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# Constructing Two Dimensional Theories of Gravity for Schwarzschild, Reissner-Nordstrom, and Non-Extremal Kerr Black Holes

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# Constructing Two Dimensional Theories of Gravity for Schwarzschild, Reissner-Nordstrom, and Non-Extremal Kerr Black Holes

A Major Qualifying Project

Submitted to the Faculty of

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By:

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## Abstract

The purpose of this project was to provide a study on the construction of 2D dilaton gravity theories for the Schwarzschild and Reissner-Nordstrom black holes while also constructing the previously unknown 2D dilaton gravity theory for the non-extremal Kerr black hole. This project was successful in reproducing known 2D dilaton gravity Einstein Hilbert actions and equations of motion for the Schwarzschild and Reissner-Nordstrom black holes and also successfully constructed a viable solution for a 2D dilaton gravity theory for the non-extremal Kerr black hole. By creating a generalized 2D Einstein Hilbert action with parameters  $\alpha$  and  $\gamma$ , finding the non-extremal Kerr equations of motion from that action, and fixing the values of  $\alpha$  and  $\gamma$ , the equations of motion were solved to find the components of the generalized 2D Einstein Hilbert action for the non-extremal Kerr black hole.

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# 1 Introduction

Soon after Albert Einstein published his Theory of General Relativity in 1915, others found that there existed exact solutions to the Einstein Field Equations that describe how spacetime curves due to the presence of energy and matter. Some of these exact solutions describe regions of spacetime that are so curved, nothing can move fast enough to escape their gravitational attraction, not even light. This extremely high curvature is caused by large collections of mass-energy that curve spacetime. These regions are black holes. Black holes are regions of spacetime that cannot be mapped to conformal infinity.

The four dimensional (3 spatial dimensions and 1 time dimension) equations that are generally used to describe black holes are extremely complex and often involve tedious calculations. By making use of angular isometries, four dimensional theories can be reduced to two dimensional theories. These two dimensional descriptions are useful because they can provide new insights and physical intuitions that their four dimensional descriptions cannot.

2D Dilaton gravity is one such two dimensional theory. 2D Dilaton gravity is a near-horizon 4D to 2D dimensional reduction that can be performed in very strong gravitational fields. It formulates pure gravity as 2D gravity coupled to scalars and U(1) gauge fields [1]. 2D Dilaton gravity theory uses isotropy of black holes to simplify equations immensely, and gravity is also renormalizable in two dimensions, so 2D theories can serve as a test bed for quantum gravity [2]. Certain formulations of 2D Dilaton gravity even allow for exact black hole solutions that contain no singularities, which are unavoidable in four dimensions [3].

Schwarzschild, Reissner-Nordstrom, and non-extremal Kerr black holes all have known four dimensional metrics and four dimensional gravity theories (i.e., their 4D metrics come from general relativity, the 4D gravity theory). All three of these black holes also have known two dimensional metrics; however, only the Schwarzschild and Reissner-Nordstrom have a corresponding known 2D gravity theory. For the non-extremal Kerr, the 2D gravity theory is unknown.

The goal of our project was to reproduce the construction of 2D Dilaton gravity theories for the Schwarzschild and Reissner-Nordstrom black holes in order to construct the previously unknown 2D Dilaton gravity theory for the non-extremal Kerr black hole.

## 2 Background

### 2.1 Mathematical Notation

#### 2.1.1 Dimensionality

General relativity is usually described within the context of one time dimension and three space dimensions,  $(t, x, y, z)$ . Describing spacetime in this way is referred to as the "3 + 1 formalism" [4]. These coordinates are also frequently written in the ADM notation as  $(x^0, x^1, x^2, x^3)$ , where partial derivatives are written as  $\partial_i$  and covariant derivatives are written as  $\nabla_i$  [5]. This paper will use the ADM notation too, as is customary in general relativity.

The specification of three spacial dimensions and one time dimension is due to the solutions to the Einstein equations being covariant; meaning irrespective of coordinates. Because of this, a time coordinate needs to be added to determine the time evolution of the metric tensor.

#### 2.1.2 Notation

Einstein notation, or Einstein summation convention, is a concise mathematical notation that implies summation over a set of indexed terms in an expression [6]. For example,

$$c_i x^i = \sum_{i=0}^3 c_i x^i = c_0 x^0 + c_1 x^1 + c_2 x^2 + c_3 x^3,$$

where the range of the indices depends on the dimensionality. Rather than traditionally representing exponents, the superscripts denote an index [6]. Einstein adopted this summation convention and index notation from Ricci calculus and applied it to physics for the first time in 1916 which allowed him to complete his theory of general relativity.

In addition to using Einstein notation, this paper will use general relativity units as summarized in [7]. Most notably:  $G$  (gravitational constant) = 1 and  $c$  (speed of light) = 1.

### 2.2 Review of 4D Geometry and Gravity

#### 2.2.1 Black Holes

The formation of black holes is a natural consequence of the existence of gravity. Black holes are extremely dense regions in space that curve spacetime so much that not even light can escape their gravitational attraction.

