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CONSTRAINING PLANE CONFIGURATIONS IN COMPUTER-AIDED DESIGN: COMBINATORICS OF DIRECTIONS AND LENGTHS *

BRIGITTE SERVATIUS[†] AND WALTER WHITELEY[‡]

Abstract. Configurations of points in the plane constrained by directions only or by lengths alone lead to equivalent theories known as parallel drawings and infinitesimal rigidity of plane frameworks. We combine these two theories by introducing a new matroid on the edge set of the complete graph with doubled edges to describe the combinatorial properties of direction-length designs.

Key words. computer-aided design, constraint frameworks, generic rigidity, matroid, parallel drawings, plane configurations

AMS subject classifications. Primary, 68U07, 05B35; Secondary, 05C50, 51N05, 52C25

PII. S0895480196307342

1. Introduction. A *plane configuration* in computer-aided design (CAD) is a collection of geometric objects such as points, line segments, and circular arcs in the plane, together with constraints on and between these objects [7, 13]. Naturally the designer wants to know if a realization of the configuration exists and is uniquely determined. A realization of a plane configuration is called a *plane design*. Beyond simple uniqueness of design, there are other fundamental design questions: If global uniqueness is not achieved, is the design locally unique? If the design permits continuous deformations, which additional constraints would give the appropriate uniqueness? Are all constraints essential in producing the design or are there constraints which are forced by the remaining ones?

Given a design, the constraints can be written as a system of algebraic equations whose variables are the coordinates and parameters of the geometric objects [12, 15]. Some of the above questions may be answered by computing the rank of the Jacobian of the system of constraint equations [13, 15]. Because of the size of the system and possible degeneracies, computation may be slow and unstable. Therefore a mathematical theory which answers these questions purely combinatorially is desirable [3, 12, 18].

The classical problem of Euclidean construction may be stated in the language of plane designs, as well as other familiar geometric problems. Much is known about *length designs*, where the objects are points and the distances between certain pairs of points are prescribed, forming the familiar mathematical model for a bar and joint framework [8]. On the other hand, *direction designs*, in which the constraints prescribe directions instead of distances between points, are also well understood as the problem of parallel drawings [17]. We present a combinatorial solution for the Jacobian of direction-length designs, which incorporate both of these cases.

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These results are a contribution to the more basic open case of lengths and angles, a problem which arises in geodesy (making maps).

We will start out by summarizing results for frameworks and parallel drawings in section 2, then define direction-length designs in section 3. Our main goal is to characterize robust designs (defined in section 4), which have independent constraints and locally unique realizations. Limiting designs are used as tools in our proofs and are explicitly described in section 5. In section 6 we describe a direction-length construction and prove that the construction produces robust designs. The converse is demonstrated in section 6, where the combinatorial properties of direction-length designs are explored. Finally we indicate problems arising from mixing lengths, directions, and angles and outline other topics for further research.

2. Frameworks and parallel drawings.

2.1. Frameworks. Consider the set $V = \{1, \dots, n\}$ and a function \mathbf{p} from V into \mathbb{R}^2 . We call \mathbf{p} a *configuration* and we will denote $\mathbf{p}(i)$ by \mathbf{p}_i . A configuration \mathbf{p} is *generic* if the coordinates in \mathbf{p} are algebraically independent over the rationals. (For convenience here, we will assume that all points in a configuration are distinct, $\mathbf{p}_i \neq \mathbf{p}_j, i \neq j$. In certain limiting cases, we will bring vertices into coincidence and redefine the associated constraint.)

If \mathbf{p} is an embedding, we can associate with every graph $G = (V, E)$ a *framework* $G(\mathbf{p})$, where the edge set E is interpreted as the collection of those pairs of vertices whose images under \mathbf{p} are joined by rigid bars. We call two frameworks $G(\mathbf{p})$ and $G(\mathbf{q})$ equivalent if corresponding bars have the same length.

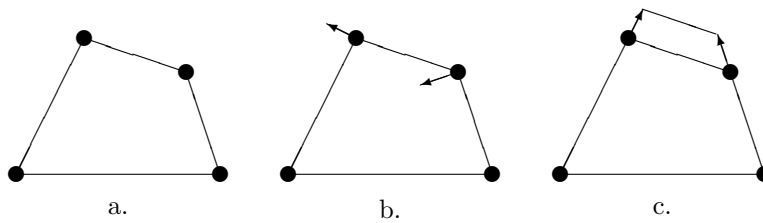
We may identify the configuration \mathbf{p} with a point in \mathbb{R}^{2n} , and measure the distance between pairs of vertices by evaluating the *rigidity function* $\rho : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{n(n+1)/2}$ defined by $\rho(\mathbf{p})_{i,j} = (\mathbf{p}_i - \mathbf{p}_j)^2$ for $i < j \leq |V|$. Clearly ρ is continuously differentiable with respect to \mathbf{p} , and we define $R(\mathbf{p})$, the rigidity matrix for the configuration \mathbf{p} , by $\rho'(\mathbf{p}) = 2R(\mathbf{p})$. With every framework $G(\mathbf{p})$ we can associate the matrix $R(G, \mathbf{p})$ consisting of those rows of $R(\mathbf{p})$ corresponding to E . A solution, \mathbf{u} , of the system $R(G, \mathbf{p})\mathbf{u} = \mathbf{0}$ consists of vectors \mathbf{u}_i in \mathbb{R}^2 , one for each point \mathbf{p}_i satisfying

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{u}_i - \mathbf{u}_j) = 0$$

for each $(i, j) \in E$. \mathbf{u} is called an *infinitesimal motion* of the framework. If $|V| \geq 2$ and $R(G, \mathbf{p})$ has rank $2n - 3$, or equivalently if all solutions to $R(G, \mathbf{p})\mathbf{x} = \mathbf{0}$ correspond to derivatives of congruences (translations or rotations), the framework is called *infinitesimally rigid*. An infinitesimally rigid framework with independent rows of the rigidity matrix is called *isostatic*.

A configuration \mathbf{p} is said to be *generic* if any length design whose constraints are dependent with respect to \mathbf{p} are in fact dependent with respect to any embedding. It is straightforward to show that almost all embeddings are generic (see [2]). If the coordinates of \mathbf{p} are algebraically independent over the rational field, then \mathbf{p} is generic. For a generic embedding, the linear independence of the rows of the rigidity matrix depends only on the graph whose edges correspond to the rows, and consequently, the *generic rigidity* of a framework depends on the graph alone.

