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# Choosing Investment Strategies by their Outcomes

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# Choosing Investment Strategies by their Outcomes

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A Major Qualifying Project submitted to the Faculty of

WORCESTER POLYTECHNIC INSTITUTE

in partial fulfilment of the requirements for the  
Degree of Bachelor of Science  
in the Mathematical Sciences

by

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Approved: \_\_\_\_\_

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# Abstract

Sharpe has shaped recent thinking in the financial sciences with his paper on the Distribution Builder, a tool designed to elicit investor preferences as probability distributions. We examine pricing such distributions in simple financial models. The no-arbitrage pricing method is a well studied technique for pricing derivative securities in financial market models. We apply this method to price arbitrary probability distributions on finite outcome spaces. This work surfaces constraints on the technique as the number of possible investment outcomes increases, and examines these constraints from multiple perspectives.

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# 1 Introduction

## 1.1 Background

In this paper we examine desired outcome distributions of investors in financial markets and attempt to ascertain the cost to achieve them. Pricing in financial markets is well studied under various models, such as the Black-Scholes-Merton model, but the object of interest is usually a derivative security. We apply these pricing techniques, particularly the no-arbitrage pricing method, to arbitrary distributions representing investor preferences.

The study of investor preferences in terms of probability distributions can be attributed largely to William F. Sharpe. Sharpe and his co-authors designed and experimented with a tool called “The Distribution Builder”. The tool is designed to elicit an individual’s investment preferences in terms of their preferred distribution of wealth at a given time horizon, such as retirement [1]. By distribution of wealth we mean the set of possible outcomes and their relative probabilities. This probability distribution encapsulates the states they wish to be possible, and perhaps more importantly impossible, such as losing everything.

The most important detail of how this works in practice for our purposes is the cost constraint. In a real market in order to possibly gain money an investor must take on some kind of risk. Thus, the more weight an investor places on higher valued outcomes in the Distribution Builder, necessarily the more wealth they need a priori in order to achieve that outcome. The limitation on resources at an investors disposal is called the cost constraint.

Our aim in this paper is to examine the cost constraint in simple models of financial markets. This then is what we mean by choosing investment strategies by their outcomes; using a tool such as Sharpe’s we can obtain an investors preferred distribution of wealth and then, using the techniques we present, identify both how much wealth you need today to achieve that outcome as well as identify a strategy to allocate your wealth appropriately.

We require a method by which to calculate how much a distribution costs. In the financial sciences two such tools are: the no-arbitrage pricing model, and the capital asset pricing method. These tools are typically used in order to price derivative securities, financial instruments whose value derives from some underlying asset and whose role is to hedge against movements in the underlyings price. For us the security will be random variables whose law is our desired wealth distribution, and

the underlying will be the stock market.

That is to say we determine how much money an investor must invest into the market to hedge the probability distribution, to always have the value of the probability distribution in stock and bond holdings at the time horizon. The question is how much stock is required to meet the investors goals in every market outcome, and both of these methods provide an answer. These pricing models will then tell us the price, the amount of wealth necessary, in order to realize this distribution. We will use the no-arbitrage approach for it's precise numerical results in the kinds of idealized models that we will deal with exclusively in this paper [2].

## 1.2 Mathematical Concepts

In order to effectively model stock markets we first recall some fundamental concepts in probability, both to fix a notation and to highlight some subtleties which will come up often throughout the report. First, since the value of a stock is an uncertain quantity it is natural to model it as a random variable. A random variable is a function from the outcome space into the real numbers. Note, it does not matter how likely an outcome is, even the most unlikely of outcomes is given a value, and the assignment of values to outcomes, that is to say the random variable, encodes no information about how probable or improbable an outcome is. The space of possible outcomes however is intrinsic to the definition of a particular random variable. In all of our cases this space shall be finite.

**Definition 1.** Let  $\Omega$  be the set of all possible outcomes then a random variable is a function  $\xi : \Omega \rightarrow \mathbb{R}$ .

An important related object to random variables is a probability distribution. A probability distribution is a tabulation of probabilities. Specifically it tabulates the probability that a random variable takes on a given quantity. A probability distribution captures information about a random variable in relation to a probability measure, and elides information about the outcome space the variable was defined upon. Since it encodes no information about how the variable relates to the outcome space it is possible for two distinct random variables to have the same probability distribution. We will exhibit an example of this shortly.

**Definition 2.** Let  $\mathbb{P}$  be a probability measure on  $\Omega$ , a probability distribution is a map  $P : \mathbb{R} \rightarrow [0, 1]$  satisfying

$$P(\omega) = \mathbb{P}(\xi^{-1}(\omega))$$

where  $\xi^{-1}(\omega)$  is the inverse image of  $\omega$  under  $\xi$ .

If we let  $\Omega = \{\omega_1, \omega_2\}$  with probability measure  $\mathbb{P}(\omega_1) = \mathbb{P}(\omega_2) = \frac{1}{2}$ , then we can exhibit two random variables,  $\xi$  and  $\bar{\xi}$ , with the same distribution:

$$\begin{aligned}\xi(\omega_1) &= 1, & \xi(\omega_2) &= 0; \\ \bar{\xi}(\omega_1) &= 0, & \bar{\xi}(\omega_2) &= 1.\end{aligned}$$

Since the probabilities of the two outcomes are equal, and probability distributions elide information about the outcome space these two variables have the same probability distribution given by:

$$P(1) = \frac{1}{2}, \quad P(0) = \frac{1}{2}.$$

This example suggests a broader result which we now present and prove.

**Theorem 1.** *Let  $\mathbb{P}$  be a probability measure on  $\Omega_k$ , a finite set with  $k$  elements, given by  $\mathbb{P}(\omega_i) = \frac{1}{n}$  for all  $i$ ,  $1 \leq i \leq k$ . Then any two random variables  $\xi : \Omega_k \rightarrow \mathbb{R}$  and  $\bar{\xi} : \Omega \rightarrow \mathbb{R}$  have the same probability distribution iff there exists a permutation  $\sigma$  of  $n$  elements such that for every  $\omega_i \in \Omega_k$ ,  $\xi(\omega_i) = \bar{\xi}(\omega_{\sigma(i)})$*

In other words, two variables have the same probability distribution if we can permute the labels on the outcome space to make them agree.

*Proof.* Let  $\xi$  and  $\bar{\xi}$  be two such random variables related by  $\sigma$ . Then the probability distribution of  $\xi$  is given by

$$P(\omega) = \mathbb{P}(\xi^{-1}(\omega))$$

for each  $\omega$  in the range of  $\xi$ . Since  $\bar{\xi}$  is related to  $\xi$  by a permutation it has the same range and hence its distribution is given by

$$P'(\omega) = \mathbb{P}(\bar{\xi}^{-1}(\omega))$$

and has the same domain. Then these distributions are the same so long as

$$\mathbb{P}(\bar{\xi}^{-1}(\omega)) = \mathbb{P}(\xi^{-1}(\omega))$$

for each value  $\omega$ . Since the probability measure is uniform this is the case.

Next, let  $\xi$  and  $\bar{\xi}$  be random variables with the same distribution,  $P$ . Select a permutation in  $S_n$ , the symmetric group on  $n$  elements, whose orbits are  $\xi^{-1}(\omega)$  for each  $\omega$  in the range of  $\xi$ . Such a permutation satisfies the theorem. □

We have exhibited these two basic concepts with the express purpose of highlighting that *a random variable is not a probability distribution*. Random variables are tightly coupled with the outcome space and unrelated to the probability measure. Probability



distributions are tightly coupled with the measure. Sure, given a particular measure there exists a map of random variables to probability distributions, but under a different measure this map changes. This paper is mostly interested in pricing the distribution relative to the uniform probability measure, every outcome being equally likely, but as we will see this sometimes requires examining other measures.

Finally, we visit the concept of martingales, and conditional expectation. The definition of expectation is the familiar one

**Definition 3.** Let  $\mathbb{P}$  be a probability measure on  $\Omega_k$ , and  $\xi : \Omega_k \rightarrow \mathbb{R}$  be a random variable. Then the *expectation* of  $\xi$  is denoted  $\mathbb{E}[\xi]$  and given by

$$\mathbb{E}[\xi] = \sum_{\omega \in \Omega_k} \xi(\omega) \mathbb{P}(\omega)$$

We also will find use for the conditional expectation, the expectation of a random variable when some information is known about the outcome but not all.

**Definition 4.** Let  $\Omega^n$  be the set of all strings of length  $n$  constructed from alphabet  $\Omega$ , a finite set. Then let  $\xi : \Omega^n \rightarrow \mathbb{R}$  be a random variable. The conditional expectation of  $\xi$  at time  $k$ ,  $0 < k < n$  is denoted  $\mathbb{E}_k[\xi]$  and given by

$$\mathbb{E}_k[\xi](\omega_1 \omega_2 \cdots \omega_k) = \sum_{\omega_{k+1}, \omega_{k+2}, \dots, \omega_n \in \Omega} \mathbb{P}(\omega_1 \omega_2 \cdots \omega_k \omega_{k+1} \omega_{k+2} \cdots \omega_n) \xi(\omega_1 \omega_2 \cdots \omega_k \omega_{k+1} \omega_{k+2} \cdots \omega_n)$$

with extremal cases given by

$$\mathbb{E}_0[\xi] = \mathbb{E}[\xi],$$

$$\mathbb{E}_n[\xi](\omega_1 \omega_2 \cdots \omega_n) = \xi(\omega_1 \omega_2 \cdots \omega_n).$$

Conditional expectations have four fundamental properties which we summarize here

**Theorem 2.** Let  $X, Y$  be random variables defined on  $\Omega^n$ , and let  $0 \leq k \leq n$  be given.

Linearity: for all constants  $c_1, c_2$  we have

$$\mathbb{E}_k[c_1 X + c_2 Y] = c_1 \mathbb{E}_k[X] + c_2 \mathbb{E}_k[Y].$$

Taking out what is known: If  $X$  only depends on the first  $k$  letters of the string then we have

$$\mathbb{E}_k[XY] = X \mathbb{E}_k[Y].$$

Iterated Conditioning: Let  $k \leq j \leq n$  then

$$\mathbb{E}_k[\mathbb{E}_j[X]] = \mathbb{E}_k[X].$$

Independence: If  $X$  depends only on the letters  $k + 1$  through  $n$  then

$$\mathbb{E}_k[X] = \mathbb{E}[X].$$

*Proof.* A proof is given [2]. □

Utilizing these definitions we are now equipped to introduce martingales. Models of markets under which the stock price is a martingale are especially easy to work with as we shall see later so this is an important definition.

