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Finite-dimensional Lorentz covariant bifurcations

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In this paper finite-dimensional Lorentz covariant bifurcation equations are constructed and their properties, solutions, and gradient structures are examined. The possible applications of these ideas and techniques to elementary particle physics are considered.

I. INTRODUCTION

In the last few years extensive research was done on bifurcations covariant with respect to the rotation group in three dimensions and their applications in various physical contexts.[1-5] In view of these efforts the study of Lorentz covariant bifurcations seems to be both natural and interesting from physical and mathematical considerations.

First, from a mathematical point of view, the Lorentz group is a simple noncompact group [as compared to O(3) which is compact] and hence all its nontrivial finite-dimensional representations are nonunitary, thus possibly introducing a new element in the analysis of the bifurcation equations. More important from a physical point of view is the fact that the Lorentz group is the invariance group of all local relativistic physical phenomena and hence covariant bifurcations with respect to this group should govern all bifurcations of relativistic processes. In particular we wish to point out that the production of new (elementary) particles through a collision of other particles at relativistic velocities can be viewed as a bifurcation process. Thus in this instance the original (stable) state of the system (consisting of the particles before the collision) becomes, above certain energy threshold, unstable due to the collision and the system bifurcates to new states or particles. It follows then that the detailed study of these Lorentz covariant bifurcations, which are independent of the explicit form of the interaction, might lead to better understanding of these processes, which goes beyond those consisting of spin and energy alone.

The plan of the paper is as follows: in Sec. II we summarize briefly the general setting for covariant bifurcations as discussed by Sattinger[1,2] and comment on the possible difficulties in its application to noncompact groups. In Sec. III the construction of Lorentz covariant bifurcation equations is carried out and in Sec. IV we present an explicit example of these equations and their solutions. In Sec. V we prove that Lorentz covariant bifurcations of the second order have a gradient structure even though the corresponding representations are nonunitary. Some possible implications of these techniques to the physics of elementary particles are considered in Sec. VI. Finally in the Appendix we show the need for a minor modification in the formula for the Clebsch–Gordan coefficients of the Lorentz group.

II. A SHORT REVIEW OF BIFURCATION THEORY WITH SYMMETRY[1,2]

We are considering the bifurcations of a nonlinear functional equation $G(u,\lambda) = 0$. Under proper conditions on $G(\lambda,u)$ we can reduce this equation at a bifurcation point (λ_0,u_0) via the Lyapunov–Schmidt method to a finite-dimensional problem:

$$F_i(\lambda,\mathbf{v}) = 0, \quad i = 1,\ldots,n, \tag{1}$$

$\mathbf{v}\in R^n$ and $n = \dim\ker G_u(\lambda_0,u_0)$. Expanding $\mathbf{F}(\lambda,\mathbf{v})$ in a power series in $\mathbf{v}$ we obtain,

$$\mathbf{F}(\lambda,\mathbf{v}) = A(\lambda)\mathbf{v} + \mathbf{B}_2(\lambda,\mathbf{v},\mathbf{v}) + \mathbf{B}_3(\lambda,\mathbf{v},\mathbf{v},\mathbf{v}) + \cdots. \tag{2}$$

On can infer then that if the original problem is covariant with respect to a representation $\boldsymbol{\Gamma}$ of a group G then the same holds for each term in the expansion (2). Furthermore, since $\mathbf{B}_2(\lambda,\mathbf{v},\mathbf{w})$ must be symmetric in $\mathbf{v},\mathbf{w}$ it follows that $\mathbf{B}_2$ must belong to the subspace of symmetric second-order tensors which transform as Γ under the action of G.

For the rest of this work we approximate $\mathbf{F}(\lambda,\mathbf{v})$ by the first two terms in (2) (obviously, if $\mathbf{B}_2\equiv 0$ one must consider $\mathbf{B}_3$, etc.) and denote $\mathbf{B}_2$ by B.

