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Distributing vertices along a Hamiltonian cycle in Dirac graphs

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Abstract

A graph $G$ on $n$ vertices is called a Dirac graph if it has minimum degree at least $n/2$. The distance $dist_G(u, v)$ is defined as the number of edges in a shortest subpath of $G$ joining $u$ and $v$. In this paper we show that in a Dirac graph $G$, for every small enough subset $A$ of the vertices, we can distribute the vertices of $A$ along a Hamiltonian cycle $C$ of $G$ in such a way that all but two pairs of subsequent vertices of $A$ have prescribed distances (apart from a difference of at most 1) along $C$. More precisely we show the following. There are $\varepsilon, n_0 > 0$ such that if $G$ is a Dirac graph on $n \geq n_0$ vertices, $d$ is an arbitrary integer with $3 \leq d \leq \varepsilon n/2$ and $A$ is an arbitrary subset of the vertices of $G$ with $2 \leq |A| = k \leq \varepsilon n/d$, then for every sequence $d_i$ of integers with $3 \leq d_i \leq d, 1 \leq i \leq k - 1$, there is a Hamiltonian cycle $C$ of $G$ and an ordering of the vertices of $A, a_1, a_2, \ldots, a_k$, such that the vertices of $A$ are visited in this order on $C$ and we have

$$|dist_C(a_i, a_{i+1}) - d_i| \leq 1, \text{ for all but one } 1 \leq i \leq k - 1.$$
1 Introduction

1.1 Notation and definitions

For basic graph concepts see the monograph of Bollobás [2].
+ will sometimes be used for disjoint union of sets. $V(G)$ and $E(G)$ denote the vertex-
set and the edge-set of the graph $G$. $(A, B, E)$ denotes a bipartite graph $G = (V, E)$,
where $V = A + B$, and $E \subseteq A \times B$. For a graph $G$ and a subset $U$ of its vertices, $G|_U$
the restriction to $U$ of $G$. $N(v)$ is the set of neighbours of $v \in V$. Hence the size of
$N(v)$ is $|N(v)| = deg(v) = deg_G(v)$, the degree of $v$. $\delta(G)$ stands for the minimum,
and $\Delta(G)$ for the maximum degree in $G$, $\nu(G)$ is the size of a maximum matching in
$G$. The distance $d_G(u, v)$ is defined as the number of edges in a shortest subpath of
$G$ joining $u$ and $v$. For $A \subseteq V(G)$ we write $N(A) = \cap_{v \in A} N(v)$, the set of common
neighbours. $N(x, y, z, \ldots)$ is shorthand for $N(\{x, y, z, \ldots\})$. For a vertex $v \in V$ and
set $U \subseteq V - \{v\}$, we write $deg(v, U)$ for the number of edges from $v$ to $U$. When
$A, B$ are disjoint subsets of $V(G)$, we denote by $e(A, B)$ the number of edges of $G$
with one endpoint in $A$ and the other in $B$. For non-empty $A$ and $B$,
$$d(A, B) = \frac{e(A, B)}{|A||B|}$$
is the density of the graph between $A$ and $B$. In particular, we write $d(A) = d(A, A) = 2|E(G|_A)|/|A|^2$.

Definition 1. The bipartite graph $G = (A, B, E)$ is $\varepsilon$-regular if

$X \subset A$, $Y \subset B$, $|X| > \varepsilon|A|$, $|Y| > \varepsilon|B|$ imply $|d(X, Y) - d(A, B)| < \varepsilon$,

otherwise it is $\varepsilon$-irregular.

We will often say simply that “the pair $(A, B)$ is $\varepsilon$-regular” with the graph $G$ implicit.
We will also need a stronger version.

Definition 2. $(A, B)$ is $(\varepsilon, \delta)$ super-regular if it is $\varepsilon$-regular and

$$deg(a) > \delta|B| \forall a \in A, \ deg(b) > \delta|A| \forall b \in B.$$ 

1.2 Distributing vertices along a Hamiltonian cycle in Dirac graphs

Let $G$ be a graph on $n \geq 3$ vertices. A Hamiltonian cycle (path) of $G$ is a cycle
(path) containing every vertex of $G$. A Hamiltonian graph is a graph containing
a Hamiltonian cycle. A Hamiltonian-connected graph is a graph in which every pair of vertices can be connected with a Hamiltonian path. A classical result of Dirac [3] asserts that if $\delta(G) \geq n/2$, then $G$ is Hamiltonian. This result of Dirac has generated an incredible amount of research, it has been generalized and strengthened in numerous ways (see the excellent survey of Gould [4]).

In a recent, interesting strengthening of Dirac’s Theorem, Kaneko and Yoshimoto [5] showed that in a Dirac graph small subsets of vertices can be somewhat uniformly distributed along a Hamiltonian cycle.

**Theorem 1.** Let $G$ be a graph of order $n$ with $\delta(G) \geq n/2$ and let $d$ be a positive integer with $d \leq n/4$. Then for any vertex set $A$ with at most $n/2d$ vertices, there exists a Hamiltonian cycle $C$ with $\text{dist}_C(u,v) \geq d$ for every $u$ and $v$ in $A$.

Note that this result is sharp; the bound on the cardinality of $A$ cannot be increased.

