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Interior solutions to Einstein-Maxwell equations in spherical and plane symmetry when $p = n\rho$

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We obtain various explicit solutions to the interior Einstein-Maxwell equations in spherical and plane symmetry when the pressure is proportional to the mass density.

I. INTRODUCTION

Several closed-form solutions to the equations of general relativity were found in the past by several authors¹⁻³ and they played key roles in our understanding of the physical implications of this theory. More recently, however, new interior solutions to these equations in spherical symmetry were found by introducing various assumptions on the equations of state or *ad hoc* functional relations between the metric's coefficients.⁴⁻⁶ It is our purpose in this paper to generalize these results and derive new explicit solutions to Einstein-Maxwell equations in spherical and plane symmetries.

In Sec. II we derive the appropriate equations in spherical symmetry and use some functional relationships between the metric coefficients to solve them explicitly when $p = n\rho$, $n \in [0, 1]$. A second alternate technique which reduces these equations to a single linear ordinary differential equation is described in the Appendix.

Similarly in Sec. III the same problem is considered in plane symmetry. Implicit solutions for this symmetry were found in the past⁷ for zero pressure; however, we now show that new explicit solutions to these equations can also be derived when $p = n\rho$.

Finally in Sec. IV we discuss the imposition of the appropriate boundary conditions on the solutions derived in Secs. II and III and examine their admissibility from a physical point of view.

II. EINSTEIN-MAXWELL EQUATIONS FOR STATIC SPHERICALLY SYMMETRIC MASS

The general form of the Einstein-Maxwell equations for a perfect fluid is

$$G_{\mu\nu} = C(T_{\mu\nu} + E_{\mu\nu}), \tag{2.1}$$

$$T_{\mu\nu} = \rho u_\mu u_\nu - p(g_{\mu\nu} - u_\mu u_\nu), \tag{2.2}$$

$$E_{\mu\nu} = F_\mu^\alpha F_{\alpha\nu} + \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}, \tag{2.3}$$

$$(\sqrt{-g} F^{\mu\nu})_{|\nu} = \sqrt{-g} S^\mu, \tag{2.4}$$

$$\{F_{\mu\nu|\lambda}\} = 0, \tag{2.5}$$

where all symbols have their usual meaning¹ and $C = -8\pi\kappa/c^2$.

For static spherical symmetry with uniform charge density

$$S^\mu = (\sigma, 0, 0, 0),$$

$$e^{-\nu/2}u_\mu = (1, 0, 0, 0),$$

and the line element is given by

$$ds^2 = e^\nu(dt)^2 - e^\lambda(dr)^2 - r^2(d\theta)^2 - r^2\sin^2\theta(d\phi)^2. \tag{2.6}$$

By normalizing C to -1 Eqs. (2.1)–(2.5) then lead to the following system of equations (where $E = F_{01}$):

$$\rho + \frac{E^2 e^{-(\nu+\lambda)}}{2} = \frac{1}{r^2} - e^{-\lambda} \left[\frac{1}{r^2} - \frac{\lambda'}{r} \right], \tag{2.7}$$

$$p - \frac{E^2 e^{-(\nu+\lambda)}}{2} = e^{-\lambda} \left[\frac{1}{r^2} + \frac{\nu'}{r} \right] - \frac{1}{r^2}, \tag{2.8}$$

$$\frac{e^\lambda}{r^2} = \frac{1}{r^2} - \frac{\nu''}{2} + \frac{\nu'\lambda'}{4} - \frac{\nu'^2}{4} + \frac{\nu'+\lambda'}{2r} + E^2 e^{-\nu}, \tag{2.9}$$

$$(e^{-(\nu+\lambda)/2} r^2 E)' = \sigma r^2 e^{(\nu+\lambda)/2}. \tag{2.10}$$

Assuming $p = n\rho$ we now use Eqs. (2.7) and (2.8) to eliminate the term containing E from (2.9). The result can be written in the form²

$$\begin{aligned} \frac{d}{dr} \left[\frac{e^{-\lambda}-1}{r^2} \right] + \frac{d}{dr} \left[\frac{e^{-\lambda}\nu'}{2r} \right] + e^{-(\nu+\lambda)} \frac{d}{dr} \left[\frac{e^\nu\nu'}{2r} \right] \\ = \frac{4}{r^3} + e^{-\lambda} \left[2\frac{n-1}{n+1} \frac{\nu'+\lambda'}{r^2} + \frac{2\lambda'}{r^2} - \frac{2\nu'}{r^2} - \frac{4}{r^3} \right]. \end{aligned} \tag{2.11}$$

