists  $b_i, c_i \in \mathbb{N}_0$  such that for any two vertices  $x, y \in V(G)$  at distance  $i = d(x, y)$ , there are exactly  $c_i$  vertices  $z$  such that  $d(y, z) = 1$  and  $d(x, z) = i - 1$ , while there are  $b_i$  vertices  $w$  such that  $d(y, w) = 1$  and  $d(x, w) = i + 1$ . The sequence

$$(b_0, b_1, \dots, b_{d-1}; c_1, \dots, c_d)$$

is called the intersection array of  $G$ .

We can see that the 4-cycle is a distance-regular graph. The distance between any two vertices is 0, 1, or 2. For the sake of example, we choose any two vertices  $x$  and  $y$  such that  $d(x, y) = 1$ . There is exactly  $c_1 = 1$  vertex,  $z$ , such that  $d(y, z) = 1$  and  $d(x, z) = 0$ , while there is  $b_1 = 1$  vertex,  $w$ , such that  $d(y, w) = 1$  and  $d(x, w) = 2$ . In a similar manner, we can apply the same argument for  $i = 0$  and  $i = 2$  to find that the intersection array of the 4-cycle is  $(2, 1; 1, 2)$ . Note that  $b_d$  and  $c_0$  are always zero and are omitted from the intersection array.

We will use the Petersen graph to illustrate how to obtain the intersection array from a graph which is known to be distance-regular. For the Petersen graph,  $d = 2$ , so we should expect that the intersection array for the Petersen graph is of the form  $(b_0, b_1; c_1, c_2)$ . To calculate  $b_1$ , let  $x, y$  be two vertices of the Petersen graph such that  $d(x, y) = 1$  and we find that there are 2 vertices  $w$  such that  $d(y, w) = 1$  and  $d(x, w) = 2$ . Thus  $b_1 = 2$  and in a similar manner, we can find that the intersection array of Petersen graph is  $(3, 2; 1, 1)$ .

A Taylor graph is a distance-regular with intersection array  $(k, \mu, 1; 1, \mu, k)$ . Some Taylor graphs are depicted in Figure 12.

Here our goal is to construct Taylor graphs from regular two-graphs, so first we define regular two-graphs. A two-graph is a pair  $(\Omega, \Delta)$  consisting of a finite vertex set  $\Omega$  and a set  $\Delta$  of unordered triples of vertices from  $\Omega$  such that each set of four vertices in  $\Omega$  contains an even number of triples in  $\Delta$ . A subset  $X$  of  $\Omega$  is called coherent if all triples contained in  $X$  belong to  $\Delta$ . A two-graph is said to be regular if every 2-subset of  $\Omega$  is contained in a constant number  $\lambda$  of coherent triples.

Taylor graphs are in a bijection with regular two-graphs [22]. Let  $(\Omega, \Delta)$  be a regular two-graph such that  $|\Omega| = n$  and  $|\Delta| = a$ , and choose a vertex  $\infty \in \Omega$ . Let  $G$  be a graph where the vertices of  $G$ ,  $V(G)$ , are labeled by the elements  $x^+$  and  $x^-$ , for all  $x \in \Omega$  (including  $\infty$ ). For all  $x \in \Omega \setminus \{\infty\}$ , the vertex labeled  $x^+$  (respectively  $x^-$ ) is adjacent to the vertex labeled  $\infty^+$  (respectively  $\infty^-$ ). Now, let  $x, y \in \Omega$  different from  $\infty^\pm$ . If  $\{\infty, x, y\}$  is coherent in  $(\Omega, \Delta)$ , then we join the vertices labeled  $x^+$  to  $y^-$  and  $x^-$  to  $y^+$  in  $G$ . If  $\{\infty, x, y\}$  is not coherent in  $(\Omega, \Delta)$ , then we join the vertices labeled  $x^+$  to  $y^+$  and  $x^-$  to  $y^-$  in  $G$ . This construction gives rise to a Taylor graph with parameters  $k = n - 1$  and  $\mu = n - 2 - a$ . It is a useful exercise to check that we can obtain a regular two-graph from a Taylor graph by reversing the method above.

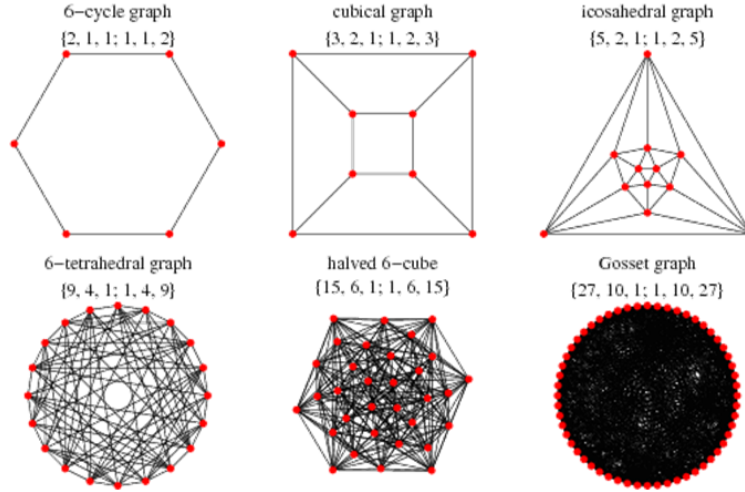


Figure 12: Some Taylor graphs [35].