In [4] Gould called for further studies on density conditions that allow the distribution of “small” subsets of vertices along a Hamiltonian cycle. In this paper we show that with similar conditions we can not only achieve that the distance between two subsequent vertices of $A$ along $C$ is at least $d$, but actually we can prescribe the exact distances (apart from a difference of at most 1) between all but two pairs of subsequent vertices of $A$ along $C$. More precisely we show the following.

**Theorem 2.** There are $\kappa, n_0 > 0$ such that if $G$ is a graph on $n \geq n_0$ vertices with $\delta(G) \geq n/2$, $d$ is an arbitrary integer with $3 \leq d \leq \kappa n/2$ and $A$ is an arbitrary subset of the vertices of $G$ with $2 \leq |A| = k \leq \kappa n/d$, then for every sequence $d_i$ of integers with $3 \leq d_i \leq d$, $1 \leq i \leq k - 1$, there is a Hamiltonian cycle $C$ of $G$ and an ordering of the vertices of $A, a_1, a_2, \ldots, a_k$, such that the vertices of $A$ are visited in this order on $C$ and we have

$$|\text{dist}_C(a_i, a_{i+1}) - d_i| \leq 1, \text{ for all but one } 1 \leq i \leq k - 1.$$  

We need the discrepancies by 1 between $\text{dist}_C(a_i, a_{i+1})$ and $d_i$ because of parity reasons. Indeed, consider the complete bipartite graph between $U$ and $V$, where $|U| = |V| = n/2$. Take $A \subset U$, then the distance between subsequent vertices of $A$ along a Hamiltonian cycle is even, and if we have an odd $d_i$ we cannot obtain a distance with that $d_i$.

To see that we might need an exceptional $i$ for which $|\text{dist}_C(a_i, a_{i+1}) - d_i| > 1$, consider the following construction. Take two complete graphs on $U$ and $V$ with $|U| = |V| = n/2$. Let $A = A' \cup A''$ with $A' \subset U$, $A'' \subset V$ and $|A'| = |A''| = |A|/2$, and add the complete bipartite graphs between $A'$ and $V$, and between $A''$ and $U$. Clearly on any Hamiltonian cycle we will have two distances much greater than $d$.

We believe that our theorem remains true for greater $|A|$’s as well, but we were unable to prove a stronger statement.
2 The main tools

In the proof the following lemma of Szemerédi plays a central role.

Lemma 1 (Regularity Lemma [15]). For every positive \( \varepsilon \) and positive integer \( m \) there are positive integers \( M \) and \( n_1 \) with the following property: for every graph \( G \) with \( n \geq n_1 \) vertices there is a partition of the vertex set into \( l + 1 \) classes (clusters)

\[
V = V_0 + V_1 + V_2 + \ldots + V_l
\]

such that

- \( m \leq l \leq M \)
- \( |V_1| = |V_2| = \ldots = |V_l| \)
- \( |V_0| < \varepsilon n \)
- at most \( \varepsilon l^2 \) of the pairs \( \{V_i, V_j\} \) are \( \varepsilon \)-irregular.

We will use the following simple consequence of Lemma 1.

Lemma 2 (Degree form). For every \( \varepsilon > 0 \) there is an \( M = M(\varepsilon) \) such that if \( G = (V, E) \) is any graph and \( \delta \in [0, 1] \) is any real number, then there is a partition of the vertex-set \( V \) into \( l + 1 \) clusters \( V_0, V_1, \ldots, V_l \), and there is a subgraph \( G' = (V, E') \) with the following properties:

- \( l \leq M \),
- \( |V_0| \leq \varepsilon |V| \),
- all clusters \( V_i, i \geq 1 \), are of the same size \( L \leq [\varepsilon |V|] \).
- \( \text{deg}_{G'}(v) > \text{deg}_G(v) - (\delta + \varepsilon)|V| \) for all \( v \in V \),
- \( G'|_{V_i} = \emptyset \) (\( V_i \) are independent in \( G' \)),
- all pairs \( G'|_{V_i \times V_j}, 1 \leq i < j \leq l \), are \( \varepsilon \)-regular, each with a density either 0 or exceeding \( \delta \).

The other main tool asserts that if \((A, B)\) is a super-regular pair with \( |A| = |B| \) and \( x \in A, y \in B \), then there is a Hamiltonian path starting with \( x \) and ending with \( y \). This is a very special case of the Blow-up Lemma ([8], [9]). More precisely.
Lemma 3. For every $\delta > 0$ there are $\varepsilon_0, n_2 > 0$ such that if $\varepsilon \leq \varepsilon_0$ and $n \geq n_2$, $G = (A, B)$ is an $(\varepsilon, \delta)$ super-regular pair with $|A| = |B| = n$ and $x \in A$, $y \in B$, then there is a Hamiltonian path in $G$ starting with $x$ and ending with $y$.

We will also use two simple Pósa-type lemmas on Hamiltonian-connectedness. The second one is the bipartite version of the first one.

Lemma 4 (see [1]). Let $G$ be a graph on $n \geq 3$ vertices with degrees $d_1 \leq d_2 \leq \ldots \leq d_n$ such that for every $2 \leq k \leq \frac{n}{2}$ we have $d_{k-1} > k$. Then $G$ is Hamiltonian-connected.