**Definition 5.** Let  $\xi_0, \xi_1, \dots, \xi_n$  be a sequence of random variables where  $\xi_0$  is a scalar, and  $\xi_1, \dots, \xi_n$  are defined on  $\Omega, \dots, \Omega^n$  respectively. In other words each of these random variables,  $\xi_k$ , depends only on the first  $k$  letters of a string of length  $n$ . Then this sequence is called a martingale if they satisfy

$$\xi_k = \mathbb{E}_k[\xi_{k+1}] \quad k = 0, 1, \dots, n - 1$$

It is worth noting as before with distributions that expectations and martingales are intrinsically tied to a certain probability measure. A sequence of random variables may be a martingale under one measure but not another. For this reason, while we will find need to change measure to make a sequence of random variables a martingale for analysis we must find a way to relate these results back to the original measure, and hence the original distribution.

With all these preliminary concepts in hand we are now equipped to begin examining market models in particular. We will begin by defining and examining the binomial model in one period, and continue by extending this to multiple periods. The multiple period case will also give us insight into a technique that will be especially useful in the multinomial case.

# 2 Pricing Probability Distributions

## 2.1 Pricing Investment Strategies

We examine a market comprising one stock, and one bond. We assume that the *bond* exhibits no interest and has constant value. The *stock* has positive value at time zero, and random positive values at further times. Time is discrete, divided into sections called *periods*. The simplest models have only one period, so-called *single-period models*. At the start of a period an investor purchases or sells stocks and bonds and at the end we examine what their holdings are worth. We will in later sections examine models where there are multiple periods, and the investor can move their investments between the stock and bond at the start of each period. In each case we will be interested in an investor's distribution of possible wealth at the end of the final period.

We now fix some notation that will help us illuminate theorems and proofs throughout the report. We denote by  $\Omega_k$  the set of all possible outcomes at the end of the period,  $k$  equal to the number of outcomes. First we examine  $k = 2$ ;  $\Omega_2 = \{\omega_1, \omega_2\}$ . This is called the *single-period binomial model*. We also assume throughout the paper that the probability measure, denoted  $\mathbb{P}$ , on  $\Omega_k$  is uniform;  $\mathbb{P}(\omega_i) = \mathbb{P}(\omega_j)$  for all  $\omega_i, \omega_j \in \Omega_k$ . The random variable  $S_1 : \Omega_2 \rightarrow \mathbb{R}$  maps outcomes onto the value that the stock takes in that outcome. We also assume that  $S_1$  takes on different values in each outcome and that these values come in order;  $S_1(\omega_1) > S_1(\omega_2)$ , relabeling if necessary. The quantity  $S_0$  represents the value of the stock at time zero, that is to say the price, and is positive by assumption.

Of particular interest is the question of in what ways an investor might invest in this market, and how those investments could possibly pay out. We call these allocations *investment strategies*.

**Definition 6.** An *investment strategy* is an allocation of funds into stocks and bonds, represented by the ordered pair  $(\alpha, \beta)$  where  $\alpha$  is the amount of stock, and  $\beta$  is the amount of bond.

**Definition 7.** The *price* of an investment strategy is a real number representing how much the investment strategy is worth at the start of the period

$$\alpha S_0 + \beta.$$

**Definition 8.** The *value* of an investment strategy is a random variable representing the amount the investment strategy is worth at the end of the period;

$$\alpha S_1 + \beta.$$

Keep in mind that while intuitively  $\alpha$  and  $\beta$  are positive values, they need not be, and the negative values also have meaningful financial interpretation. For example, consider a strategy with zero price,  $\beta = -\alpha S_0$ , with  $\alpha$  positive, hence  $\beta$  negative. This strategy represents a loan from the bond to fund the purchase of stock. If the value of the stock goes up we can pay off this loan and pocket the difference, but if the value goes down we have gambled and lost and owe money; our investment strategy takes on a negative value. Conversely, we can take a loan from the stock market to buy bonds, but if those stocks go up in value our loan will correspondingly take more wealth to pay off; again this investment strategy takes on a negative value in this outcome. On the other hand if the stock goes down it will be cheaper to pay off than it cost to purchase, the investment strategy has positive value in this outcome. These zero-price strategies motivate concerns we cover presently.

As in reality we will require that money must come from somewhere, it cannot spontaneously appear in an investor's pockets without a little risk.

**Definition 9.** A market is said to exhibit *arbitrage* if there exists an investment strategy,  $(\alpha, \beta)$ , with zero price,  $\alpha S_0 + \beta = 0$ , a positive probability of taking on a positive value, and zero probability of taking on a negative one,  $\alpha S_1 + \beta \geq 0$ , and  $\mathbb{P}[\alpha S_1 + \beta \geq 0] > 0$ .

Such a strategy requires no funds to start, has no chance to incur loss, yet still has a chance to profit. As an example, take the binomial model whose stock satisfies  $0 < S_0 < S_1(\omega_1) < S_1(\omega_2)$ . In this model the stock can only increase in value. By borrowing some funds from the bond,  $\beta < 0$ , we purchase some stock,  $\alpha > 0$ , such that the price is zero. But, the stock is guaranteed to grow in value from what it cost us to purchase, and so we can pay back our interestless debt at the end of the period with money to spare, regardless of the outcome. Such markets are pathological and in order to avoid them we introduce the No-Arbitrage Condition:

**Definition 10.** The *No-Arbitrage Condition* states that valid markets do not exhibit arbitrage; there does not exist an investment strategy with zero price, a positive probability of taking on a positive value and zero probability of taking on a negative one.

In the binomial model this condition is equivalent to  $S_1(\omega_1) > S_0 > S_1(\omega_2)$ . In other terms, there must be a chance that the stock decreases in value, and there must be a chance it increases.

**Theorem 3.** *The No-Arbitrage Condition is equivalent to the condition*

$$S_1(\omega_1) > S_0 > S_1(\omega_2)$$

*in the single-period binomial model.*

*Proof.* A proof is given in [2]. □

Finally, the thrust of this report is that we can select investment strategies based on their outcomes; by their outcomes we of course mean their probability distribution. As a stepping stone let's first introduce another definition:

**Definition 11.** Let  $\xi : \Omega_2 \rightarrow \mathbb{R}$  be an arbitrary random variable. Then the *price* of  $\xi$  is the price of the investment strategy satisfying the system of equations

$$\begin{aligned}\alpha S_1(\omega_1) + \beta &= \xi(\omega_1), \\ \alpha S_1(\omega_2) + \beta &= \xi(\omega_2)\end{aligned}$$

if such an investment strategy exists.

Correspondingly we define price for distributions:

**Definition 12.** Let  $P$  be a probability distribution. Then the price of  $P$  is the minimum price of all the random variables which have probability distribution  $P$ .

The primary goal of this report is to find results on pricing probability distributions, this is called portfolio optimization, and indentifying the investment strategies that correspond to them. The methods by which we achieve this, and the extent to which we can achieve this, both depend on the particular market model we are studying. In each of the following sections we examine a different model.

## 2.2 Pricing in the Binomial Model

We first examine the single-period, binomial model. We denote by  $\xi$  an arbitrary random variable defined on  $\Omega_2$  and describe a price for it in general.

**Theorem 4.** *Let  $\xi : \Omega_2 \rightarrow \mathbb{R}$  be a arbitrary random variable, then the price of  $\xi$  is given by*

$$\frac{\xi(\omega_1)(S_0 - S_1(\omega_2)) + \xi(\omega_2)(S_1(\omega_1) - S_0)}{S_1(\omega_1) - S_1(\omega_2)}$$

*with corresponding investment strategy*

$$(\alpha, \beta) = \left( \frac{\xi(\omega_1) - \xi(\omega_2)}{S_1(\omega_1) - S_1(\omega_2)}, \frac{S_1(\omega_1)\xi(\omega_2) - S_1(\omega_2)\xi(\omega_1)}{S_1(\omega_1) - S_1(\omega_2)} \right).$$

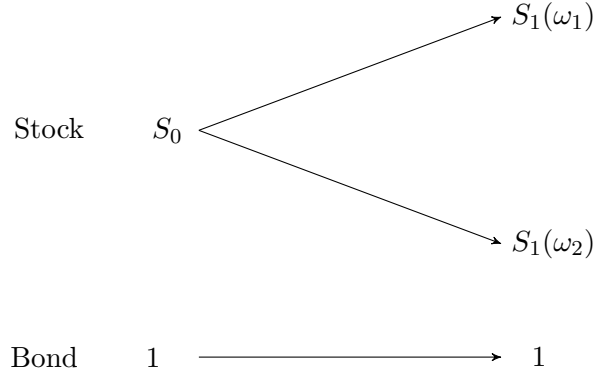


Figure 2.1: Stock and bond dynamics in two outcome markets.

*Proof.* By definition of the price of a random variable we have the following system of equations

$$\begin{aligned}\alpha S_1(\omega_1) + \beta &= \xi(\omega_1), \\ \alpha S_1(\omega_2) + \beta &= \xi(\omega_2).\end{aligned}$$

Since this system of equations is so small it is easily be solved by inverting its associated matrix.

$$\begin{bmatrix} S_1(\omega_1) & 1 \\ S_1(\omega_2) & 1 \end{bmatrix}^{-1} = \frac{1}{S_1(\omega_1) - S_1(\omega_2)} \begin{bmatrix} 1 & -1 \\ -S_1(\omega_2) & S_1(\omega_1) \end{bmatrix}.$$

Recall our earlier assumption that the stock value be random ensures that  $S_1(\omega_1) - S_1(\omega_2) \neq 0$ . Next applying the inverted matrix to the right hand side column vector we find

$$\begin{aligned}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} &= \frac{1}{S_1(\omega_1) - S_1(\omega_2)} \begin{bmatrix} 1 & -1 \\ -S_1(\omega_2) & S_1(\omega_1) \end{bmatrix} \begin{bmatrix} \xi(\omega_1) \\ \xi(\omega_2) \end{bmatrix} \\ &= \frac{1}{S_1(\omega_1) - S_1(\omega_2)} \begin{bmatrix} \xi(\omega_1) - \xi(\omega_2) \\ S_1(\omega_1)\xi(\omega_2) - S_1(\omega_2)\xi(\omega_1) \end{bmatrix}.\end{aligned}$$

Note, we have already achieved one half of our claim and described in general the investment strategy which achieves  $\xi$ .