If we introduce now

$$u = \frac{\nu'}{2r}, \quad v = e^{-\lambda}, \tag{2.12}$$

Eq. (2.11) will take the form

$$\frac{dv}{dr} + f(r)v = \frac{2}{r(3+2N+ur^2)}, \tag{2.13}$$

where $N = (n-1)/(n+1)$ and

$$f(r) = \frac{2r^2(du/dr + u^2r + 1/r^3) + 4(1-N)ur}{3+2N+ur^2}. \tag{2.14}$$

Equation (2.13) can be considered as a linear first-order equation for v [if $f(r)$ is known] and its solution can be written explicitly as

$$v \exp \left[\int f(r)dr \right] = \int \frac{2dr}{r(3+2N+ur^2)} \exp \left[\int f(r)dr \right]. \tag{2.15}$$

To obtain explicit solutions for our problem we now make the assumption⁵ that

$$\exp \left[\int f(r) dr \right] = r^a(3+2N+ur^2), \quad (2.16)$$

which, after some algebra, gives the following differential equation for u :

$$u' + 2ru^2 + u \frac{2-4N-a}{r} = \frac{a(3+2N)-2}{r^3}. \quad (2.17)$$

The general solution of (2.17) is

$$u = \frac{1}{r^2} \left[K_i - \frac{\alpha}{Cr^\alpha + 2} \right], \quad (2.18)$$

where C is an arbitrary constant and

$$K_i = \frac{(4N+a) \pm \{(4N+a)^2 + 8[a(3+2N)-2]\}^{1/2}}{4}, \quad (2.19)$$

$$i = 1, 2$$

$$\alpha = 4K_i - (4N+a). \quad (2.20)$$

From Eqs. (2.12) and (2.15) we finally obtain the explicit solutions

$$e^v = Dr^{2K_1} + (D/C)r^{2K_2} \quad (2.21)$$

and

$$e^\lambda = \frac{(Cr^\alpha + 2)(3+2N+K_i) - \alpha}{(Cr^\alpha + 2)(2/a + B/r^a)}, \quad (2.22)$$

where D and B are additional arbitrary integration constants.

We shall discuss the admissibility of these solutions from a physical point of view as well as the imposition of the boundary conditions in Sec. IV.

III. INTERIOR STATIC SOLUTIONS TO EINSTEIN-MAXWELL EQUATIONS IN PLANE SYMMETRY

In plane symmetry the line element is given by

$$ds^2 = e^\nu[(dt)^2 - (dx)^2] - e^\lambda[(dy)^2 + (dz)^2], \quad (3.1)$$

where $\nu = \nu(x)$, $\lambda = \lambda(x)$ and the only nonvanishing components of $F_{\mu\nu}$ are⁸

$$F_{23} = c_1, \quad F_{01} = c_2 e^{\nu-\lambda}. \quad (3.2)$$

Einstein-Maxwell equations (2.1)–(2.5) reduce under these conditions to

$$\lambda'' + \frac{1}{2}\lambda'^2 - \frac{1}{2}\nu'\lambda' = -[\rho e^\nu + \frac{1}{2}(c_1^2 + c_2^2)e^{\nu-2\lambda}], \quad (3.3)$$

$$-\frac{1}{4}\lambda'^2 - \frac{1}{2}\nu'\lambda' = -[\rho e^\nu - \frac{1}{2}(c_1^2 + c_2^2)e^{\nu-2\lambda}], \quad (3.4)$$

$$-\frac{1}{2}(\nu'' + \lambda'' + \frac{1}{2}\lambda'^2) = -[\rho e^\nu + \frac{1}{2}(c_1^2 + c_2^2)e^{\nu-2\lambda}]. \quad (3.5)$$

To solve this system for $p = n\rho$ we combine (3.5) with (3.4) and (3.4) with (3.3) to obtain the following system for λ, ν :

$$\nu'' + \lambda'' - \nu'\lambda' = 2Ae^{\nu-2\lambda}, \quad (3.6)$$

$$\left[-n\lambda'' - \frac{2n+1}{4}\lambda'^2 \right] + \frac{n-1}{2}\nu'\lambda' = \frac{n+1}{2}Ae^{\nu-2\lambda}. \quad (3.7)$$

Eliminating $Ae^{\nu-2\lambda}$ between these two equations, we obtain

$$(5n+1)\lambda'' + (n+1)\nu'' + (2n+1)\lambda'^2 - (3n-1)\nu'\lambda' = 0. \quad (3.8)$$