Black holes fit into one of three categories, depending on their mass: primordial black holes (the smallest), stellar black holes (medium-sized and most common), and supermassive black holes (the

most massive by far). Primordial black holes are theorized to have formed in the early universe and supermassive black holes are theorized to have formed when the galaxy they inhabit formed. Stellar black holes, the most common and well understood, form due to gravitational collapse [8].

According to the no-hair theorem, "the external gravitational and electromagnetic fields of a [...] black hole [...] are determined uniquely by the [black] hole's mass  $M$ , charge  $Q$ , and intrinsic angular momentum  $S$ " [9]. In other words, all black holes can be completely characterized by the three observable quantities: mass, electric charge, and angular momentum.

There are four types of asymptotically flat black holes, the Schwarzschild, Reissner-Nordstrom, Kerr, and Kerr-Newman black holes, as shown in Table 1 below. Black holes do not necessarily remain of the same type their whole lives. Rather, they can become charged by happening to attract more positive or negative charge than the other, and inversely they can lose their charge by attracting matter that neutralizes their charge. Similarly, black holes can gain angular momentum by absorbing matter non-spherically-symmetrically or by non-spherically symmetric masses gravitationally collapsing into black holes.

	Non-rotating ( $J = 0$ )	Rotating ( $J \neq 0$ )
Uncharged ( $Q = 0$ )	Schwarzschild	Kerr
Charged ( $Q \neq 0$ )	Reissner-Nordstrom	Kerr-Newman

Table 1: types of asymptotically flat black holes and their physical characteristics

In Table 1 above,  $Q$  represents the body's electric charge and  $J$  represents its angular momentum. The other property that all black holes share is  $M$ , the mass of the black hole. This project focused on the Schwarzschild, Reissner-Nordstrom, and Kerr black holes only.

### 2.2.2 Einstein's Field Equations

Einstein's Field Equations are a set of 10 equations that describe gravitational force as a result of spacetime curvature due to the presence of mass-energy [10].

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \quad (1)$$

where  $T_{\mu\nu}$  is the energy-momentum or stress-energy tensor,  $\Lambda$  is the cosmological constant,  $R$  is the Ricci Curvature Scalar:

$$R = g^{\mu\nu}R_{\mu\rho\nu}^{\rho} \quad (2)$$

and  $R_{\mu\nu}$  is the contracted Riemann Curvature Tensor ( $R_{\beta\mu\nu}^\alpha$ ):

$$R_{\beta\mu\nu}^\alpha = \partial_\mu \Gamma_{\beta\nu}^\alpha - \partial_\nu \Gamma_{\beta\mu}^\alpha + \Gamma_{\mu\gamma}^\alpha \Gamma_{\beta\nu}^\gamma - \Gamma_{\nu\sigma}^\alpha \Gamma_{\beta\mu}^\sigma, \quad (3)$$

where  $\partial_1$  (for example) denotes the partial derivative with respect to the coordinate  $x^1$ .

The Riemann Curvature Tensor describes the failure of the covariant derivative to commute ( $[\nabla_\mu, \nabla_\nu]A^\sigma = R_{\rho\mu\nu}^\sigma A^\rho$ ). If the surface is completely flat, the Riemann Curvature Tensor (and hence  $R_{\mu\nu}$  and  $R$ ) will be zero. When no matter is present, the energy-momentum tensor  $T_{\mu\nu}$  vanishes, yielding  $R_{\mu\nu} = 0$ . The  $\Gamma$ 's (Christoffel Symbols of the second kind) describe how vectors parallel transport on a curved surface.  $\Gamma_{\mu\nu}^\alpha$  measures a vector's deviation from its original orientation when the vector is parallel transported in a loop on a surface.

$$\Gamma_{\mu\nu}^\alpha = \frac{1}{2} g^{\alpha\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \quad (4)$$

All of these quantities depend on the metric tensor, a tensor that describes the geometry of spacetime. The metric can be written as the line element, which gives the spacetime interval between two points or as a matrix whose components are the gravitational field components.

Flat space metric in Cartesian coordinates:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \quad (5)$$

The Minkowski (flat space) metric in Cartesian coordinates is represented by the matrix  $\eta_{\mu\nu}$ :

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6)$$

The spacetime interval between two points in Spherical, flat space:

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad (7)$$

The flat space metric in Spherical coordinates:

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{bmatrix} \quad (8)$$

The 4D Einstein Hilbert Action,

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R \quad (9)$$

where  $R$  is the Ricci Curvature Scalar and  $g$  is the determinant of the metric matrix, yields the Einstein Field Equations when the principle of stationary action is applied [11].

### 2.3 2D Dilaton Gravity

Often, calculations involving four dimensional spacetime and actions can be complex and tedious, such as in the case of black holes. In these situations, it can be useful to use a two dimensional theory of gravity to simplify these calculations and sometimes make finding analytic solutions possible. 2D dilaton gravity theory is one gravitational theory that can be used to reduce four dimensional theories to two dimensions. Two dimensional dilaton gravity is quite relevant now, given its relation to string theory and its aid in examining hawking radiation of black holes [12]. 2D dilaton gravity theory represents a 1 + 1 dimensional theory of gravity coupled to a scalar dilaton field, denoted  $\psi$ .