We next substitute these values into the price expression to find the price of  $\xi$

given by:

$$\begin{aligned}
& \frac{\xi(\omega_1) - \xi(\omega_2)}{S_1(\omega_1) - S_1(\omega_2)} S_0 + \frac{S_1(\omega_1)\xi(\omega_2) - S_1(\omega_2)\xi(\omega_1)}{S_1(\omega_1) - S_1(\omega_2)} \\
&= \frac{\xi(\omega_1)S_0 - \xi(\omega_2)S_0 + S_1(\omega_1)\xi(\omega_2) - S_1(\omega_2)\xi(\omega_1)}{S_1(\omega_1) - S_1(\omega_2)} \\
&= \frac{\xi(\omega_1)(S_0 - S_1(\omega_2)) + \xi(\omega_2)(S_1(\omega_1) - S_0)}{S_1(\omega_1) - S_1(\omega_2)}.
\end{aligned}$$

□

Since a binomial model only has two outcomes there are only two random variables which share the same distribution. The complementary random variable,  $\bar{\xi}$ , is defined by the equalities:

$$\begin{aligned}
\bar{\xi}(\omega_1) &= \xi(\omega_2), \\
\bar{\xi}(\omega_2) &= \xi(\omega_1).
\end{aligned}$$

These two variables have identical probability distributions, and no other random variable shares it with them. Since the distribution is our focus, we would like to know which of these investment strategies is cheaper, or if they cost the same.

**Theorem 5.** *Let  $\xi : \Omega_2 \rightarrow \mathbb{R}$  be an arbitrary random variable, and  $\bar{\xi}$  defined as above. Then the price of  $\bar{\xi}$  is given by*

$$\frac{\xi(\omega_2)(S_0 - S_1(\omega_2)) + \xi(\omega_1)(S_1(\omega_1) - S_0)}{S_1(\omega_1) - S_1(\omega_2)}$$

with corresponding investment strategy

$$(\bar{\alpha}, \bar{\beta}) = \left( \frac{\xi(\omega_2) - \xi(\omega_1)}{S_1(\omega_1) - S_1(\omega_2)}, \frac{S_1(\omega_1)\xi(\omega_1) - S_1(\omega_2)\xi(\omega_2)}{S_1(\omega_1) - S_1(\omega_2)} \right).$$

*Proof.* Pricing  $\bar{\xi}$  follows in exactly the same way as pricing  $\xi$ . Using the definition of  $\bar{\xi}$  we find an expression for  $\bar{\xi}$  in terms of  $\xi$ . □

This leads us to a sufficient condition for their prices to be equivalent

**Theorem 6.**  *$\xi$  and  $\bar{\xi}$  have the same price iff*

$$S_1(\omega_1) - S_0 = S_0 - S_1(\omega_2)$$

*Proof.* Denote the quantity  $S_1(\omega_1) - S_0$  by  $S$ . Then the price of  $\xi$  and  $\bar{\xi}$  are both

$$\frac{\xi(\omega_1) + \xi(\omega_2)}{S_1(\omega_1) - S_1(\omega_2)} S$$

by substitution. □

This condition has a nice economic interpretation; the two strategies cost the same if the stock moves up by the same amount as it moves down. Further, the two strategies have economic interpretations. First, notice that  $\alpha = -\bar{\alpha}$ , and  $\alpha \neq 0$  since then  $\xi$  wouldn't be random. Without loss of generality, relabeling if necessary, let  $\alpha > 0 > \bar{\alpha}$ . Then  $\xi$  represents a strategy where one buys stock at the start of the period, and sells it afterwards, whereas  $\bar{\xi}$  represents a complementary strategy, where one sells stock at the start of the period, and replaces it when the period ends. This illustrates the equation  $\xi(\omega_1) = \bar{\xi}(\omega_2)$ , as  $\omega_1$  represents the outcome where the stock price increases, so  $\xi$  increases as well. On the other hand  $\bar{\xi}$  increases when the stock goes down in price, and accordingly is equal in value in outcome  $\omega_2$  to  $\xi$  in  $\omega_1$ .

Given this condition then we have successfully priced any probability distribution, all the random variables sharing a distribution have the same price and so any one of their investment strategies will provide us our desired distribution at the minimal price. Failing this condition, however, we need some way to determine which strategy is cheaper in order to price their probability distribution.

Fix  $\xi$  as above as the strategy with  $\alpha > 0$ .

**Theorem 7.**  $\xi$  has lower price than  $\bar{\xi}$  if and only if

$$S_0 - S_1(\omega_2) < S_1(\omega_1) - S_0$$

*Proof.* By definition the statement  $\xi$  has lower price than  $\bar{\xi}$  is equivalent to the following inequality:

$$\frac{\xi(\omega_1)(S_0 - S_1(\omega_2)) + \xi(\omega_2)(S_1(\omega_1) - S_0)}{S_1(\omega_1) - S_1(\omega_2)} < \frac{\xi(\omega_1)(S_1(\omega_1) - S_0) + \xi(\omega_2)(S_0 - S_1(\omega_2))}{S_1(\omega_1) - S_1(\omega_2)}$$

The following are equivalent to the above.

$$\begin{aligned} \xi(\omega_1)(S_0 - S_1(\omega_2)) + \xi(\omega_2)(S_1(\omega_1) - S_0) &< \xi(\omega_1)(S_1(\omega_1) - S_0) + \xi(\omega_2)(S_0 - S_1(\omega_2)) \\ \Leftrightarrow 0 &< \xi(\omega_1)(S_1(\omega_1) + S_1(\omega_2) - 2S_0) - \xi(\omega_2)(S_1(\omega_1) + S_1(\omega_2) - 2S_0) \\ \Leftrightarrow 0 &< (\xi(\omega_1) - \xi(\omega_2))(S_1(\omega_1) + S_1(\omega_2) - 2S_0) \\ \Leftrightarrow 0 &< S_1(\omega_1) + S_1(\omega_2) - 2S_0 \\ \Leftrightarrow S_0 - S_1(\omega_2) &< S_1(\omega_1) - S_0 \end{aligned}$$

□

This is the same as the condition for equality, but now as an inequality. With similar interpretation, if the stock goes up by more than it goes down then it is cheaper to invest in stock than to sell and replace it. Thus, we can price probability distributions in the single-period binomial model regardless of which particular model we are working in. Depending on which condition holds we know whether to look for an investment strategy with positive  $\alpha$ , negative  $\alpha$ , or if it does not matter. Since  $\alpha$  is positive exactly when  $\xi(\omega_1) > \xi(\omega_2)$  then we can easily construct the random variable which has minimum price as well.



## 2.3 Pricing in the Multi-Period Binomial Model

In this section we utilize the results of the previous section to examine the case where the model has multiple periods. We start by examining the case where the model has two periods and take the simplifying assumption that the stock moves up by the same amount that it moves down, and that these moves are the same regardless of period. We are able to extend this readily to the case where the market has more than one period, and where the stock moves up by a different amount than it moves down.

We will need to extend our notation first though. Since there are now multiple periods before which an investor can buy stock we must record the value of the stock during each period. If the market has  $k$  periods then we write  $\Omega_2^k$  for the set of all strings of length  $k$  consisting of  $\omega_1$  and  $\omega_2$ , and  $S_k$  is the value of the stock at the end of the  $k$ -th period:  $S_k : \Omega_2^k \rightarrow \mathbb{R}$ . Similarly for any  $i$ ,  $1 \leq i \leq k$  denote by  $\Omega_2^i$  the set of strings of length  $i$  and let  $S_i : \Omega_2^i \rightarrow \mathbb{R}$  be the random variable representing the value of the stock after  $i$  periods.  $S_0$  continues to be the real valued, deterministic value of the price before the first period. In the two-period binomial model then we have two random variables,  $S_1$  and  $S_2$ , and one scalar  $S_0$ .

Our goal as always is to price a probability distribution, but as a first step we need to examine how to price an arbitrary random variable,  $\xi : \Omega_2^k \rightarrow \mathbb{R}$ . That is we want to find an investment strategy  $(\alpha, \beta)$  which agrees with  $\xi$ . However, an investment strategy in a multi-period market may be dynamic, it might reallocate funds based on previous outcomes. That is to say, instead of an ordered pair it is instead a process:  $(\alpha_0, \beta_0), (\alpha_1, \beta_1), \dots, (\alpha_{k-1}, \beta_{k-1})$ . For each  $i$ ,  $1 \leq i \leq k-1$ , we have that  $\alpha_i, \beta_i : \Omega_2^i \rightarrow \mathbb{R}$ . The price of this investment strategy remains the real number given by  $\alpha_0 S_0 + \beta_0$ , and its value is now given by  $\alpha_{k-1} S_k + \beta_{k-1}$ .

It is helpful to track how this strategy performs after each period so we introduce intermediates,  $\xi_i : \Omega_2^i \rightarrow \mathbb{R}$ , for all  $i$ ,  $1 \leq i \leq k$ , and require  $\xi_k = \xi$ . These values track the value of our investment strategy over time,  $\xi_i = \alpha_{i-1} S_i + \beta_{i-1}$ . Further, we require that each period we utilize the entirety of what our holdings are worth, for  $0 \leq i \leq k-1$ :

$$\xi_i = \alpha_i S_i + \beta_i$$

We start with  $k = 2$  the two-period binomial model to illustrate how the results from the last section come in handy. Since at each juncture there are still only two paths we group together the following equations to illuminate the similarities to the binomial model. First, assume that period one has outcome  $\omega_1$  then we find the following system:

$$\begin{aligned} \alpha_1(\omega_1) S_2(\omega_1 \omega_1) + \beta_1(\omega_1) &= \xi_2(\omega_1 \omega_1), \\ \alpha_1(\omega_1) S_2(\omega_1 \omega_2) + \beta_1(\omega_1) &= \xi_2(\omega_1 \omega_2). \end{aligned}$$

Next, assume period one had outcome  $\omega_2$  then we have instead this system:

$$\begin{aligned}\alpha_1(\omega_2)S_2(\omega_2\omega_1) + \beta_1(\omega_2) &= \xi_2(\omega_2\omega_1), \\ \alpha_1(\omega_2)S_2(\omega_2\omega_2) + \beta_1(\omega_2) &= \xi_2(\omega_2\omega_2).\end{aligned}$$

We solve each of these systems according to our previous methods to find values for  $\alpha_1(\omega_1), \beta_1(\omega_1)$  and values for  $\alpha_1(\omega_2), \beta_1(\omega_2)$ . With these values in hand we then substitute into the equation  $\xi_1 = \alpha_1 S_1 + \beta_1$  to determine that variable. Then pricing our random variable,  $\xi$  is now the same problem as pricing  $\xi_1$  as is made clear by the equations:

$$\begin{aligned}\alpha_0 S_1(\omega_1) + \beta_0 &= \xi_1(\omega_1), \\ \alpha_0 S_1(\omega_2) + \beta_0 &= \xi_1(\omega_2).\end{aligned}$$