2.2. Parallel drawings. If \mathbf{u} is an infinitesimal motion of $R(G)$, $\mathbf{u}_i = (u_i, v_i)$, then $\mathbf{u}_i^\perp - \mathbf{u}_j^\perp$ is parallel to $\mathbf{p}_i - \mathbf{p}_j$ for every edge (i, j) , where $\mathbf{u}_i^\perp = (v_i, -u_i)$; so $G(\mathbf{p} + \mathbf{u}^\perp)$ is a framework whose edges are all parallel to edges in $G(\mathbf{p})$ (see Figure 2.1). $G(\mathbf{p} + \mathbf{u}^\perp)$ is said to be a *parallel redrawing* of $G(\mathbf{p})$. If \mathbf{t} is an infinitesimal translation, then $G(\mathbf{p} + \mathbf{t}^\perp)$ is congruent to $G(\mathbf{p})$. If \mathbf{r} is an infinitesimal rotation, $G(\mathbf{p} + \mathbf{r}^\perp)$ is a

FIG. 2.1. *Infinitesimal motions and parallel redrawings.*TABLE 2.1
First order terminology.

Plane design	Bar frameworks	Parallel drawings
Locally unique solution	rigid	tight
Locally unique solution with independent constraints	isostatic	minimally tight
Infinitely many nontrivial solutions	flexible	loose

dilation or contraction of $G(\mathbf{p})$, and if \mathbf{u} is a nontrivial infinitesimal motion, $G(\mathbf{p} + \mathbf{u}^\perp)$ will not be similar to $G(\mathbf{p})$.

Conversely, every parallel redrawing of a framework in the plane induces an infinitesimal motion of the framework. More directly, given a graph $G = (V, E)$, we can interpret the edges as line segments in the plane whose direction is to be fixed and thereby obtain the theory of *parallel drawings*, or *direction designs*, which is equivalent to the linearized problem obtained from interpreting the edges of G as length constraints. In Table 2.1 we compare the corresponding terminology used in these two theories.

3. Direction-length designs. The equivalent theories of parallel drawings and infinitesimal analysis of frameworks make tractable plane designs of lengths alone, and directions alone. We now mix these two types of constraints into a single system with an inclusive theory of designs with both kinds of constraints.

To distinguish the two kinds of constraints in figures of designs, we will follow the convention of indicating a length constraint between two points as an ordinary edge, and a direction constraint between two points as an edge with two arrowheads along its interior (see Figure 3.1).

DEFINITION 3.1. *A direction-length design is a double graph $FG = (V; D, L)$, where D, L are two sets of edges (no loops), and an assignment \mathbf{p} of points $\mathbf{p}_i \in \mathbb{R}^2$ for each vertex $i \in V$. We call elements of D direction constraints and elements of L length constraints. Together, these are written as the design $FG(\mathbf{p})$.*

The edges L represent pairs of points whose lengths are held fixed. The edges D represent pairs whose directions are fixed. Since D and L need not be disjoint, a particular pair may have both types of connections. We also speak of the *direction graph* $F = (V, D)$ and the *length graph* $G = (V, L)$. We say that a direction-length design is *pure* if it only has edges of one type, and *mixed* otherwise. Two direction-length designs are said to be equivalent if they differ by a translation (see Figure 3.1).

We recall that for lengths the first-order constraints on “infinitesimal motions” (derivatives of the point positions) are

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{u}_i - \mathbf{u}_j) = 0.$$

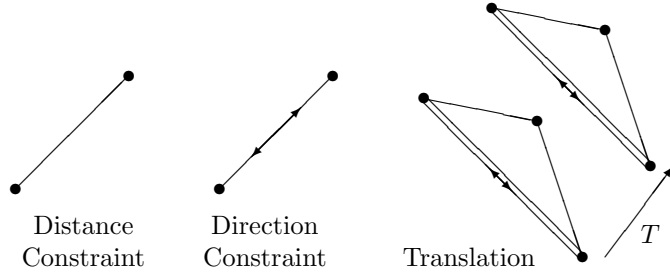


FIG. 3.1.

For plane directions, the constraint $(\mathbf{q}_i - \mathbf{q}_j) = \alpha(\mathbf{p}_i - \mathbf{p}_j)$ can also be rewritten in derivative form. The first step is to recall that the vector $(\mathbf{p}_i - \mathbf{p}_j)$ can be replaced by a constant normal $\mathbf{n}_{ij} = (\mathbf{p}_i - \mathbf{p}_j)^\perp$, and the equation becomes

$$\mathbf{n}_{ij} \cdot (\mathbf{p}(t)_i - \mathbf{p}(t)_j) = 0.$$

Taking derivatives, we obtain

$$\mathbf{n}_{ij} \cdot (\mathbf{u}_i - \mathbf{u}_j) = 0,$$

or equivalently

$$(\mathbf{p}_i - \mathbf{p}_j)^\perp \cdot (\mathbf{u}_i - \mathbf{u}_j) = 0.$$

Together, these produce a homogeneous linear system $R(FG, \mathbf{p}) \times \mathbf{u} = \mathbf{0}$. The matrix $R(FG, \mathbf{p})$ is the *constraint matrix* of the design. A set of constraints is *independent* if the corresponding rows of the matrix are independent. A solution to this system of constraints is called a *shake*. The design (with distinct vertices) is *stiff* if and only if this system has only the translations as solutions. Otherwise it is *shaky*. A set of constraints is *spanning* on the configuration \mathbf{p} if it creates a stiff subdesign on these points. Equivalently, a spanning set of constraints spans the row space for the complete design on the configuration \mathbf{p} , with the complete graph on these vertices as both length and direction constraints.

Example 1. Consider the simple design $FG = (\{1, 2\}; \{(1, 2)\}, \{(1, 2)\})$. The equations

$$|\mathbf{q}_1 - \mathbf{q}_2| = |\mathbf{p}_1 - \mathbf{p}_2| \text{ and } \mathbf{q}_1 - \mathbf{q}_2 = \alpha(\mathbf{p}_1 - \mathbf{p}_2)$$

are equivalent to the matrix equation

$$\begin{bmatrix} x_1 - x_2 & y_1 - y_2 & x_2 - x_1 & y_2 - y_1 \\ y_2 - y_1 & x_1 - x_2 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \times \begin{bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

If the points are distinct, it is easy to see that this system reduces to

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \times \begin{bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

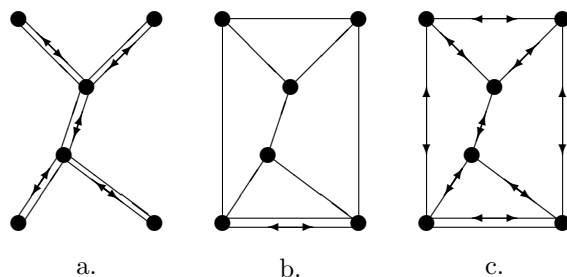


FIG. 3.2. Three robust designs on 6 points.