The construction and anlysis of second-order G-covariant bifurcations proceeds as follows: first, we identify those representations for which Γ appears as a symmetric tensor in the decomposition of $\Gamma\times\Gamma$. Then to construct B explicitly we can either use the Clebsh–Gordan coefficients of G or apply the Lie generators of G directly on some "ground state" of B. Furthermore, if Γ is irreducible it follows then from Schur's Lemma that $A(\lambda) = \lambda I$. Once the solutions of

$$B(\mathbf{v},\mathbf{v}) + A(\lambda)\mathbf{v} = 0 \tag{3}$$

have been found (usually there are several solutions) one can infer the stability of each bifurcating state by introducing the parametrization

$$\lambda = -\epsilon, \quad \mathbf{v} = -\epsilon\zeta \tag{4}$$

and calculating the eigenvalues of $J - \lambda I$, where J is the Jacobian of B at the solution. For $\epsilon > 0$, negative and positive eigenvalues correspond then to stable and unstable subcritical branching states, respectively.

Although the results of bifurcation theory reviewed above are rather general, proper care should be exercised in their application to the Lorentz group since it is a noncompact group. Due to this fact there exist some open mathematical questions as to whether the Fredholm alternative and the Lyapunov–Schmidt procedure hold under these conditions. While these problems should be addressed formally, we would like to observe that from a physical point of view the Lorentz group is a local symmetry group. Accordingly, in our analysis of the bifurcation equations we have to consider only a proper neighborhood of the identity in this group. It is our contention then that under these restrictions

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the bifurcation theory as developed in Refs. 1 and 2 holds for Lorentz covariant bifurcations.

Another possible source of trouble in applying the general theory to the finite-dimensional representations of the Lorentz group is that these are nonunitary.

However, a close examination of the theory developed in Ref. 1 (and Theorem 4.1 in particular) shows that the unitarity assumption is not needed and we can apply these results in our context.

III. CONSTRUCTION OF LORENTZ COVARIANT BIFURCATION EQUATIONS

We first observe that there are two ways to characterize the spinor representations of the (proper) Lorentz group (which we denote henceforth by G.) These are $(k,n)_s$ and (j_0,j_1). The first of these notations relates to the spinor contents of the representation while the second relates to its decomposition with respect to O(3).[6]

Lemma 1: Let $\Gamma = (k,n)_s$ be an irreducible representation of G and let $F(\lambda,\mathbf{v})$ be covariant with respect to Γ, then $B\equiv 0$ unless k,n are even.

Proof: For B to be different from zero it is necessary (but not sufficient) for Γ to appear in the decomposition of $\Gamma\times\Gamma$. However,

$$\Gamma\times\Gamma = \sum \oplus (k',n')_s, \tag{5}$$

where

$$k' = 0,2,\ldots,2k, \quad n' = 0,2,\ldots,2n.$$

Hence, we infer that $(k,n)_s$ appears in this decomposition only if k,n are even.

Proposition 1: If $\Gamma = (k,n)_s$, then B does not vanish if and only if k,n and $(k+n)/2$ are even integers.

Proof: In view of Lemma 1 Γ appears (once) in the decomposition of $\Gamma\times\Gamma$ under the conditions of this proposition. We must prove, however, that it appears as a symmetric tensor. To show this we denote the states of Γ by $x(k,n,j,m)$. The states of a representation $(k',n')_s$, which appear in the decomposition of $\Gamma\times\Gamma$, are then given by

$$x(k',n',j,m) = \sum H^{k',n',j,m}_{k,n,j_1,m_1;k,n,j_2,m_2} x(k,n,j_1,m_1)x(k,n,j_2,m_2), \tag{6}$$

where the H's are the Clebsch–Gordan (CG) coefficients of G. Hence Γ appears as a symmetric tensor in the decomposition if and only if

$$H^{k,n,j,m}_{k,n,j_1,m_1;k,n,j_2,m_2} = H^{k,n,j,m}_{k,n,j_2,m_2;k,n,j_1,m_1}. \tag{7}$$