To illustrate, we will construct the Hamming scheme  $H(3, 2)$ , the 3-cube, which is a Taylor graph. We first construct the regular-two graph  $(\Omega, \Delta)$  by letting  $\Omega = \{1, 2, 3, 4\}$  and  $\Delta = \{\{123\}, \{124\}, \{134\}, \{234\}\}$ . Then we construct the Taylor graph  $\Gamma$  from the regular two-graph  $(\Omega, \Delta)$  following the construction in the previous paragraph. Let  $1 \in \Omega$  be the vertex  $\infty$ . All 2-sets not containing  $\infty^\pm$  are coherent and Figure 13 depicts the Taylor graph corresponding to this construction.

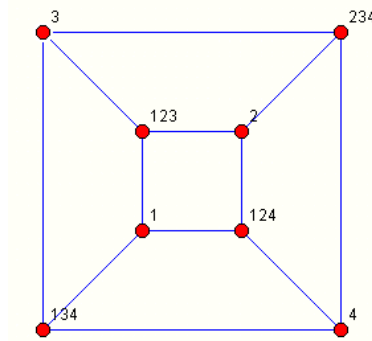


Figure 13:  $H(3,2)$ : A Taylor graph arising from a regular two-graph.

We introduce some terminology before we present a theorem about regular two-graphs. Let  $\Gamma$  be a graph with vertex set  $V(\Gamma)$ , and let  $x \in V(\Gamma)$ . The neighborhood of  $x$ , denoted  $N(x)$ , is the induced subgraph of  $\Gamma$  where  $V(N(x)) = \{y \in V(\Gamma) : (x, y) \text{ is an edge in } \Gamma\}$ . The switching graph  $Sw(\Gamma)$  of  $\Gamma$  is the graph with vertex set  $V(\Gamma) \times \{0, 1\}$ , in which  $(v, i)$  is adjacent to  $(w, i)$  if and only if  $v$  and  $w$  are connected in  $\Gamma$  and  $(v, i)$  is adjacent to  $(w, 1 - i)$  if and only if  $v$  and  $w$  are disconnected in  $\Gamma$ . We say a regular two-graph is trivial if this



regular two-graph is a switching graph of the complete graph or the empty graph.

**Theorem 3.9.** [38, Theorem 11.6.1] *Let  $\Gamma$  be a nontrivial two-graph on  $n+1$  vertices. Then the following are equivalent:*

- (i)  $\Gamma$  is a regular two-graph
- (ii) For all  $x \in V(\Gamma)$ ,  $N(x)$  is a regular graph
- (iii) For all  $x \in V(\Gamma)$ ,  $N(x)$  is a  $(n, k, a, c)$  strongly regular graph with  $k = 2c$
- (iv) There exists  $x \in V(\Gamma)$  such that  $N(x)$  is a  $(n, k, a, c)$  strongly regular graph with  $k = 2c$

Recall that a conference graph is a  $(v, \frac{(v-1)}{2}, \frac{(v-5)}{4}, \frac{(v-1)}{4})$  strongly regular graph. Note that  $k = 2c$ . It is known that every conference graph yields a regular two-graph, while it is clear from the above theorem that every regular two-graph yields a conference graph [38, p 258].

It is also possible to construct regular two-graphs in terms of equiangular lines intersecting at the origin. Suppose that we have  $n$  equiangular lines in  $\mathbb{R}^d$ , at angle  $\theta$ . If  $n = \frac{d - \cos^2(\theta)}{1 - d\cos^2(\theta)}$ , then these lines give rise to a regular two-graph [38]. It is interesting to note that the regular two-graph associated with a Paley graph with valency  $k$  corresponds to the regular two-graph constructed from  $4k+2$  equiangular lines in  $\mathbb{R}^{2k+1}$ . More information about the equiangular lines can be found in Chapter 11 of [38].

Here we present the intersection matrices, dual intersection matrices, and eigenmatrices of the association scheme arising from a Taylor graph with intersection array  $(k, \mu, 1; , 1, \mu, k)$ .

$$L_0 = I_4,$$

$$L_1 = \begin{bmatrix} 0 & k & 0 & 0 \\ 1 & k - \mu - 1 & \mu & 0 \\ 0 & \mu & k - \mu - 1 & 1 \\ 0 & 0 & k & 0 \end{bmatrix},$$

$$L_2 = \begin{bmatrix} 0 & 0 & k & 0 \\ 1 & \mu & k - \mu - 1 & 1 \\ 0 & k - \mu - 1 & \mu & 0 \\ 0 & k & 0 & 0 \end{bmatrix},$$

$$L_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

$$P = \begin{bmatrix} 1 & k & k & 1 \\ 1 & \theta_1 & -\theta_1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & \theta_3 & -\theta_3 & -1 \end{bmatrix},$$

$$Q = \begin{bmatrix} 1 & m & k & k+1-m \\ 1 & \frac{k+1}{\sqrt{\Delta}} & -1 & -\frac{k+1}{\sqrt{\Delta}} \\ 1 & -\frac{k+1}{\sqrt{\Delta}} & -1 & \frac{k+1}{\sqrt{\Delta}} \\ 1 & -m & k & m-1-k \end{bmatrix}.$$

$$L_0^* = I_4,$$

$$L_1^* = \begin{bmatrix} 0 & m & 0 & 0 \\ 1 & 0 & m-1 & 0 \\ 0 & \frac{m(m-1)}{k} & 0 & \frac{m^2}{k} \\ 0 & 0 & m & 0 \end{bmatrix}, L_2^* = \begin{bmatrix} 0 & 0 & k & 0 \\ 0 & m-1 & k-2m+1 & m \\ 1 & 0 & k-1 & 0 \\ 0 & m & k-2m+1 & m-1 \end{bmatrix},$$

$$L_3^* = \begin{bmatrix} 0 & 0 & 0 & m \\ 0 & 0 & m & 0 \\ 0 & \frac{k}{m_2} & 0 & \frac{m(m-1)}{k} \\ 1 & 0 & m-1 & 0 \end{bmatrix}.$$