Lemma 5 (see [1]). Let $G = (A, B)$ be a bipartite graph with $|A| = |B| = n \geq 2$ with degrees $d_1 \leq d_2 \leq \ldots \leq d_n$ from $A$ and with degrees $d'_1 \leq d'_2 \leq \ldots \leq d'_n$ from $B$. Suppose that for every $2 \leq j \leq \frac{n+1}{2}$ we have $d_{j-1} > j$ and that for every $2 \leq k \leq \frac{n+1}{2}$ we have $d'_{k-1} > k$. Then $G$ is Hamiltonian-connected.

Finally we will use the following simple fact.

Lemma 6 (Erdős, Pósa, see [2]). Let $G$ be a graph on $n$ vertices. Then

$$\nu(G) \geq \min\{\delta(G), \frac{n-1}{2}\}.$$ 

In case we have a good upper bound on the maximum degree of $G$, we can strengthen this lemma in the following way.

Lemma 7. In a graph $G$ of order $n$

$$\nu(G) \geq \delta(G) \frac{n}{2(\delta(G) + \Delta(G))} \geq \delta(G) \frac{n}{4\Delta(G)}.$$ 

In fact, let us take a maximal matching $M$ with $m$ edges. Then for the number of edges $E$ between $M$ and $V(G) \setminus M$ we get $\delta(G)(n - 2m) \leq E \leq 2m\Delta(G)$, which proves the lemma.

3 Outline of the proof

In this paper we use the Regularity Lemma-Blow-up Lemma method again (see [6]-[12], [14]). The method is usually applied to find certain spanning subgraphs in dense graphs. Typical examples are spanning trees (Bollobás-conjecture, see [6]), Hamiltonian cycles or powers of Hamiltonian cycles (Pósa-Seymour conjecture, see [10, 11]) or $H$-factors for a fixed graph $H$ (Alon-Yuster conjecture, see [12]).
Let us consider a graph $G$ of order $n$ with
\[ \delta(G) \geq \frac{n}{2}. \] (1)
We will assume throughout the paper that $n$ is sufficiently large. We will use the following main parameters
\[ 0 < \kappa \ll \varepsilon \ll \delta \ll \alpha \ll 1, \] (2)
where $a \ll b$ means that $a$ is sufficiently small compared to $b$. For simplicity we do not compute the actual dependencies, although it could be done.

Let $d$ be an arbitrary integer with $4 \leq d \leq \kappa n/2$ and let $A$ be an arbitrary subset of the vertices of $G$ with
\[ 2 \leq |A| = k \leq \kappa n/d. \] (3)
Consider an arbitrary sequence $\mathbf{d} = \{d_i\}_{3 \leq d_i \leq d, 1 \leq i \leq k-1}$. A cycle $C$ in $G$ (or a path $P$) is called an $(A, \mathbf{d})$-cycle (or an $(A, \mathbf{d})$-path) if there is an ordering of the vertices of $A, a_1, a_2, \ldots, a_k$, such that the vertices of $A$ are visited in this order on $C$ (on $P$) and we have
\[ |\text{dist}_C(a_i, a_{i+1}) - d_i| \leq 1, \quad 1 \leq i \leq k-1. \]
We must show that there is a Hamiltonian cycle that is almost an $(A, \mathbf{d})$-cycle, namely we can have
\[ |\text{dist}_C(a_i, a_{i+1}) - d_i| > 1 \]
for only one $1 \leq i \leq k-1$.

First in the next section, in the non-extremal part of the proof, we show this assuming that the following extremal condition does not hold for our graph $G$. We show later in Section 5 that Theorem 2 is true in the extremal case as well.

**Extremal Condition (EC):** There exist (not necessarily disjoint) $A, B \subset V(G)$ such that
- $|A| = |B| = \left\lfloor \frac{n}{2} \right\rfloor$, and
- $d(A, B) < \alpha$.

In the non-extremal case we apply Lemma 2 for $G$, with $\varepsilon$ and $\delta$ as in (2). We get a partition of $V(G') = \cup_{1 \leq i \leq l} V_i$. We define the following **reduced graph $G_r$**: The vertices of $G_r$ are the clusters $V_i, 1 \leq i \leq l$, and we have an edge between two clusters if they form an $\varepsilon$-regular pair in $G'$ with density exceeding $\delta$. Since in $G'$, $\delta(G') > (\frac{1}{2} - (\delta + \varepsilon))n$, an easy calculation shows that in $G_r$ we have
\[ \delta(G_r) \geq \left( \frac{1}{2} - 3\delta \right)l. \] (4)
Indeed, because the neighbors of \( u \in V_i \) in \( G' \) can only be in \( V_0 \) and in the clusters which are neighbors of \( V_i \) in \( G_r \), then for a \( V_i, 1 \leq i \leq l \) we have:

\[
\left( \frac{1}{2} - (\delta + \varepsilon) \right) nL \leq \sum_{u \in V_i} deg_G(u) \leq \varepsilon nL + deg_{G_r}(V_i)L^2.
\]

From this we get inequality (4):

\[
deg_{G_r}(V_i) \geq \left( \frac{1}{2} - \delta - 2\varepsilon \right) \frac{n}{L} \geq \left( \frac{1}{2} - 3\delta \right) l.
\]

Applying Lemma 6 we can find a matching \( M \) in \( G_r \) of size at least \( \left( \frac{1}{2} - 3\delta \right) l \). Put \( |M| = m \). Let us put the vertices of the clusters not covered by \( M \) into the exceptional set \( V_0 \). For simplicity \( V_0 \) still denotes the resulting set. Then

\[
|V_0| \leq 6\delta l + \varepsilon n \leq 7\delta n. \tag{5}
\]

Denote the \( i \)-th pair in \( M \) by \((V'_1, V'_2)\) for \( 1 \leq i \leq m \).