However, it is well known that the CG coefficients of G are given by[6,7] (see, however, the discussion in the Appendix)

$$H^{k,n,j,m}_{k_1,n_1,j_1,m_1;k_2,n_2,j_2,m_2} = (-1)^{(k+n)/2}[(k+1)(n+1)]^{1/2}\alpha(j_1,j_2,j) \times\begin{pmatrix} j_1,j_2,j \\ m_1,m_2,m\end{pmatrix}\begin{Bmatrix} k_1/2 & n_1/2 & j_1 \\ k_2/2 & n_2/2 & j_2 \\ k/2 & n/2 & j\end{Bmatrix}, \tag{8}$$

where α is symmetric in j_1,j_2. Hence, using the symmetry properties of the $3j$ and the $9j$ symbols[8] we infer that (7) is true if and only if $(-1)^{(k+n)/2} = 1$, which proves our statement.

We now note that the explicit form of B is known if the CG coefficients of G are known. In fact for $\Gamma = (k,n)_s$ we have

$$B(j,m) = \sum H^{j,m}_{j_1,m_1;j_2,m_2} x(j_1,m_1)x(j_2,m_2) \tag{9}$$

(for brevity we dropped the designation of the representation and shall do so henceforth whenever its meaning is clear). However, since the actual calculation of the H's is tedious we describe a direct method to do so for irreducible spinor representations of the form $(n,n)_s \equiv (0,n+1)$, i.e., the ladder representations whose decomposition with respect to O(3) contains the irreducible representations $j = 0,1,\ldots,n$. (The method to be described can be applied to other spinor representations with minor modifications.)

Proposition 2: The action of the operator

$$F^2 = -(B_1^2 + B_2^2 + B_3^2) = F_- F_+ + F_3^2 - J_3 \tag{10}$$

on $x(k,n,j,m)$ (here k,n are arbitrary) is given by

$$F^2 x(k,n,j,m) = (j_1^2 - j_0^2 + j^2 + j + 1)x(k,n,j,m). \tag{11}$$

Proof: The proof of this proposition proceeds through direct (and long) computation using the results in Ref. 6 regarding the matrix elements of the operators F_+, F_-, and F_3.

At this point we would like to note that the operator F^2 seems to have an intrinsic importance from a group theoretical point of view. Thus, as is obvious from (11), any state $x(k,n,j,m)$ of $(k,n)_s$ is an eigenstate of F^2. Moreover, the corresponding eigenvalues are independent of m. However, we found no reference to this operator in the classical literature on G.

We start the construction of the quadratic form $B(j,m)$ with $B(0,0)$. To this end we observe that the representation $j=0$ appears only in the decomposition of $j\times j$. Hence we attempt to write

$$B(0,0) = \sum_{j=0}^{n} \sum_{m=-j}^{j} a(j,m)x(j,m)x(j,-m). \tag{12}$$

To determine the coefficients $a(j,m)$ we use the fact that

$$J_+ B(0,0) = 0 \tag{13}$$

[or equivalently $J_- B(0,0) = 0$]. This yields after some simple algebra

$$B(0,0) = \sum_{j=0}^{n} b(j) \sum_{m=-j}^{j} (-1)^m x(j,m)x(j,-m), \tag{14}$$

where $b(j)$ are constants which depend on j only. To determine these coefficients we now apply F^2 to $B(0,0)$ using (11) with $j_0 = 0, j_1 = n+1$,

$$F^2 B(0,0) = -(n^2 + 2n)B(0,0). \tag{15}$$

Evaluating the left-hand side of this equation by direct application of $F_3^2 + F_- F_+ - J_3$ to (14) yields a system of linear equations for $b(j)$ (note that F^2 is not a derivation) which when solved determines $B(0,0)$. The other components of B can be obtained then by repeated aplications of F_+ and J_- or F_- and J_+.