The general 2D dilaton gravity Einstein Hilbert action is as follows:

$$S_{dilaton} = \frac{1}{2\pi} \int d^2x \sqrt{-g} e^{-2\psi} [R + 4(\nabla\psi)^2 + 4\lambda^2] \quad (10)$$

where  $\lambda^2 = \frac{\pi}{2G}$  and is used to achieve the desired units.



### 3 The Schwarzschild Black Hole

The Schwarzschild solution to the Einstein Field Equations describes the spacetime around a non-rotating, uncharged, spherically symmetric body of mass  $M$ . The four dimensional Schwarzschild metric in spherical coordinates:

$$g_{\mu\nu} = \begin{bmatrix} -(1 - \frac{2GM}{r}) & 0 & 0 & 0 \\ 0 & (1 - \frac{2GM}{r})^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{bmatrix} \quad (11)$$

has the property of having a coordinate singularity when  $r = 2GM$  and a curvature singularity at  $r = 0$ .  $r = 2GM$  is known as the Schwarzschild radius, the radius of the event horizon for a Schwarzschild mass. For objects with relatively low densities, the Schwarzschild radius is within the object itself; for objects with relatively high densities, the Schwarzschild radius is above the surface of the object. Those objects with high densities that have a Schwarzschild radius greater than their surface radius are Schwarzschild black holes [13]. As  $r \rightarrow \infty$ , the Schwarzschild metric approaches the Minkowski (flat space) metric  $\eta_{\mu\nu}$ , so the Schwarzschild black hole is asymptotically flat. According to Birkhoff's theorem, the Schwarzschild solution is the most general spherically symmetric solution to Einstein's field equations.

#### 3.1 Schwarzschild 4D Theory

The four dimensional Schwarzschild solution can be found by assuming a spherically symmetric, static, vacuum solution and solving the Einstein Field Equations for the metric  $g_{\mu\nu}$ . Since a spherically symmetric, static solution is assumed, the time and radial components must depend on  $r$  only, not  $t$ , so the metric must be of the form:

$$g_{\mu\nu} = \begin{bmatrix} A(r) & 0 & 0 & 0 \\ 0 & B(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{bmatrix} \quad (12)$$

Assuming a vacuum solution means  $T_{\mu\nu} = 0$  and  $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$ , which implies  $R_{\mu\nu} = 0$ . Using

this fact and the form of the metric, the Christoffel symbols of the second kind can be calculated. Since  $R_{\mu\nu} = 0$ ,

$$R_{\mu\nu} = R_{\mu\rho\nu}^{\rho} = \partial_{\rho}\Gamma_{\nu\mu}^{\rho} - \partial_{\nu}\Gamma_{\rho\mu}^{\rho} + \Gamma_{\rho\lambda}^{\rho}\Gamma_{\nu\mu}^{\lambda} - \Gamma_{\nu\lambda}^{\rho}\Gamma_{\rho\mu}^{\lambda} = 0 \quad (13)$$

This creates a system of differential equations that can be solved for  $A(r)$  and  $B(r)$  to find that  $A(r) = C(1 + \frac{1}{Dr})$  and  $B(r) = (1 + \frac{1}{Dr})^{-1}$ , where C and D are real valued constants. From the weak field approximation, it can be determined that  $C = -1$  and  $D = -\frac{1}{2GM}$  in general relativity units, which results in the Schwarzschild metric presented in equation 11 [14].

The Einstein Hilbert action (equation 9) can be used to find the equation of motion for the Schwarzschild black hole by varying the action with respect to the inverse metric and applying the principle of stationary action. This yields the vacuum Einstein Field Equation:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \quad (14)$$

## 3.2 Schwarzschild 2D Dilaton Theory

### 3.2.1 Schwarzschild 2D Dilaton Theory Action

To construct the 2 (1+1) dimensional dilaton gravity theory for the Schwarzschild black hole, we begin with the 4 (3+1) dimensional Einstein Hilbert action introduced previously (equation 9):

$$S_{4D} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R \quad (15)$$

where  $d^4x$  is the four dimensional volume element in spherical coordinates such that:

$$S_{4D} = \frac{1}{16\pi G} \int \sqrt{-g} R r^2 \sin \theta dt dr d\theta d\phi \quad (16)$$

From here,  $r^2$  may be rewritten as an exponential in terms of the dilaton scalar field  $\psi$ : [15]

$$S = \frac{1}{16\pi G} \int \frac{e^{-2\psi(r)}}{\lambda^2} \sqrt{-g} R \sin \theta dt dr d\theta d\phi \quad (17)$$

where  $\lambda$  is a constant such that the field  $\psi$  is dimensionless:  $\lambda^2 = \frac{\pi}{2G}$  [15]. Using the generalized black hole metric,

$$g_{\mu\nu} = \begin{bmatrix} -f(r) & 0 & 0 & 0 \\ 0 & f(r)^{-1} & 0 & 0 \\ 0 & 0 & \frac{e^{-2\psi(r)}}{\lambda^2} & 0 \\ 0 & 0 & 0 & \frac{e^{-2\psi(r)}}{\lambda^2} \sin^2 \theta \end{bmatrix} \quad (18)$$

the four dimensional Ricci scalar can be calculated:

$$R^{4D} = 4f'(r)\psi'(r) - f''(r) + f(r) [-6\psi'(r)^2 + 4\psi''(r)] \quad (19)$$