In the above matrices,

$$\Delta = (k - 2\mu - 1)^2 + 4k,$$

$$\theta_1 = \frac{k - 2\mu - 1 + \sqrt{\Delta}}{2},$$

$$\theta_3 = \frac{k - 2\mu - 1 - \sqrt{\Delta}}{2}$$

and

$$m = \frac{-\theta_3(k+1)}{\sqrt{\Delta}}.$$

From  $L_1^*$ , we see that an association scheme arising from a Taylor graph is Q-bipartite, and from  $Q$  we see that the scheme contains irrational eigenvalues if and only if  $\Delta$  is not a perfect square. In the following section we study Hadamard graphs, which give rise to an infinite family of 4-class Q-bipartite association schemes with irrational eigenvalues.

### 3.4.2 Hadamard Graphs

A Hadamard graph is a distance-regular graph with intersection array

$$(2\mu, 2\mu - 1, \mu, 1; 1, \mu, 2\mu - 1, 2\mu).$$

In order to construct Hadamard graphs, we first introduce Hadamard matrices. An  $n \times n$  matrix  $H$  is a Hadamard matrix of order  $n$  if  $H_{ij} = \pm 1$  and the rows of  $H$  are pairwise orthogonal. Now, define a set of  $4n$  symbols  $A = \{r_i^+, r_i^-, c_i^+, c_i^- : i = 1, \dots, n\}$ , where  $r_i$  corresponds to the  $i^{\text{th}}$  row and  $c_i$  corresponds to the  $i^{\text{th}}$  column. Let  $G$  be a graph and label the vertex set,  $V(G)$ , by the elements of  $A$ . If  $h_{ij} = 1$  then include edges labeled  $(r_i^+, c_j^+)$ . If  $h_{ij} = -1$  then join the vertices in this manner:  $(r_i^+, c_j^-)$ . This construction gives rise to a Hadamard graph and conversely, every Hadamard graph arises in this way [8, Theorem 1.8.1].

We will construct the Hadamard graph of order 8 from the  $2 \times 2$  Hadamard matrix. Let

$$H = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

It is easy to check that  $H$  is a Hadamard matrix. Now we will construct the Hadamard graph,  $\Gamma$ . Let  $V(\Gamma)$  be labeled by the set  $\{r_1^+, r_1^-, r_2^+, r_2^-, c_1^+, c_1^-, c_2^+, c_2^-\}$ . We join two vertices as stated above. For example, since  $H_{11} = 1$  we join the vertices labeled  $r_1^-$  and  $c_1^-$  as well as  $r_1^+$  and  $c_1^+$  but not  $r_1^+$  and  $c_1^-$  or  $r_1^-$  and  $c_1^+$ . After applying this construction for  $1 \leq i, j \leq 2$  we find that the Hadamard graph of order 8 is the 8-cycle.

It is known that a  $(4n + 3, 2n + 1, n)$  symmetric design is equivalent to a Hadamard matrix of order  $4n + 4$ . Also, The Menon design is a  $(4t^2, 2t^2 - t, t^2 - t)$  symmetric design where  $n = t^2$ . A Menon design exists if and only if there exists a regular Hadamard matrix of order  $4t^2$  [14]. A Hadamard matrix of order  $4t^2$  exists whenever:

- (i) there is a Hadamard matrix of order  $2t$ ,
- (ii)  $2t - 1$  and  $2t + 1$  are both prime powers,
- (iii)  $t$  is an odd integer, or
- (iv) there exists a Hadamard difference set of order  $t^2$  [14, pg 116].

It has been conjectured that a Menon design exists for all  $t \in \mathbb{Z}^+$ .

It is known that Hadamard graph of order  $n$  exists for each value of  $n = 4, 8, 12, \dots, 664$  and it is conjectured that a Hadamard graph exists for all  $n$  divisible by 4. The Hadamard graphs induce the only known example of irrational Q-polynomial association schemes with  $d = 4$ .

To end this section, we note that any 4-class Q-polynomial association scheme that is Q-bipartite and Q-antipodal with  $w \geq 3$  fibres is rational. LeCompte, Martin and Owens [26] derived that the second eigenmatrix of such a scheme is of the form:

$$Q = \begin{bmatrix} 1 & d & w(d-1) & d(w-1) & w-1 \\ 1 & \sqrt{d} & 0 & -\sqrt{d} & -1 \\ 1 & 0 & -w & 0 & w-1 \\ 1 & -\sqrt{d} & 0 & \sqrt{d} & -1 \\ 1 & -d & w(d-1) & -d & w-1 \end{bmatrix}.$$

Boykin, Sitharam, Tarifi and Wocjan [10] proved that in this case the equation  $H_1 H_2 = \sqrt{d} H_3$  must be satisfied where  $H_1, H_2$ , and  $H_3$  are Hadamard matrices. But all the entries of  $H_1, H_2$ , and  $H_3$  are integers which indicates that  $d$  must be a square. Thus all the entries in  $Q$  are rational so that the association scheme is rational.

## 4 Our Results

Let  $(X, \{R_i\}_{i=0}^d)$  be a  $d$ -class Q-polynomial association scheme with at least one irrational eigenvalue and  $m_1 > 2$ . The splitting field of  $(X, \{R_i\}_{i=0}^d)$  is defined to be  $\mathbb{F} = \mathbb{Q}(P_{11}, P_{11}, \dots, P_{dd})$ , the smallest subfield of  $\mathbb{R}$  containing all entries of the first eigenmatrix  $P$ . Recall, the Krein field of  $(X, \{R_i\}_{i=0}^d)$  is the smallest subfield of  $\mathbb{R}$  containing all of the Krein parameters  $q_{ij}^k$  of the scheme. Since  $q_{ij}^k = \frac{m_i m_j}{v} \sum_{l=0}^d \frac{1}{k_l^2} P_{li} P_{lj} P_{lk}$ , this is a subfield of  $\mathbb{F}$  [2, Theorem II.3.6].