In order to solve the second-order bifurcation equations we shall use the following results, which are completely analogous to those for O(3).[1]

Lemma 2: The states $x(j,m)$ in an irreducible spinor representation of G can be chosen so that

$$\bar{x}(j,m) = (-1)^m x(j,-m). \tag{16}$$

Proof: An irreducible spinor representation of G can be decomposed into irreducible representations of O(3), each of which appears once. It was shown by Sattinger[1] that due to the uniqueness of $x(j,m)$ in an irreducible representation of O(3) it is possible to satisfy condition (14). This proves the lemma.

Proposition 3: Let the reduced second-order bifurcation equations for $(n,n)_s$

$$B(j,m) + \lambda x(j,m) = 0 \tag{17}$$

be restricted to the subclass of solutions with the symmetry

$$x(j,m) = (-1)^m x(j,-m), \tag{18}$$

then

$$B(j,-m) = (-1)^m B(j,m), \tag{19}$$

i.e., the bifurcation equations for $m<0$ are redundant.

Proof: From the previous lemma it follows that $B(j,m)$ can be chosen so that

$$\bar{B}(j,m) = (-1)^m B(j,-m). \tag{20}$$

However, by construction $B(j,m)$ for $(n,n)_s$ is a quadratic form with real coefficients, hence

$$\bar{B}(j,m)(x(j,m)) = B(j,m)(\bar{x}(j,m)). \tag{21}$$

But from Lemma 2 and condition (18) we obtain

$$\bar{x}(j,m) = (-1)^m x(j,-m) = x(j,m). \tag{22}$$

Thus,

$$\begin{aligned} B(j,-m)(x(j,m)) &= (-1)^m \bar{B}(j,m)(x(j,m)) \\ &= (-1)^m B(j,m)(jx(j,m)) \\ &= (-1)^m B(j,m)(x(j,m)), \end{aligned} \tag{23}$$

which is the desired result.

Finally we note that when Γ is reducible the additional considerations that are necessary to construct and solve the bifurcation equations are completely analogous to the O(3) case[1] and will not be discussed further here.

Remark: The representations $(j,j+1)$ of G are equivalent to irreducible representations of O(3). It follows then that the construction of covariant bifurcation equations which are related to these representations proceeds exactly as in the O(3) case.

IV. AN EXAMPLE

In this section we construct and solve explicitly the bifurcation equations for $(2,2)_s$. To begin with we apply F^2 to $B(0,0)$ and use (9) and (11). We obtain the following equations for $b(j)$:

$$b(0) = 2b(1),\ b(1) + b(2) = 0. \tag{24}$$

Hence, up to a multiplicative constant, $B(0,0)$ in this case is given by

$$\begin{aligned} B(0,0) = {}& 2x^2(0,0) + x^2(1,0) \\ & - 2x(1,1)x(1,-1) - 2x(2,-2)x(2,2) \\ & + 2x(2,1)x(2,-1) - x^2(2,0). \end{aligned}$$

The other components of B can now be evaluated by repeated application of F_+ and J_-,