Thus $u_1 = u_2$ and $w_1 = w_2$, so the infinitesimal translation (u_1, w_1) is the only solution.

We are essentially interested in the rank (and independence) of the constraint matrix. The rank of the constraint matrix depends on both the double graph FG and the configuration \mathbf{p} . However, all generic \mathbf{p} give the same rank for $R(FG, \mathbf{p})$, maximal over all configurations. A set of constraints is *generically independent* if it is independent for some (hence all) generic configurations. A set of edges is *generically spanning* if it is spanning for some (hence all) generic configurations.

Since any nonempty design has a two-dimensional space of translations in the plane, the maximum rank that the matrix can have is $2|V| - 2$. A unique solution will therefore require $2|V| - 2$ independent constraints, or equivalently $2|V| - 2$ spanning constraints. Such sets, which are independent and spanning, induce a *robust design*. We may observe the following.

LEMMA 3.2.

1. An independent set of $|L| = 2|V| - 3$ lengths plus any single direction constraint is an independent set of $2|V| - 2$ constraints; see Figure 3.2b.
2. An independent set of $|D| = 2|V| - 3$ directions plus any single length constraint is an independent set of $2|V| - 2$ constraints; see Figure 3.2c.
3. A spanning tree, used once as L for lengths and a second time as D for directions, is a spanning set of $2|V| - 2$ constraints; see Figure 3.2a.
4. If there are only length constraints, then every infinitesimal rotation is a shake.
5. If there are only direction constraints, then any infinitesimal dilation is a shake.
6. A spanning set of constraints must contain both direction and length constraints.

3.1. Swapping. The form of the constraint matrix implies that lengths and directions play symmetric roles in the theory. In fact, we have a basic “duality” between these two constraints.

DEFINITION 3.3. Given a double graph $FG = (V; D, L)$, the swapped double graph is $FG^s = GF = (V; L, D)$, where the roles of lengths and directions have been switched.

In Figure 3.2a the swapped design is identical to the original, while b swaps to c.

THEOREM 3.4 (swapping theorem). A direction-length design $FG(\mathbf{p})$ and the swapped design (the swapped double graph at the same points) $FG^s(\mathbf{p}) = GF(\mathbf{p})$ have isomorphic solution spaces of shakes.

In particular, a direction-length design $FG(\mathbf{p})$ is stiff (robust) if and only if the swapped design $GF(\mathbf{p})$ is stiff (robust).

Proof. Consider the constraint matrix $R(FG, \mathbf{p})$ for the first design. If we rotate the design 90 degrees clockwise to form \mathbf{q} , then the independence of any set of constraints is unchanged and the matrices $R(FG, \mathbf{q})$ and $R(FG^s, \mathbf{p})$ are identical up to the sign of the rows. \square

4. Robust designs. If a direction-length design has $2|V| - 2$ independent constraints, then the design is stiff, and the removal of any constraint introduces a shake. We called such a design robust. If a double graph FG has a configuration \mathbf{p} for which the design $FG(\mathbf{p})$ is robust, we say the FG is *robust*. Equivalently, FG is robust if $FG(\mathbf{p})$ is robust for all generic configurations \mathbf{p} .

The term robust is used to indicate that small changes in the parameters of a design yield a “nearby” design with identical stiffness properties, which is highly desirable for ease of rendering and computability. This is indeed the case for robust double graphs, since the generic configurations comprise an open dense set of configurations.

5. Limiting designs. For our analysis, it is useful to expand the allowable designs to include typical limiting cases. For a given direction-length design $FG(\mathbf{p})$, the *normalized constraint matrix*, $R_n(FG, \mathbf{p})$, is obtained from $R(FG, \mathbf{p})$ by scaling the rows; multiplying row (i, j) by $|\mathbf{p}_i - \mathbf{p}_j|^{-1}$. The advantage of the normalized constraint matrix is that it has the same row dependencies as the original matrix, while its entries remain finite and nonzero under the limits $\lim_{\mathbf{p}_i \rightarrow \infty}$ and $\lim_{\mathbf{p}_i \rightarrow \mathbf{p}_j}$.

5.1. Vertices at infinity. Let \mathbf{p} be a configuration of FG , and consider the limit of $R_n(FG, \mathbf{p})$ as $\mathbf{p}_i \xrightarrow{\mathbf{q}} \infty$ in the direction of a unit vector \mathbf{q} . Then the limit of a row corresponding to length constraint $l(i, j)$ of R_n has entries \mathbf{q} in the columns corresponding to i , and $-\mathbf{q}$ in the columns corresponding to j , and the limit of a row corresponding to direction constraint $d(i, j)$ of R_n has entries \mathbf{q}^\perp in the columns corresponding to i , and $-\mathbf{q}^\perp$ in the columns corresponding to j .

If the vertex i has two distinct neighbors, then $\lim_{\mathbf{p}_i \xrightarrow{\mathbf{q}} \infty} R_n(FG, \mathbf{p})$ is not the constraint matrix of a direction-length design, since the vertex i has no possible location. We will indicate a vertex at infinity as in Figure 5.1.

As a vertex tends to infinity, the edges in its star tend to parallelism, and so if a vertex has only direction constraints or only length constraints, then the limiting design has an infinitesimal motion even if none of the ordinary direction-length designs of the configuration do.

Example 2. Suppose we consider the complete graph on four vertices, $\mathbf{p}_0 = (-1, -1)$, $\mathbf{p}_1 = (+1, -1)$, $\mathbf{p}_2 = (0, 0)$, and $\mathbf{p}_3 = (0, 1)$ (see Figure 5.1a). The constraint matrix is

$$R(FG, \mathbf{p}) = \begin{bmatrix} 0 & -2 & 0 & 2 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

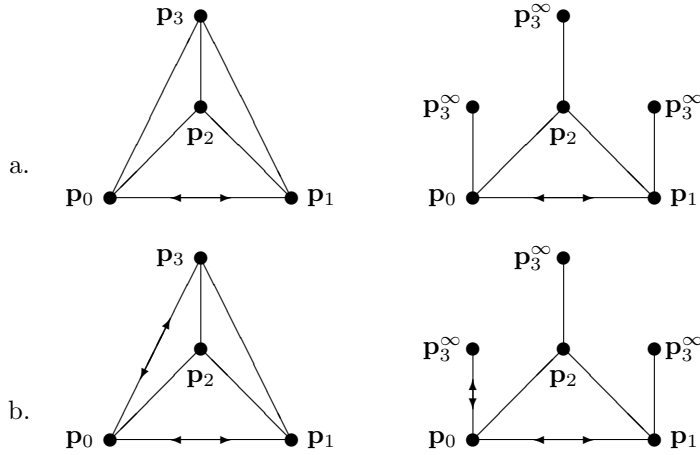


FIG. 5.1. A point passing to infinity.