Working from right to left we unravel the two period model as a sequence of problems that are essentially single-period binomial model problems, each connected to the others by the price expression. In particular, using the exact method we used for the one period model, we find the following closed forms for the constants  $\alpha_1(\omega_1)$  and  $\beta_1(\omega_2)$ :

$$\begin{aligned}\alpha_1(\omega_1) &= \frac{\xi_2(\omega_1\omega_1) - \xi_2(\omega_1\omega_2)}{S_2(\omega_1\omega_1) - S_2(\omega_1\omega_2)}, \\ \beta_1(\omega_1) &= \frac{S_2(\omega_1\omega_1)\xi_2(\omega_1\omega_2) - S_2(\omega_1\omega_2)\xi_2(\omega_1\omega_1)}{S_2(\omega_1\omega_1) - S_2(\omega_1\omega_2)}.\end{aligned}$$

Which via substitution yields a price for  $\xi_1(\omega_1)$ :

$$\frac{\xi_2(\omega_1\omega_1)(S_1(\omega_1) - S_2(\omega_1\omega_2)) + \xi_2(\omega_1\omega_2)(S_1(\omega_1) - S_2(\omega_1\omega_1))}{S_2(\omega_1\omega_1) - S_2(\omega_1\omega_2)}.$$

The process to determine  $\xi_1(\omega_2)$  is identical. With these values in hand, writing down a price for the general model amounts to substitution,

$$\begin{aligned}&\frac{\xi_1(\omega_1)(S_0 - S_1(\omega_2)) + \xi_1(\omega_2)(S_1(\omega_1) - S_0)}{S_1(\omega_1) - S_1(\omega_2)} = \\ &\frac{1}{S_1(\omega_1) - S_1(\omega_2)} \left( \frac{\xi_2(\omega_1\omega_1)(S_1(\omega_1) - S_2(\omega_1\omega_2)) + \xi_2(\omega_1\omega_2)(S_2(\omega_1\omega_1) - S_1(\omega_1))}{S_2(\omega_1\omega_1) - S_2(\omega_1\omega_2)} (S_0 - S_1(\omega_2)) \right. \\ &\quad \left. + \frac{\xi_2(\omega_2\omega_1)(S_1(\omega_2) - S_2(\omega_2\omega_2)) + \xi_2(\omega_2\omega_2)(S_2(\omega_2\omega_1) - S_1(\omega_2))}{S_2(\omega_2\omega_1) - S_2(\omega_2\omega_2)} (S_1(\omega_1) - S_0) \right).\end{aligned}$$

A fraction which the reader surely must find very enlightening. All joking aside, while it is instructive to work directly with the two-period model to demonstrate how the price expression allows us to unwind the model into interconnected one-period

models, it will be far more productive to study it further in general, which we shall do shortly.

First, if the two-period model demonstrated anything, it surely demonstrated the need for terser notation. We now introduce two constants which will aid us in simplifying fractions like the above. The *up factor* and the *down factor* are two scalars, denoted  $u$  and  $d$  respectively, and defined as follows:

$$u = \frac{S_1(\omega_1)}{S_0} \quad d = \frac{S_1(\omega_2)}{S_0}$$

It follows from the ordering of the values of  $S_1$  and that  $S_0$  is positive, that  $u > d$ , further the no-arbitrage condition gives us  $u > 1 > d$ . Importantly, since the stock moves in the same manner each period, for any given  $\omega \in \Omega_2^i$ , we have that

$$u = \frac{S_1(\omega_1)}{S_0} = \frac{S_{i+1}(\omega\omega_1)}{S_i(\omega)} \quad d = \frac{S_1(\omega_2)}{S_0} = \frac{S_{i+1}(\omega\omega_2)}{S_i(\omega)}$$

Hence, if  $\omega \in \Omega_2^i$  contains  $m$  many  $\omega_1$  and  $n$  many  $\omega_2$  we can write  $S_i(\omega) = u^m d^n S_0$ .

Our first step towards general results on pricing random variables in multi-period binomial markets is to relate our intermediates.

**Theorem 8.** *For  $i$ , an integer between 0 and  $n - 1$ , we have that*

$$\xi_i(\omega) = \xi_{i+1}(\omega\omega_1) \frac{1-d}{u-d} + \xi_{i+1}(\omega\omega_2) \frac{u-1}{u-d}.$$

*Proof.* Fix  $\omega \in \Omega_2^i$ . Then by Theorem 4 we have that the price of  $\xi_{i+1}$  is given by

$$\xi_i(\omega) = \frac{\xi_{i+1}(\omega\omega_1)(S_i(\omega) - S_{i+1}(\omega\omega_2)) + \xi_{i+1}(\omega\omega_2)(S_{i+1}(\omega\omega_1) - S_i(\omega))}{S_{i+1}(\omega\omega_1) - S_{i+1}(\omega\omega_2)}.$$

Next we take advantage of our new notation, let  $m, n$  be integers representing the number of occurrences of  $\omega_1$  in  $\omega$  and  $\omega_2$  in  $\omega$  respectively. Then we can rewrite the above to get

$$\begin{aligned} \xi_i(\omega) &= \frac{\xi_{i+1}(\omega\omega_1)(u^m d^n S_0 - u^m d^{n+1} S_0) + \xi_{i+1}(\omega\omega_2)(u^m d^n S_0 - u^{m+1} d^n S_0)}{u^{m+1} d^n S_0 - u^m d^{n+1} S_0} \\ &= \xi_{i+1}(\omega\omega_1) \frac{u^m d^n S_0 - u^m d^{n+1} S_0}{u^{m+1} d^n S_0 - u^m d^{n+1} S_0} + \xi_{i+1}(\omega\omega_2) \frac{u^m d^n S_0 - u^{m+1} d^n S_0}{u^{m+1} d^n S_0 - u^m d^{n+1} S_0} \\ &= \xi_{i+1}(\omega\omega_1) \frac{1-d}{u-d} + \xi_{i+1}(\omega\omega_2) \frac{u-1}{u-d}. \end{aligned}$$

□

Since this relation holds for arbitrary  $\xi_i$  and arbitrary  $\omega$  it certainly holds for  $\xi_{i+1}(\omega\omega_1)$  and  $\xi_{i+1}(\omega\omega_2)$ , allowing us to find a relation between  $\xi_i$  and  $\xi_{i+2}$ , which by substitution looks like this:

$$\begin{aligned}
\xi_i(\omega) &= \xi_{i+1}(\omega\omega_1)\frac{1-d}{u-d} + \xi_{i+1}(\omega\omega_2)\frac{u-1}{u-d} \\
&= \left( \xi_{i+2}(\omega\omega_1\omega_1)\frac{1-d}{u-d} + \xi_{i+2}(\omega\omega_1\omega_2)\frac{u-1}{u-d} \right) \frac{1-d}{u-d} \\
&\quad + \left( \xi_{i+2}(\omega\omega_2\omega_1)\frac{1-d}{u-d} + \xi_{i+2}(\omega\omega_2\omega_2)\frac{u-1}{u-d} \right) \frac{u-1}{u-d} \\
&= \xi_{i+2}(\omega\omega_1\omega_1) \left( \frac{1-d}{u-d} \right)^2 \\
&\quad + \xi_{i+2}(\omega\omega_1\omega_2) \frac{(1-d)(u-1)}{(u-d)^2} \\
&\quad + \xi_{i+2}(\omega\omega_2\omega_1) \frac{(1-d)(u-1)}{(u-d)^2} \\
&\quad + \xi_{i+2}(\omega\omega_2\omega_2) \left( \frac{u-1}{u-d} \right)^2.
\end{aligned}$$

But why stop there? First, for notational convenience we define  $\sigma$  to be the function that takes a string of  $\omega_1$  and  $\omega_2$  to the number of occurrences of  $\omega_1$  in that string.

**Definition 13.** Let  $\sigma : \Omega_2^n \rightarrow \mathbb{R}$  be a function which maps a string onto the number of occurrences of  $\omega_1$  in that string.

**Theorem 9.** *The price of the random variable  $\xi_n$  is given by the following weighted sum of its outcomes:*

$$\xi_0 = \sum_{\omega' \in \Omega_2^n} \xi_n(\omega') \frac{(1-d)^{\sigma(\omega')} (u-1)^{n-\sigma(\omega')}}{(u-d)^n}.$$

*Proof.* We proceed by induction. We have already shown that

$$\xi_0 = \xi_1(\omega_1)\frac{1-d}{u-d} + \xi_1(\omega_2)\frac{u-1}{u-d}.$$

Next assume that for some  $k$ ,  $1 \leq k < n$ , that

$$\xi_0 = \sum_{\omega' \in \Omega_2^k} \xi_k(\omega') \frac{(1-d)^{\sigma(\omega')} (u-1)^{k-\sigma(\omega')}}{(u-d)^k}.$$

Note that in the case  $k = 1$  we recover our base case, though we have adopted new notation to make the  $k > 1$  case terser. We apply our previous theorem to get

$$\begin{aligned}
& \sum_{\omega' \in \Omega_2^k} \xi_k(\omega') \frac{(1-d)^{\sigma(\omega')}(u-1)^{k-\sigma(\omega')}}{(u-d)^k} \\
&= \sum_{\omega' \in \Omega_2^k} \left( \xi_{k+1}(\omega'\omega_1) \frac{1-d}{u-d} + \xi_{k+1}(\omega'\omega_2) \frac{u-1}{u-d} \right) \frac{(1-d)^{\sigma(\omega')}(u-1)^{k-\sigma(\omega')}}{(u-d)^k} \\
&= \sum_{\omega' \in \Omega_2^{k+1}} \xi_{k+1}(\omega') \frac{(1-d)^{\sigma(\omega')}(u-1)^{k+1-\sigma(\omega')}}{(u-d)^{k+1}}.
\end{aligned}$$

□

Since  $\xi_0$  is the price of  $\xi_n$  this theorem then yields a simple closed form expression for pricing arbitrary random variables in the multi-period binomial model. It does not, however, address the pricing of distributions. In order to examine pricing a distribution lets first take a simplifying assumption. In the single-period model we found both random variables for a given distribution had the same price when we had

$$S_1(\omega_1) - S_0 = S_0 - S_1(\omega_2).$$

Dividing through by  $S_0$  gives us an equivalent condition in terms of  $u$  and  $d$ :

$$u - 1 = 1 - d.$$

**Theorem 10.** *Assuming  $u - 1 = 1 - d$ , every random variable sharing the same probability distribution has the same price.*

*Proof.* Taking this additional assumption our price expression simplifies to

$$\xi_0 = \sum_{\omega' \in \Omega_2^n} \xi(\omega') \left( \frac{u-1}{u-d} \right)^n = \left( \frac{u-1}{u-d} \right)^n \sum_{\omega' \in \Omega_2^n} \xi(\omega')$$

But we can do more:

$$\begin{aligned}
\frac{u-1}{u-d} &= \frac{u-1}{u-1+1-d} \\
&= \frac{u-1}{(u-1)+(1-d)} \\
&= \frac{u-1}{2(u-1)} \\
&= \frac{1}{2}.
\end{aligned}$$

Note that we used our simplifying assumption again in this derivation. This turns our price expression into

$$\xi_0 = \frac{1}{2^n} \sum_{\omega' \in \Omega_2^n} \xi(\omega')$$

Since any random variable with the same distribution as  $\xi$  will sum to the same value, this assumption guarantees that the price of any two random variables with the same distribution have the same price.  $\square$

In order to compare various random variables with the same distribution we will need to first have a handle on the coefficients.