$$\begin{aligned} B(1,1) = {}& -2\sqrt{3}x(1,-1)x(2,2) + \sqrt{6}x(1,0)x(2,1) \\ & -\sqrt{2}x(1,1)x(2,0) + 2x(0,0)x(1,1), \\ B(1,0) = {}& -\sqrt{6}x(1,-1)x(2,1) + 2\sqrt{2}x(1,0)x(2,0) \\ & -\sqrt{6}x(1,1)x(2,-1) + 2x(0,0)x(1,0), \\ B(1,-1) = {}& -2\sqrt{3}x(1,1)x(2,-2) + \sqrt{6}x(1,0)x(2,-1) \\ & -\sqrt{2}x(1,-1)x(2,0) + 2x(0,0)x(1-1), \\ B(2,2) = {}& -2x(2,2)[\sqrt{2}x(2,0) + x(0,0)] + \sqrt{3}x^2(2,1) \\ & +\sqrt{3}x^2(1,1), \\ B(2,1) = {}& -2\sqrt{3}x(2,-1)x(2,2) + \sqrt{2}x(2,0)x(2,1) \\ & -2x(0,0)x(2,1) + \sqrt{6}x(1,0)x(1,1), \\ B(2,0) = {}& -2\sqrt{2}x(2,-2)x(2,2) - \sqrt{2}x(2,-1)x(2,1) \\ & +\sqrt{2}x^2(2,0) - 2x(0,0)x(2,0) \\ & +\sqrt{2}x(1,-1)x(1,1) + \sqrt{2}x^2(1,0), \\ B(2,-1) = {}& -2\sqrt{3}x(2,-2)x(2,1) + \sqrt{2}x(2,-1)x(2,0) \\ & -2x(0,0)x(2,-1) + \sqrt{6}x(1,0)x(1,-1), \\ B(2,-2) = {}& -2x(2,-2)[\sqrt{2}x(2,0) + x(0,0)] \\ & +\sqrt{3}x^2(2,-1) + \sqrt{3}x^2(1,-1). \end{aligned}$$

Solving the second-order bifurcation equations $B(\mathbf{v},\mathbf{v}) + \lambda\mathbf{v} = 0$ under the restrictions of Proposition 3 we found the following four solutions for $(2,2)_s$ [note that for an irreducible representation $A(\lambda) = \lambda I$ in (21)]. The nonzero components of these solutions are

$$\begin{aligned} &(1)\ x(0,0) = -\lambda/2; \quad (2)\ x(0,0) = \lambda/6, \\ &x(2,0) = -(\sqrt{2}/3)\lambda; \quad (3,4)\ x(2,2) = \pm(\sqrt{3}/6)\lambda, \\ &x(2,0) = (\sqrt{2}/6)\lambda, \quad x(0,0) = \lambda/6. \end{aligned}$$

To determine the stability of these bifurcating solutions we let $\lambda = 1$ and evaluate the eigenvalues of J–I, where J is the Jacobian of $B(\mathbf{v},\mathbf{v})$ at the solution. It is interesting to note that for all the four solutions given above these eigenvalues are $(-3, -2, 0)$ (the corresponding multiplicities are 3, 1, 5). Hence if we introduce the scaling $\lambda = -\epsilon$, $x(j,m) = \epsilon z(j,m)$ then for $\epsilon > 0$ we obtain subcritical branching with one neutral and two stable modes. (From a physical point of view the most natural interpretation of λ is as the total energy of the system.)

V. GRADIENT STRUCTURE OF THE BIFURCATION EQUATIONS

In this section we show that for $\gamma = (n,n)_s$ there exists a Lorentz invariant polynomial $p(x(j,m))$ so that

$$B(j,m) = \frac{\partial p}{\partial x(j,m)} \tag{25}$$

if the $B(j,m)$ satisfy the condition (19). To prove this assertion we note that p has to be a third-order polynomial in

$x(j,m)$. Taking into account the invariance of p with respect to O(3) it is easy to infer that p must be of the form

$$p=\sum_{j=0}^{n} d(j) \sum_{m=-j}^{i} (-1)^m B(j,m) x(j,-m). \tag{26}$$

To determine the coefficients $d(j)$ we use the invariance of p, which implies that $F_3(p)=0$. This immediately yields $d(j)=d(j-1)$, $j=1,...,n$, i.e., up to a multiplicative constant,

$$p=\sum_{j=0}^{n} \sum_{m=-j}^{j} (-1)^m B(j,m) x(j,-m). \tag{27}$$

Using (9) and a little algebra then yields

$$p=\sum_{j_1,j_2,j_3} \sum_{m_1,m_2,m_3} (-1)^m H^{j_3,-m_3}_{j_1m_1;j_2,m_2} \times x(j_1,m_1)x(j_2,m_2)x(j_3,m_3). \tag{28}$$