and the normalized matrix is

$$R_n(FG, \mathbf{p}) = \begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\alpha & -\alpha & 0 & 0 & \alpha & \alpha & 0 & 0 \\ 0 & 0 & \alpha & -\alpha & -\alpha & \alpha & 0 & 0 \\ -\beta & -2\beta & 0 & 0 & 0 & 0 & \beta & 2\beta \\ 0 & 0 & \beta & -2\beta & 0 & 0 & -\beta & 2\beta \\ 0 & 0 & 0 & 0 & 0 & -\alpha & 0 & \alpha \end{bmatrix},$$

where $\alpha = \frac{1}{\sqrt{2}}$ and $\beta = \frac{1}{\sqrt{5}}$. The limit as $\mathbf{p}_3 \xrightarrow{\mathbf{q}} \infty$, $\mathbf{q} = (0, 1)$, is the limit design on the right, with normalized matrix

$$\lim_{\mathbf{p}_3 \xrightarrow{\mathbf{q}} \infty} R_n(FG, \mathbf{p}) = \begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\alpha & -\alpha & 0 & 0 & \alpha & \alpha & 0 & 0 \\ 0 & 0 & \alpha & -\alpha & -\alpha & \alpha & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{matrix} 1 \\ -1 \\ 1 \end{matrix}$$

and infinitesimal motion $\mathbf{u}_3 = (1, 0)$ and $\mathbf{u}_0 = \mathbf{u}_1 = \mathbf{u}_2 = (0, 0)$. The numbers to the right of the matrix indicate the coefficients of a linear dependence of the rows.

The normalized matrix of the limit design in Figure 5.1b is

$$\lim_{\mathbf{p}_3 \xrightarrow{\mathbf{q}} \infty} R_n(FG, \mathbf{p}) = \begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\alpha & -\alpha & 0 & 0 & \alpha & \alpha & 0 & 0 \\ 0 & 0 & \alpha & -\alpha & -\alpha & \alpha & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix},$$

which allows no nontrivial motion.

Since the limit of a dependent set is a dependent set in the limit design, an independent set in the limit design implies the nearby regular designs are also independent. If the limit design is spanning, then the nearby designs are also spanning.

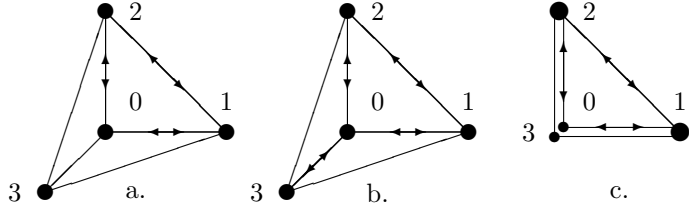


FIG. 5.2.

5.2. Infinitesimal edges. The points in a direction-length design are assumed to be distinct. However, it is sometimes useful to consider the limit design as one point \mathbf{p}_i approaches another point \mathbf{p}_j in the direction of the unit vector \mathbf{q} . The row for a length constraint $l(i, j)$ in the limit of the normalized constraint matrix will have \mathbf{q} in the columns corresponding to i and $-\mathbf{q}$ in the columns corresponding to j . The row for a direction constraint $d(i, j)$ in the limit of the normalized constraint matrix will have \mathbf{q}^\perp in the columns corresponding to i and $-\mathbf{q}^\perp$ in the columns corresponding to j .

Example 3. Consider the designs of Figure 5.2a and b. It is straightforward to check that both designs are generically independent.

If we take the limit as \mathbf{p}_3 approaches \mathbf{p}_0 along the direction $(1, 1)$, Figure 5.2c, then the limit of design 5.2a has matrix

$$\begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \alpha & \alpha & -\alpha & -\alpha & 0 & 0 \\ \alpha & \alpha & 0 & 0 & 0 & 0 & -\alpha & -\alpha \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix},$$

which has rank 6, while the limit of design 5.2b has matrix

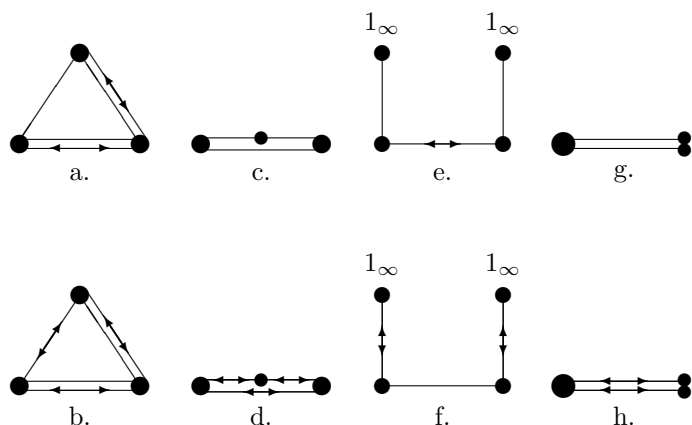
$$\begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \alpha & \alpha & -\alpha & -\alpha & 0 & 0 \\ -\alpha & \alpha & 0 & 0 & 0 & 0 & \alpha & -\alpha \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix} \begin{matrix} \alpha \\ \alpha \\ -1 \\ 1 \\ \alpha \\ -\alpha \end{matrix},$$

which has rank 5, ($\alpha = \frac{1}{\sqrt{2}}$).

Again, the limit of a dependent set is a dependent set in the limit design and an independent (spanning) set in the limit design implies the nearby regular designs are also independent (spanning).

5.3. Cycles on 3 vertices. In this section we describe small cycles which will be useful in subsequent arguments.

A *cycle* is a minimally dependent set of constraints. Among 3 vertices any set of 5 constraints is dependent, so the designs of Figures 5.3a and 5.3b are dependent. To see they are cycles, we need only observe that removing any constraint yields a robust design. These are both *generic cycles*. We can have a cycle on fewer than 5 edges if the position is special.

FIG. 5.3. *Cycles with 3 vertices.*

The design of Figure 5.3c is clearly a cycle, with matrix

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 2 & 0 & 0 & 0 & -2 & 0 \end{bmatrix} \begin{array}{l} 2 \\ 2 \\ -1 \end{array}$$

and dependence given in the right column, similarly for Figure 5.3d.

The design of Figure 5.3e has point 1 approach ∞ in the vertical direction. The matrix is

$$\begin{bmatrix} 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix} \begin{array}{l} 1 \\ -1 \\ 1 \end{array}$$

and similarly for Figure 5.3f.

