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A NON-ABELIAN 2-GROUP WHOSE ENOMORPHISMS GENERATE A RING, AND OTHER EXAMPLES OF E-GROUPS

by J. J. MALONE

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1. Introduction

Groups for which the distributively generated near-ring generated by the endomorphisms is in fact a ring are known as E-groups and are discussed in (3). R. Faudree in (1) has given the only published examples of non-abelian E-groups by presenting defining relations for a family of p-groups. However, as shown in (3), Faudree's group does not have the desired property when \( p = 2 \).

In this note, it is shown that most of the groups discussed by D. Jonah and M. Konvisser in (2) are actually E-groups. These groups, described below in Section 2 are proved by Jonah and Konvisser to be such that all their automorphisms are central. Here, it is shown that most of these groups are E-groups by proving that each strict endomorphism (i.e. an endomorphism that is not an automorphism) has its image in the centre of the group. Since one of the groups treated is a 2-group, this paper provides the only published example of a non-abelian 2-group which is an E-group.

E-groups are also discussed in (4) and (5). However, no examples are given in those papers.

2. The Groups of Jonah and Konvisser

The groups treated in (2) are described as follows. Let \( \lambda = (\lambda_1, \lambda_2) \) be a vector with integer entries at least one of which is relatively prime to \( p \). Then let \( G_\lambda = \langle a_1, a_2, b_1, b_2 \rangle \) be the \( p \)-group of class 2 with the additional relations:

\[
\begin{align*}
    a_1^p &= [a_1, b_1], \\
    a_2^p &= [a_1, b^4], \\
    b_1^p &= [a_2, b_1 b_2], \\
    b_2^p &= [a_2, b_2], \text{ and } [a_1, a_2] = [b_1, b_2] = 1.
\end{align*}
\]

It is noted in (2) that \( G = G_\lambda \) has order \( p^8 \) and exponent \( p^2 \), that \( Z(G) = G' = \langle [a_1, b_1], [a_1, b_2], [a_2, b_1], [a_2, b_2] \rangle \) is elementary abelian of order \( p^4 \), and that \( G_\mu \) and \( G_\lambda \) are isomorphic if and only if \( \mu = k\lambda \) for some \( k \) relatively prime to \( p \). Thus, for each prime \( p \), there are \( p + 1 \) non-isomorphic groups given as \( \lambda \) varies over the set \( \{(0, 0), (1, 0), (1, 1), \ldots, (1, p-1)\} \). However, there is some difficulty with the defining relations when \( \lambda = (1, 0) \) since \( a_1^p = a_2^p \) so that \( G_\lambda \) does not have order \( p^8 \). Therefore, that case will not be considered in this paper.

If \( p = 2 \) and \( \lambda = (1, 1) \), then application of the defining relations shows that \((a_1 a_2 b_2)^2 = 1\). Thus \( G^2 \neq G' \). But Theorem 2 of (3) says that, in our context, \( G^2 = G' \) is a necessary condition for \( G \) to be an E-group. Therefore, when \( p = 2 \), the case of \( \lambda = (1, 1) \) will also not be considered in this paper.
In (2) it is noted that the normal subgroups \( A = \langle a_1, a_2, Z(G) \rangle \) and \( B = \langle b_1, b_2, Z(G) \rangle \) are the only abelian subgroups of order \( p^2 \) over the centre. Furthermore, \( A^p \cong [x, G] \) for some \( x \) in \( A \) while there is no \( y \) in \( B \) such that \( B^p \cong [y, G] \). For \( F \) a \( p \)-group as given in (1) by Faudree, Lemma 6 of (1) implies that the centralisers \( C_F(a_2) \) and \( C_F(a_3) \) are the only abelian subgroups of order \( p^2 \) over the centre. Hence, if \( F \) and \( G \) are to be isomorphic, the two centralisers in \( F \) must correspond, in some order, to \( A \) and \( B \) in \( G \). From Lemma 6 we also have that any element of order \( p^2 \) in \( C_F(a_2) \) has the form \( a_2^r a_2^s \) and an element of order \( p^2 \) in \( C_F(a_3) \) has the form \( a_3^r a_3^s \) with at least one of \( r \) and \( s \) relatively prime to \( p \). But, the defining relations in \( F \) indicate there is no \( y \) in \( C_F(a_2) \) such that \( \langle a_2^r, a_2^s \rangle = \langle C_F(a_2)^p \rangle \cong [y, F] \) and no \( x \) in \( C_F(a_3) \) such that \( \langle a_3^r, a_3^s \rangle = \langle C_F(a_3)^p \rangle \cong [x, F] \). Hence \( F \) and \( G \) are not isomorphic since neither centraliser of order \( p^6 \) in \( F \) can correspond to \( A \) in \( G \).

3. The Strict Endomorphisms

The groups described in Section 2 for which \( p \) is odd and \( \lambda \) is in \( \{0, 1\}, (1, 1), (1, 2), \ldots, (1, p-1) \) or \( p = 2 \) and \( \lambda = (0, 1) \) will be referred to as \( JK \)-groups.

Lemma 1. For any \( JK \)-group, \( Z(G) = G' = U_p(G) \), where \( U_p(G) \) is the set of elements whose order divides \( p \).