Plugging the four dimensional Ricci scalar into equation 17 gives:

$$S_{4D} = \frac{1}{16\pi G} \int \frac{e^{-2\psi(r)}}{\lambda^2} \sqrt{-g^{4D}} \left[ 2e^{2\psi(r)}\lambda^2 + 4f'(r)\psi'(r) - f''(r) + f(r) [-6\psi'(r)^2 + 4\psi''(r)] \right] \sin \theta dt dr d\theta d\phi \quad (20)$$

To reduce the action from 4D to 2D, the action can be integrated over  $\theta$  from 0 to  $\pi$  and  $\phi$  from 0 to  $2\pi$  since the Schwarzschild black hole is spherically symmetric. These integrals evaluate to 2 and  $2\pi$ , respectively, resulting in an additional overall factor of  $4\pi$  and leaving the action in two dimensions ( $t$  and  $r$ ):

$$S_{2D} = \frac{1}{4G} \int \frac{e^{-2\psi(r)}}{\lambda^2} \sqrt{-g^{2D}} \left[ 2e^{2\psi(r)}\lambda^2 + 4f'(r)\psi'(r) - f''(r) + f(r) [-6\psi'(r)^2 + 4\psi''(r)] \right] dt dr \quad (21)$$

The two dimensional Ricci scalar can be calculated using the two dimensional generalized metric

$$g_{\mu\nu} = \begin{bmatrix} -f(r) & 0 \\ 0 & f(r)^{-1} \end{bmatrix} \quad (22)$$

to find that  $R^{2D} = -f''(r)$ . Substituting in the two dimensional Ricci scalar and simplifying results in the final 2D dilaton gravity Einstein Hilbert action for the Schwarzschild black hole:

$$S_{2D} = \frac{1}{4G} \int d^2x \frac{e^{-2\psi(r)}}{\lambda^2} \sqrt{-g^{2D}} \left[ 2e^{2\psi(r)}\lambda^2 + R^{2D} + 2(\nabla\psi)^2 \right] \quad (23)$$

where  $d^2x = dt dr$ .

### 3.2.2 2D Dilaton Theory Equations of Motion

Similar to the four dimensional case, the equations of motion for the 2D Dilaton theory Schwarzschild black hole may be found by varying the Einstein Hilbert action with respect to the relevant fields and applying the principle of stationary action. For the Schwarzschild black hole, there are only two fields, so there are two associated equations of motion.

To find the first equation of motion, which describes the dynamics of the gravitational field, the action must be varied with respect to the inverse metric  $g^{\mu\nu}$  (as is customary):

$$\delta_{g^{\mu\nu}} S_{2D} = \frac{1}{4G} \int d^2x \frac{e^{-2\psi(r)}}{\lambda^2} \delta \left[ \sqrt{-g} \left( 2e^{2\psi(r)} \lambda^2 + R^{2D} + 2(\nabla\psi)^2 \right) \right] \quad (24)$$

In two dimensions, there is no classical matter configuration that can source gravity, so

$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$ , simplifying the variation. Using  $\delta\sqrt{-g} = \frac{-1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$  and the Palatini identity  $\delta R_{\mu\nu} = \delta R_{\mu\alpha\nu}^\alpha = \nabla_\alpha(\delta\Gamma_{\nu\mu}^\alpha) - \nabla_\nu(\delta\Gamma_{\alpha\mu}^\alpha)$  [11] the variation of the 2D dilaton Einstein Hilbert action with respect to the inverse metric is as follows:

$$\delta_{g^{\mu\nu}} S_{2D} = \frac{1}{2\pi} \int d^2x \sqrt{-g} \left[ 2\nabla_\mu e^{-\psi} \nabla_\nu e^{-\psi} - g_{\mu\nu} (\nabla_\alpha e^{-\psi} \nabla^\alpha e^{-\psi} + \lambda^2) - \nabla_\mu \nabla_\nu e^{-2\psi} + \square e^{-2\psi} g_{\mu\nu} \right] \delta g^{\mu\nu}, \quad (25)$$

where  $\square$  is the d'Alembert operator such that  $\square = \nabla_\alpha \nabla^\alpha$  and  $\alpha$  is a dummy index.

