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A Low-Dimensional Model for Chaos in Open Fluid-Flows

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A forced Landau–Stuart equation is studied in order to derive a low-dimensional model describing the temporal behavior of a paradigm open flow, the two-dimensional forced cylinder wake. Numerical results from the model exhibit several characteristics of circle maps, and compare qualitatively to previous experimental results for an oscillating cylinder wake. The low-dimensional model is also shown to reduce to a circle map in the limit of small forcing amplitudes. Observation of circle map dynamics in the forced Landau–Stuart equation strengthens the conjecture that globally unstable fluid flows are amenable to a dynamical systems approach focusing on the study of low-dimensional iterative maps. The established connection between the Landau–Stuart equation and the circle map unifies certain aspects of spatiotemporal stability and low-dimensional chaos theory.

I. INTRODUCTION

During the last few years evidence has accumulated that the nonlinear dynamics of certain open fluid flows can be described by low-dimensional dynamical systems. For example, it has been shown experimentally that the periodically forced cylinder wake,[1-4] capillary jet,[4,5] and low-density jet[6] have dynamics reminiscent of a dissipative circle map. Focusing on the cylinder wake experiments as a paradigm open flow, a controlled modulation with specified frequency (f_e) and amplitude (K) was imposed on the wake (with natural vortex shredding frequency f_0) yielding two control parameters $\Omega=f_e/f_0$ (bare winding number), and K. Once the external modulation is imposed f_0 shifts to f_0', yielding $\omega=f_e/f_0'$ (dressed winding number). This is similar in spirit to the convection experiments of Refs. 7 and 8 and the well-studied circle map,[9]

$$\theta_{n+1}=\theta_n+\Omega-\frac{K}{2\pi}\sin(2\pi\theta_n). \quad (1)$$

The circle map has been studied in recent years as a standard model for the transition from quasiperiodicity to chaos in purely temporal dynamical systems, and its properties are believed to be universal for any map with a cubic inflection point.[9] The parameter θ represents the angular measure on a Poincaré section. For $K<1$ (subcritical behavior), iterates of the map lock on to rational ω values (in general different from Ω for nonzero K) in the Arnold tongues[10] (lock-in regions) which increase in width as K increases. At $K=1$, the critical line, a universal transition to chaos exists.

The forced cylinder wake exhibits rational frequency lock-in states in widening lock-in regions. In addition, the dynamics of the wake were shown to compare to universal predictions of the circle map at the transition to chaos from both a dynamical and statistical point of view utilizing Fourier spectra, singularity spectra, and scaling functions.[1-4]

A recent related approach[11] describing the spatiotemporal stability of open fluid flows has established two types of instabilities: convectively unstable flow and flow governed by a global instability. The existence of an unstable global mode is intimately linked to a streamwise region in the developing flow where local velocity profiles are absolutely unstable. This region of local absolute instability acts as an intrinsic resonator exciting the discrete frequency behavior. This interpretation of the dynamics of two-dimensional wakes at low Reynolds numbers near criticality is firmly established. Huerre and Monkewitz[11] have argued that globally unstable flows are describable by a Landau–Stuart equation, at least near a critical Reynolds number for instability.

Briefly, start-up transients produced by a step increase of the Reynolds number from a subcritical to supercritical value showed that the oscillatory (vortex shedding) mode at frequency f_0 is the saturated state of a time-amplified linear global instability. Landau–Stuart constants have been determined experimentally[12-14] for the cylinder wake. Provensal *et al.*[12] have studied the forced Landau–Stuart equation in a study of an acoustically forced wake. Several investigators[15,16] have described three-dimensional vortex shedding using low-dimensional models based on the Ginzburg–Landau equation. While our attention here is on the Landau–Stuart equation this does not exclude the use of other possible models, however, as Albarede and Monkewitz[15] have emphasized, most suitable temporal models reduce to the generic Landau–Stuart equation close to the onset of the instability. Other investigators[17-23] have also contributed to the effort to develop a finite-dimensional description for laminar cylinder wakes.