As  $\mathbb{F}/\mathbb{Q}$  is a Galois extension, there are exactly  $[\mathbb{F} : \mathbb{Q}]$  field automorphisms of  $\mathbb{F}$ . Each such automorphism  $\phi$  fixes each adjacency matrix  $A_i$ , since each  $A_i$  has rational entries, and permutes the set of minimal idempotents  $\{E_0, \dots, E_d\}$ . If  $\phi$  is applied entrywise, then  $(E_j)^\phi = E_{j'}$  for some  $j' \in \{0, \dots, d\}$ . It is not hard to see that if  $E_0, E_1, \dots, E_d$  is one Q-polynomial ordering, then  $E_0, E_{1'}, \dots, E_{d'}$  is another Q-polynomial ordering. Using a result of Suzuki [34] which establishes that an association scheme, other than one arising from a polygon, can have at most two Q-polynomial orderings, Cerzo and Suzuki [15] show that  $[\mathbb{F} : \mathbb{Q}] \leq 2$ . Since we only consider schemes with irrational eigenvalues in this paper, we assume  $[\mathbb{F} : \mathbb{Q}] = 2$ .

For the rest of this paper, let  $d = 3$ . From Theorem 2.11 and from applying the Galois automorphism, we know that if  $E_0, E_1, E_2, E_3$  is a Q-polynomial ordering, then there must be another and this second ordering will either be

- (i)  $E_0, E_2, E_1, E_3$  or
- (ii)  $E_0, E_3, E_2, E_1$ .

These two possibilities are explored separately in the following subsections.

### 4.1 The First Case

We will first consider case (i).

As outlined above, we know there exists a non-trivial automorphism  $\phi : \mathbb{F} \rightarrow \mathbb{F}$  and, in case (i), we have  $(E_1)^\phi = E_2$ . To simplify notation, let us denote  $\phi(a)$  by  $\bar{a}$  for  $a \in \mathbb{F}$  with

the understanding that  $\bar{a} = a$  if and only if  $a$  is rational. Then we may write  $\bar{E}_1 = E_2$ . Writing  $n = |X|$  and

$$nE_1 = mI + \sigma_1 A_1 + \sigma_2 A_2 + \sigma_3 A_3,$$

we have

$$Q = \begin{bmatrix} 1 & m & m & m_3 \\ 1 & \sigma_1 & \bar{\sigma}_1 & Q_{13} \\ 1 & \sigma_2 & \bar{\sigma}_2 & Q_{23} \\ 1 & \sigma_3 & \bar{\sigma}_3 & Q_{33} \end{bmatrix}$$

where  $Q_{i3} = -1 - \sigma_i - \bar{\sigma}_i$  for  $1 \leq i \leq 3$ . Observe that  $Q_{i3} \in \mathbb{Q}$  since  $\overline{\sigma_i + \bar{\sigma}_i} = \bar{\sigma}_i + \sigma_i$ . Also, we have the dual intersection matrices  $L_i^* = [q_{ij}^k]_{k,j}$  as follows:

$$L_1^* = \begin{bmatrix} 0 & m & 0 & 0 \\ 1 & a_1^* & b_1^* & 0 \\ 0 & b_1^* & a_2^* & b_2^* \\ 0 & 0 & c_3^* & a_3^* \end{bmatrix}, \quad L_2^* = \begin{bmatrix} 0 & 0 & m & 0 \\ 0 & b_1^* & a_2^* & b_2^* \\ 1 & a_2^* & q_{22}^2 & q_{23}^2 \\ 0 & c_3^* & q_{22}^3 & q_{23}^3 \end{bmatrix}, \quad L_3^* = \begin{bmatrix} 0 & 0 & 0 & m_3 \\ 0 & 0 & q_{32}^1 & q_{33}^1 \\ 0 & q_{31}^2 & q_{32}^2 & q_{33}^2 \\ 1 & q_{31}^3 & q_{32}^3 & q_{33}^3 \end{bmatrix}.$$

Now,

$$E_1 \circ E_3 = \frac{1}{n}(b_2^* E_2 + a_3^* E_3)$$

and

$$E_2 \circ E_3 = \frac{1}{n}(b_2^* E_1 + q_{23}^2 E_2 + a_3^* E_3).$$

Applying  $\phi$  to both sides of the first equation, we see  $q_{23}^2 = 0$ ,  $b_2^* = \bar{b}_2^*$ , and  $a_3^* = \bar{a}_3^*$ .

Since this is an irrational scheme, without loss of generality, let  $\sigma_1 \notin \mathbb{Q}$ ; we will show  $b_2^* = -Q_{13}$ . Now

$$Q_{11}Q_{13} = b_2^* \bar{\sigma}_1 + a_3^* Q_{13} \tag{4.1}$$

and

$$Q_{12}Q_{13} = b_2^* \sigma_1 + a_3^* Q_{13}.$$

Thus  $Q_{11}Q_{13} - Q_{12}Q_{13} = b_2^*(\bar{\sigma}_1 - \sigma_1)$  which implies  $Q_{13}(\sigma_1 - \bar{\sigma}_1) = b_2^*(\bar{\sigma}_1 - \sigma_1)$ . Therefore,  $Q_{13} = -b_2^*$ , which is nonzero.