To proceed we have two alternatives:

(a) Eliminate, in (3.8), the term in λ'^2 . This can be accomplished by the transformation

$$\alpha = \frac{-(2n+1)}{3n-1}\lambda' + \nu', \quad (3.9)$$

$$\beta = \lambda', \quad (3.10)$$

which leads to

$$a\beta' + b\alpha' + c\alpha\beta = 0, \quad (3.11)$$

where

$$a = (5n+1) + \frac{(n+1)(2n+1)}{3n-1},$$

$$b = n+1,$$

$$c = -(3n-1).$$

Equation (3.11) is a linear first-order equation which can be solved for α in terms of β or for β in terms of α . We now consider each of these possibilities separately.

1. Solving Eq. (3.11) for α in terms of β we obtain

$$\alpha = A_1 q_1 - \frac{a}{b} q_1 \int \beta' q_1^{-1} dx, \quad (3.12)$$

where A_1 is a constant of integration and

$$q_1 = \exp \left[-\frac{c}{b} \int \beta dx \right]. \quad (3.13)$$

We can now derive explicit solutions to our problem by introducing *ad hoc* assumptions about the value of q_1 , e.g., if we assume that $q_1 = x^{-\gamma}$ we obtain

$$e^\lambda = c_0 |x|^{(b\gamma/c)}, \quad (3.14)$$

$$e^\nu = c_1 |x|^{s_1} \exp \left[\frac{A_1}{1-\gamma} x^{1-\gamma} \right], \quad (3.15)$$

where c_0 and c_1 are constants and

$$s_1 = \frac{\gamma a}{c(\gamma-1)} + \frac{(2n+1)b\gamma}{c(3n-1)}.$$

2. Solving Eq. (3.11) for β in terms of α , we obtain

$$\beta = A_2 q_2 - \frac{b}{a} q_2 \int \alpha' q_2^{-1} dx, \quad (3.16)$$

where A_2 is a constant of integration and

$$q_2 = \exp \left[-\frac{c}{a} \int \alpha dx \right]. \quad (3.17)$$

Once again explicit solutions can be derived by introduc-

ing a proper ansatz on q_2 , e.g., if $q_2 = x^{-\gamma}$ then

$$e^\lambda = c_2 |x|^{b\gamma/c(\gamma-1)} \exp \left[\frac{A_2}{1-\gamma} x^{1-\gamma} \right], \tag{3.18}$$

$$e^\nu = c_3 |x|^{s_2} \exp \left[\frac{2n+1}{3n-1} \frac{A_2}{1-\gamma} x^{1-\gamma} \right], \tag{3.19}$$

where c_2 and c_3 are constants and

$$s_2 = \frac{a\gamma}{c} + \frac{b(2n+1)\gamma}{c(\gamma-1)(3n-1)}.$$

(b) We now eliminate in Eq. (3.8) the term in $v'\lambda'$. To this end we perform the transformation

$$\alpha = \lambda' - \nu', \tag{3.20}$$

$$\beta = -\frac{(n+3)}{3n-1} \lambda' + \nu', \tag{3.21}$$

which gives

$$a\alpha' + b\beta' + c\alpha^2 + d\beta^2 = 0, \tag{3.22}$$

where

$$e^\lambda = \left[\cosh \frac{(x+c_4)\sqrt{pc}}{|a|} \right]^{a(3n-1)/c(2n-4)} \left[\cosh \frac{(x+c_5)\sqrt{-pd}}{|b|} \right]^{-b(3n-1)/d(2n-4)} \tag{3.26}$$

and

$$e^\nu = \left[\cosh \frac{(x+c_4)\sqrt{pc}}{|a|} \right]^{a(n+3)/c(2n-4)} \left[\cosh \frac{(x+c_5)\sqrt{-pd}}{|b|} \right]^{-b(3n-1)/d(2n-4)}. \tag{3.27}$$

IV. PROPERTIES OF THE SOLUTIONS

The solutions derived in Secs. II and III must have certain properties and satisfy certain constraints in order to have a physical significance. In this section we shall prove that at least a subclass of the spherically symmetric solutions derived in Sec. II satisfy all these requirements and hence are physically admissible. (We shall discuss also, briefly, the plane-symmetric case.)