Huerre and Monkewitz[11] also conjecture that globally unstable flows are readily amenable to a dynamical systems approach focusing on the study of low-dimensional iterative maps, such as the circle map. This was the perspective used in previously described studies of oscillating cylinder wakes.[1-4] The motivation behind the present work was to further explore this conjecture by studying a forced Landau–Stuart equation from a dynamical systems perspective. Observation of circle map dynamics for this finite-dimensional model would strengthen the conjecture, while serving to unify certain aspects of spatiotemporal stability and low-dimensional chaos theory.

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The qualitative behavior of the resultant model will be compared to previous experimental results for an oscillating cylinder wake. The low-dimensional model will also be shown to reduce to a circle map in some approximation. Implications of this work in predicting the nonlinear dynamics of flows with a global instability will also be discussed.

II. THE FORCED LANDAU–STUART EQUATION

Consider the Landau–Stuart equation[12–14,24,25] which describes the supercritical state of a system undergoing a Hopf bifurcation. In this context it can be written as

$$\frac{d\mathbf{u}}{dt}=\mathbf{au}-\mathbf{c}|\mathbf{u}^2|\mathbf{u}, \tag{2}$$

where $\mathbf{u}$ is the complex velocity fluctuation $\mathbf{u}=r(t)e^{i\psi(t)}$, and $\mathbf{a}$ and $\mathbf{c}$ are the complex Landau–Stuart constants. We then have

$$\frac{dr}{dt}=a_r r-c_r r^3 \tag{3}$$

and

$$\frac{d\psi}{dt}=2\pi f_0=a_i-c_i r^2 \tag{4}$$

governing the real and imaginary parts, respectively. Here, f_0 is the frequency of oscillation, and $r(t)=|\mathbf{u}|$. Thus, the constant $\mathbf{a}$ describes the disturbance amplitude and phase in the linear regime, while $\mathbf{c}$ details the nonlinear behavior as a saturation amplitude is approached. For $dr/dt=0$ (the steady state), the saturation amplitude is given by $r_{os}=\sqrt{a_r/c_r}$. The time scale of the oscillations is assumed much smaller than the characteristic time for growth of the disturbance ($1/f_0\ll 1/a_r$). The above formulation is therefore valid in a region near the critical Reynolds number.

We next study the effect of an external complex forcing term, $\mathbf{F}=\bar{F}+F_0e^{i\omega_e t}$ with a prescribed amplitude and frequency on this saturated steady state with

$$\frac{d\mathbf{u}}{dt}=\mathbf{au}-\mathbf{c}|\mathbf{u}^2|\mathbf{u}+\mathbf{F}. \tag{5}$$

If $F_0=F'r_{os}\omega_e$, the parameter F' is then the ratio of the velocity perturbation introduced by forcing to the saturation amplitude. The term $\bar{F}=CF_0$ allows for a mean forcing on the system.

Separating the forced Landau–Stuart equation into real and imaginary parts yields the following pair of coupled differential equations:

$$\frac{dr}{dt}=a_r r-c_r r^3+F_0\cos(\omega_e t-\psi)+CF_0\cos\psi, \tag{6}$$

$$\frac{d\psi}{dt}=a_i-c_i r^2+\frac{F_0}{r}\sin(\omega_e t-\psi)+C\frac{F_0}{r}\sin\psi. \tag{7}$$

The Landau constants for the cylinder wake can be expressed as follows from experiments;[13,14]

$$a_r D^2/\nu=0.20(\mathrm{Re}-\mathrm{Re}_{cr}),$$

$$a_i D^2/\nu=34.3+0.7(\mathrm{Re}-\mathrm{Re}_{cr}), \tag{8}$$

$$c_r D^2/\nu=0.024,$$

$$c_i D^2/\nu=-0.0744,$$

and

$$f_0 D^2/\nu=5.46+0.21(\mathrm{Re}-\mathrm{Re}_{cr}). \tag{9}$$

Here, Re is the cylinder Reynolds number, D is the cylinder diameter, and ν the kinematic viscosity. Next, we take $\mathrm{Re}_{cr}=46$, and $\mathrm{Re}=55$ to match previous experiments.[1–4] Combining Eqs. (8) and (9) yields