Last, the design of Figure 5.3g has point 1 approach, point 2 in the vertical direction $(0, 1)$ with the direction edge $d(1, 2)$. The matrix is

$$\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{bmatrix} \begin{array}{l} 1 \\ -1 \\ 1 \end{array}$$

and similarly for Figure 5.3h with vertical length edge $l(1, 2)$.

6. Extendability. For plane rigidity and plane directions, the simple inductive constructions for the independent (rigid) structures are the oldest characterizations (see [5, 16, 20]). In the proof of our broader combinatorial characterization, an inductive construction for robust direction-length designs remains a key step.

6.1. 0-extensions.

DEFINITION 6.1. *Let $FG = (V; D, L)$ be a double graph. Let FG' be the double graph obtained from FG by adjoining a new vertex v whose total degree is 2. We say that FG' is a 0-extension of FG (see Figure 6.1a).*

The neighbors of the new vertex v need not be distinct vertices if the two new constraints at v are of a different type.

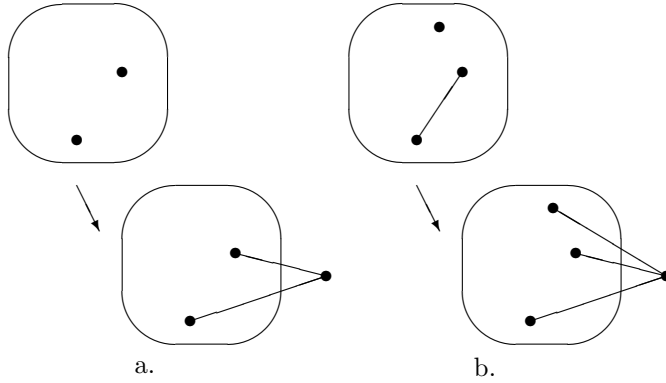


FIG. 6.1. Extensions.

Let FG' be a 0-extension of FG and let v be the new vertex. Then the matrix of FG' is in block form

$$\left[\begin{array}{c|cc} A & & \mathbf{0} \\ \hline B & (x_v - x_a) & (y_v - y_a) \\ & (x_v - x_b) & (y_v - y_b) \end{array} \right]$$

if the new edges are both lengths,

$$\left[\begin{array}{c|cc} A & & \mathbf{0} \\ \hline B & -(y_v - y_a) & (x_v - x_a) \\ & -(y_v - y_b) & (x_v - x_b) \end{array} \right]$$

if they are both directions, and

$$\left[\begin{array}{c|cc} A & & \mathbf{0} \\ \hline B & (x_v - x_a) & (y_v - y_a) \\ & -(y_v - y_b) & (x_v - x_b) \end{array} \right]$$

if there is one of each type. So the rows corresponding to the new constraints in the new matrix are independent of the other rows if the new edges are not parallel, in the first two cases, or perpendicular, in the third case, and we have the following.

LEMMA 6.2. *Let FG' be a 0-extension of FG and suppose FG is independent with respect to some configuration \mathbf{p} . Then \mathbf{p} may be extended to the new vertex so that FG' is also independent.*

In particular, if FG is generically independent or robust, then any 0-extension of FG is generically independent or robust, respectively.

6.2. 1-extensions.

DEFINITION 6.3. *Let $FG = (V; D, L)$ be a double graph with edge f . A 1-extension of FG , FG' , is obtained from FG by removing the edge f and adding a new vertex v of degree 3 so that*

1. *the neighbors of v include both endpoints of f ,*
2. *neither D nor L decrease in size.*

We can think of the new edges (v, a) and (v, b) as *splitting* the constraint $l(a, b)$ or $d(i, j)$ (see Figure 6.1b). Condition 2 is satisfied as long as a length constraint is not replaced by three direction constraints, or vice versa. A configuration is *general* if no three points are collinear.

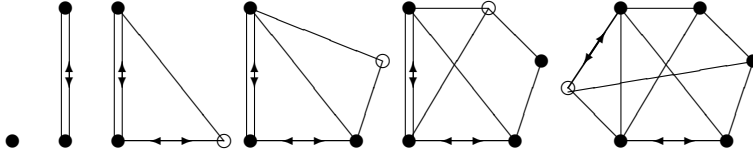


FIG. 6.2. A direction-length construction.

LEMMA 6.4. *Let FG' be a 1-extension of FG , and let FG be independent (spanning) with respect to a general configuration \mathbf{p} . Then \mathbf{p} can be extended so that FG' is also independent (spanning).*

Proof. By the swapping theorem (Theorem 3.4), we assume without loss of generality (w.l.o.g.) that $f \in L$.

Let $\{a, b\}$ be the endpoints of f and let v denote the new vertex with new edges (v, a) , (v, b) , and (v, c) .

Case 1. Let $l(v, a), l(v, b) \in L$, c distinct from a and b . We can adjoin v by a 0-extension to vertices a and c with constraints $l(v, a), l(v, b)$ with the new vertex v placed along the segment from \mathbf{p}_a to \mathbf{p}_b . Then, since a triangle of lengths with vertices on a line is a cycle, we can replace the constraint $l(a, b)$ with the constraint $l(v, b)$ so that FG' is independent (spanning).

Case 2. Let $l(v, a) \in L$ and $d(v, b) \in D$. We can adjoin v by a 0-extension with constraints $l(v, a)$ and $l(v, c)$ and take the limit $\mathbf{p}_a \xrightarrow{\mathbf{q}} \mathbf{p}_b$ in the direction \mathbf{q} perpendicular to (a, b) . Since the rows for $l(a, b)$, $l(v, a)$ form a cycle with the infinitesimal direction $d(v, b)$, we can replace $l(a, b)$ with $d(b, v)$ and the limiting design is independent (spanning). Therefore any nearby generic configuration gives an independent (spanning) design.

Case 3. Let $d(v, a), d(v, b) \in D$, c distinct from a and b . Then again adjoin v by a 0-extension, and let \mathbf{v} approach ∞ in the direction \mathbf{q} perpendicular to (a, b) . In this position, the rows for $l(a, b)$, $d(v, a)$, and $d(v, b)$ form a cycle with (v, b) , so we can replace $l(a, b)$ with $d(b, v)$ and the limiting design is independent (spanning). Again, any generic \mathbf{p} is also independent (spanning). \square

Remark. Notice that the “limiting design” argument does, indeed, break down if we try the forbidden replacements: replace a single direction with three lengths, or replace a single length by three directions. With a limiting point “at infinity,” all three directions (or lengths) will be parallel rows of the matrix, and the initial 0-extension will fail to be independent.

6.3. Direction-length constructions. In the spirit of the classical Henneberg sequences, we now describe how to obtain complex robust designs from a single vertex using only the simple extensions just developed.