From (4.1) we know  $\sigma_1 Q_{13} = -Q_{13} \bar{\sigma}_1 + a_3^* Q_{13}$  which implies  $a_3^* = \sigma_1 + \bar{\sigma}_1$  and  $b_2^* = a_3^* + 1$ . It follows that  $c_3^*, q_{33}^3 \in \mathbb{Q}$ .

Now, we aim to show that exactly one  $\sigma_i \in \mathbb{Q}$  where  $i \in \{1, 2, 3\}$ . We use the standard orthogonality relation and write  $k_i = p_{ii}^0$ , for  $1 \leq i \leq 3$ , to obtain

$$P = \begin{bmatrix} 1 & k_1 & k_2 & k_3 \\ 1 & \frac{k_1 \sigma_1}{m} & \frac{k_2 \sigma_2}{m} & \frac{k_3 \sigma_3}{m} \\ 1 & \frac{k_1 \bar{\sigma}_1}{m} & \frac{k_2 \bar{\sigma}_2}{m} & \frac{k_3 \bar{\sigma}_3}{m} \\ 1 & \frac{k_1 Q_{13}}{m_3} & \frac{k_2 Q_{23}}{m_3} & \frac{k_3 Q_{33}}{m_3} \end{bmatrix}.$$

Since the entries of row 1 of  $P$  sum to zero, we know that at least two of the  $\sigma_i$  values are irrational.

Assume, by way of contradiction, that  $\sigma_i \notin \mathbb{Q}$  for  $1 \leq i \leq 3$ . We may write  $\mathbb{F} = \mathbb{Q}[\sqrt{D}]$  where  $D$  is some positive squarefree integer. Let  $\sigma_i = \alpha + \beta_i\sqrt{D}$  where  $\alpha = \frac{a_3^*}{2}$  and  $\beta_i \in \mathbb{Q}$ . Then

$$Q = \begin{bmatrix} 1 & m & m & m_3 \\ 1 & \alpha + \beta_1\sqrt{D} & \alpha - \beta_1\sqrt{D} & -b_2^* \\ 1 & \alpha + \beta_2\sqrt{D} & \alpha - \beta_2\sqrt{D} & -b_2^* \\ 1 & \alpha + \beta_3\sqrt{D} & \alpha - \beta_3\sqrt{D} & -b_2^* \end{bmatrix}$$

which is row equivalent to

$$Q = \begin{bmatrix} 1 & m & m & m_3 \\ 1 & \alpha + \beta_1\sqrt{D} & \alpha - \beta_1\sqrt{D} & -b_2^* \\ 0 & (\beta_1 - \beta_2)\sqrt{D} & -(\beta_1 - \beta_2)\sqrt{D} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which implies  $\text{rank}(Q) < 4$ , a contradiction. So exactly one  $\sigma_i$  value is rational as desired. Without loss of generality, we will assume  $\sigma_1, \sigma_2 \notin \mathbb{Q}$  and  $\sigma_3 \in \mathbb{Q}$ .

Now we will find conditions on  $a_3^*$ .

$$Q_{13}^2 = m_3 + \frac{m_3}{m}a_3^*\sigma_1 + \frac{m_3}{m}a_3^*\bar{\sigma}_1 + (m_3 - 2a_3^* - 1)(-1 - a_3^*)$$

Simplifying, we obtain:

$$a_3^*\left(a_3^*\frac{m_3}{m} - m_3 + a_3^* + 1\right) = 0.$$

Therefore,

$$a_3^* = 0 \quad \text{or} \quad a_3^* = \frac{m(m_3 - 1)}{m_3 + m}.$$

Now we will prove that  $\sigma_3 = -1$ . We have

$$Q_{31}Q_{33} = b_2^*\sigma_3 + a_3^*(-1 - 2\sigma_3).$$

Simplifying this we obtain

$$\sigma_3 = \frac{a_3^* - 2 \pm \sqrt{a_3^{*2} - 4a_3^* + 4 + 8a_3^*}}{4}.$$

So

$$\sigma_3 = \frac{a_3^*}{2} \quad \text{or} \quad \sigma_3 = -1.$$

Also  $Q_{31}^2 = m + a_1^*\sigma_3 + b_1^*\sigma_3$ , which implies  $\sigma_3^2 = m + (a_1^* + b_1^*)\sigma_3$ . Since  $1 + a_1^* + b_1^* = m$  we obtain:

$$\sigma_3 = m \quad \text{or} \quad \sigma_3 = -1.$$

We will show  $\sigma_3 = -1$  by contradiction. Assume  $\sigma_3 \neq -1$ ; then  $\sigma_3 = m = \frac{a_3^*}{2}$ . Also  $a_3^* = 0$  or  $a_3^* = \frac{m(m_3-1)}{m_3+m}$ . If  $a_3^* = 0$ , we find  $m = 0$  which is a contradiction. If  $a_3^* = \frac{m(m_3-1)}{m+m_3}$  we obtain  $2m + m_3 = -1$  which is also impossible. Therefore we have  $\sigma_3 = -1$ .

Now,

$$E_2 \circ E_2 = \frac{1}{n}(mE_0 + a_2^*E_1 + q_{22}^2E_2)$$

and

$$E_1 \circ E_1 = \frac{1}{n}(mE_0 + a_1^*E_1 + b_1^*E_2).$$

Thus  $a_1^* = q_{22}^2$  and  $b_1^* = \bar{a}_2^*$ . Lastly, since  $q_{ij}^k = q_{ik}^j$  we find that  $a_3^* = q_{31}^3 = q_{32}^3$ .