$$a_r=0.12\omega_0/\pi=a_r^*\omega_0,$$

$$a_i=2.76\omega_0/\pi=a_i^*\omega_0, \tag{10}$$

$$c_r=1.63\times10^{-3}\omega_0/\pi=c_r^*\omega_0,$$

$$c_i=-5.03\times10^{-3}\omega_0/\pi=c_i^*\omega_0.$$

With this normalization, Eqs. (6) and (7) become the low-dimensional, autonomous, dissipative system of ordinary differential equations,

$$\frac{dr}{dt}=f(r,\psi,t)$$
$$=\omega_0 r(a_r^*-c_r^*r^2)+F_0\cos(\omega_e t-\psi)+CF_0\cos\psi, \tag{11}$$

$$\frac{d\psi}{dt}=g(r,\psi,t)$$
$$=\omega_0(a_i^*-c_i^*r^2)+\frac{F_0}{r}\sin(\omega_e t-\psi)-C\frac{F_0}{r}\sin\psi. \tag{12}$$

Provensal *et al.*[12] derived similar equations in a study of an acoustically forced wake, however, this set of equations was not studied from a "dynamical systems" viewpoint. Also, Provensal *et al.* exclude an important Landau–Stuart constant, c_i, concluding that the frequency selection in the saturation state of the cylinder wake is completely described by linear theory, i.e., $c_i=0$. Experimental observations[13,14] studying the impulsively started flow past a cylinder show that this is not the case, and that c_i is finite.

Our simple model studies the interaction of the wake velocity fluctuations with the external forcing at one spatial location. Recent work in spatiotemporal linear stability analysis[11] has suggested that most important features of globally unstable flows can be characterized by single-point measurements. Our model is consistent with this viewpoint.

III. NUMERICAL RESULTS

The low-dimensional model, Eqs. (11) and (12), was solved numerically using a fourth-order Runge–Kutta iteration scheme. Also, the LSODA (Livermore Solver of Ordinary Differential Equations) software package was used to confirm results. By setting the parameters ω_0 and ω_e in the differential equation $\Omega=\omega_0/\omega_e$ (bare winding number)

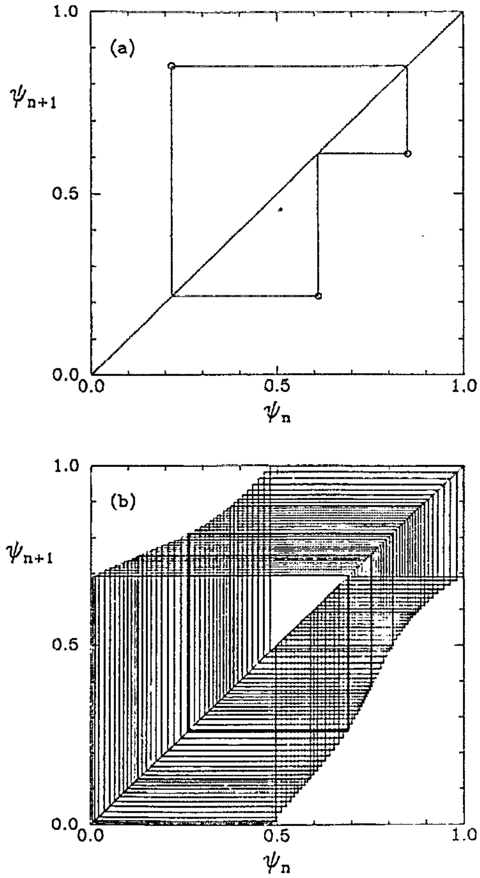

FIG. 1. Return maps for a lock-in and quasiperiodic state from the numerical solution of the forced Landau–Stuart equation. (a) The periodic trajectory of a 2/3 lock-in state. $\Omega=0.648$, $F'=0.2$, $C=1.1$. (b) The nonrepeating trajectory of a quasiperiodic state. $\Omega=0.644$, $F'=0.2$, $C=1.1$.

is specified. The parameter F' sets the forcing level and is analogous to the nonlinearity parameter K in the circle map. Initial transients are allowed to decay before the external modulation is applied.