DEFINITION 6.5. *A direction-length construction of the double graph $FG = (V; D, L)$ is a sequence of direction-length double graphs,*

$$FG_1, FG_2, \dots, FG_{|V|},$$

beginning with the single vertex graph FG_1 , ending with $FG_{|V|} = FG$, such that FG_k is a 0-extension or 1-extension of FG_{k-1} (see Figure 6.2).

From Lemmas 6.2 and 6.4 we have the following theorem.

THEOREM 6.6. *A double graph FG with a direction-length construction is generically robust.*

In section 7, the converse is demonstrated. Since the class of constructions is

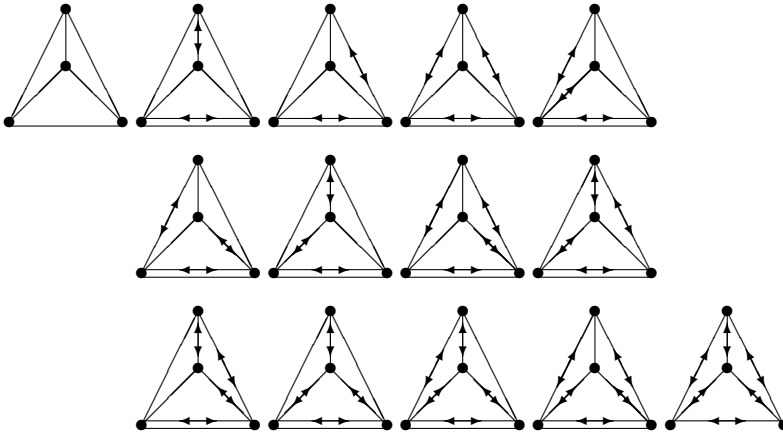


FIG. 6.3. *Generic 3-connected cycles on 4 vertices.*

closed under swapping, the class of constructible designs is closed under swapping.

6.4. Generic cycles on 4 vertices. Let us enumerate the generic cycles on 4 vertices, that is, those double graphs whose edges correspond to minimally dependent sets of constraints.

A generic cycle cannot have a vertex of total valence 2 (or less) since that would be a 0-extension of an independent set, or a 0-extension of a smaller cycle. On the other hand, on 4 vertices, a set of 6 directions or 6 lengths must be dependent, as well as a set of 7 edges of mixed type. Thus a cycle on 4 vertices is either

1. a tetrahedron of lengths;
2. a tetrahedron of directions;
3. the edges of both types form a tetrahedron with a doubled edge (the graph is vertex 3-connected);
4. the edges of both types form two attached triangles, with a doubled edge in each (not the shared edge) (the graph is vertex 2-connected).

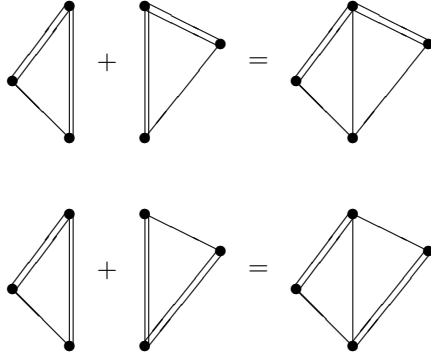
Moreover, the third type must have at least 2 edges of each kind, since if there was only one, then deleting it would leave a pure tetrahedron which is dependent. Also cycles of type 4 must have at least two edges of each kind, since there is a pair of doubled edges.

All candidates of types 1–3 are listed in Figure 6.3. To see that the mixed graphs are all in fact generic cycles, one may easily give a direction-length construction for each of the graphs with any one edge deleted.

The circuits of type 4 can be constructed from two of the cycles on three vertices by cycle exchange. Figure 6.4 illustrates this process. The single lines represent constraints of either kind, while the double lines indicate that constraints of both kinds are present. Altogether there are 12 circuits of type 4.

With the exception of the 2 pure cycles, all generic cycles on 4 vertices may be obtained from the generic cycles on 3 vertices by either 1-extension or cycle-exchange. It seems plausible that all generic cycles may be obtained from the generic cycles on 3 and 4 vertices by a sequence of extensions and cycle exchanges, but to date no proof is known, not even in the case of pure designs.

7. The generic matroid. Consider a complete double graph $K_n^2 = (V; D_c, L_c)$ on $V = \{1, \dots, n\}$ together with a generic configuration \mathbf{p} . Since we are interested

FIG. 6.4. *Generic 2-connected cycles on 4 vertices.*

in the combinatorial properties of the matrix $R(K_n^2) = R(K_n^2, \mathbf{p})$, we examine the matroid $\text{CAD}_{\text{dl}}(n)$ defined by the rows of $R(K_n^2)$, which we call the *generic dl-cadroid on n vertices*. Theorem 6.6 states that every double graph on n vertices with a direction-length construction is a basis of $\text{CAD}_{\text{dl}}(n)$.

We know that the rank of the full constraint matrix for a generic configuration of n points in \mathbb{R}^2 has rank $2n - 2$. Also, for all $k < n$, $\text{CAD}_{\text{dl}}(k)$ may be viewed as a restriction of $\text{CAD}_{\text{dl}}(n)$. Therefore we can offer clear necessary conditions for a basis B of $\text{CAD}_{\text{dl}}(n)$.

CAD_{dl}1: $|B| = 2n - 2$;

CAD_{dl}2: for all nonempty subsets $E \subseteq B$

$$|E| \leq 2|V(E)| - 2;$$

CAD_{dl}3: for all pure nonempty subsets $E \subseteq B$,

$$|E| \leq 2|V(E)| - 3.$$

Theorem 7.4 will show that these are also sufficient.

We first show that CAD_{dl}1, ..., CAD_{dl}3 define the bases of a matroid $\text{Count}(n)$ on $D_c \cup L_c$ and then show that this matroid is isomorphic to $\text{CAD}_{\text{dl}}(n)$.

THEOREM 7.1. *Let $K^2 = (V; L_c, D_c)$ denote the complete double graph on $|V|$ vertices. Then the collection of subsets $B \subseteq L_c \cup D_c$ which satisfy CAD_{dl}1, ..., CAD_{dl}3 are the bases of a matroid on $L_c \cup D_c$.*

Proof. We will show that the collection \mathcal{C} of minimal sets which violate CAD_{dl}1, ..., CAD_{dl}3 satisfy the cycle axioms for a matroid.

If $C \in \mathcal{C}$ is pure, then $|C| = 2|V(C)| - 2$ and $|C'| \leq 2|V(C')| - 3$ for all proper nonempty subsets C' of C .

If $C \in \mathcal{C}$ and C is mixed, then C must contain at least two elements from both D_c and L_c . We have $|C| = 2|V(C)| - 1$ and all proper subsets of C must be independent, i.e., satisfy CAD_{dl}3.