At this point, we may express the parameters as follows:

$$L_1^* = \begin{bmatrix} 0 & m & 0 & 0 \\ 1 & a_1^* & b_1^* & 0 \\ 0 & b_1^* & \bar{b}_1^* & b_2^* \\ 0 & 0 & c_3^* & a_3^* \end{bmatrix} L_2^* = \begin{bmatrix} 0 & 0 & m & 0 \\ 0 & b_1^* & \bar{b}_1^* & b_2^* \\ 1 & b_1^* & a_1^* & 0 \\ 0 & c_3^* & 0 & a_3^* \end{bmatrix} L_3^* = \begin{bmatrix} 0 & 0 & 0 & m_3 \\ 0 & 0 & b_2^* & q_{33}^1 \\ 0 & b_2^* & 0 & q_{33}^1 \\ 1 & a_3^* & a_3^* & q_{33}^3 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & m & m & m_3 \\ 1 & \sigma & \bar{\sigma} & -b_2^* \\ 1 & \sigma_2 & \bar{\sigma}_2 & -b_2^* \\ 1 & -1 & -1 & 1 \end{bmatrix} P = \begin{bmatrix} 1 & k_1 & k_2 & k_3 \\ 1 & \frac{k_1\sigma_1}{m} & \frac{k_2\sigma_2}{m} & -\frac{k_3}{m} \\ 1 & \frac{k_1\bar{\sigma}_1}{m} & \frac{k_2\bar{\sigma}_2}{m} & -\frac{k_3}{m} \\ 1 & -\frac{k_1b_2^*}{m_3} & -\frac{k_2b_2^*}{m_3} & \frac{k_3}{m_3} \end{bmatrix}$$

**Theorem:** For any 3-class Q-polynomial association scheme with irrational eigenvalues, the following are equivalent

- (i)  $a_3^* = 0$
- (ii)  $m_3 = 1$
- (iii)  $a_1^* = \bar{b}_1^*$ .

(Pf) First,  $a_3^* = 0$  is equivalent to  $m_3 = 1$  follows from  $a_3^* = 0$  or  $a_3^* = \frac{(m_3-1)m}{m_3+m}$ .

We obtain,  $(a_1^* - b_1^*) + (\bar{a}_1^* - \bar{b}_1^*) = 2a_3^*$  from the irrational part of the equation

$$\sigma_1^2 = m + a_1^*\sigma_1 + b_1^*\bar{\sigma}_1$$

Thus  $a_3^* = 0$  if and only if  $a_1^* = \bar{b}_1^*$

**Conjecture:**  $a_3^* = 0$

Note that the above conjecture states that any irrational 3-class Q-polynomial association scheme with second ordering  $E_0, E_2, E_1, E_3$  arises from a symmetric design. There is evidence to suggest that this conjecture is true. There are no counterexamples for any association schemes with less than 1024 vertices [8, pg 425-432]. Also, note that this conjecture

would imply that an association scheme arising from case (i) is P-polynomial, since each symmetric design gives rise to a P-polynomial scheme. We saw earlier that each linked system of symmetric designs gives rise to a rational association scheme but is not necessarily P-polynomial. Once we relax the condition that the association scheme must be irrational, we find examples of Q-polynomial schemes that are not P-polynomial. We could conjecture that when we restrict ourselves to irrational schemes, then the only examples that arise are also P-polynomial. Our previous conjecture supports this idea for irrational 3-class Q-polynomial association schemes with second ordering  $E_0, E_2, E_1, E_3$ .

**Example:**  $IG(7, 3, 1)$  has parameters

$$P = \begin{bmatrix} 1 & 3 & 4 & 6 \\ 1 & \sqrt{2} & -\sqrt{2} & -1 \\ 1 & -\sqrt{2} & \sqrt{2} & -1 \\ 1 & -3 & -4 & 6 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 6 & 6 & 1 \\ 1 & 2\sqrt{2} & -2\sqrt{2} & -1 \\ 1 & -\frac{3}{2}\sqrt{2} & \frac{3}{2}\sqrt{2} & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$L_1^* = \begin{bmatrix} 0 & 6 & 0 & 0 \\ 1 & \frac{5}{2} + \frac{\sqrt{2}}{4} & \frac{5}{2} - \frac{\sqrt{2}}{4} & 0 \\ 0 & \frac{5}{2} - \frac{\sqrt{2}}{4} & \frac{5}{2} + \frac{\sqrt{2}}{4} & 1 \\ 0 & 0 & 6 & 0 \end{bmatrix} \quad L_2^* = \begin{bmatrix} 0 & 0 & 6 & 0 \\ 0 & \frac{5}{2} - \frac{\sqrt{2}}{4} & \frac{5}{2} + \frac{\sqrt{2}}{4} & 1 \\ 1 & \frac{5}{2} + \frac{\sqrt{2}}{4} & \frac{5}{2} - \frac{\sqrt{2}}{4} & 0 \\ 0 & 6 & 0 & 0 \end{bmatrix} \quad L_3^* = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

## 4.2 The Second Case

We will now consider case (ii). As in case (i), we know there exists a non-trivial automorphism  $\phi : \mathbb{F} \rightarrow \mathbb{F}$  which in this case is defined by  $(E_1)^\phi = E_3$