Applying the principle of stationary action gives the stress energy tensor:

$$-2\nabla_\mu \nabla^\mu \psi + 4\nabla_\nu \psi \nabla^\nu \psi - 2e^{2\psi} \lambda^2 = 0 \quad (26)$$

To find the second equation of motion, which describes the dynamics of the dilaton scalar field, the action must be varied with respect to the dilaton field  $\psi$ . Similar to above:

$$\delta_\psi S_{2D} = \frac{1}{2\pi} \int d^2x \sqrt{-g} \left[ -2\delta\psi e^{-2\psi} R + 4\nabla_\mu e^{-\psi} \nabla^\mu (e^{-\psi} (-\delta\psi)) \right] \quad (27)$$

After simplifying and applying the principle of stationary action, the following and final equation of motion is found for the Schwarzschild black hole in 2D dilaton gravity:

$$\frac{R}{2} + \nabla_\mu \psi \nabla^\mu \psi + \square \psi = 0 \quad (28)$$

## 4 The Reissner-Nordstrom Black Hole

The Reissner-Nordstrom solution to the Einstein Field Equations describes the spacetime around a *charged*, non-rotating, spherically symmetric body of mass  $M$ . The four dimensional Reissner-Nordstrom metric is as follows:

$$g_{\mu\nu} = \begin{bmatrix} -(1 - \frac{2GM}{r} + \frac{r_Q^2}{r^2}) & 0 & 0 & 0 \\ 0 & (1 - \frac{2GM}{r} + \frac{r_Q^2}{r^2})^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{bmatrix} \quad (29)$$

where

$$r_Q^2 = \frac{Q^2 G}{4\pi\epsilon_0} \quad (30)$$

and  $\frac{1}{4\pi\epsilon_0}$  is the Coulomb force constant. Similar to the Schwarzschild metric, the Reissner-Nordstrom metric has the property of having coordinate singularities at  $r = \frac{r_s \pm \sqrt{r_s^2 - 4r_Q^2}}{2}$  and a curvature singularity at  $r = 0$ . As  $Q \rightarrow 0$ , the Reissner-Nordstrom metric approaches the Schwarzschild metric. As  $r \rightarrow \infty$ , the Reissner-Nordstrom metric approaches the Minkowski (flat space) metric  $\eta_{\mu\nu}$ , so the Reissner-Nordstrom black hole is also asymptotically flat.

### 4.1 Reissner-Nordstrom 4D Theory

The four dimensional Reissner-Nordstrom solution can be found, similar to the Schwarzschild solution, by assuming a spherically symmetric, static solution and solving the Einstein Field Equations for the metric  $g_{\mu\nu}$ . This derivation is recited in detail in [16].

To find the equations of motion for the Reissner-Nordstrom black hole, the Einstein Hilbert action (equation 9) must be varied with respect to the inverse metric and the electromagnetic potential  $A_\mu$ . In this case, since the Reissner-Nordstrom black hole is charged, the Einstein Hilbert action must include a Maxwell field coupling, in addition to the usual gravitational term:

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R - \frac{1}{16\pi} \int d^4x \sqrt{-g} F_{\mu\nu} F^{\mu\nu} \quad (31)$$

where  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  is the electromagnetic tensor.

To find the first equation of motion, the action must be varied with respect to the inverse metric

and the principle of stationary action must be applied:

$$\delta_{g^{\mu\nu}} S = \frac{1}{16\pi G} \int d^4x \left[ \frac{-1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} R_{\mu\nu} \right] - \frac{1}{16\pi} \int d^4x \left[ \frac{-1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} F_{\mu\alpha} F_{\nu}^{\alpha} \right] \quad (32)$$

Again, using  $\delta\sqrt{g} = \frac{-1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}$ , the Palatini identity  $\delta R_{\mu\nu} = \delta R_{\mu\alpha\nu}^{\alpha} = \nabla_{\alpha}(\delta\Gamma_{\nu\mu}^{\alpha}) - \nabla_{\nu}(\delta\Gamma_{\alpha\mu}^{\alpha})$ , and the principle of stationary action, the resulting equation of motion is:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - 2G \left[ F_{\mu\alpha} F_{\nu}^{\alpha} - \frac{1}{4} F^2 \right] = 0 \quad (33)$$

To find the second equation of motion, the above must be repeated, varying with respect to the electromagnetic potential  $A_{\mu}$ :

$$\nabla_{\mu} F_{\beta\nu} = 0 \quad (34)$$

## 4.2 Reissner-Nordstrom 2D Dilaton Theory

### 4.2.1 Reissner-Nordstrom 2D Dilaton Theory Action

To dimensionally reduce the 4D Einstein Hilbert action for the Reissner-Nordstrom black hole, the same steps must be followed as for the Schwarzschild Einstein Hilbert action. After rewriting  $r^2$  as an exponential in terms of the dilaton scalar field  $\psi$  and integrating over the angular components, the 2D dilaton gravity action for the Reissner-Nordstrom black hole is found:

$$S_{2D} = \frac{1}{4G} \int d^2x \frac{e^{-2\psi(r)}}{\lambda^2} \sqrt{-g} \left[ 2e^{2\psi(r)} \lambda^2 + R + 2(\nabla\psi)^2 - F_{\mu\nu} F^{\mu\nu} \right] \quad (35)$$

where  $\lambda^2 = \frac{\pi}{2G}$ .