A Poincaré section of the continuous time traces in $r(t)$ and $\psi(t)$ was extracted by sampling the numerical solution at discrete time intervals separated by the period of forcing, $T=2\pi/\omega_e$. The Poincaré sections were summarized through return maps, which is a mapping of the variables at the beginning of the nth forcing period, $T=2\pi n/\omega_e$, to the values at the end of that period. The dressed winding number ω (average rotation per iteration) can be extracted by studying the return map of the variable ψ.

We next show that results from the numerical solution of Eqs. (11) and (12) exhibit several features of circle maps and qualitatively match observations in forced open fluid flows. Figure 1(a) shows a return map exhibiting a periodic functional iteration, with period $q=3$, for a typical lock-in state ($\omega=p/q=\frac{2}{3}$). Figure 1(a) highlights the functional iteration $\psi_{n+1}=F(\psi_n)$ achieved by reflecting each iterate through the line $y=x$ to obtain the next iterate, ψ_n. For comparison, a quasiperiodic return map is pre-

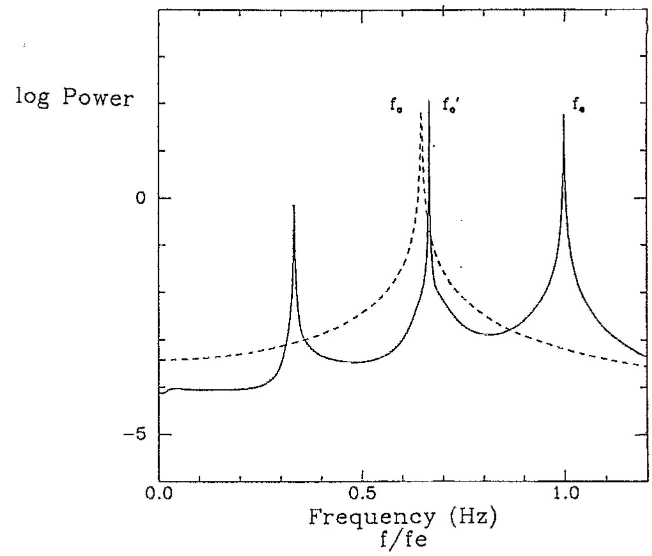

FIG. 2. The power spectra of the real part, u_r, of the complex velocity fluctuations, **u**, from numerical solution of the forced Landau–Stuart equation for the 2/3 lock-in state of Fig. 1.

sented in Fig. 1(b), here the trajectory is not periodic and the iterates fill the entire ψ_n axis.

A spectral analysis of the continuous variable u_r, the real part of the complex velocity fluctuation **u**, for the $\omega=p/q=\frac{2}{3}$ lock-in state is shown in Fig. 2. The unforced case with natural shedding frequency, f_0, is shown dashed, while the forced case is shown solid. Equally spaced spectral peaks, at linear combinations of the frequencies f_0' and f_e, characteristic of a lock-in state are observed.

Figure 3 shows a small region of the F'-Ω plane from our numerical study. This figure is derived from studying return map iterates (as in Fig. 1) for a matrix of (F',Ω)

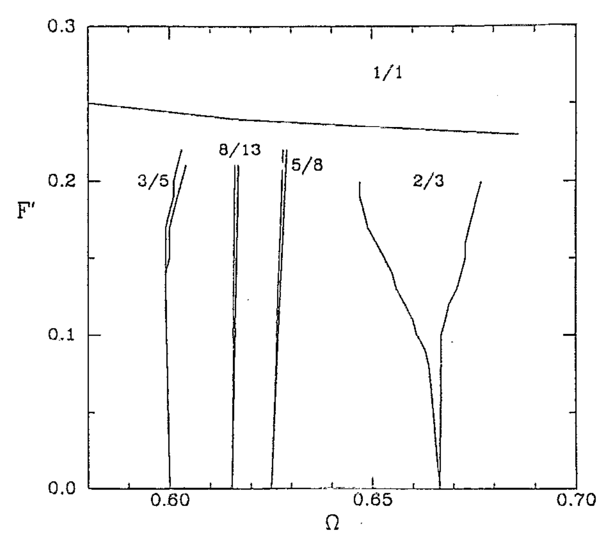

FIG. 3. A portion of the F'-Ω plane from the numerical solution. Arnold tongues where lock-in states occur are shown. The tongues widen in extent as the level of forcing F' increases. Only tongues of significant extent are shown.