The rest of the non-extremal case is organized as follows. In Section 4.1 first we find an \((A, d)\)-path \( P \). Then in Section 4.2 we find short connecting paths \( P_i \) between the consecutive edges in the matching \( M \) (for \( i = m \) the next edge is \( i = 1 \)). The first connecting path \( P_1 \) between \((V'_1, V'_2)\) and \((V'_2, V'_3)\) will also contain \( P \), the others have length exactly 3. In Section 4.3 we will take care of the exceptional vertices and make some adjustments by extending some of the connecting paths so that the distribution of the remaining vertices inside each edge in \( M \) is perfect, i.e. there are the same number of vertices left in both clusters of the edge. Finally applying Lemma 3 we close the Hamiltonian cycle in each edge and thus giving a Hamiltonian \((A, d)\)-cycle.

## 4 The non-extremal case

Throughout this section we assume that the extremal case EC does not hold.

### 4.1 Finding an \((A, d)\)-path

We are going to use the following fact several times.

**Fact 1.** If \( x, y \in V(G) \) then there are at least \( \delta n \) internally disjoint paths of length 3 connecting \( x \) and \( y \).
Indeed, if we choose \( A \subset N_G(x) \) with \( |A| = \lfloor \frac{n}{2} \rfloor \) and \( B \subset N_G(y) \) with \( |B| = \lfloor \frac{n}{2} \rfloor \), then the fact that EC does not hold implies \( d(A, B) \geq \alpha \) and Fact 1 follows.

We construct an \((A, \mathcal{A})\)-path \( P = Q_1 \cup \ldots \cup Q_k \) in the following way. Let \( a_1, \ldots, a_k \) be the vertices of \( A \) in an arbitrary order (so note that here actually we can prescribe the order of the vertices of \( A \) as well). First we construct a path \( Q_1 \) of length \( d_1 \) connecting \( a_1 \) and \( a_2 \). For this purpose first we construct greedily a path \( Q_1' \) starting from \( a_1 \) that has length \( d_1 - 3 \) ((1) makes this possible). Denote the other endpoint of \( Q_1' \) by \( a_1' \). Applying Fact 1, we connect \( a_1' \) and \( a_2 \) by a path \( Q_1'' \) of length 3 that is internally disjoint from \( Q_1' \). Then \( Q_1 = Q_1' \cup Q_1'' \) is a path connecting \( a_1 \) and \( a_2 \) with length \( d_1 \).

We iterate this procedure. For the construction of \( Q_2 \), first we greedily construct a path \( Q_2' \) starting from \( a_2 \) that is internally disjoint from \( Q_1 \) and has length \( d_2 - 3 \). Denote the other endpoint of \( Q_2' \) by \( a_2' \). Applying Fact 1, we connect \( a_2' \) and \( a_3 \) by a path \( Q_2'' \) of length 3 that is internally disjoint from \( Q_1 \cup Q_2' \). Then \( Q_2 = Q_2' \cup Q_2'' \) is a path connecting \( a_2 \) and \( a_3 \) with length \( d_2 \).

By iterating this procedure we get an \((A, \mathcal{A})\)-path \( P \). (1), (2), (3) and Fact 1 imply that we never get stuck since

\[
|V(P)| = \sum_{i=1}^{k-1} d_i \leq (k - 1)d \leq \kappa n \ll \delta n.
\]

Observe that here in the non-extremal case there is no discrepancy between \( \text{dist}(a_i, a_{i+1}) \) and \( d_i \) for all \( 1 \leq i \leq k - 1 \), and furthermore we can specify the order of the vertices of \( A \) as well.

### 4.2 Connecting paths

For the first connecting path \( P_1 \) between \((V_1^1, V_2^1)\) and \((V_1^2, V_2^2)\), first we connect a typical vertex \( u \) of \( V_2^1 \) (more precisely a vertex \( u \) with \( \deg(u, V_1^1) \geq (\delta - \varepsilon)|V_1^1| \), most vertices in \( V_2^1 \) satisfy this) and \( a_1 \) with a path of length 3, and then we connect \( a_k \) and a typical vertex \( w \) of \( V_1^2 \) (so \( \deg(w, V_2^2) \geq (\delta - \varepsilon)|V_2^2| \)) with a path of length 3. To construct the second connecting path \( P_2 \) between \((V_1^2, V_2^2)\) and \((V_1^3, V_2^3)\) we just connect a typical vertex of \( V_2^2 \) and a typical vertex \( V_1^3 \) with a path of length 3. Continuing in this fashion, finally we connect a typical vertex of \( V_2^m \) with a typical vertex of \( V_1^1 \) with a path of length 3. Thus \( P_1 \) has length at most \( \kappa n + 6 \), all other \( P_i \)s have length 3.

We remove the vertices on these connecting paths from the clusters, but for simplicity we keep the notation for the resulting clusters. These connecting paths will be parts of the final Hamiltonian cycle. If the number of remaining vertices (in the clusters and in \( V_0 \)) is odd, then we take another typical vertex \( w \) of \( V_1^2 \) and we extend
$P_1$ by a path of length 3 that ends with $w$. So we may always assume that the number of remaining vertices is even.