Since this action relies on three fields (gravitational, dilaton, and electromagnetic), there will be three equations of motion. To find the equation of motion that describes the dynamics of the dilaton scalar field  $\psi$ , the action must be varied with respect to  $\psi$  and the principle of stationary action must be applied, resulting in the equation of motion for  $\psi$ :

$$R + 2(\square\psi + \nabla_{\mu}\psi\nabla^{\mu}\psi) = 0 \quad (36)$$

The above is repeated with respect to the inverse metric  $g^{\mu\nu}$  to find the equation of motion that

describes the dynamics of the gravitational field:

$$-2\Box\psi + 4\nabla_\alpha\psi\nabla^\alpha\psi - 2\lambda^2 e^{2\psi} - F_{\mu\nu}F^{\mu\nu} = 0 \quad (37)$$

and again with respect to the electromagnetic potential  $A_\mu$  to find the equation of motion that describes the dynamics of the electromagnetic field:

$$\nabla_\nu(e^{-2\psi}F^{\nu\mu}) = 0 \quad (38)$$

## 5 The Kerr Black Hole

The Kerr solution to the Einstein Field Equations describes the spacetime around a *rotating*, uncharged, axially symmetric body of mass  $M$ . Note that rotation necessarily means the body is not spherically symmetric, just axially so. It turns out that this one difference complicates the Kerr solutions (and derivations) significantly. The original four dimensional Kerr metric presented in Kerr's paper is as follows:

$$\begin{aligned} ds^2 = & - \left( 1 - \frac{2Mr}{r^2 + a^2 \cos^2(\theta)} \right) (dv - a \sin^2(\theta) d\tilde{\phi})^2 \\ & + 2(dv - a \sin^2(\theta) d\tilde{\phi})(dr - a \sin^2(\theta) d\tilde{\phi}) \\ & + (r^2 + a^2 \cos^2(\theta))(d\theta^2 + \sin^2(\theta) d\tilde{\phi}^2) \end{aligned} \quad (39)$$

in ingoing Eddington-Finkelstein coordinates.

When  $a = 0$ , the Kerr metric reduces to the Schwarzschild metric. Unlike the Schwarzschild and Reissner-Nordstrom metrics, this form of the Kerr metric has three off-diagonal terms. The  $\tilde{\phi}$  is used to differentiate the ingoing Eddington-Finkelstein coordinate  $\phi$  from (more commonly) the Boyer-Lindquist coordinate  $\phi$ . The Kerr metric has a singularity where  $r^2 + a^2 \cos^2(\theta) = 0$ , so where  $r = 0$  and  $\theta = \frac{\pi}{2}$ . The meaning of this singularity is unclear in polar coordinates, however a more intuitive understanding of the Kerr singularity can be obtained by switching to the 'Kerr-Schild' form of the metric [17].

For the sake of brevity, the form and related equations will not be presented in this paper. The important insight the Kerr-Schild form yields is that the singularity "  $r = 0$  corresponds to the ring  $x^2 + y^2 = a^2, z = 0$ ," meaning the singularity of the Kerr black hole is in the shape of a ring, rather than a point like the Schwarzschild and Reissner-Nordstrom black holes [17].

To best convey the physical intuition behind the Kerr metric curvature singularity, the Kerr metric is most commonly written in Boyer-Lindquist coordinates (though this form was discovered after the original form of the Kerr metric):

$$\begin{aligned}
ds^2 = & - \left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar \sin^2(\theta)}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 \\
& + \left(r^2 + a^2 + \frac{2Ma^2r \sin^2(\theta)}{\Sigma}\right) \sin^2(\theta) d\phi^2,
\end{aligned} \tag{40}$$

where  $\Sigma \equiv r^2 + a^2 \cos^2(\theta)$  and  $\Delta \equiv r^2 - 2Mr + a^2$ . The transformation from ingoing Eddington-Finkelstein coordinates to Boyer-Lindquist coordinates can be found in [17].

This form only has one off-diagonal term, which makes it easier to work with than the aforementioned forms. Notably, it is easier to see how this form is asymptotically flat and it is easier to identify  $M$  as the mass and  $J = aM$  as the angular momentum by letting  $r \rightarrow \infty$ . When  $a = 0$ , the Kerr metric simplifies to the Schwarzschild metric. This form has a coordinate singularity at  $\Delta = 0$ , and the roots of  $\Delta$  identify the inner and outer horizons, with  $r_{\pm} = M \pm \sqrt{M^2 - a^2}$ .  $r_+$  corresponds to the event horizon and  $r_-$  likely has no physical importance according to [17].

There are two types of Kerr black holes: extremal and non-extremal. The extremal Kerr solution has maximal angular momentum  $J_{max} = a_{max}M = M^2$  where the spin  $a = \frac{J}{M}$  [18]. In the extremal case, the Kerr's event horizon and Cauchy horizon meet, simplifying the metric immensely. The non-extremal Kerr has any angular momentum  $a < GM$ , so it has two distinct horizons and is more complex. While the extremal Kerr has a known four dimensional *and* two dimensional theory of gravity, the non-extremal Kerr does not have a known two dimensional theory of gravity, only a two dimensional metric.

