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On a Turán-type hypergraph problem of Brown, Erdős and T. Sós

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February 11, 2003

Note

Abstract

We let $G^{(r)}(n, m)$ denote the set of $r$-uniform hypergraphs with $n$ vertices and $m$ edges, and $f^{(r)}(n, p, s)$ is the smallest $m$ such that every member of $G^{(r)}(n, m)$ contains a member of $G^{(r)}(p, s)$. In this paper we are interested in the growth of $f^{(r)}(n, p, s)$ for fixed values $r, p$ and $s$. Brown, Erdős and T. Sós ([2]) proved that for $r > k \geq 2$ and $s \geq 3$ we have $f^{(r)}(n, s(r - k) + k + 1, s) = \Theta(n^k)$. This suggests the difficult question whether $f^{(r)}(n, s(r - k) + k + 1, s) = o(n^k)$. This was first established for $r = s = 3$ and $k = 2$ by Ruzsa and Szemerédi ([11]). Then for $s = 3$ and $k = 2$ Erdős, Frankl and Rödl ([6]) extended this result for any $r$, and they conjectured that it also holds for $k = 2$ and any $s$. In this note we show that

$$f^{(r)}(n, s(r - k) + k + \lfloor \log_2 s \rfloor, s) = o(n^k)$$

for all $k \geq 2$.

In addition we show that

$$f^{(r)}(n, 4(r - 3) + 4, s) = o(n^3).$$

1 Introduction

1.1 Notation and definitions

For basic graph and hypergraph concepts see the monograph of Bollobás [1].

A hypergraph $\mathcal{F}$ is called $r$-uniform if $|F| = r$ for every edge $F \in \mathcal{F}$. An $r$-uniform hypergraph $\mathcal{F}$ on the set $X$ is $r$-partite if there exists a partition $X = X_1 \cup \ldots \cup X_r$ with $|F \cap X_i| = 1$ for every edge $F \in \mathcal{F}$ and $1 \leq i \leq r$. $|\mathcal{F}|$ denotes the number of edges of $\mathcal{F}$. In this paper $\log n$ denotes the base 2 logarithm.
1.2 Turán-type hypergraph problems

We let $G^{(r)}(n, m)$ denote the set of $r$-uniform hypergraphs with $n$ vertices and $m$ edges, and $f^{(r)}(n, p, s)$ is the smallest $m$ such that every member of $G^{(r)}(n, m)$ contains a member of $G^{(r)}(p, s)$. The determination of $f^{(r)}(n, p, s)$ has been a longstanding open problem. Special cases of this problem appeared in [3], [5]. For more about Turán-type hypergraph results consult the surveys by Füredi [9] and Sidorenko [13]. In this note we are interested in the growth of $f^{(r)}(n, p, s)$ for fixed values $r, p$ and $s$.

Brown, Erdős and T. Sós ([2]) proved that for $r > k \geq 2$ and $s \geq 3$ we have

$$f^{(r)}(n, s(r - k) + k, s) = \Theta(n^k).$$

This suggests the following difficult question.

**Conjecture 1.**

$$f^{(r)}(n, s(r - k) + k + 1, s) = o(n^k).$$

This was first established for $r = s = 3$ and $k = 2$ by the celebrated result of Ruzsa and Szemerédi ([11]). Then for $s = 3$ and $k = 2$ Erdős, Frankl and Rödl ([6]) extended this result for any $r$, and they conjectured that it also holds for $k = 2$ and any $s$. In this direction in [12] we showed that

$$f^{(r)}(n, s(r - 2) + 2 + \lfloor \log s \rfloor, s) = o(n^2).$$

In this note we extend this result for $k > 2$, showing that Conjecture 1 is not far from being true.

**Theorem 1.** For all integers $r > k \geq 2$ and $s \geq 3$,

$$f^{(r)}(n, s(r - k) + k + \lfloor \log s \rfloor, s) = o(n^k).$$

Thus roughly speaking Conjecture 1 is true apart from a $\lfloor \log s \rfloor$ term. However, it still remains open whether one can replace this term with $1$ and prove Conjecture 1.