In this case we start with the second eigenmatrix,

$$Q = \begin{bmatrix} 1 & m & m_2 & m \\ 1 & \sigma & Q_{12} & \bar{\sigma} \\ 1 & \sigma_2 & Q_{22} & \bar{\sigma}_2 \\ 1 & \sigma_3 & Q_{32} & \bar{\sigma}_3 \end{bmatrix}$$

and utilizing  $q_{ij}^k m_k = q_{ik}^j m_j$ , we obtain the dual intersection matrices,  $L_i^*$ , as follows:

$$L_1^* = \begin{bmatrix} 0 & m & 0 & 0 \\ 1 & a_1^* & b_1^* & 0 \\ 0 & c_2^* & a_2^* & b_2^* \\ 0 & 0 & c_3^* & a_3^* \end{bmatrix} \quad L_2^* = \begin{bmatrix} 0 & 0 & m_2 & 0 \\ 0 & b_1^* & q_{22}^1 & c_3^* \\ 1 & a_2^* & q_{22}^2 & q_{23}^2 \\ 0 & c_3^* & q_{22}^3 & q_{23}^3 \end{bmatrix} \quad L_3^* = \begin{bmatrix} 0 & 0 & 0 & m \\ 0 & 0 & c_3^* & q_{33}^1 \\ 0 & q_{31}^2 & q_{32}^2 & q_{33}^2 \\ 1 & q_{31}^3 & q_{32}^3 & q_{33}^3 \end{bmatrix}$$

We have that

$$nE_1 \circ E_2 = b_1^* E_1 + a_2^* E_2 + c_3^* E_3$$



and

$$nE_3 \circ E_2 = q_{32}^1 E_1 + q_{32}^2 E_2 + q_{32}^3 E_3.$$

Applying  $\phi$  to both sides of the second equation, we obtain

$$nE_1 \circ E_2 = \overline{q_{32}^1} E_3 + \overline{q_{32}^2} E_2 + \overline{q_{32}^3} E_1.$$

Therefore  $b_1^* = \overline{q_{32}^3}$ ,  $a_2^* = \overline{q_{32}^2}$ , and  $c_3^* = \overline{q_{32}^1}$ . Thus  $c_3^*, a_3^*, b_2^* \in \mathbb{Q}$ .

Without loss of generality, assume  $\sigma \notin \mathbb{Q}$ . Now, consider

$$\sigma \bar{\sigma} = b_2^* Q_{12} + a_3^* \bar{\sigma}.$$

From here we obtain,  $\bar{\sigma} a_3^* \in \mathbb{Q}$  so that  $a_3^* = 0$  and  $c_3^* = m$ .

Also, we have that

$$nE_1 \circ E_1 = mE_0 + a_1^* E_1 + c_2^* E_2$$

and

$$nE_3 \circ E_3 = mE_0 + q_{33}^1 E_1 + q_{33}^2 E_2 + q_{33}^3 E_3.$$

Applying  $\phi$  in a similar manner as before, we obtain  $q_{33}^1 = 0$ ,  $\overline{c_2^*} = q_{33}^2$ , and  $\overline{a_1^*} = q_{33}^3$ .

Also, we can see that  $\overline{q_{22}^1} = q_{22}^3$  and  $q_{22}^2 = m_2 - \overline{a_2^*} - a_2^* - 1 \in \mathbb{Q}$ .

Thus we obtain the dual intersection matrices are the following:

$$L_1^* = \begin{bmatrix} 0 & m & 0 & 0 \\ 1 & a_1^* & b_1^* & 0 \\ 0 & c_2^* & a_2^* & \frac{m^2}{m_2} \\ 0 & 0 & m & 0 \end{bmatrix}, L_2^* = \begin{bmatrix} 0 & 0 & m_2 & 0 \\ 0 & b_1^* & q_{22}^1 & m \\ 1 & a_2^* & \frac{q_{22}^2}{a_2^*} & \frac{a_2^*}{a_2^*} \\ 0 & m & \frac{q_{22}^1}{q_{22}^1} & \frac{b_1^*}{b_1^*} \end{bmatrix}, L_3^* = \begin{bmatrix} 0 & 0 & 0 & m \\ 0 & 0 & m & 0 \\ 0 & b_2^* & \frac{a_2^*}{a_2^*} & \frac{c_2^*}{c_2^*} \\ 1 & 0 & \frac{b_1^*}{b_1^*} & \frac{a_1^*}{a_1^*} \end{bmatrix}.$$

**Example:** The icosahedron has the following parameters:

$$P = \begin{bmatrix} 1 & 5 & 5 & 1 \\ 1 & \sqrt{5} & -\sqrt{5} & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -\sqrt{5} & \sqrt{5} & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 3 & 5 & 3 \\ 1 & \frac{3}{5}\sqrt{5} & -1 & -\frac{3}{5}\sqrt{5} \\ 1 & -\frac{3}{5}\sqrt{5} & -1 & \frac{3}{5}\sqrt{5} \\ 1 & -3 & 5 & -3 \end{bmatrix},$$

$$L_1^* = \begin{bmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & \frac{6}{5} & 0 & \frac{9}{5} \\ 0 & 0 & 3 & 0 \end{bmatrix}, \quad L_2^* = \begin{bmatrix} 0 & 0 & 5 & 0 \\ 0 & 2 & 0 & 3 \\ 1 & 0 & 4 & 0 \\ 0 & 3 & 0 & 2 \end{bmatrix}, \quad L_3^* = \begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 \\ 0 & \frac{9}{5} & 0 & \frac{6}{5} \\ 1 & 0 & 2 & 0 \end{bmatrix}.$$