**Theorem 11.** *The sequence*

$$\frac{(1-d)^n(u-1)^0}{(u-d)^n}, \frac{(1-d)^{n-1}(u-1)^1}{(u-d)^n}, \dots, \frac{(1-d)^0(u-1)^n}{(u-d)^n}$$

*is increasing when  $u-1 > 1-d$ , decreasing when  $u-1 < 1-d$  and constant otherwise.*

*Proof.* We have already shown that the sequence is constant if  $u-1 = 1-d$ . Next, fix  $s$  in  $0, 1, \dots, n-1$ . Then the ratio of two consecutive terms gives

$$\frac{(1-d)^{n-s-1}(u-1)^{s+1}(u-d)^n}{(1-d)^{n-s}(u-1)^s(u-d)^n} = \frac{(u-1)}{(1-d)},$$

indicating that the sequence is increasing when

$$\frac{u-1}{1-d} > 1$$

and decreasing when

$$\frac{u-1}{1-d} < 1.$$

In other terms, it is increasing when

$$u-1 > 1-d$$

and decreasing when

$$u-1 < 1-d.$$

$\square$

We will need an additional smaller result to tackle the pricing of distributions in general.

**Theorem 12.** Let  $(a_n), (b_n)$  be monotone sequences. Then the sum

$$\sum_{k=0}^n a_k b_k$$

has minimal value when  $(a_n), (b_n)$  are anticomotonic, one is increasing and the other decreasing, and maximal value when they are comotonic, both increasing or both decreasing

*Proof.* We first prove the case where  $(a_n)$  and  $(b_n)$  are sequences of two elements and proceed by induction. Let  $a_1 > a_2$  and  $b_1 > b_2$ , then

$$(a_1 - a_2)(b_1 - b_2) > 0,$$

or,

$$a_1 b_1 + a_2 b_2 > a_2 b_1 + a_1 b_2.$$

Hence the result is true if the sequences are of length two. Assume this holds for sequences of length  $n - 1$ . Let

$$a_1 > \cdots > a_n, \quad \text{and} \quad b_1 > \cdots > b_n.$$

Select a permutation  $\sigma$  for which the arrangement

$$a_{\sigma(1)} b_1 + \cdots + a_{\sigma(n)} b_n$$

gives rise a maximal result. If  $\sigma(n)$  were different from  $n$ , say that  $\sigma(n) = k$ , then there must be some  $j$  less than  $n$  such that  $\sigma(j) = n$ . However, by what was just proved  $a_n > a_k$  and  $b_n > b_j$  implies that  $a_n b_n + a_k b_j > a_k b_n + a_n b_j$ . This would suggest another permutation  $\tau$  which agreed with  $\sigma$  except at  $j$  and  $n$  where  $\tau(j) = k$  and  $\tau(n) = n$  gives rise to a larger result, a contradiction. This implies that  $\sigma(n) = n$ . By the induction hypothesis we know that  $\sigma(i) = i$  for each  $i = 1, \dots, n - 1$  and this concludes the proof of the upper bound. Note also that the same argument goes through with non-strict inequalities.

In order to show that the reverse order is minimal apply the upper bound to the sequences  $b_n$  and  $-a_n, -a_{n-1}, \dots, -a_1$ .

□

**Theorem 13.** The price of a random variable has minimal price among all random variables with the same distribution, only if it is monotonic in the number of occurrences of  $\omega_1$ . Specifically, when  $u - 1 > 1 - d$  it is minimal if  $\xi_n$  is increasing and when  $u - 1 < 1 - d$  it is only minimal if  $\xi_n$  is decreasing.

*Proof.* Let  $\xi_n$  be a random variable. Combining the results of the last two theorems we know the price of  $\xi_n$  is given by

$$\xi_0 = \sum_{\omega' \in \Omega_2^n} \xi_n(\omega') \frac{(1-d)^{\sigma(\omega')} (u-1)^{n-\sigma(\omega')}}{(u-d)^n}.$$

It is minimized when  $\xi_n$  and the second factor are anticomonotonic; when one is increasing while the other decreases. Further, we have showed that as the number of  $\omega_2$  increases the coefficients increase when  $u-1 > 1-d$ , and decrease otherwise. Thus the minimal priced random variable will have the opposite property.  $\square$

## 2.4 Pricing in Three-Outcome Markets

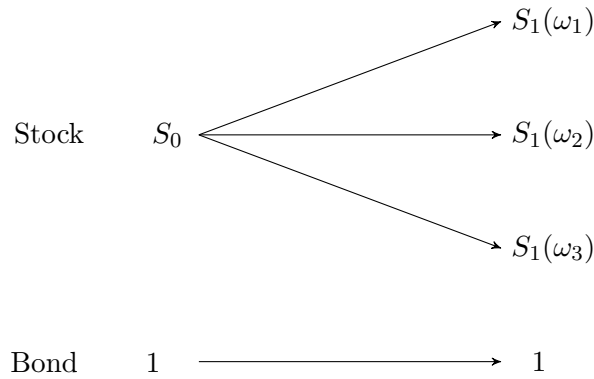


Figure 2.2: Stock and bond dynamics in three outcome markets.

Consider now the market with three outcomes;  $\Omega_3 = \{\omega_1, \omega_2, \omega_3\}$ . We again assume that  $S_1$  takes on distinct values for each outcome and that they come in order,  $S_1(\omega_1) > S_1(\omega_2) > S_1(\omega_3)$ . Consider again a random variable as above,  $\xi$ . The associated system of equations is overdetermined. This system is given by:

$$\begin{aligned} \alpha S_1(\omega_1) + \beta &= \xi(\omega_1), \\ \alpha S_1(\omega_2) + \beta &= \xi(\omega_2), \\ \alpha S_1(\omega_3) + \beta &= \xi(\omega_3). \end{aligned}$$

Its associated matrix is given by

$$\begin{bmatrix} S_1(\omega_1) & 1 \\ S_1(\omega_2) & 1 \\ S_1(\omega_3) & 1 \end{bmatrix}.$$



It is clear from this matrix and our distinctness assumption on the values that  $S_1$  can possibly take on that this matrix has linearly independent rows, and hence no solutions exist to our system of linear equations. We can't find an investment strategy or price for  $\xi$  in the manner we did for the two outcome market. In order to study the three outcome market we can instead attempt to find a strategy that takes on values at least as great as  $\xi$  does. This changes our system of linear equations to a system of linear constraints:

$$\begin{aligned}\alpha S_1(\omega_1) + \beta &\geq \xi(\omega_1), \\ \alpha S_1(\omega_2) + \beta &\geq \xi(\omega_2), \\ \alpha S_1(\omega_3) + \beta &\geq \xi(\omega_3).\end{aligned}$$

In light of these constraints we could find many strategies, but we are interested only in the cheapest. Thus, we will attempt to satisfy those constraints while minimizing the price:

$$\alpha S_0 + \beta$$

This transforms our problem from a system of linear equations into a linear programming problem.

First, a note on feasibility, no matter what values  $\xi$  or  $S_1$  might take on we can always find a feasible solution. Set  $\alpha = 0$  and  $\beta$  to be the maximum of  $\xi(\omega_1)$ ,  $\xi(\omega_2)$  and  $\xi(\omega_3)$ . This satisfies our constraints, though it takes no advantage of the possibility that  $S_1$  changes and hence is probably not optimal.

Now, knowing that there are certainly feasible solutions we can more confidently look for optimal ones. Since this is a system in two unknowns we apply the graphical method in order to solve it. It will be helpful to rewrite our constraints in slope-intercept form:

$$\begin{aligned}\beta &\geq -\alpha S_1(\omega_1) + \xi(\omega_1), \\ \beta &\geq -\alpha S_1(\omega_2) + \xi(\omega_2), \\ \beta &\geq -\alpha S_1(\omega_3) + \xi(\omega_3).\end{aligned}$$

Each of these inequalities specifies a half space bounded by the line where both sides are equal and directed up. By our initial assumptions they come in order of decreasing slope,  $-S_1$  determines the slope, but we can say nothing about their  $\beta$ -intercepts, determined by  $\xi$ . Similarly if we denote the price by  $p$  then we have:

$$\beta = -\alpha S_0 + p$$

In this case the objective function has slope  $S_0$  and  $\beta$ -intercept equal to its price. This function partitions the space into parallel lines, each pair of collinear points having

the same price. The superhedging investment strategy is described by the point which has minimal price and lies in the feasible region described by our constraints. We know that  $S_1(\omega_1) > S_0 > S_1(\omega_3)$  since otherwise the market would admit arbitrage; this is an important restriction as it tells us that points lying on the lines with those slopes are all different prices. Further since  $S_0 > S_1(\omega_3)$  we know that the price increases along the line with slope  $S_1(\omega_3)$ , and similarly for  $S_1(\omega_1)$ . Thus, where we once had a large configuration space to consider we can reduce our focus to merely the intersections of boundary lines, and the line with slope  $S_1(\omega_2)$ . For some diagrams, some of these points lie outside of the feasible region and may be ignored. If  $S_0 = S_1(\omega_2)$  it is worth noting that every point on that line is equally priced.