In addition, by using a recent, deep result of Frankl and Rödl ([8]) we show that Conjecture 1 is true for $k = 3$ and $s = 4$.

**Theorem 2.** For all integers $r \geq 4$,

$$f^{(r)}(n, 4(r - 3) + 4, 4) = o(n^3).$$

In the next section we provide the tools, then we prove the theorems.
2 Tools

We will use a simple but useful result of Erdős and Kleitman ([7], see also on page 1300 in [10]).

**Lemma 1.** Every $k$-uniform hypergraph $\mathcal{F}$ contains a $k$-partite $k$-uniform hypergraph $\mathcal{H}$ with

$$\frac{|\mathcal{H}|}{|\mathcal{F}|} \geq \frac{k!}{k^k}.$$

We will also need a recent result of Frankl and Rödl. Following their notation from [8], let $A_i = \{a_i, b_i\}$ be pairwise disjoint 2-element sets for $1 \leq i \leq k$. Define $F_i = \{a_1, \ldots, a_k, b_i\} \setminus \{a_i\}$ and $\mathcal{F}(k) = \{F_1, \ldots, F_k\}$. Let $ex^*(n, \mathcal{F}(k))$ denote max $|\mathcal{H}|$ such that $\mathcal{H}$ is a $k$-partite hypergraph on $n$ vertices that is $\mathcal{F}(k)$-free, and $|H \cap H'| \leq k - 2$ holds for all distinct $H, H' \in \mathcal{H}$. In [8] the following deep result is shown.

**Lemma 2.**

$ex^*(n, \mathcal{F}(4)) = o(n^3)$.

3 Proof of Theorem 1

Let $r > k \geq 2$, $s \geq 3$, $p = s(r - k) + k + \lfloor \log s \rfloor$ and $l = \lfloor \log s \rfloor$. For $k = 2$ we showed that the theorem is true in [12]; thus we may assume $k > 2$.

Assume indirectly that there is a constant $c > 0$ such that

$$f^{(r)}(n, p, s) > \left\lfloor cn^k \right\rfloor. \quad (1)$$

From this assumption we will get a contradiction. (1) means that there exists an $r$-uniform hypergraph $\mathcal{F}$ with

$$f^{(r)}(n, p, s) - 1 \geq \left\lfloor cn^k \right\rfloor \geq cn^k$$

edges that does not contain a member of $G^{(r)}(p, s)$, i.e. a set of $p$ vertices spanning at least $s$ edges. Let us assume that $n$ is sufficiently large.

Using the Erdős-Kleitman theorem (Lemma 1) we find an $r$-partite subhypergraph $\mathcal{H}$ of $\mathcal{F}$ with at least

$$\frac{r!c}{r^r n^k}$$

edges. Let $X_1, \ldots, X_r$ be the vertex classes of this $r$-partite hypergraph $\mathcal{H}$. Consider the $(k + 1)$-uniform hypergraph $\mathcal{H}^*$ which is defined by the removal of $X_1, \ldots, X_{r-k-1}$ from the vertex set of $\mathcal{H}$ and from all edges of $\mathcal{H}$. If a $(k + 1)$-edge of $\mathcal{H}^*$ has multiplicity greater than 1, then we keep only one edge. Note that every $(k+1)$-edge has multiplicity less than $s$. Indeed, otherwise taking a $(k + 1)$-edge with multiplicity at least $s$ and $s$ $r$-edges of $\mathcal{H}$ containing this edge, we get a set of at most

$$s(r - k - 1) + k + 1 \leq s(r - k) + k + \lfloor \log s \rfloor = p$$

edges.
vertices that span at least \( s \) \( r \)-edges, a contradiction. Then if in \( \mathcal{H}^* \) we keep only one edge from each multiple \((k + 1)\)-edge we still have at least
\[
\frac{r!c}{r^s} n^k
\]
edges.