A careful analysis of this expression using the symmetry properties of the H's (Ref. 8) and (A7) shows that the coefficient of

$$x(j_1,m_1)x(j_2,m_2)x(j_3,m_3)$$

(for fixed j_i,m_i, $i=1,2,3$) is given by

$$6(-1)^{m_3} H^{j_3,-m_3}_{j_1m_1;j_2m_2}.$$

Hence when we differentiate p with respect to $x(j_3,m_3)$ we obtain

$$\frac{1}{3}\frac{\partial p}{\partial x(j_3,m_3)}=(-1)^{m_3}\sum_{m_1m_2} H^{j_3,-m_3}_{j_1,m_1;j_2,m_2} x(j_1,m_1)x(j_2,m_2)$$
$$=(-1)^m B(j_3,-m_3)=B(j_3,m_3)$$

[where we took into account the fact that $x(j_1,m_1)x(j_2,m_2)$ appears twice in this summation].

We verified this result explicitly for $(2,2)_s$.

VI. POSSIBLE PHYSICAL APPLICATIONS

To summarize the results of this paper from a physical point of view we observe that without any reference to the explicit form of the interaction involved in the bifurcating process we were able to deduce, using Lorentz covariance alone, the number of bifurcating modes and their stability. It follows then that if the proposed application of bifurcation theory to colliding beams of particles is correct one should be able to predict *a priori* certain experimental facts which were derived so far on a phenomenological basis only.

The major obstacle for such a direct application is that particles at relativistic speeds are "dressed particles." Thus it is not *a priori* clear that the same representation which is related to a given particle at low energies is appropriate for its description at high energies. In particular, one should not rule out the use of the ladder representations of G for the description of bifurcating processes involving elementary particles.[9]

A possible objection to this point of view is that in treating elementary particles one should consider the quantized fields rather than the classical equations of motion. However, regardless of the paradigm one adopts for the study of these bifurcating processes G covariance must hold and our calculations should be applicable to it. (The same applies to any physical process which is G covariant.)

Another open question that is related to this bifurcation theory is the determination of the isotropy group for the bifurcating modes [this is open even for O(3)]. The identification of such an isotropy group should in principle lead to additional quantum numbers which characterize the bifurcating states.

APPENDIX: ON THE CG COEFFICIENTS OF *G*

The CG coefficients for the spinor representations of the Lorentz group appear in various references (see Refs. 6 and 7). The one due to Gel'fand *et al.*[6] is equivalent to

$$H^{k,n,j,m}_{k_1,n_1,j_1,m_1;k_2,n_2,j_2,m_2}=\sum_{m_1',m_2'}\left(\frac{k}{2},m_1'+m_2';\frac{n}{2},m-m_1'-m_2'\middle|j,m\right)\left(\frac{k_1}{2},m_1';\frac{k^2}{2},m_2'\middle|\frac{k}{2},m_1'+m_2'\right)$$
$$\times\left(\frac{n_1}{2},m_1-m_1';\frac{n_2}{2},m_2-m_2'\middle|\frac{n}{2},m-m_1'-m_2'\right)\left(\frac{k_1}{2},m_1';\frac{n_1}{2},m_1-m_1'\middle|j_1,m_1\right)\left(\frac{k_2}{2},m_2';\frac{n_2}{2},m_2-m_2'\middle|j_2,m_2\right), \tag{A1}$$

while the other[7] is given by

$$H^{k,n,j,m}_{k_1,n_1,j_1,m_1;k_2,n_2,j_2,m_2}=(-1)^{(k+n)/2}[(k+1)(n+1)(2j_1+1)(2j_2+1)]^{1/2}(j_1,m_1;j_2,m_2|j,m)\begin{Bmatrix} k_1/2 & n_1/2 & j_1\\ k_2/2 & n_2/2 & j_2\\ k/2 & n/2 & j\end{Bmatrix}. \tag{A2}$$