The conditions that the spherically symmetric solution given by Eqs. (2.21) and (2.22) must satisfy to be physically admissible are as follows:

1. The mass density ρ must be zero on the boundary $r=r_0$.
2. The metric coefficients must be continuous across the boundary hypersurface, i.e., must match the coefficients of the appropriate Reissner-Nordstrom metric¹ at $r=r_0$.
3. The mass density must be non-negative for $r < r_0$ and the pressure must be monotonically decreasing outward.
4. e^ν, e^λ must be positive, nonsingular, and continuous for $r < r_0$.

We start by making the following observations.

Lemma: For $a > 2/(3+2N)$ and $K = K_2$ [as given in Eq. (2.19)] the following holds:

$$K < 0, \quad a + 2K > 0, \quad \text{and} \quad 3 + 2N + K > 0.$$

$$a = \frac{(5n+1)(3n-1) + (n+1)(n+3)}{2n-4},$$

$$b = \frac{(5n+1)(3n-1) + (n+1)(3n-1)}{2n-4},$$

$$c = \frac{(n-2)(3n-1)^2}{(2n-4)^2} = -d.$$

To solve (3.22) we now split this equation by introducing the ansatz

$$a\alpha' + c\alpha^2 = p \quad \text{and} \quad b\beta' + d\beta^2 = -p, \tag{3.23}$$

where p is some negative constant. Solutions for α and β are then given by

$$\alpha = \frac{|a| |p|}{a\sqrt{pc}} \tanh \frac{(x+c_4)\sqrt{pc}}{|a|}, \tag{3.24}$$

$$\beta = \frac{|b| |p|}{b\sqrt{-pd}} \tanh \frac{(x+c_5)\sqrt{-pd}}{|b|}, \tag{3.25}$$

where c_4 and c_5 are constants of integration. Hence we infer from (3.20) and (3.21) that the metric coefficients are given by

Proof: All these inequalities follow directly from the definition of K and the fact that $n \in (0,1)$ implies $N \in (-1,0)$.

Theorem 1: Let $a > 2/(3+2N)$, $n \in (0,1)$, and $K = K_2$. Then

$$e^\nu = C_1 r^{2K}, \tag{4.1}$$

$$e^\lambda = \frac{ar^a(3+2N+K)}{2r^a + C_2} \tag{4.2}$$

(where C_1 and C_2 are arbitrary constants) are solutions of Eqs. (2.7)–(2.10) satisfying the ansatz (2.16).

Proof: We showed in Sec. II that u has to satisfy Eq. (2.17). A particular solution of this equation is

$$u = \frac{K}{r^2}. \tag{4.3}$$

From (2.12) we obtain $\nu' = 2K/r$ which leads to (4.1). Substituting (4.3) in (2.15) we now readily obtain Eq. (4.2).

It should be noted that the solutions (4.1) and (4.2) contain only two arbitrary constants C_1 and C_2 while the more general solutions given in Sec. II contain three such constants (which allow greater freedom in adjusting for the requirements of physical admissibility). However, we now concentrate our attention only on this subclass of solutions since in the general case we have to solve a set of cumbersome algebraic equations to adjust for the boun-

dary conditions and this task can be carried out only numerically. On the other hand the admissibility of the subclass of solutions under consideration can be proved analytically.

Theorem 2: For $a > 2/(3+2N)$, $n \in (0,1)$, and $K = K_2$ there exist certain star radii [whose values are given by Eq. (4.9)] for which the solutions (4.1) and (4.2) is physically admissible.

Proof: From Eqs. (2.7) and (2.8) and the assumption $p = n\rho$ we obtain

$$\rho = \frac{1}{(n+1)}(p + \rho) = \frac{e^{-\lambda}}{(n+1)r}(\nu' + \lambda'). \tag{4.4}$$

Hence

$$\rho = \frac{e^{-\lambda}}{(n+1)r} \left[2K + \frac{aC_2}{2r^a + C_2} \right]. \tag{4.5}$$

Since $\rho = 0$ on the boundary we infer that

$$C_2 = \frac{-4r_0^a K}{a + 2K}. \tag{4.6}$$

(Note that $C_2 > 0$ since $K < 0$ and $a + 2K > 0$.)

Similarly the requirement that e^ν is continuous across the boundary is satisfied by adjusting C_1 to

$$C_1 = \frac{1}{r_0^{2K}} \left[1 - \frac{2m}{r_0} + \frac{\epsilon^2}{2c^2 r_0^2} \right] \tag{4.7}$$

(once again note that $C_1 > 0$). However, since e^λ must also be continuous across the boundary we have to satisfy

$$C_1 r_0^{2K} = \frac{2r_0^a + C_2}{a r_0^a (3 + 2N + K)}, \tag{4.8}$$

which can be simplified using (4.6) and (4.7) to

$$1 - \frac{2m}{r_0} + \frac{\epsilon^2}{2c^2 r_0^2} = \frac{2}{(a + 2K)(3 + 2N + K)} \tag{4.9}$$

and this equation has real positive roots whenever $m > (\epsilon/2c)^{1/2}$. Thus e^λ can be adjusted to be continuous across the boundary only if r_0 is chosen to satisfy Eq. (4.9) [alternatively Eq. (4.9) can be used to determine a].