4.3 Adjustments and the handling of the exceptional vertices

We already have an exceptional set $V_0$ of vertices in $G$. We add some more vertices to $V_0$ to achieve super-regularity. From $V^i_1$ (and similarly from $V^i_2$) we remove all vertices $u$ for which $deg(u, V^i_1) < (\delta - \varepsilon)|V^i_1|$, $\varepsilon$-regularity guarantees that at most $\varepsilon|V^i_1| \leq \varepsilon L$ such vertices exist in each cluster $V^i_1$.

Thus using (5), we still have

$$|V_0| \leq 7\delta n + 2\varepsilon n \leq 9\delta n.$$ 

Since we are looking for a Hamiltonian cycle, we have to include the vertices of $V_0$ on the Hamiltonian cycle as well. We are going to extend some of the connecting paths $P_i$, so now they are going to contain the vertices of $V_0$. Let us consider the first vertex (in an arbitrary ordering of the vertices in $V_0$) $w$ in $V_0$. We find a pair $(V^i_1, V^i_2)$ such that either

$$deg(w, V^i_1) \geq \delta |V^i_1|,$$

or

$$deg(w, V^i_2) \geq \delta |V^i_2|.$$ 

We assign $w$ to the pair $(V^i_1, V^i_2)$. We extend $P_{i-1}$ (for $i = 1, P_m$) in $(V^i_1, V^i_2)$ by a path of length 3 in case (6) holds, and by a path of length 4 in case (7) holds, so that now the path ends with $w$. To finish the procedure for $w$, in case (6) holds we add one more vertex $w'$ to $P_{i-1}$ after $w$ such that $(w, w') \in E(G)$ and $w'$ is a typical vertex of $V^i_1$, so $deg(w', V^i_1) \geq (\delta - \varepsilon)|V^i_1|$. In case (7) holds we add two more vertices $w', w''$ to $P_{i-1}$ after $w$ such that $(w, w'), (w', w'') \in E(G)$, $w'$ is a typical vertex of $V^i_2$ and $w''$ is a typical vertex of $V^i_1$.

After handling $w$, we repeat the same procedure for the other vertices in $V_0$. However, we have to pay attention to several technical details. First, of course in repeating this procedure we always consider the remaining free vertices in each cluster; the vertices on the connecting paths are always removed. Second, we make sure that we never assign too many vertices of $V_0$ to one pair $(V^i_1, V^i_2)$. It is not hard to see (using (1) and $\delta \ll 1$) that we can guarantee that we always assign at most $\sqrt{\delta |V^i_1|}$ vertices of $V_0$ to a pair $(V^i_1, V^i_2)$. Finally, since we are removing vertices from a pair $(V^i_1, V^i_2)$, we might violate the super-regularity. Note that we never violate the $\varepsilon$-regularity. Therefore, we do the following. After handling (say) $[\delta^2 n]$ vertices from $V_0$, we update $V_0$ as follows. In a pair $(V^i_1, V^i_2)$ we remove all vertices $u$ from $V^i_1$ (and similarly from $V^i_2$) for which $deg(u, V^i_2) < (\delta - \varepsilon)|V^i_2|$ (again, we consider only the
remaining vertices). Again, we added at most $2\varepsilon n$ vertices to $V_0$. In $V_0$ we handle these vertices first and then we move on to the other vertices in $V_0$.

After we are done with all the vertices in $V_0$, we might have a small discrepancy ($\leq 2\sqrt{\varepsilon} V_i'$) among the remaining vertices in $V_i^i$ and in $V_i^j$ in a pair. Therefore, we have to make some adjustments. Let us take a pair $(V_1^i, V_2^j)$ with a discrepancy $\geq 2$ (if one such pair exists), say $V_1^i \geq |V_2^j| + 2$ (only remaining vertices are considered). Using the fact that EC does not hold we find an alternating path (with respect to $M$) in $G_r$ of length 6 starting with $V_1^i$ and ending with $V_2^j$. Let us denote this path by

$$V_1^i, V_1^{i_1}, V_1^{i_2}, V_1^{i_3}, V_1^{i_4}, V_1^{i_5}, V_1^i$$

(the construction is similar if the clusters in $(V_1^{i_1}, V_2^{i_1})$ or in $(V_1^{i_2}, V_2^{i_2})$ are visited in different order). We remove a typical vertex from $V_1^i$ and we add it to $V_1^{i_1}$, then we remove a typical vertex from $V_1^{i_1}$ and we add it to $V_1^{i_2}$, finally we remove a typical vertex from $V_2^{i_2}$ and we add it to $V_2^i$. When we add a new vertex to a pair $(V_1^i, V_2^j)$, we extend the connecting path $P_{j-1}$ by a path of length 4 in the pair so that it now includes the new vertex.

Now we are one step closer to the perfect distribution, and by iterating this procedure we can assure that the discrepancy in every pair is $\leq 1$. We consider only those pairs for which the discrepancy is exactly 1, so in particular the number of remaining vertices in one such a pair is odd. From the construction it follows that we have an even number of such pairs. We pair up these pairs arbitrarily. If $(V_1^i, V_2^j)$ and $(V_1^j, V_2^i)$ is one such pair with $|V_1^i| = |V_2^j| + 1$ and $|V_1^j| = |V_2^i| + 1$ (otherwise similar), then similar to the construction above, we find an alternating path in $G_r$ of length 6 between $V_1^i$ and $V_2^i$, and we move a typical vertex of $V_1^i$ through the intermediate clusters to $V_2^j$.