Now, pricing an individual random variable is easy and follows exactly as any linear programming problem in two variables proceeds. Further, for a given distribution we can brute force its lowest price, investment strategy and corresponding random variable by solving for the price of the six different random variables which admit such a distribution. However, one might want a more direct approach to calculating the price of a distribution, similar to the case of the single-period binomial model where we found a simple discriminant to determine which of  $\xi$  and  $\bar{\xi}$  is cheaper. Now however we have six different possible random variables. If we select a canonical representation for a distribution, say that described by  $\xi(\omega_1) > \xi(\omega_2) > \xi(\omega_3)$ , we would like to be able to describe in terms of the market what transpositions will lower the price. If we let  $i, j \in \{1, 2, 3\}$  then define  $\xi_{(ij)}$  by the following

$$\begin{aligned}\xi_{(ij)}(\omega_i) &= \xi(\omega_j), \\ \xi_{(ij)}(\omega_j) &= \xi(\omega_i).\end{aligned}$$

We would like a simple discriminant that tells us if  $\xi$  or  $\xi_{(ij)}$  admits a lower priced investment strategy. This would be the first step in understanding in general how to price distributions instead of random variables, but unfortunately even this is not quite possible. The issue being that a point which is feasible for one random variable need not be feasible for another. Thus, the price of a random variable, relative to another with the same distribution, depends on their values, as these determine the feasible region. More particularly, if we looked only at the critical points of  $\xi$ , and  $\xi_{(ij)}$ , we could certainly compare the prices of these points, and order these prices only according to the values of  $S_1$ , however without also knowing the explicit values of  $\xi$  we would be unable to determine which ones are feasible, and hence determine which of  $\xi$  and  $\xi_{(ij)}$  is cheaper.

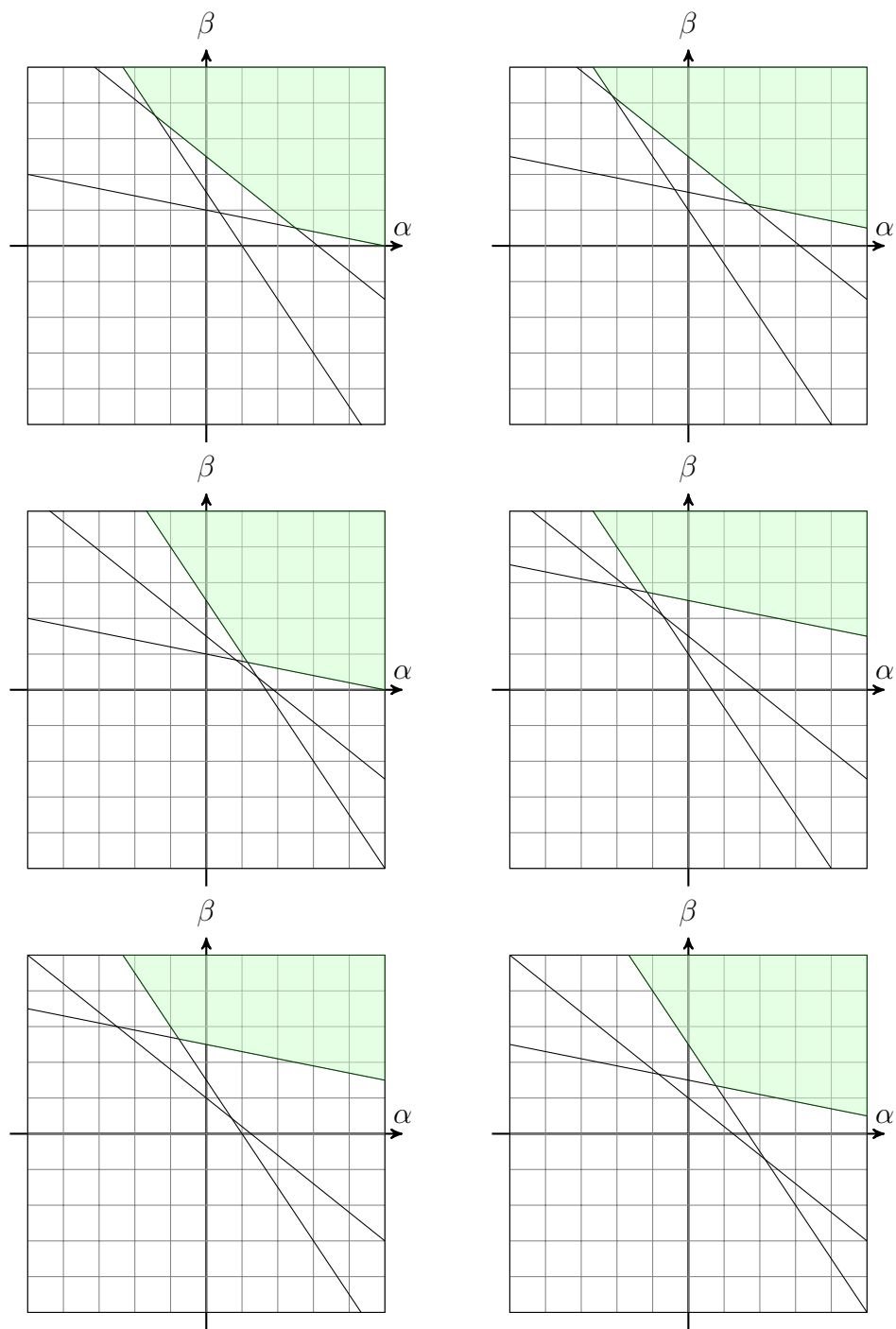


Figure 2.3: Examples of all optimization problems with a given distribution. The green area is the set of feasible solutions.

## 3 Pricing via State Prices

While in the last few sections we have successfully managed to examine the prices of distributions in binomial models, our technique relied heavily on the invertibility of the associated matrix. If we are to generalize to models with three or more outcomes we will need alternative methods when the corresponding matrix is not invertible. Shreve [2] describes a method that will be useful to recapitulate here, state pricing. The idea behind this technique is to associate to each possible market state a price, much as we did with the  $\xi_k(\omega)$ , but this time the price is based on the conditional expectation of  $\xi_n$  under what is called the risk-neutral probability measure. Further, under the risk neutral measure  $\xi_n$  may have a different distribution than it does under the original measure. We relate the risk-neutral conditional expectation with the original conditional expectation so that we can price probability distributions.

### 3.1 Risk-Neutral Probability Measures

We begin by introducing the risk-neutral probability measures. In the binomial model these represent alternative probabilities for the outcomes  $\omega_1$  and  $\omega_2$ . We will denote the risk neutral probability measure by  $\tilde{\mathbb{P}}$ , and in the binomial case we have

$$\tilde{\mathbb{P}}(\omega_1) = q_1, \quad \tilde{\mathbb{P}}(\omega_2) = q_2.$$

These values are such that the stock is a martingale under the risk-neutral expectation, denoted  $\tilde{\mathbb{E}}$ . Note that they do not reflect the actual probabilities, these are estimated from historical data, but rather a fictitious construct to aid in the mathematics. We will later relate the two. First, let us see how to calculate these numbers.

There are two key constraints on these numbers which both make them useful, and let us determine suitable values for them. The first is obvious, we called these numbers a probability measure and hence require that they sum to one. The next is that the stock must be a martingale under them. This amounts to the following

$$\begin{aligned} \tilde{\mathbb{E}}[S_1] &= \tilde{\mathbb{E}}[S_0] = S_0 \\ \Leftrightarrow q_1 S_1(\omega_1) + q_2 S_1(\omega_2) &= S_0. \end{aligned}$$

Finally, in order for these values to be sensible they must agree on what outcomes are possible with the uniform measure;  $q_1 > 0$  and  $q_2 > 0$ . Implicitly this also means that  $q_1 < 1$  and  $q_2 < 1$ . We summarize these considerations in the following definition

**Definition 14.** The risk-neutral probabilities on  $\Omega_2$  are two real positive numbers,  $q_1, q_2$  satisfying

$$\begin{aligned} q_1 + q_2 &= 1, \\ q_1 S_1(\omega_1) + q_2 S_1(\omega_2) &= S_0. \end{aligned}$$

Scaling the second condition by  $S_0$  we find an equivalent condition:

$$uq_1 + dq_2 = 1.$$

Solving the associated matrix yields a closed form expression for each of these terms:

$$q_1 = \frac{1-d}{u-d}, \quad q_2 = \frac{u-1}{u-d}.$$

These numbers importantly encode the price of derivative securities quite succinctly.

**Theorem 14.** *The price of a random variable  $\xi : \Omega_2 \rightarrow \mathbb{R}$  is given by*

$$\tilde{\mathbb{E}}[\xi] = q_1 \xi(\omega_1) + q_2 \xi(\omega_2).$$

*Proof.* The price of  $\xi$  is given by, by prior result

$$\frac{\xi(\omega_1)(S_0 - S_1(\omega_2)) + \xi(\omega_2)(S_1(\omega_1) - S_0)}{S_1(\omega_1) - S_1(\omega_2)}$$

Scaling by  $S_0$  and rearranging we find that this is exactly

$$\begin{aligned} & \frac{\xi(\omega_1)(S_0 - S_1(\omega_2)) + \xi(\omega_2)(S_1(\omega_1) - S_0)}{S_1(\omega_1) - S_1(\omega_2)} \\ &= \frac{\xi(\omega_1)(1-d) + \xi(\omega_2)(u-1)}{u-d} \\ &= \xi(\omega_1) \frac{1-d}{u-d} + \xi(\omega_2) \frac{u-1}{u-d} \\ &= \tilde{\mathbb{E}}[\xi]. \end{aligned}$$

□

Next, we calculate  $q_1, q_2$  in the same manner as in Definition 14 and define  $\tilde{\mathbb{P}}_1 : \Omega_2^1 \rightarrow \mathbb{R}$  by

$$\tilde{\mathbb{P}}_1(\omega_1) = q_1, \quad \tilde{\mathbb{P}}_1(\omega_2) = q_2.$$

Then we define  $\tilde{\mathbb{P}}_k : \Omega_2^k \rightarrow \mathbb{R}$  by, for some  $\omega \in \Omega_2^{k-1}$ ,

$$\tilde{\mathbb{P}}_k(\omega\omega_1) = q_1\tilde{\mathbb{P}}_{k-1}(\omega), \quad \tilde{\mathbb{P}}_k(\omega\omega_2) = q_2\tilde{\mathbb{P}}_{k-1}(\omega).$$

for  $1 < k \leq n$  where  $n$  is the number of periods as usual. This can be summarized in terms of our earlier  $\sigma$  map as follows, for some  $\omega \in \Omega_2^k$

$$\tilde{\mathbb{P}}_k(\omega) = q_1^{\sigma(\omega)} q_2^{n-\sigma(\omega)}$$

for  $0 < k \leq n$ . Note that whether we define  $\tilde{\mathbb{E}}_k$  as the conditional expectation of  $\tilde{\mathbb{P}}$  in the  $k$ -th period, or as the expectation under the measure  $\tilde{\mathbb{P}}_k$  these two definitions coincide because of the way we constructed  $\tilde{\mathbb{P}}$ . Using these conditional expectations we recover something akin to the previous result in the multi-period model.