## References

- [1] S. Bang, A. Dubickas, J. H. Koolen and V. Moulton. There are only finitely many distance-regular graphs of fixed valency greater than two. arXiv.org: 0909.5253.
- [2] E. Bannai and T. Ito. Algebraic Combinatorics I: Association Schemes. Benjamin-Cummings, Menlo Park (1984).
- [3] E. Bannai and T. Ito. On finite Moore graphs. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **20** (1973), 191–208.
- [4] E. Bannai and T. Ito. On distance-regular graphs with fixed valency, III. *J. Algebra.* **107** (1987), 43–52.
- [5] E. Bannai and T. Ito. On distance-regular graphs with fixed valency. *Graphs Combin.* **3** (1987), 95–109.
- [6] E. Bannai and T. Ito. On distance-regular with fixed valency, II. *Graphs Combin.* **3** (1988), 219–228.
- [7] E. Bannai and T. Ito. On distance-regular with fixed valency, IV. *Europ. J. Combin.* **10** (1989), 137–148.
- [8] A. E. Brouwer, A. M. Cohen and A. Neumaier. Distance-Regular Graphs. Springer-Verlag, Berlin (1989).
- [9] R. H. Bruck and H.J. Ryser. The nonexistence of certain finite projective planes. *Can. J. Math.* **1** (1949), 88–93.
- [10] P. O. Boykin, M. Sitharam, M. Tarifi and P. Wocjan. Real mutually unbiased bases. arXiv: quant-ph/0502024.
- [11] P. J. Cameron, C. E. Praeger, J. Saxl and G. M. Seitz. On the Sims conjecture and distance-transitive graphs. *Bull. London Math. Soc.* **15** (1983), 499–506.
- [12] P. J. Cameron. On groups with several doubly transitive permutation representations. *Math. Z.* **128** (1972), 1–14.
- [13] S. Chowla and H. J. Ryser. Combinatorial problems. *Can. J. Math.* **2** (1950), 93–99.
- [14] C. J. Colbourn and J. H. Dinitz. The CRC Handbook of Combinatorial Designs. Chapman and Hall/CRC (2nd ed.), Boca Raton (2006).
- [15] D. Cerzo and H. Suzuki. Non-existence of imprimitive Q-polynomial schemes of exceptional type with  $d=4$ . *Europ. J. Combin.* **30** (2009), 674–681.
- [16] R. M. Damerell. On Moore Graphs. *Proc. Cambridge Philos. Soc.* **74** (1973), 227–236.

- [17] P. Delsarte and V. I. Levenshtein. Association schemes and coding theory. *IEEE Trans. Inform. Theory.* **44** (no. 6) (1998), 2477–2504.
- [18] P. Dita. Hadamard matrices from mutually unbiased bases. arXiv: 1002.4933.
- [19] W. Feit and G. Higman. The nonexistence of certain generalized polygons. *J. Algebra.* **1** (1964), 114–131.
- [20] S. Gosselin. Regular two-graphs and equiangular lines. Master Thesis, University of Waterloo (2004).
- [21] W. H. Haemers. Matrix techniques for strongly regular graphs and related geometries. Reader with Addendum for a series of lectures given in Ghent (2000).
- [22] E. Kuijken and C. Tonesi. Distance-regular graphs and  $(\alpha, \beta)$ -geometries. *J. Geom.* **82** (2005), 135–145.
- [23] J. H. Koolen and V. Moulton. On a conjecture of Bannai and Ito: There are finitely many distance-regular graphs with degree 5, 6 or 7. *Europ. J. Combin.* **23** (2002), 987–1006.
- [24] J. H. Koolen and V. Moulton. There are finitely many triangle-free distance regular graphs with degree 8, 9 or 10. *J. Alg. Combin.* **19** (no. 2) (2004), 205–217.
- [25] J. A. Gallian. Contemporary Abstract Algebra. Brooks/Cole, Belmont (2010).
- [26] N. LeCompte, W. J. Martin and W. Owens. On the equivalence between real mutually unbiased bases and a certain class of association schemes. In press, *Europ. J. Combin.* **31** (no. 6) (2010), 1499–1512.
- [27] D. A. Marcus. Number Fields. Springer-Verlag, New York (1977).
- [28] W. J. Martin and J. Williford. There are finitely many  $\mathbb{Q}$ -polynomial association schemes with given first multiplicity at least three. *Europ. J. Combin.* **30** (no. 3) (2009), 698–704.
- [29] W. J. Martin, M. Muzychuk and J. Williford. Imprimitve cometric association schemes: constructions and analysis. *J. Alg. Combin.* **25** (no. 4) (2007), 399–415.
- [30] W. J. Martin, E. van Dam and M. Muzychuk. Uniformity in association schemes and coherent configurations: cometric  $\mathbb{Q}$ -antipodal schemes and linked systems. arXiv: 1001.4928.
- [31] W. J. Martin, M. Muzychuk and J. Williford. Some new constructions of imprimitve cometric association schemes. Proceedings, Algebraic Combinatorics, an international conference in honour of Eiichi Bannai's 60th birthday, June 26-30, 2006, Sendai, Japan.
- [32] A. Munemasa. Splitting fields of association schemes. *J. Combin. Theory Ser. A.* **57** (no. 1) (1991), 157–161.

- [33] H. Suzuki. Imprimitve Q-polynomial association schemes. *J. Alg. Combin.* **7** (no. 2) (1998), 165–180.
- [34] H. Suzuki. Association schemes with multiple Q-polynomial structures. *J. Alg. Combin.* **7** (no.2) (1998), 181–196.
- [35] Taylor Graph from Wolfram MathWorld. Wolfram MathWorld (2011). <http://mathworld.wolfram.com/TaylorGraph.html>.
- [36] H. Tanaka and R. Tanaka. Nonexistence of exceptional imprimitve Q-polynomial association schemes with six classes. *Europ. J. Combin.* **32** (2011), 155–161.
- [37] E. R. van Dam, W. H. Haemers, J. H. Koolen and E. Spence. Characterizing distance-regularity of graphs by the spectrum. *J. Combin. Theory Ser. A.* **113** (2006), 1805–1820.
- [38] C. Godsil and G. Royle. Algebraic Graph Theory. Springer-Verlag, New York (2001).