Thus we may assume that the distribution is perfect, in every pair $(V_1^i, V_2^j)$ we have the same number of vertices left. In this case Lemma 3 closes the Hamiltonian cycle in every pair.

5 The extremal case

First we assume that we have the following special case.

**Case 1:** There is a partition $V(G) = A_1 \cup A_2$ with $|A_1| = \lfloor \frac{n}{2} \rfloor$ and $d(A_1) < \alpha^{1/3}$.

Note that in this case from (1) we also have $d(A_1, A_2) > 1 - \alpha^{1/3}$. Thus, roughly speaking in this case we have very few edges in $G\mid_{A_1}$, and we have an almost complete bipartite graph between $A_1$ and $A_2$.

A vertex $v \in A_i, i \in \{1, 2\}$, is called **exceptional** if it is not connected to most of
the vertices in the other set, more precisely if we have

\[ \deg(v, A_v) \leq \left(1 - \alpha^{1/6}\right) |A_v|, \{i, i'\} = \{1, 2\}. \]

Note that (1) implies that if \( v \in A_i \) is exceptional, then

\[ \deg(v, A_i) \geq \alpha^{1/6}|A_i|. \]

But then since \( d(A_1, A_2) > 1 - \alpha^{1/3} \), we get that the number of exceptional vertices
in \( A_i \) is at most \( \alpha^{1/3}|A_i| \). We remove the exceptional vertices from each set and add
them to \( A_2 \) if they have more neighbors in \( A_1 \), and add them to \( A_1 \) if they have
more neighbors in \( A_2 \). We still denote the resulting sets by \( A_1 \) and \( A_2 \). Assume that
\( |A_1| \leq |A_2| \), so \( |A_2| - |A_1| = r \), where \( 0 \leq r \leq 2\alpha^{1/6}|A_2| \). In \( G|_{A_1 \times A_2} \) apart from at
most \( 2\alpha^{1/6}|A_2| \) exceptional vertices all the degrees are at least \( (1 - 3\alpha^{1/6})|A_2| \), and
the degrees of the exceptional vertices are at least \( |A_2|/3 \).

Our goal is to achieve \( r = 0 \). If there is a vertex \( x \in A_2 \) for which

\[ \deg(x, A_2) \geq \alpha^{1/7}|A_2|, \tag{8} \]

then we remove \( x \) from \( A_2 \) and add it to \( A_1 \). We iterate this procedure until either
there are no more vertices in \( A_2 \) satisfying (8) or \( |A_1| = |A_2| \). Assume that we have
the first case. Since we have \( \Delta(G|_{A_2}) < \alpha^{1/7}|A_2| \), (1) and Lemma 7 imply that \( G|_{A_2} \)
has an \( r \)-matching \( M \) denoted by \( \{u_1, v_1\}, \ldots, \{u_r, v_r\} \). Furthermore, for every edge
in \( M \) we can guarantee that at least one of the endpoints (say \( u_i \)) is not in \( A \). This
matching \( M \) will be used to balance the discrepancy between \( |A_1| \) and \( |A_2| \).

Note that in \( G|_{A_1 \times A_2} \) the degrees of the exceptional vertices are still much more
than the number of these exceptional vertices. These degree conditions and (2) imply
the following fact (similar to Fact 1).

**Fact 2.** If \( x, y \in A_1 \) then in \( G|_{A_1 \times A_2} \) there are at least \( \delta n \) internally disjoint paths
of length 4 connecting \( x \) and \( y \). If \( x, y \in A_2 \) then in \( G|_{A_1 \times A_2} \) there are at least \( \delta n \)
internally disjoint paths of length 2 connecting \( x \) and \( y \). If \( x \in A_i, y \in A_r \) then in
\( G|_{A_1 \times A_2} \) there are at least \( \delta n \) internally disjoint paths of length 3 connecting \( x \) and \( y \).

Let \( A \) be an arbitrary subset of the vertices of \( G \) satisfying (3). In this case we
construct the desired Hamiltonian cycle in the following way. First by using Fact 2
and a similar procedure as in Section 4.1 we find in \( G|_{A_1 \times A_2} \) an \( (A, d) \)-path

\[ P = P(a_1, a_k) = Q_1 \cup \ldots \cup Q_k \]

connecting the vertices \( a_1 \) and \( a_k \). The only difference from Section 4.1 is that here
because of parity reasons we might have \( \text{dist}_G(a_i, a_{i+1}) = d_i + 1 \). Indeed, first we
construct a path $Q_1$ of length $d_1$ or $d_1 + 1$ connecting $a_1$ and $a_2$. If $a_1$ is covered by an edge of $M$, say $a_1 = v_i$, then we start $Q_1$ with the edge $\{v_i, u_i\}$ (note that $u_i \notin A$). If $d_1 = 3$, then to get $Q_1$ we connect $u_i$ and $a_2$ in $G|_{A_1 \times A_2}$ by a path of length 2 in case $a_2 \in A_2$, and by a path of length 3 in case $a_2 \in A_1$. If $d_1 > 3$, then we greedily construct a path $Q'_1$ that has length $d_1 - 3$, starts with the edge $\{v_i, u_i\}$ and continues in $G|_{A_1 \times A_2}$. Denote the other endpoint of $Q'_1$ by $a'_1$. Applying Fact 2, we connect $a'_1$ and $a_2$ by a path $Q''_1$ of length 3 in case they are in different sets, and by a path of length 4 in case they are in the same set. Then $Q_1 = Q'_1 \cup Q''_1$ is a path connecting $a_1$ and $a_2$ with length $d_1$ or $d_1 + 1$.