**Theorem 15.** *Let  $\omega \in \Omega_2^k$  then the time  $k$  value of a random variable  $\xi_{k+1}$  is given by*

$$\tilde{\mathbb{E}}_k[\xi_{k+1}] = q_1\xi_{k+1}(\omega\omega_1) + q_2\xi_{k+1}(\omega\omega_2)$$

*Proof.* This proof follows in exactly the same manner as above. □

It remains to be verified that  $S_0, S_1, \dots, S_n$  is a martingale under  $\tilde{\mathbb{P}}_n$ . We do that presently

**Theorem 16.** *The sequence  $S_0, S_1, \dots, S_n$  is a martingale.*

*Proof.* Let  $\omega \in \Omega_2^k$ , for some  $k = 0, 1, \dots, n-1$ , then

$$\begin{aligned} \tilde{\mathbb{E}}_k[S_{k+1}](\omega) &= q_1S_{k+1}(\omega\omega_1) + q_2S_{k+1}(\omega\omega_2) \\ &= q_1uS_k(\omega) + q_2dS_k(\omega) \\ &= S_k(\omega)(uq_1 + dq_2) \\ &= S_k(\omega). \end{aligned}$$

□

We have more than that however.

**Theorem 17.** *Let  $\xi : \Omega_2^n \rightarrow \mathbb{R}$  be a random variable, and  $\xi_0, \xi_1, \dots, \xi_n$  be the sequence of intermediates given by*

$$\xi_k(\omega) = \xi_{k+1}(\omega\omega_1)q_1 + \xi_{k+1}(\omega\omega_2)q_2,$$

*then this sequence is a martingale under  $\tilde{\mathbb{P}}$ .*

*Proof.* This follows in exactly the same way as the above. □

This particular process is familiar to us from the previous section; it is the investment strategy for the multi-period model that ends in  $\xi$ . Indeed, we could have stated that the process is defined by

$$\xi_i = \alpha_i S_i + \beta_i$$

and then applied a prior theorem to get the same result.

Now, there was a particular case in the binomial model where our probability distributions were especially easy to price;  $u - 1 = 1 - d$ . One might wonder what makes this case special. Now that we have described the risk-neutral measure, and demonstrated the risk-neutral pricing method, we are well equipped to demonstrate what makes this case special:

$$\begin{aligned} q_1 &= \frac{1 - d}{u - d} \\ &= \frac{1 - d}{u - 1 + 1 - d} \\ &= \frac{1 - d}{2(1 - d)} \\ &= \frac{1}{2}. \end{aligned}$$

This is a familiar trick we have already used, but this then means that we have

$$\mathbb{P}(\omega_1) = \tilde{\mathbb{P}}(\omega_1), \quad \mathbb{P}(\omega_2) = \tilde{\mathbb{P}}(\omega_2).$$

The uniform measure and the risk-neutral measure coincide! In this case we have that

$$\mathbb{E}[\xi] = \tilde{\mathbb{E}}[\xi] = \xi_0.$$

Since under the uniform measure any two variables with the same distribution have the same expectation, this is another way of saying something we already know; they also have the same price. We now present a way to extend this concept to other cases.

**Definition 15.** The *Radon-Nikodym derivative* of  $\tilde{\mathbb{P}}$  with respect to  $\mathbb{P}$ , denoted  $Z(\omega)$  is given by

$$Z(\omega) = \frac{\tilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)} \quad \forall \omega \in \Omega.$$

It has the following fundamental properties

**Theorem 18.** Let  $\mathbb{P}$  and  $\tilde{\mathbb{P}}$  be probability measures on  $\Omega$ , such that for all  $\omega \in \Omega$  we have  $\mathbb{P}(\omega) > 0$  and  $\tilde{\mathbb{P}}(\omega) > 0$ .

- We have  $Z(\omega) > 0$  for all  $\omega \in \Omega$ .

- We have that  $\mathbb{E}[Z] = 1$ .
- We have that for any random variable  $Y : \Omega \rightarrow \mathbb{R}$  that

$$\tilde{\mathbb{E}}[Y] = \mathbb{E}[ZY]$$

*Proof.* A proof is given in [2]. □

In the particular case of the binomial model we have  $Z$  given by:

$$Z(\omega_1) = 2q_1, \quad Z(\omega_2) = 2q_2.$$

In the multi-period case we simply extend it in the same manner as before with the exponents depending on the number of occurrences of  $\omega_1$  and  $\omega_2$ . Now, the third property of the Radon-Nikodym derivative should be very suggestive of what is to follow, it will let us relate the risk-neutral expectation which accurately prices random variables with the actual expectation. Before we get there though we have some more definitions to get through in order to be precise.

**Definition 16.** Let  $\mathbb{P}$ ,  $\tilde{\mathbb{P}}$ , and  $Z$  be defined as above, then the *Radon-Nikodym Derivative Process* is a sequence of random variables given by

$$Z_k = \mathbb{E}_k[Z] \quad k = 0, 1, \dots, n.$$

**Theorem 19.** *The process  $(Z_n)$  is martingale under  $\mathbb{P}$*

*Proof.* This follows directly from the iterated conditioning property. □

We further have the following results

**Theorem 20.** *Let  $Y : \Omega^n \rightarrow \mathbb{R}$  be a random variable depending on only the first  $k$  letters of the string of length  $n$ , then we have*

$$\tilde{\mathbb{E}}[Y] = \mathbb{E}[Z_k Y].$$

*Proof.* A proof is given in [2]. □

**Theorem 21.** *Let  $j < k$  be positive integers less than  $n$  and let  $Y : \Omega^n \rightarrow \mathbb{R}$  be a random variable depending only on the first  $k$  letters of the string of length  $n$ . Then we have*

$$\tilde{\mathbb{E}}[Y] = \frac{1}{Z_j} \mathbb{E}_j[Z_k Y].$$

*Proof.* A proof is given in [2]. □

Finally this lets us reformulate our definition of the intermediates  $\xi_k$  in terms of expectation.



**Theorem 22.** Let  $\xi_0, \xi_1, \dots, \xi_n$  be random variables as before where  $\xi_k$  represents the time at price  $k$  to realize the random variable  $\xi_n$  then these prices are given by

$$\xi_k = \frac{1}{Z_k} \mathbb{E}_k[Z_n \xi_n].$$

*Proof.* By definition the time  $k$  value of the random variable is its conditional expectation under the risk neutral measure. Further, we have that

$$\tilde{\mathbb{E}}[\xi_n] = \frac{1}{Z_k} \mathbb{E}_k[Z_n \xi_n],$$

by Theorem 21. □

Importantly, so long as we can indentify a suitable risk-neutral measure we can define the state prices and hence price the random variable. We now apply this to the three outcome case.

## 3.2 Revisiting the Three-Outcome Model

Consider the three-outcome model,  $\Omega_3 = \{\omega_1, \omega_2, \omega_3\}$ . Since the stock now takes on three values we need three parameters

$$u = \frac{S_1(\omega_1)}{S_0}, \quad m = \frac{S_1(\omega_2)}{S_0}, \quad d = \frac{S_1(\omega_3)}{S_0},$$

and hence we need three risk-neutral probabilities,  $q_1, q_2, q_3$ , which satisfy as before the equations

$$\begin{aligned} q_1 + q_2 + q_3 &= 1, \\ uq_1 + mq_2 + dq_3 &= 1. \end{aligned}$$

Which unfortunately leaves us with exactly the same problem we had with the other approach; a non-invertible matrix. This system of equations, however, is underspecified, and hence has an infinite family of solutions. We describe those solutions presently and see how they suggest a risk-neutral measure that will aid us in our analysis

$$\begin{aligned} q_1 + q_2 &= 1 - q_3, \\ uq_1 + dq_3 &= 1 - mq_3. \end{aligned}$$

in terms of  $q_2$  as our free parameter. We find the following expressions

$$q_1 = \frac{(1-d) + (d-m)q_2}{u-d} \quad q_3 = \frac{(u-1) + (m-u)q_2}{u-d}$$

This then describes a line in three-dimensional Euclidean space of points satisfying our equations. However, much of this line makes no sense in the context of finding a risk-neutral measure. We want that  $q_1, q_2, q_3 > 0$  so that the measure agrees with the uniform measure on which paths are possible, and thus implicitly we must have that  $q_1, q_2, q_3 < 1$ . Intersecting our line with this feasible region, an open box with unit side length, we find an interval of feasible solutions in terms of  $q_2$ .

**Theorem 23.**  $q_2$  is in  $(0, 1)$  when  $m = 1$ , in  $(0, \frac{1-d}{m-d})$  when  $m > 1$ , and in  $(0, \frac{u-1}{u-m})$  when  $m < 1$ .

*Proof.* Starting with the bounds on  $q_1$  we make equivalent transformations:

$$\begin{aligned}
0 &< q_1 < 1 \\
\Leftrightarrow 0 &< \frac{1-d+(d-m)q_2}{u-d} < 1 \\
\Leftrightarrow 0 &< 1-d+(d-m)q_2 < u-d \\
\Leftrightarrow d-1 &< (d-m)q_2 < u-1 \\
\Leftrightarrow \frac{d-1}{d-m} &> q_2 > \frac{u-1}{d-m}.
\end{aligned}$$

The left bound is written with positive denominator and numerator as  $\frac{1-d}{m-d}$  and the right bound is less than zero and less restrictive than our feasibility assumptions. Similarly by looking at  $q_3$  we find the following are equivalent:

$$\begin{aligned}
0 &< q_3 < 1 \\
\Leftrightarrow 0 &< \frac{u-1+(m-u)q_2}{u-d} < 1 \\
\Leftrightarrow 0 &< u-1+(m-u)q_2 < u-d \\
\Leftrightarrow 1-u &< (d-m)q_2 < 1-d \\
\Leftrightarrow \frac{1-u}{m-u} &> q_2 > \frac{1-d}{m-u}.
\end{aligned}$$

Here we have a left bound with positive denominator and numerator written as  $\frac{u-1}{u-m}$  and again a negative left bound. This tells us that  $q_2$  is contained in an interval whose infimum is zero. The supremum of the interval of possible  $q_2$  values is the smallest of  $\frac{1-d}{m-d}$  and  $\frac{u-1}{u-m}$  and 1. Note that the following are equivalent:

$$\begin{aligned}
\frac{1-d}{m-d} &< 1 \\
\Leftrightarrow 1-d &< m-d
\end{aligned}$$

$$\Leftrightarrow 1 < m.$$

Also, the following are equivalent:

$$\frac{u-1}{u-m} < 1$$

$$\Leftrightarrow u-1 < u-m$$

$$\Leftrightarrow -1 < -m$$

$$\Leftrightarrow m < 1.$$

Finally, by substitution  $m = 1$  implies that each of the three possible supremums is equal to 1.  $\square$

These inequalities show that there are three cases to consider depending on the value of  $m$ , but that in each case we can find an open interval for  $q_2$  over which the parameters are an equivalent risk-neutral measure to the actual probabilities in our model.