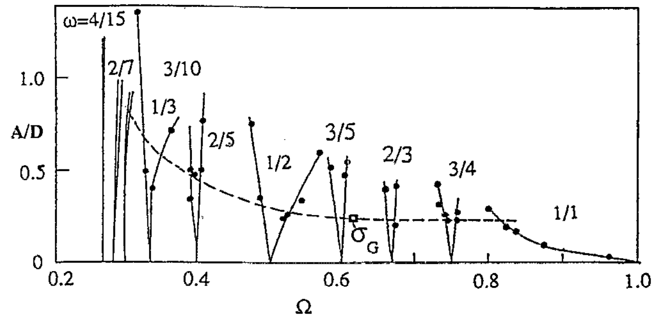

FIG. 4. The experiment A/D-Ω plane for the oscillating cylinder–wake system. The regions labeled with p/q values are lock-in tongues. About 30 such tongues were observed in experiments, but only those with appreciable widths are shown. The critical line for transition to chaos is shown dashed. Reproduced from Ref. 1.

values. The width of the lock-in tongues increases with F' in qualitative agreement with circle map theory. The decrease in width at a given F' as q increases is also in agreement. Lock-in tongues with denominators q as large as $q=115$ have been observed, although not all tongues are shown for clarity. Fourier spectra and lock-in regions with similar qualitative characteristics were observed in experiments on forced cylinder wakes,[1–4] low-density jets,[4,5] and capillary jets.[6] In Fig. 4 the A/D-Ω plane from the oscillating cylinder experiments of Refs. 1–4 is shown for comparison. Here, A/D is the nondimensional cylinder oscillation amplitude, analogous to F' in the numerical work.

Figure 5 shows typical return maps as F' is increased to values near a critical line associated with a transition to chaos. A noninvertible return map, containing a cubic inflection point with zero slope is clearly observed in Fig. 5(d). This suggests the existence of a critical point for chaos in our numerical phase plane. Presently we have not analyzed universal features of transition to chaos at a critical golden mean point, largely due to the difficulty of determining the precise critical point location. The extreme sensitivity of universal predictions of circle maps to small deviations from criticality has been discussed.[26] This is an area for future work.

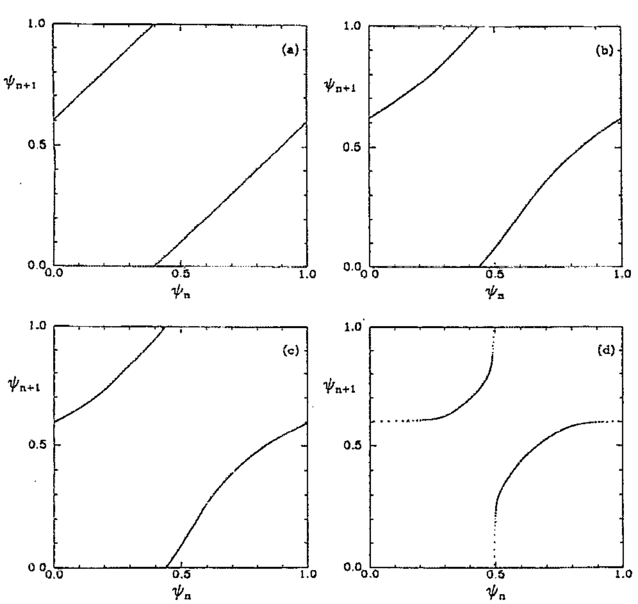

FIG. 5. Return maps from the numerical solution as forcing amplitude F' is increased. $\Omega=0.623$. (a) $F'=0$, (b) $F'=0.1$, (c) $F'=0.15$, and (d) $F'=0.22$ approximating the critical transition to chaos. Note the appearance of a cubic inflection point with zero slope at $\psi_n=0$ in (d).