Proof. From the defining relations it is immediate that \( G' = \langle a_1^p, a_2^p, b_1^p, b_2^p \rangle = G^p \), \( G' \cong Z(G) \), and \( G' \cong U_p(G) \) with \( |G| = p^k \) and \( |G'| = p^l \). If \( |Z(G)| \geq p^3 \), then \( |G/Z(G)| \leq p^3 \). But \( G/Z(G) \) is generated by \( \langle a_1Z(G), a_2Z(G), b_1Z(G), b_2Z(G) \rangle \). So two generators of \( G \) are congruent mod \( Z(G) \) and \( G/Z(G) \) is generated by at most three elements. Thus \( |G'| \leq p^3 \), a contradiction. Hence \( Z(G) = G' \). Also, for \( x \) and \( y \) in \( G \), \( (xy)^p = x^p y^p \) by Proposition VI.1.k(4) of (6). Since, for odd \( p \), \( G' \) has exponent \( p \), it follows that \( (xy)^p = x^p y^p \). Then \( (1) = (a_1^r a_2^s b_1^t b_2^u)^p = a_1^p a_2^p b_1^p b_2^p \) implies \( p \) divides each of \( r, s, t, u \) so that \( a_1^r a_2^s b_1^t b_2^u \) is in \( G^p \). Hence \( U_p(G) = G^p \) for odd \( p \). For \( p = 2 \) we show directly from the generating relations that \( G^2 = U_2(G) \). Let \( g = a_1^{2n_1} a_2^{2n_2} b_1^{2n_1} b_2^{2n_2} \) be an element of order 2 in \( G \). Then \( e = g^2 = a_1^{2n_1 (1 + n_2)} a_2^{2n_2 + n_2} b_1^{2n_1 (1 - m_2)} b_2^{2n_2 + n_2 m_2 + n_1 m_2} \). Since \( g \) has order 2, each exponent in \( g^2 \) must be divisible by 4. If \( m_2 \) is odd, then the exponent of \( a_2 \) shows that \( n_2 \) and \( m_1 \) must also be odd. Then, from the exponent of \( a_1 \), \( n_1 \) must be odd. But now, the \( b_2 \) exponent is not divisible by 4. If \( m_2 \) is even, then \( m_1 \), \( n_1 \), and \( n_2 \) must also be even. Hence \( U_2(G) = G^2 \) and the Lemma is proved.

Lemma 2. Let \( m_1, m_2, n_1, n_2, \lambda_1, \lambda_2 \) be integers modulo \( p \) and \( \lambda_2^{-1} \) be the inverse of \( \lambda_2 \) in the field of order \( p \). Then the matrix \( A \) has rank 2 or greater over the field of order \( p \) if at least one of \( m_1, m_2, n_1, n_2 \) is not congruent to 0 modulo \( p \).

\[
A = \begin{bmatrix}
-m_1 \lambda_2^{-1} m_2 & \lambda_2^{-1} m_2 & 0 & 0 \\
0 & 0 & m_1 & -m_1 + m_2 \\
-n_1 & 0 & -n_2 & n_2 \\
\lambda_1 \lambda_2^{-1} n_1 & -\lambda_2^{-1} n_1 & 0 & -n_2
\end{bmatrix}
\]
Proof. If either \( m_1 \) or \( m_2 \not\equiv 0 \pmod{p} \), the first two rows of \( A \) are linearly independent. If either \( n_1 \) or \( n_2 \not\equiv 0 \pmod{p} \), the last two rows of \( A \) are linearly independent.

**Theorem.** Any JK-group is an E-group.

**Proof.** In (2) it is shown that all automorphisms of JK-groups are central. The theorem will be established if it is demonstrated that any strict endomorphism of one of these groups has its image in the centre of the group. This will be shown by arguing as was done in Lemma 5 of (1).

Let \( \theta \) be a strict endomorphism of \( G \). Then \( (h)\theta \in G' \) for some \( h \not\in G' \) and \( h = a^1_1 a^2_2 b^m_1 b^m_2 \) with at least one exponent \( \not\equiv 0 \pmod{p} \). Also, \( [(c, h)]\theta = [(c)\theta, (h)\theta] = 1 \) so that

\[
\langle [c, h] : c \in \{a_1, a_2, b_1, b_2\} \rangle \subseteq \ker \theta. \]

Note that

\[
\begin{align*}
[a_1, h] &= a_1^{pm_1-pm_1}a_2^{pm_1}, \\
[a_2, h] &= b_1^{pm_1}b_2^{pm_2}, \\
b_1, h &= a_1^{pm_1}b_1^{pm_2}, \quad b_2, h = a_2^{pm_2}b_2^{pm_2}. \\
\end{align*}
\]

The matrix of the powers of \( a^i_1, a^i_2, b^i_1, b^i_2 \) in (*) is \( A \). Hence \( |\ker \theta \cap G'| \geq p^2 \) and there exists \( \{h_i : 1 \leq i \leq 4\} \) such that \( G = \langle h_1, h_2, h_3, h_4 \rangle \) and \( (h_1)\theta = (h_2)\theta = 1 \). Thus for \( i = 1 \) or \( 2 \), \( (h_i)\theta \in G' \) and \( (G)\theta' = ((h_3)\theta, (h_4)\theta)' \). Hence \( |(G')\theta| = |((G)\theta)'| \leq p \) and \( |\ker \theta \cap G'| \geq p^3 \). We can then additionally assume that \( (h_3)\theta = 1 \) and \( (h_4)\theta = G' \). But now, \( ((G)\theta)' = 1 \). Hence \( (G)\theta \) is abelian, \( G' \subseteq \ker \theta \) and \( (G)\theta \subseteq G' = Z(G) \).

**REFERENCES**


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