Under these measures  $S_1$  will be a martingale, and it will agree with our original measure on the possible paths. However, as we saw last time there is not necessarily an investment strategy, with any price at all which achieves exactly  $\xi$  in every outcome. Last time we resorted to finding the cheapest strategy which outperforms  $\xi$  in each outcome and here we will have to do the same.

Now as before the no-arbitrage price of a random variable is its expectation under the risk-neutral measure, we write it as a function of  $q_2$  here:

$$p(q_2) = q_1(q_2)\xi(\omega_1) + q_2\xi(\omega_2) + q_3(q_2)\xi(\omega_3).$$

Since we described above bounds for  $q_2$  in each of the three cases we know that this function has an open interval for its domain, and, as it is continuous, an open interval of images.

The interval of images under  $p$  is the set of prices we can select for  $\xi$  which do not introduce arbitrage. If the random variable is priced less than this interval it is replicable for less than the price. Priced any higher than this interval implies that the initial value is overestimated, and a loan in  $\xi$  will be worth less than the stock purchased. In that case however, we would be describing strategies that are expensively priced, and consistently outperform  $\xi$ , hence the short position arbitrage opportunity. Since any one of these strategies would outperform  $\xi$  the cheapest would be the most desirable superhedging strategy. We have shown that the arbitrage-free prices form an open interval, and that any strategy with greater price than any of the arbitrage-free strategies superhedges. The least upper bound of an open interval is its supremum, and this price is the cheapest superhedging price.

**Theorem 24.** *The price of a random variable is the larger of  $p(0)$ , and one of  $p(1)$ ,  $p(\frac{1-d}{m-d})$ , or  $p(\frac{u-1}{u-m})$  depending on whether  $m = 1$ ,  $m > 1$ , or  $m < 1$  respectively.*

*Proof.* Since  $p$  is continuous the Extreme Value Theorem tells us that the supremum of  $p$  occurs at either end of its domain;  $q_2 = 0$  or one of  $q_2 = 1, \frac{u-1}{u-m}, \frac{1-d}{m-d}$ . The supremum as discussed above is the price.  $\square$

Thus pricing a random variable amounts to comparing 1 and  $m$  to determine which of three cases we are in, and to comparing the endpoints of  $p$ . We will denote by  $q_2^*$  the pre-image of the super-hedging price. We will utilize  $q_1^*, q_3^*$  as shorthand for  $q_1(q_2^*)$  and  $q_3(q_2^*)$  respectively.

We know that at least one of  $q_1, q_2, q_3$  is surely zero, since positivity is how we selected the boundary of the interval. This is useful for pricing a probability distribution rather than a random variable. Pricing a distribution is done by comparing the six possible permutations of the values assignment to outcomes. Each permutation of the values of  $\xi$  yields a price function defined over  $q_2$ . Each permutation has its own price which is the higher endpoint of its price function. The lowest such price is the price of the distribution.

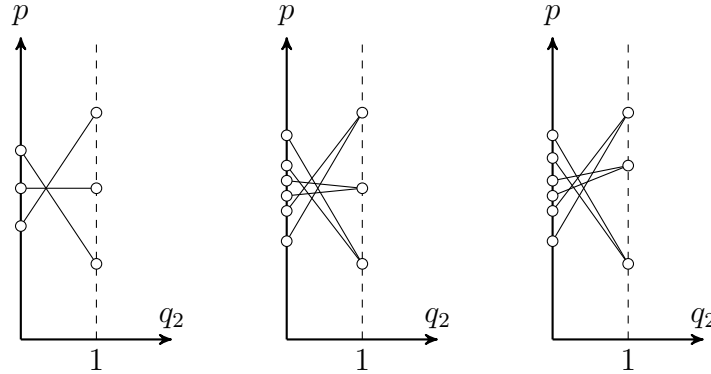


Figure 3.1: Examples of pricing distributions. Each graph represents a different market model and a different distribution. Each line represents the function of all possible prices for a different random variable.

In order to illustrate the technique we will examine the case where  $m = 1$ .

**Theorem 25.** *Let  $\xi$  be the random variable of a given distribution such that  $\xi(\omega_1) > \xi(\omega_2) > \xi(\omega_3)$ . If  $m = 1$ , then the price of the distribution is one of  $\xi(\omega_2)$ ,  $q_1^*\xi(\omega_1) + q_3^*\xi(\omega_3)$ , or  $q_1^*\xi(\omega_3) + q_3^*\xi(\omega_1)$ .*

*Proof.* When  $m = 1$  we know for sure that  $q_2 \in (0, 1)$  and that  $q_2^* \in \{0, 1\}$ , where  $p$  is the price function for  $\xi$ . We have that

$$p(1) = \xi(\omega_2),$$

$$p(0) = q_1^* \xi(\omega_1) + q_3^* \xi(\omega_3).$$

Equivalently,

$$p(0) = \frac{1-d}{u-d} \xi(\omega_1) + \frac{u-1}{u-d} \xi(\omega_3).$$

The price of the distribution is the price of the variable whose maximum of  $p(0)$  and  $p(1)$  is the lowest. That is we take some permutation of  $\xi$ , call it  $\bar{\xi}$  and examine the quantities  $q_1^* \bar{\xi}(\omega_1) + q_3^* \bar{\xi}(\omega_3)$  and  $\bar{\xi}(\omega_2)$ , the higher is the price of  $\bar{\xi}$ , and the particular  $\bar{\xi}$  with the lowest price determines the price of the distribution. Notationally we have

$$\bigwedge_{\sigma \in \mathcal{S}_3} (q_1^* \xi(\omega_{\sigma(1)}) + q_3^* \xi(\omega_{\sigma(3)})) \vee \xi(\omega_{\sigma(2)})$$

Note that the left hand quantities are weighted averages, thus the sum is no larger than the larger of the two, and no smaller than the smaller. In the cases

$$\begin{aligned} (q_1^* \xi(\omega_1) + q_3^* \xi(\omega_2)) \vee \xi(\omega_3), \\ (q_1^* \xi(\omega_2) + q_3^* \xi(\omega_1)) \vee \xi(\omega_3), \end{aligned}$$

we have that the price is  $q_1^* \xi(\omega_1) + q_3^* \xi(\omega_2)$  and  $q_1^* \xi(\omega_2) + q_3^* \xi(\omega_1)$  respectively, since  $\xi(\omega_2)$  and  $\xi(\omega_1)$  are both larger than  $\xi(\omega_3)$ . In the cases

$$\begin{aligned} (q_1^* \xi(\omega_2) + q_3^* \xi(\omega_3)) \vee \xi(\omega_1), \\ (q_1^* \xi(\omega_3) + q_3^* \xi(\omega_2)) \vee \xi(\omega_1), \end{aligned}$$

we have that the price is  $\xi(\omega_1)$ . This is certainly not the price of the distribution since it is the largest quantity out of all the candidates. Thus we have only four cases to compare. The remaining possibilities are

$$\begin{aligned} (q_1^* \xi(\omega_1) + q_3^* \xi(\omega_3)) \vee \xi(\omega_2), \\ (q_1^* \xi(\omega_3) + q_3^* \xi(\omega_1)) \vee \xi(\omega_2), \\ (q_1^* \xi(\omega_1) + q_3^* \xi(\omega_2)), \\ (q_1^* \xi(\omega_2) + q_3^* \xi(\omega_1)). \end{aligned}$$

Certainly

$$\begin{aligned} q_1^* \xi(\omega_1) + q_3^* \xi(\omega_2) &> q_1^* \xi(\omega_1) + q_3^* \xi(\omega_3), \\ q_1^* \xi(\omega_1) + q_3^* \xi(\omega_2) &> \xi(\omega_2) \end{aligned}$$

and

$$\begin{aligned} q_1^* \xi(\omega_2) + q_3^* \xi(\omega_1) &> q_1^* \xi(\omega_3) + q_3^* \xi(\omega_1), \\ q_1^* \xi(\omega_2) + q_3^* \xi(\omega_1) &> \xi(\omega_2) \end{aligned}$$

eliminating both of those cases as possibly being the minimum. This reduces our original problem to the following:

$$((q_1^*\xi(\omega_1) + q_3^*\xi(\omega_3)) \vee \xi(\omega_2)) \wedge ((q_1^*\xi(\omega_3) + q_3^*\xi(\omega_1)) \vee \xi(\omega_2))$$

These remaining possibilities can't be narrowed down further without knowing the particular values of  $q_1^*$ ,  $q_3^*$  and each value  $\xi$  takes on. While the first consideration can be addressed solely in terms of the market, the second prevents us from generally pricing the distribution. We must manually compare these quantities to determine which is the solution.  $\square$

Unfortunately, we can't find a general solution for pricing distributions. We must write out and price each permutation, since the ordering of the prices depends on the particular values of the distribution. Indeed even in the simplest cases the price of a random variable depends on the values it takes on: although we can in that case write out a discriminant instead of calculating the two possible prices and comparing them. More of these discriminants might exist in the general cases but regardless calculating and comparing the twelve possible prices will always succeed in pricing the distribution. Importantly since the discriminant for pricing a random variable depends on the values it takes on it is not possible to get results comparable to the binomial model. In the binomial model our discriminant depends on the market, not the distribution, but in the trinomial model this is not the case and the particular distribution is the determining factor: no general result across all distributions is possible.

## 4 Conclusion

While we have successfully analyzed the binomial model using the usual no-arbitrage pricing techniques the trinomial model is not readily solved. Indeed, both approaches to the trinomial model fail to admit a pricing strategy for general distributions in our analysis. In order to tackle these kinds of problems without the general tools that the binomial case provides requires comparing every random variable with a given distribution. This grows combinatorially with the number of outcomes and limits the usefulness of the pricing strategy.

In the binomial model the price of the distributions depends only on the market parameters while in the trinomial model it depends also on characteristics of the distribution we wish to price. Further, in the binomial model given a distribution we can not only price it, but also exhibit an appropriate investment strategy with that price. In the trinomial model while we did have some success in pricing investment strategies, we did not exhibit any techniques for constructing investment strategies. This is possibly complicated by the fact many investment strategies may exist with a given price that superhedge the distribution.

Pricing in these simple models is an important part of analysing and pricing distributions of wealth. Results about simple models sometimes lead to conclusions about more realistic models. However, as we saw moving from the binomial to the trinomial model, sometimes changing models introduces new complications.

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