We iterate this procedure; we construct $Q_2, \ldots, Q_k$ similarly and thus we get $P = Q_1 \cup \ldots \cup Q_k$. Say the remaining edges of $M$ which are not traversed by $P$ are

$$\{u_{i_1}, v_{i_1}\}, \ldots, \{u_{i_{r'}}, v_{i_{r'}}\} \quad \text{for} \quad 0 \leq r' \leq r.$$

Then we connect the endpoint $a_k$ of $P$ and $u_{i_1}$ by a path $Q_1$ of length 2 or 3, connect $v_{i_1}$ and $u_{i_2}$ by a path $Q_2$ of length 2, etc. Finally connect $v_{i_{r'-1}}$ and $u_{i_r}$ by a path $Q_r'$ of length 2. Consider the following path.

$$P' = (P, Q_1, \{u_{i_1}, v_{i_1}\}, Q_2, \{u_{i_2}, v_{i_2}\}, \ldots, Q_{r'}, \{u_{i_{r'}}, v_{i_{r'}}\}).$$

In case $a_1 \in A_2$, add one more vertex from $A_1$ to the end of the path. Remove $P'$ from $G|_{A_1 \times A_2}$ apart from the endvertices $a_1$ and $v_{i_r}$. From (2), (3) and the degree conditions we get that the resulting graph satisfies the conditions of Lemma 5 and thus it is Hamiltonian-connected. This closes the desired Hamiltonian cycle. For this purpose we could also use Lemma 3, the remaining bipartite graph is super-regular with the appropriate choice of parameters, but here the much simpler Lemma 5 also suffices. Note also that here we have no exceptional $i$, so we have

$$|\text{dist}_C(a_i, a_{i+1}) - d_i| \leq 1 \quad \text{for all} \quad 1 \leq i \leq k - 1.$$

**Case 2:** Assume next that we have a partition $V(G) = A_1 \cup A_2$ with $|A_1| = \lfloor \frac{n}{2} \rfloor$ and $d(A_1, A_2) < \alpha^{1/3}$. Thus roughly speaking, $G|_{A_1}$ and $G|_{A_2}$ are almost complete and the bipartite graph between $A_1$ and $A_2$ is sparse.

Again we define exceptional vertices $v \in A_i, i \in \{1, 2\}$, as

$$\deg(v, A_{i'}) \geq \alpha^{1/6}|A_{i'}|, \quad \{i, i'\} = \{1, 2\}.$$

Note that again the number of exceptional vertices in $A_i$ is at most $\alpha^{1/6}|A_i|$. We remove the exceptional vertices from each set and add them to the set where they have more neighbors. We still denote the sets by $A_1$ and $A_2$. Thus in $G|_{A_i}$, $i \in \{1, 2\}$, apart from at most $2\alpha^{1/6}|A_i|$ exceptional vertices all the degrees are at least $(1 - 2\alpha^{1/6})|A_i|$, and the degrees of the exceptional vertices are at least $|A_i|/3$. These degree conditions and (2) imply the following fact (similar to Facts 1 and 2).
Fact 3. If $x, y \in A_i$ then in $G|_{A_i}$ there are at least $d_i$ internally disjoint paths of length \(3\) connecting $x$ and $y$. Furthermore, if at least one of the vertices $x$ and $y$ is non-exceptional then there are at least $d_i$ internally disjoint paths of length $2$ connecting $x$ and $y$.

Assume that $|A_1| \leq |A_2|$. Let $A$ be an arbitrary subset of the vertices of $G$ satisfying (3). Put
\[
A' = A \cap A_1, A'' = A \cap A_2, k' = |A'|, k'' = |A''|,
\]
\[
\mathcal{d}' = \{d_i \mid 1 \leq i \leq k' - 1\} \quad \text{and} \quad \mathcal{d}'' = \{d_i \mid k' + 1 \leq i \leq k - 1\}.
\]
We show that we can find two vertex disjoint edges (called bridges) \(\{u_1, v_1\}, \{u_2, v_2\}\) in $G|_{A_1 \times A_2}$ such that for both of these bridges at least one of the endpoints (say $u_i$) is non-exceptional and it is not in $A$. This is trivial if $|A_1| < |A_2|$, since then for every $u \in A_1$ we have $deg(u, A_2) \geq 2$. Thus we may assume that $|A_1| = |A_2|$. But then for every $u \in A_1$ we have $deg(u, A_2) \geq 1$ and for every $v \in A_2$ we have $deg(v, A_1) \geq 1$, and thus again we can pick the two bridges.

We distinguish two subcases.