We need to show that if $C_1, C_2 \in \mathcal{C}$, and $x \in C_1 \cap C_2$, then there exists $C_3 \in \mathcal{C}$, $C_3 \subseteq C_1 \cup C_2 - x$.

Let the supports of C_1 and C_2 have cardinalities m and n , respectively, and let the support of $C_1 \cap C_2$ be i .

If C_1 and C_2 are both mixed, then we have

$$|C_1 \cup C_2 - e| = |C_1| + |C_2| - |C_1 \cap C_2| - 1$$

$$\begin{aligned} &\geq 2n - 1 + 2m - 1 - (2i - 2) - 1 = 2(m + n - i) - 1 \\ &= 2|V(C_1 \cup C_2)| - 1, \end{aligned}$$

so $C_1 \cup C_2 - e$ contains an element of \mathcal{C} since it violates $\text{CAD}_{\text{dl}2}$.

If C_1 is mixed and C_2 is pure, then their intersection has at most $2i - 3$ edges and C_2 also has one edge fewer than before, so we arrive at the following conclusion:

$$|C_1 \cup C_2 - e| = 2(m + n - i) - 1 = 2|V(C_1 \cup C_2)| - 1.$$

If C_1 and C_2 are both pure (of the same type, since they have nonempty intersection), then

$$|C_1 \cup C_2 - e| = 2(m + n - i) - 2 = 2|V(C_1 \cup C_2)| - 2.$$

Since $C_1 \cup C_2$ is also pure, this gives the dependence. \square

This result is a particular case of a more general construction of matroids from “submodular counts” described in [23].

Observe that the generic cycles of $\text{CAD}_{\text{dl}}(n)$ listed in Figures 5.3 and 6.3 are also cycles in $\text{Count}(n)$ and these cycles are in fact all the cycles of $\text{Count}(n)$ on 3 or 4 vertices. Notice also that the symmetry of the definition of $\text{Count}(n)$ directly demonstrates the invariance of all matroidal properties under swapping.

We need the following lemmas.

LEMMA 7.2. *If B is a basis of $\text{Count}(n)$, then the double graph induced by B is edge 2-connected.*

Proof. If $B - e$ is disconnected with two components on k and l vertices, then the rank of B is at most $2(k + l) - 3$. \square

LEMMA 7.3. *Let I be independent in a matroid and let C be a cycle in this matroid. Then for each element $e \in I \cap C$ there is an element $f \in C - I$ so that $I - e + f$ is independent.*

Proof. Let $e \in I \cap C$. Assume that for each $f \in C - I$, $I - e + f$ is dependent. Then $C - e$ is a subset of the closure of $I - e$. Since e is in the closure of $C - e$, e is in the closure of $I - e$. Since $e \in I$ and I is independent, this is a contradiction. \square

THEOREM 7.4. *For any set B of edges in K_n^2 the following are equivalent:*

1. B is a basis of $\text{Count}(n)$;
2. B is a basis of $\text{CAD}_{\text{dl}}(n)$;
3. B has a direction-length construction.

Proof. (3) \Rightarrow (2). By Theorem 6.6, every set with a direction-length construction is a basis of $\text{CAD}_{\text{dl}}(n)$.

(2) \Rightarrow (1). Every basis of $\text{CAD}_{\text{dl}}(n)$ satisfies $\text{CAD}1, \dots, \text{CAD}3$ and so is a basis for $\text{Count}(n)$.

(1) \Rightarrow (3). The proof is by induction on the number of vertices. It is trivial for 2 vertices.

Assume it is true for $n - 1$ vertices. Since the average valence of a basis in $\text{Count}(n)$ is $4(1 - 1/n) < 4$, there is some vertex of total valence ≤ 3 . By the 2-connectivity, this vertex must have valence either 2 or 3. If the valence is 2, then the robust set is the 0-extension of a smaller independent set, and we are done.

Assume B has a vertex v of valence 3. If $\text{star}(v)$, the set of constraints with endpoint v , is mixed (has constraints of both types), we add constraints among the neighbors of v to create a $\text{Count}(k)$, $k = 2$ or 3 , basis B_v for these neighbors. Adding the three valent vertex v , we have a dependent set in $\text{Count}(n)$ and therefore a small cycle C containing v . We have $C \not\subseteq B$, but $\text{star}(v) \subseteq C \cap B$. By Lemma 7.3, for any

edge e in $\text{star}(v)$, there is an $f \in C - B$ such that $B - e + f$ is independent, in fact, a basis, B' . Therefore, $B_{n-1} = B' - \text{star}(v)$ is a basis of $\text{Count}(n-1)$, and by induction it has a construction. Since every replacement of a constraint f by a mixed vertex is a valid 1-extension, B is a 1-extension of B_{n-1} . Therefore B has a construction.

If $\text{star}(v)$ is pure (say all lengths up to swapping), then it has 3 distinct neighbors. Adding length constraints among these neighbors will produce a unique pure cycle C – the complete graph on 4 vertices. As before, for any edge $e \in \text{star}(v)$, we can find a length constraint $f \in C - B$ such that $B - e + f$ is independent. The replacement of a length f in B_{n-1} by the 3-lengths at v is a valid 1-extension.

This completes the induction. \square

Remark. The characterization of $\text{Count}(n)$, by the count, appears to be exponential: “for all subsets $B' \dots$ ” However, by a theorem of Nash-Williams [10, 11], independent sets are decomposable into two spanning forests with the additional condition that no two subtrees that both contain only edges of D or edges of L do not have the same span. A general matroidal algorithm by Edmonds can be used to provide such a decomposition in polynomial time. Also Sugihara [15] and Imai [6] have general polynomial time algorithms to verify such conditions.

For length designs (and therefore also direction designs) Crapo has adapted Edmonds’s algorithm to also give a low degree polynomial algorithm for the tree structures which correspond to the counts $2|V(E)| - 3$. It is clear that this approach could be modified for our closely related counts, giving polynomial time algorithms to confirm a basis (or extract a basis from a spanning set). This algorithm would have the additional advantage that its output (the two trees mentioned above) could be displayed for rapid visual verification.

Remark. There are some additional results on both necessary and sufficient connectivity for spanning sets. All of these results are, in some form, the direct analogues of results for length designs (plane frameworks). All of the proofs are based on the counting properties of $\text{Count}(n)$.

1. Circuits in $\text{CAD}_{\text{dl}}(n)$ are vertex 2-connected and edge 3-connected.
2. All circuits of $\text{CAD}_{\text{dl}}(n)$ are spanning on their vertices.
3. If a direction-length design is vertex 6-connected and mixed, then it is spanning. This is a direct analogue of a result of [9] for frameworks. Their proof (also based on counts) extends with small modifications.