IV. AN APPROXIMATE RETURN MAP

Further evidence that the underlying dynamics of the system of equations (11) and (12) are governed by a circle map is gained through derivation of an approximate return map valid at low forcing amplitudes. The lowest-order approximate solution from a Taylor series expansion to these equations is

$$r_{n+1} \approx r_n + \Delta t f(r_n, \psi_n, t_n),$$
$$\psi_{n+1} \approx \psi_n + \Delta t g(r_n, \psi_n, t_n). \tag{13}$$

For small F', $dr/dt \approx 0$ and $d\psi/dt \approx \omega_0$, therefore we expect this will yield an accurate approximation for the "true" return map from the numerical studies for low forcing amplitudes. The concept of universality implies that the level of complexity of the model chosen does not matter as long as it abstracts the underlying dynamics of the proper universality class. This means that less complex models can yield proper physics.

Applying Eq. (13), and setting $\Delta t = 2\pi/\omega_e$ yields the lowest-order two-dimensional iterative map

$$r_{n+1} \approx r_n + 2\pi\Omega r_n (a_r^* - c_r^* r_n^2) + K r_{os} \cos(2\pi\theta_n), \tag{14}$$

$$\theta_{n+1} \approx \theta_n + \Omega(a_i^* - c_i^* r_n^2) - \frac{K}{2\pi} \frac{r_{os}}{r_n} \sin(2\pi\theta_n), \tag{15}$$

where $\Omega = \omega_0/\omega_e$, $\theta_n = 2\pi\psi_n$, and $F'(1+C) = K/2\pi$. If we assume that $r_{n+1} \approx r_n \approx r_{os} = \sqrt{a_r^*/c_r^*}$ for small F', and since $a_i^* = (c_i^* a_r^*/c_r^*) + 1$ from (4) and (10), the map (14) and (15) reduces to the sine circle map

$$\theta_{n+1} \approx \theta_n + \Omega - \frac{K}{2\pi} \sin(2\pi\theta_n) \tag{16}$$

for small F'.

V. CONCLUSIONS

While our focus has been primarily on the cylinder wake, the derivation of the approximate map is not dependent on particular values of the Landau–Stuart equations constants, and is therefore applicable to any system describable by a Landau–Stuart equation. The results of this work suggest that the underlying dynamics of the forced Landau–Stuart equation are of the same universality class as the circle map. We have shown that an approximate return map valid in the low forcing limit yields a circle map. At larger forcing amplitudes the full dynamical system yields widening mode-locked tongues and noninvertible return maps with cubic inflection points in qualitative agreement with circle maps. These numerical results are specific to the forced cylinder wake as the Landau–Stuart constants have been specified. The qualitative behavior of the full dynamical system compares well to previous experimental results for an oscillating cylinder wake.

Circle map dynamics have not been previously observed in a forced Landau–Stuart equation. The current work establishes connections between cylinder wake descriptions focusing on the Landau–Stuart equation and the circle map. This work strengthens the earlier conjectures[11] (and observations[1–4]) that globally unstable open flows are amenable to a dynamical systems approach focusing on low-dimensional iterative maps. The established connections between the forced Landau–Stuart equation and the circle map unify certain aspects of spatiotemporal stability[11] and low-dimensional chaos theory. Predictive ability has been gained on the class of open fluid systems describable by circle map theory, i.e., systems undergoing a Hopf bifurcation governed by a Landau–Stuart equation, a behavior associated with a global instability. This predictive ability has been preliminarily tested in several other globally unstable flows, including capillary jets[4,5] and low-density jets.[6] Observation of universal behavior in different fluid systems has implications transcending the current results. Further study of this finite-dimensional model, particularly behavior near critical points for transition to chaos, is ongoing.

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