Subcase 2.1: $u_1$ and $u_2$ are in different sets, say $u_1 \in A_1 \setminus A'$ and $u_2 \in A_2 \setminus A''$. Here we construct the desired Hamiltonian cycle in the following way. First by using Fact 3 and a similar procedure as in Section 4.1 we find in $G|_{A_1}$ an $(A', \mathcal{d}')$-path $P' = P'(a_1, v_2)$ with endpoints $a_1 \in A$ and $v_2$ (if $v_2 \in A'$ then this is just the last vertex $v_2 = a_{k'}$ from $A$ on the path, otherwise we connect the last vertex $a_{k'}$ and $v_2$ by a path of length 3). Similarly we find in $G|_{A_2}$ an $(A'', \mathcal{d}'')$-path $P'' = P''(a_{k'+1}, v_1)$ with endpoints $a_{k'+1} \in A$ and $v_1$. Then in $G|_{A_1}$ we remove the path $P'$ apart from the endvertex $a_1$. From (2), (3) and the degree conditions we get that the resulting graph satisfies the conditions of Lemma 4 and thus it is Hamiltonian-connected. Take a Hamiltonian path $P_1 = P_1(u_1, a_1)$ with endpoints $u_1$ and $a_1$. Similarly in $G|_{A_2}$ we remove the path $P''$ apart from the endvertex $a_{k'+1}$ and we find a Hamiltonian path $P_2 = P_2(u_2, a_{k'+1})$ with endpoints $u_2$ and $a_{k'+1}$. Then in this case the desired Hamiltonian cycle $C$ is the following.

\[
C = (P', \{v_2, u_2\}, P_2, P'', \{v_1, u_1\}, P_1).
\]
Note that here actually in $C$ we have
\[
dist_C(a_i, a_{i+1}) = d_i \quad \text{for all} \quad 1 \leq i \leq k' - 1 \quad \text{and} \quad k' + 1 \leq i \leq k - 1.
\]
However, $dist_C(a_{k'}, a_{k'+1})$ could be very different from $d_{k'}$.

Subcase 2.2: $u_1$ and $u_2$ are in the same set (say $A_1$). Here we do the following. We may assume that $v_1, v_2 \in A''$, since otherwise we are back to Subcase 2.1. We
denote \( v_2 \) by \( a_{k'+1} \) and \( v_1 \) by \( a_k \). First we find in \( G_{A_1} \) again an \((A', \vec{d}')\)-path \( P'' = P'(a_1, a_{k'}) \) with endpoints \( a_1 \) and \( a_{k'} \). We connect \( a_{k'} \) and \( u_2 \) with a path \( Q = Q(a_{k'}, u_2) \) of length \( d_{k'} - 1 \) that is internally disjoint from \( P'' \) and \( u_1 \). The degree conditions guarantee that this is possible (even if \( d_{k'} = 3 \), since \( u_2 \) is non-exceptional). Then we remove \( P'' \) and \( Q \) from \( G_{A_2} \) apart from the endvertex \( a_1 \) and we find a Hamiltonian path \( P_1 = P_1(u_1, a_1) \) with endpoints \( u_1 \) and \( a_1 \). Define

\[
A''' = A'' \setminus \{a_k\} \quad \text{and} \quad d''' = \{d_i \mid k' + 1 \leq i \leq k - 2\} = \vec{d}'' \setminus \{d_{k-1}\}.
\]

We find in \( G_{A_2} \) an \((A''', \vec{d}''')\)-path \( P''' = P'''(a_{k'+1}, a_{k-1}) \) with endpoints \( a_{k'+1} \) and \( a_{k-1} \). We remove \( P''' \) from \( G_{A_2} \) apart from the endvertex \( a_{k-1} \) and we find a Hamiltonian path \( P_2 = P_2(a_{k-1}, v_1) \) with endpoints \( a_{k-1} \) and \( v_1 = a_k \). Then in this case the Hamiltonian cycle \( C \) is the following.

\[
C = (P', Q, \{u_2, v_2\}, P'', P_2, \{v_1, u_1\}, P_1).
\]

Note that here actually in \( C \) we have

\[
dist_C(a_i, a_{i+1}) = d_i \quad \text{for all} \quad 1 \leq i \leq k - 2,
\]

but \( dist_C(a_{k-1}, a_k) \) could be very different from \( d_{k-1} \).

**Case 3:** Assume finally that the extremal case EC holds, so we have \( A, B \subset V(G), |A| = |B| = \lfloor \frac{n}{2} \rfloor \) and \( d(A, B) < \alpha \). We have three possibilities.

- \( |A \cap B| < \sqrt{\alpha n} \). The statement follows from Case 2. Indeed, let \( A_1 = A, A_2 = V(G) \setminus A_1 \), then clearly \( d(A_1, A_2) < \alpha^{1/3} \) if \( \alpha \ll 1 \) holds.

- \( \sqrt{\alpha n} \leq |A \cap B| < (1 - \sqrt{\alpha}) \frac{n}{2} \). This case is not possible under the given conditions. In fact, otherwise we would get

\[
|A \cap B| \frac{n}{2} \leq \sum_{u \in A \cap B} deg_G(u) = \sum_{u \in A \cap B} deg_G(u, A \cup B) + \sum_{u \in A \cap B} deg_G(u, V(G) \setminus (A \cup B)) \leq 2\alpha n^2 + |A \cap B| (|A \cap B| + 1),
\]

or

\[
|A \cap B| \left( \frac{n}{2} - |A \cap B| - 1 \right) \leq 2\alpha n^2,
\]

a contradiction under the given conditions.

- \( |A \cap B| \geq (1 - \sqrt{\alpha}) \frac{n}{2} \). The statement follows from Case 1 by choosing \( A_1 = A, A_2 = V(G) \setminus A_1 \), and then \( d(A_1) < \alpha^{1/3} \).

This finishes the extremal case and the proof of Theorem 2.
References