So far we showed that the integration constants and r_0 can be chosen so that the first two requirements on the solution to be physically acceptable are satisfied. The third and fourth requirements however can be easily seen to be satisfied also since $C_1, C_2 > 0$ and by virtue of Eq. (4.5) and the lemma. This proves the theorem.

To complete our discussion we observe that in the solution given above both e^ν and ρ have a singularity at $r = 0$. However, this singularity disappears when one evaluates related physical quantities such as the four-dimensional volume element or the mass in a three-dimensional volume element. Thus we infer that this singularity is not important from a physical point of view.

Finally we note that the plane-symmetric solutions derived in Sec. III must satisfy similar requirements as those enumerated for the spherical ones to be physically admissible. The only difference will be that e^ν, e^λ must now be adjusted on the boundary to match the Patnaik exterior

solution.⁸ Once again however the adjustment of the integration constants for the general solution can be carried out only numerically.

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APPENDIX

In this appendix we outline a second approach to the solution of Einstein-Maxwell equations in spherical symmetry in general (i.e., without assuming $p = n\rho$). We shall show that at least in one instance the method yields explicit solutions (up to quadratures). However, a detailed investigation of the properties of these solutions (and others related to them) must be carried out by approximations and therefore will be deferred to a later publication.

To begin with, we integrate Eq. (2.10) to obtain

$$E = \frac{\sigma}{r^2} e^{(\nu+\lambda)/2} \int r^2 e^{(\nu+\lambda)/2} dr. \tag{A1}$$

However, if we substitute this expression for E in the other equations we obtain as a result an integrodifferential system. To overcome this difficulty we shall now follow the line of reasoning in Sec. II and introduce the ansatz

$$e^{(\lambda+\nu)/2} = e^{ar}. \tag{A2}$$

Substituting (A2) and the relation

$$\lambda' = 2a - \nu',$$

which is a direct consequence thereof in (A1) and (2.9), we obtain

$$E = \sigma \left[\frac{C e^{ar}}{r^2} + e^{2ar} \left[\frac{1}{a} - \frac{2}{a^2 r} + \frac{2}{a^3 r^2} \right] \right], \tag{A3}$$

$$\frac{e^{2ar}}{r^2} = e^\nu \left[\frac{1}{r^2} - \frac{\nu''}{2} - \frac{(\nu')^2}{2} + \frac{a}{2} \nu' + \frac{a}{r} \right] + E^2, \tag{A4}$$

where C is an integration constant. To linearize Eq. (A4) we introduce $\alpha = e^\nu$ and obtain

$$\alpha'' - a\alpha' - 2\alpha \left[\frac{1}{r^2} + \frac{a}{r} \right] = h(r), \tag{A5}$$

where

$$h(r) = 2 \left[E^2 - \frac{e^{2ar}}{r^2} \right].$$

The homogeneous part of (A5) has a fundamental set of solutions in the form

$$\alpha_1 = r^2 e^{ar}, \tag{A6}$$

$$\alpha_2 = \frac{1}{3r} - \frac{a}{6} - \frac{a^2 r}{6} + \frac{a^3 r^2 e^{-ar}}{6} \int \frac{e^{-ar}}{r} . \quad (\text{A7})$$

Hence the general solution of (A5) is given explicitly (up to quadratures) by

$$e^\nu = \alpha = C_1 \alpha_1 + C_2 \alpha_2 + \alpha_p , \quad (\text{A8})$$

where C_1 and C_2 are arbitrary constants and α_p is formally given by

$$\alpha_p = u_1 \alpha_1 + u_2 \alpha_2 ,$$

$$u_1 = - \int \frac{h(r) \alpha_2(r) dr}{\Delta(\alpha_1, \alpha_2)} , \quad (\text{A9})$$

$$u_2 = \int \frac{h(r) \alpha_1(r) dr}{\Delta(\alpha_1, \alpha_2)} ,$$

where $\Delta(\alpha_1, \alpha_2)$ is the Wronskian of α_1, α_2 .

We note that e^λ can be easily computed now from (A2) using (A8).

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