## 5.1 Non-Extremal Kerr

### 5.1.1 Non-Extremal Kerr 4D Theory

Based on our studies of the Schwarzschild and Reissner-Nordstrom 2D dilaton gravity theories, we created a generalized 2D Einstein Hilbert action for the non-extremal Kerr black hole which had components from the Schwarzschild 2D dilaton action and was coupled to a gauge field (as in the Reissner-Nordstrom action):

$$S = \frac{l^2}{4G} \int d^2x \sqrt{-g} \left[ e^{-2\psi(r)} R + \frac{\alpha}{l^2} + 2\nabla_{\mu} e^{-\psi(r)} \nabla^{\mu} e^{-\psi(r)} - l^2 \beta e^{\gamma\psi(r)} F^2 \right] \tag{41}$$



where  $l^2 = \frac{1}{\lambda^2}$ . In this action,  $\alpha, \beta$ , and  $\gamma$  are constants that serve as parameters to generalize the action. The first three terms come from the Schwarzschild 2D dilaton gravity action, though the cosmological coupling term ( $\frac{\alpha}{l^2}$ ) has been further generalized with the parameter  $\alpha$ . The last term is the Maxwell field coupling, where  $\gamma$  determines the strength of the coupling to the field and  $F_{\mu\nu}$  is the electromagnetic tensor such that  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ . In electrodynamics, the gauge transformations are U(1) transformations, so angular momentum can be represented by the electromagnetic potential in two dimensions, where the extra dimensions are compactified [19]. Therefore, it is reasonable to suppose that the 2D dilaton gravity action for the non-extremal Kerr black hole would be in the form of equation 41.

To fully determine what the 2D dilaton gravity action is comprised of,  $\psi$ ,  $A_\mu$ , and  $\alpha$ ,  $\beta$ , and  $\gamma$  must be determined using the 2D non-extremal Kerr metric and the associated 2D Ricci scalar. The 2D non-extremal Kerr metric is known to be of the form:

$$g_{\mu\nu} = \begin{bmatrix} -f(r) & 0 \\ 0 & f(r)^{-1} \end{bmatrix}, f(r) = \frac{r^2 - 2GMr + a^2}{r^2 + a^2}, \quad (42)$$

where  $f(r)$  comes from setting  $\theta$  and  $\phi$  equal to zero in the four dimensional Kerr metric (equation 40).

Finding the equations of motion for the three associated fields gives three differential equations that may be solved simultaneously to find  $\psi$ ,  $A_\mu$ , and  $\alpha$ ,  $\beta$ , and  $\gamma$ . To find the equations of motion, we vary with respect to the inverse metric, the dilaton scalar field  $\psi$ , and the electromagnetic potential  $A_\mu$ , as in the case of the Reissner-Nordstrom 2D dilaton gravity equations of motion. Varying the action with respect to the inverse metric  $g^{\mu\nu}$  and applying the principle of stationary action yields the first equation of motion:

$$-2\nabla_\nu \nabla^\nu \psi(r) + 4\nabla_\mu \psi(r) \nabla^\nu \psi(r) - \alpha \lambda^2 e^{2\psi(r)} - \beta e^{(\gamma+2)\psi(r)} F^2 = 0 \quad (43)$$

Varying with respect to the scalar field  $\psi(r)$  and applying the principle of stationary action yields the second equation of motion:

$$R^{2D} + 2[\nabla_\nu \psi(r) \nabla^\nu \psi(r) - \nabla_\mu \psi(r) \nabla^\mu \psi(r)] + \frac{\beta\gamma}{2} e^{(\gamma+2)\psi(r)} F^2 = 0 \quad (44)$$

Varying with respect to the electromagnetic potential field  $A_\mu$  and applying the principle of

stationary action yields the third equation of motion:

$$\nabla_\nu [e^{\gamma\psi(r)} F^{\nu\mu}] = 0 \quad (45)$$

Where  $F^2 = F_{\mu\nu}F^{\mu\nu}$ .

Solving these three differential equations simultaneously was extremely difficult, however, by isolating  $\beta e^{(\gamma+2)\psi(r)} F^2$  in both equation 43 and equation 44, we were able to set them equal to each other and reduce the three differential equations to two differential equations:

$$R + (2 - \gamma)\nabla_\mu \nabla^\mu \psi(r) + 2(\gamma - 1)\nabla_\nu \psi(r)\nabla^\nu \psi(r) - \frac{\gamma\alpha}{2\gamma^2} e^{2\psi(r)} = 0 \quad (46)$$

$$\nabla_\nu [e^{\gamma\psi(r)} F^{\nu\mu}] = 0 \quad (47)$$

Fortunately, equation 46 is a function of  $\psi(r)$  only (not  $A(r)$ ), so it was theoretically much easier to solve for  $\psi(r)$  than before.

Solving these differential equations simultaneously proved computationally difficult, so we were unable to find a general solution for  $\psi(r)$  and  $A(r)$ . However, by setting  $\alpha = 0$  we were able to get equation 46 in the form of a differential equation with a known solution. We chose  $\gamma = 1$  because  $\gamma$  must be non-zero for dilaton gravity (otherwise  $\psi$  will vanish in the Maxwell field term) and we wanted to avoid  $\gamma = -4$ , the extremal solution [20]. Using these specific values, we were able to find analytic solutions for  $\psi(r)$  and  $A(r)$  as follows:

$$\psi(r) = \frac{2C_1 G^2 M^2 \tan^{-1}\left(\frac{r-GM}{\sqrt{a^2-G^2M^2}}\right)}{\sqrt{a^2-G^2M^2}} + (C_1 GM + 1) \log(a^2 - 2GMr + r^2) - \log(a^2 + r^2) + C_1 r + C_2 \quad (48)$$

$$A(r) = C_2 - (a^2 + r(r - 2GM))^{-C_1 GM} \exp\left(-\frac{2C_1 G^2 M^2 \tan^{-1}\left(\frac{r-GM}{\sqrt{a^2-G^2M^2}}\right)}{\sqrt{a^2-G^2M^2}} - C_1 r\right) \quad (49)$$

where  $C_1$  and  $C_2$  are arbitrary, real-valued constants.