In the 5-connected 5-regular frameworks example of Lovasz and Yemini [9], we can double one of the 5-cliques to get an example of a 5-connected double graph which is not stiff.

8. Concluding remarks. Our entire analysis of constraints has been “local,” with robustness guaranteeing local uniqueness for *small* changes in the configuration, up to congruence. As we mentioned in the introduction, the problem of global uniqueness up to congruence, for all configurations is more difficult. This is no longer a matter of linear algebra and matroids; it is quadratic algebra with all the attendant difficulties. For frameworks, this global uniqueness is called “global rigidity” [1]. For pure lengths, any basis of the generic rigidity matroid will *not* be globally rigid, except in special singular (nongeneric) configurations, where the design is dependent [4].

On the other hand, for pure directions, both global and local transformations are described by linear equations, and the design is globally unique, up to translations and dilations, if and only if it is locally unique.

For direction-length designs, we have both types of cases.

1. A robust direction-length design with one length and $2n - 3$ directions will

be globally unique, up to translation and dilation by -1 , if and only if it is locally unique.

2. A direction-length design with one direction and the remaining constraints lengths will be globally unique, up to translation, dilation by -1 , and reflection in the line of the single direction, if and only if the length design is globally unique, up to congruence.
3. A direction-length design which is globally unique, up to translation and dilation by -1 , is 2-connected in a vertex sense. (Otherwise, we can take the point of disconnection, and dilate one of the components by -1 in this center.)

An inspection of a result and proof of Hendrickson [4] indicates that the following result also holds.

PROPOSITION 8.1. *A robust direction-length design FG with more than one length is not globally unique.*

As we mentioned in the introduction, our work with lengths and directions was motivated by a broader unsolved problem in plane CAD. Consider a design constrained by lengths between pairs of points and angles between lines. This angle constraint could involve two edges sharing a vertex or simply be the angle between to disjoint edges (“the following two lines are parallel”). The problem of a polynomial time algorithm, or direct combinatorial algorithm, for these constraints is unsolved and difficult [24]. (We do have the corresponding constraint matrices (which have nonzero vector entries under up to four vertices per angle row). By using variables for the coordinates of points, we have a well-defined generic matroid for the constraints $CAD_{dl}(n)$. Taking determinants, we get a superexponential “combinatorial” algorithm to check for bases, or independence in $CAD_{dl}(n)$.)

Writing A for the set of angle constraints (actually partially ordered triples and quadruples), and L for the length constraints, there is a necessary set of counts for B to be a basis of the matroid $CAD_{sfda}(n)$:

$CAD_{sfda}1$: $|B| = 2n - 3$;

$CAD_{sfda}2$: for all nonempty subsets $E \subseteq B$

$$|E| \leq 2|V(E)| - 3;$$

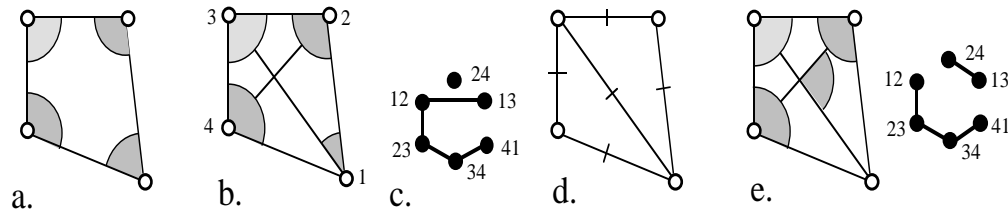
$CAD_{sfda}3$: for all nonempty subsets of angles $E \subseteq B$, $E \subseteq A$

$$|E| \leq 2|V(E)| - 4.$$

The subtracted constant 3 in $CAD_{sfda}(n)$ corresponds to the translations and rotations of a robust design. The subtracted constant 4 in $CAD_{sfda}3$ corresponds to the translations, rotations, and dilation permitted by a maximal pure angle design.

However, these conditions are not enough: any “polygon of angles” will be dependent, and in a quadrilateral, these four angles on four points will not violate the condition $CAD_{sfda}3$. Even if we carefully insert this “polygon condition” (by adding variables directly for the edges, etc.) the added count will not be sufficient to define a matroid (as occurred for $\text{Count}(n)$). In practice, the appearance of such “nonspanning” circuits is a sign that the techniques employed in this paper, adapted from the study of plane frameworks, will be inadequate.

However, if we have an angle design in which the angles are linked together as a connected set among the attached edges (ideally a tree since any polygon is dependent; see Figure 8.1a), the design can be analyzed with our theory. Taking any one of the edges in these angles, and defining an arbitrary direction to it, we can work through the

FIG. 8.1. *Generic cycles with angles.*

attached angles to assign a direction constraint for each of the angle constraints. This induces a direction-length design whose properties of robustness, independence, etc. directly correspond to the robustness, independence, etc. of the original angle-length design. The reader can check that, with one added direction and each angle converted to a direction constraint, the three conditions $CAD_{sfd a1}$, $CAD_{sfd a2}$, $CAD_{sfd a3}$ convert to the axioms for $Count(n)$. We have solved this special case of the general unsolved problem of angle-length designs.

If the angles form a forest of several trees (see Figure 8.1), the combinatorial analysis becomes difficult and unsolved. One key difficulty is that we do not yet have an adequate list of inductive constructions which are guaranteed to generate all bases of the matroid $CAD_{sfd a}(n)$. Moreover, this list will have to involve inductive principles for vertices attached to up to 7 constraints, since each angle may involve up to four vertices. It is unclear whether there will be any polynomial time algorithm for general bases in $CAD_{sfd a}(n)$.

More generally, the lines could contain many points (not just two) and we would have additional incidence constraints for vertices lying on lines. This takes us into several other unsolved problems, both for incidences alone and for mixes of incidences, lengths, and angles [23].

Finally, we could convert “direction constraints” into directions for lines, but replace incidences with possibly nonzero distances from points to lines. Again certain special cases of this can be solved [14] and other extensions are unsolved.

We have focused on constraints in plane CAD because we have some substantial results. Many of the related problems in 3-space are substantially more difficult. For example, the problem of independent length constraints alone in 3-space is unsolved. While there is a corresponding matrix, and a partial list of inductive constructions, there is no combinatorial characterization (beyond the constraint matrix with variable entries and the associated superexponential algorithm).

For direction constraints in 3-space, there are substantial results. A “direction” for a line segment becomes two rows in the constraint matrix, corresponding to two planes, with assigned normals, containing the line. The entire theory of plane directions has an appropriate extension to this “polymatroid” (two rows for each edge). This approach is described in more detail in [17, 19, 24].

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