However, it is easy to show that these two formulas agree up to an unimportant factor of $(-1)^c$, where c depends only on k_i,n_i, $i=1,2,3$. We contend, nevertheless, that in both of these equivalent formulas a factor of

$$\alpha(j_1,j_2,j)=(-1)^{(j_1+j_2-j_3)/2} \tag{A3}$$

should be inserted on the right-hand side.

To begin with we observe that (qualitatively) both (A1) and (A2) are incompatible with the matrix elements of the Lie algebra of G as they appear in Refs. 6 and 7. In fact, according to (A1) and (A2) the CG coefficients of G are all real. Hence the states of Γ' in the decomposition of $\Gamma\times\Gamma$ should all appear with real coefficients. However, when one applies F_+ or F_3 on such a state of Γ' one obtains, in general (i.e., when $j_0 \neq 0$), an expression with both real and complex coefficients. Such a state cannot be a multiple of the one constructed using (A1) or (A2).

More concretely we computed the Clebsch–Gordan coefficients for the ground state $B(0,0)$ of $(2,2)_s$ in the decomposition of $(2,2)_s \times (2,2)_s$ using both (A1) and (A2) and found in both cases that

$$B(0,0) = (1/2\sqrt{3})\{2x^2(0,0) + 2x(1,1)x(1,-1) - x^2(1,0) - 2x(2,2)x(2,-2) + 2x(1,1)x(1,-1) - x^2(2,0)\}. \quad \text{(A4)}$$

However, (A3) is incorrect. In fact from Refs. 6 and 7 we infer that

$$F_+ B(0,0) = (4/\sqrt{3})B(1,1). \quad \text{(A5)}$$

But if we apply F_+ to (A3) we obtain

$$B(1,1) = 6x(0,0)x(1,1). \quad \text{(A6)}$$

This expression for $B(1,1)$ is wrong since from (A1) we obtain, e.g.,

$$H^{1,1}_{2,2;1,-1} = \tfrac{1}{2},$$

i.e., a term with $x(2,2)x(1,-1)$, should appear in (A6).

Similarly if we construct the highest weight $B(2,2)$ of $(2,2)_s$ using (A1) we obtain

$$B(2,2) = (1/2\sqrt{3})\{-2x(0,0)x(2,2) + 2\sqrt{2}(2,0)x(2,2) + \sqrt{3}x^2(1,1) - \sqrt{3}x^2(2,1)\}.$$

However, the application of F_+ on this state yields

$$F_+ B(2,2) = \tfrac{4}{3}x(1,1)x(2,2),$$

rather than zero as it should.

By a little algebra one finds that the required adjustment in (A1) [or (A2)] for $(2,2)_s$ is given by (A3). We conjecture, however, that this is true for all spinor representations of G since (A3) is independent of (k,n). We in fact verified this statement directly for $(4,4)_s$. The general proof of this conjecture will require a separate publication (which is outside our main thrust in this paper). The general formula for the CG coefficients of G is given therefore by

$$H^{k,n,j,m}_{k_1,n_1,j_1,m_1;k_2,n_2,j_2,m_2} = (-1)^{(k+n+j_1+j_2-j_2)/2}[(k+1)(n+1)(2j_1+1) \times (2j_2+1)]^{1/2}(j_1m_1;j_2,m_2|jm) \times \begin{Bmatrix} k_1/2 & n_1/2 & j_1 \\ k_2/2 & n_2/2 & j_2 \\ k/2 & n/2 & j \end{Bmatrix}. \quad \text{(A7)}$$

We would like to note, however, that the main results of this paper are independent of the proposed adjustment in (A1).

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