## 6 Conclusion

The goal of this project was to construct the known 2D dilaton gravity theories for the Schwarzschild and Reissner-Nordstrom black holes in order to construct the previously unknown 2D dilaton gravity theory for the non-extremal Kerr black hole. We successfully reproduced the construction of the 2D dilaton gravity theory for the Schwarzschild black hole in section 3.2 and for the Reissner-Nordstrom black hole in section 4.2. We were able to construct the 2D dilaton gravity Einstein Hilbert actions and find the 2D dilaton gravity equations of motion for both the Schwarzschild and Reissner-Nordstrom black holes.

For the non-extremal Kerr black hole, we were successful in constructing a viable 2D dilaton gravity theory which was previously unknown. Though we were unable to find a general analytic solution and had to fix  $\alpha$  and  $\gamma$ , the values we chose are plausible for the non-extremal Kerr black hole. Choosing  $\alpha = 0$  does mean that the cosmological coupling in the Einstein Hilbert action ( $\frac{\alpha}{l^2}$ ) vanishes, however that does not necessarily mean the solution is non-physical, it does however mean that taking the limit as  $a \rightarrow 0$  no longer results in the Schwarzschild 2D dilaton gravity theory.

As of now, the physical interpretation of the solution for the dilaton scalar field  $\psi(r)$  (equation 48) and  $A(r)$  (equation 49) is not known. Interestingly however,  $\psi(r)$  seems to depend on the logarithm of the two dimensional metric ( $\log(\frac{a^2-2GMr+r^2}{a^2+r^2}) = \log(f(r))$ ) for the non-extremal Kerr black hole, which is not the case for the Schwarzschild and Reissner-Nordstrom black holes.

### 6.1 Moving Forward

In the future, we intend to work towards developing a physical intuition as to what  $\alpha = 0$  may mean in our specific solution. Additionally, we intend to identify tests that can be performed on both our equations of motion and specific result to verify our findings are indeed physical. Simultaneously, we could also pursue other specific results, especially those with nonzero  $\alpha$  values. Lastly, and most significantly, we hope to work towards a general solution for  $\psi(r)$  and  $A_\mu(r)$  that does not require fixing the values of  $\alpha$  and  $\gamma$ .

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## References

- [1] H. Pelzer, T. Strobl, *Generalized 2d dilaton gravity with matter fields*, (1998).  
arXiv:gr-qc/9805059v2
- [2] D. Grumiller, W. Kummer, D.V. Vassilevich, *Dilaton Gravity in Two Dimensions*, (2008).  
arXiv:hep-th/0204253v9
- [3] M. Trodden, *A Nonsingular Two Dimensional Black Hole*, (1993).  
arXiv:hep-th/9305111
- [4] E.ourgoulhon, *3+1 Formalism and Bases of Numerical Relativity*, (2007).  
arXiv:gr-qc/0703035v1
- [5] R. Arnowitt, S. Deser, C.W Misner, *Dynamical Structure and Definition of Energy in General Relativity*, (1959).
- [6] T.A. Moore, *A General Relativity Workbook* Chapter 4, (2013).
- [7] T.A. Moore, *A General Relativity Workbook* Chapter 2, (2013).
- [8] F. Wild, *What is a black hole?*, (2014).  
<https://www.nasa.gov/audience/forstudents/5-8/features/nasa-knows/what-is-a-black-hole-58.html>
- [9] C.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation* pg. 876, (2017).
- [10] C.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation* pg. 433 (2017).
- [11] E. Poisson, *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics*, pg. 121-124, (2004).
- [12] D. Louis-Martinez, J. Gegenberg, G. Kunstatter, *Exact Dirac Quantization of All 2-D Dilaton Gravity Theories*, (1993).  
arXiv:gr-qc/9309018
- [13] T.A. Moore, *A General Relativity Workbook* Chapter 14, (2013).
- [14] S.M. Carroll, *Lecture Notes on General Relativity* pg. 167, (1997).  
arXiv:gr-qc/9712019
- [15] A. Strominger, *Les Houches Lectures on Black Holes*, (1995).  
arXiv:hep-th/9501071
- [16] G. Mammadov, *Reissner-Nordstrom Metric*, (2009).
- [17] S.A. Teukolsky, *The Kerr Metric*, (2015).  
arXiv:1410.2130 [gr-qc]
- [18] C.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation* pg. 878 (2017).
- [19] A. Zalognik, *Seminar 4: Kaluza-Klein Theory* pg. 9, (2012).
- [20] A. Castro, F. Larsen, *Near Extremal Kerr Entropy from AdS2 Quantum Gravity*, (2009).  
arXiv:0908.1121v2 [hep-th]