Define for every \( x_1 \in X_{r-k}, x_2 \in X_{r-k+1}, \ldots, x_{k-2} \in X_{r-3} \) the following hypergraph:
\[
\mathcal{H}^*(x_1, \ldots, x_{k-2}) = \{ G \setminus \{ x_1, \ldots, x_{k-2} \} | \{ x_1, \ldots, x_{k-2} \} \subset G \in \mathcal{H}^* \}.
\]
There are \( x_1, \ldots, x_{k-2} \) for which we have
\[
|\mathcal{H}^*(x_1, \ldots, x_{k-2})| \geq \frac{r!c}{r^s} n^2.
\]
By Theorem 1 for \( k = 2 \) ([12]), we have a \( G^{(3)}(s + 2 + \lfloor \log s \rfloor, s) \) in this 3-uniform \( \mathcal{H}^*(x_1, \ldots, x_{k-2}) \). Then in the original \( \mathcal{H} \) we have a set of at most
\[
s(r - (k + 1)) + (k - 2) + s + 2 + \lfloor \log s \rfloor = s(r - k) + k + \lfloor \log s \rfloor = p
\]
vertices spanning at least \( s \) \( r \)-edges, a contradiction.

This completes the proof of Theorem 1. \( \square \)

4 Proof of Theorem 2

Let \( r \geq 4 \) and \( p = 4(r - 3) + 4 \).

Proceeding similarly as above, assume indirectly that there is a constant \( c > 0 \) such that
\[
f^{(r)}(n, p, 4) > \lceil cn^3 \rceil. \tag{2}
\]
From this assumption we will get a contradiction. (2) means that there exists an \( r \)-uniform hypergraph \( \mathcal{F} \) with
\[
f^{(r)}(n, p, 4) - 1 \geq \lceil cn^3 \rceil \geq cn^3
\]
edges that does not contain a member of \( G^{(r)}(p, 4) \), i.e. a set of \( p \) vertices spanning at least 4 edges. Let us assume that \( n \) is sufficiently large.

Similarly as above, first by using Lemma 1 we find an \( r \)-partite subhypergraph \( \mathcal{H} \) of \( \mathcal{F} \) with at least
\[
\frac{r!c}{r^r} n^3
\]
edges and with partite sets \( X_1, \ldots, X_r \). Then we reduce \( \mathcal{H} \) to \( \{ X_{r-3}, X_{r-2}, X_{r-1}, X_r \} \) to get \( \mathcal{H}^* \) with at least
\[
\frac{r!c}{r^r} n^3
\]
4-edges.
Now consider an arbitrary 4-edge \( H \in \mathcal{H}^* \), and \( H_1 \subseteq H \) with \( |H_1| = 3 \). There can be at most 3 \( H' \in \mathcal{H}^* \) edges with \( H \cap H' = H_1 \), since otherwise we get a set of at most

\[
4(r - 4) + 7 = 4(r - 3) + 3 < p
\]

vertices spanning at least 4 \( r \)-edges, a contradiction.

Since we can choose \( H_1 \) in 4 different ways, altogether there can be at most 12 \( H' \in \mathcal{H}^* \) edges with \( |H \cap H'| = 3 \). We remove all these at most 12 \( H' \) edges from \( \mathcal{H}^* \). In the remaining hypergraph again we consider an arbitrary 4-edge \( H \) and we remove all other edges \( H' \) for which \( |H \cap H'| = 3 \). We continue in this fashion until we have no two 4-edges \( H \) and \( H' \) with \( |H \cap H'| = 3 \). Denote the resulting hypergraph by \( \mathcal{H}^{**} \), then

\[
|\mathcal{H}^{**}| \geq \frac{r!c}{13^r s^3} n^3. \tag{3}
\]

Furthermore, \( \mathcal{H}^{**} \) is \( \mathcal{F}(4) \)-free, since otherwise we get a set of at most

\[
4(r - 4) + 8 = 4(r - 3) + 4 = p
\]

vertices spanning at least 4 \( r \)-edges, a contradiction.

However, then (3) is in contradiction with Lemma 2.

This completes the proof of Theorem 2. \( \square \)

References


